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Statistical Characteristics of the Damped Vibrations of a String Excited by Stochastic Forces

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Our theoretical study aims at finding some statistic parameters characterizing the damped vibrations of a string excited by stochastic impulses. We derive the dependence of these parameters on the parameters of the string as well as on the stochastic distributions of the impulse magnitude, and on the place of the action. We also carry out a numerical simulation verifying the derived mathematical model and interpret the differences between the results obtained in simulation and the mathematical calculations.

This study is the fourth stage of a research aimed at designing a probe that facilitates the process of measuring of the parameters, determining the quality of a technological process.

Keywords: string, stochastic impulses, statistic parameters characterizing the vibrations.

1. Introduction

The work was inspired by attempts at constructing a measuring device that would control the granularity of the medium in a dust pipeline. The device had to signal the appearance of big particles in excessive quantity in the transported dust. The difficulties that arose then in connection with interpretation of the statistical data, forced us to search for a mathematical model that would explain its causes.

The developed model described below is based on the theorem proved in [3]. This theorem allows for calculation of the basic statistical parameters such as mathematical expectation and variance, that characterize the movement of continuous and discrete systems stimulated by stochastic impulses. In this theorem we assume that the probability that an impulse will occur in a short time interval is proportional to its duration and the moments of impulse occurrence; their magnitude and places of occurrence are probabilistically independent. These assumptions seem to be quite natural in regard to the actual working conditions of the above-mentioned measuring device.

This study is the fourth stage of the research aimed at designing a probe that facilitates measuring of the parameters determining the quality of a technological process.

At the first stage of the research, the Theorem 1 was proved in [3]. In the same paper it was also manifested that the graph of the mean value of the amplitude of an oscillator, excited by stochastic impulses without damping, is identical with the graph of the free movement of an oscillator satisfying certain initial conditions (Fig. 1). The variance of the random variable representing the amplitude of an oscillator is shown in Fig. 2.

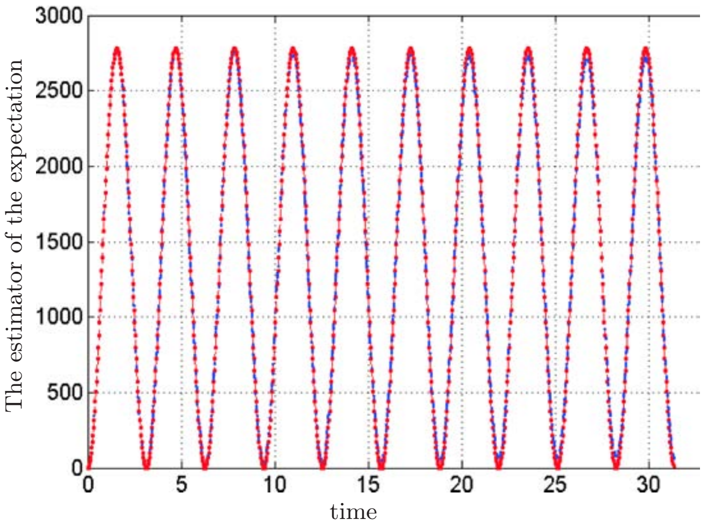

Fig. 1. Estimators of the expectation for the oscillators without damping. Key: dashed red – the theoretical expectation, dashed blue – the estimator of the expectation, 10 000 elements in a statistical sample.

At the second stage of the research, the behavior of an oscillator with damping excited by stochastic impulses with the same distribution as that in [3] was discussed in [4]. The graph of the mean value of the movement (Fig. 3) of the oscillator is identical with the graph of the free movement of an oscillator satisfying certain initial conditions. In both cases (the oscillator with damping and that

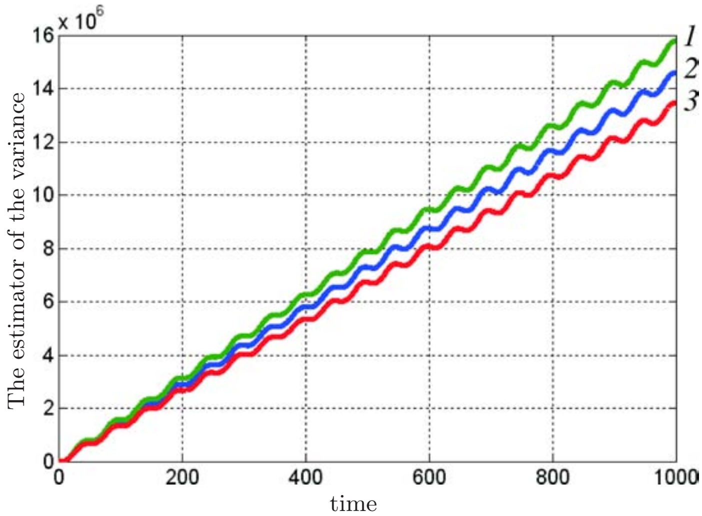

Fig. 2. Estimators of the variance for three different distributions η for the oscillators without damping. $\lambda E(\eta) = \text{const} = 5566.7$. Key: *1* – $\lambda E(\eta^2) = 397\,943.11$, *2* – $E(\eta^2) = 368\,588.00$, *3* – $E(\eta^2) = 339\,231.60$.

without it) the graphs are increased by the mean value of stochastic impulses acting on the oscillator. The variance of the movement (Fig. 4) of the oscillator with damping tends to a certain constant, unlike the variance of the oscillator without damping.

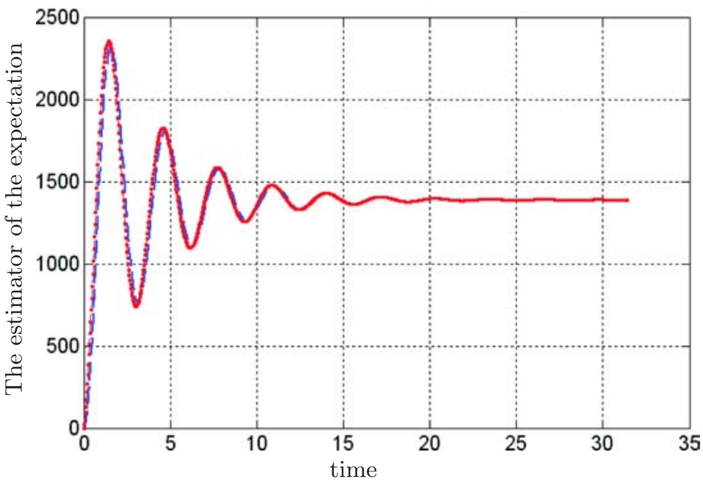

Fig. 3. Estimators of the expectation for the oscillators with damping. Key: dashed red – the theoretical expectation, dashed blue – the estimator of the expectation, 10 000 elements in a statistical sample.

In the paper [5], the movement of a string without dumping excited by stochastic impulses distributed in the same way as those exciting the oscillator is

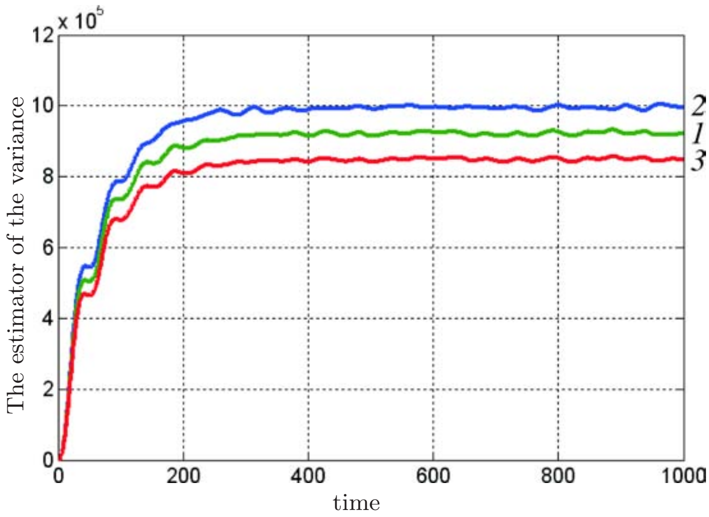

Fig. 4. Estimators of the variance for three different distributions η for the oscillators with damping. $\lambda E(\eta)$ = const = 5566.7 Key: *1* – $\lambda E(\eta^2) = 397\,943.11$, *2* – $E(\eta^2) = 368\,588.00$, *3* – $E(\eta^2) = 339\,231.60$.

discussed. However, the place they act on is a stochastic value, which is a significant difference as compared with the oscillator. Yet, it turns out that particular points of the string in its motion behave like an oscillator without damping (Fig. 5 and Fig. 6). This follows from the fact that the amplitudes of the higher, odd harmonic components are small while the even ones are equal to zero.

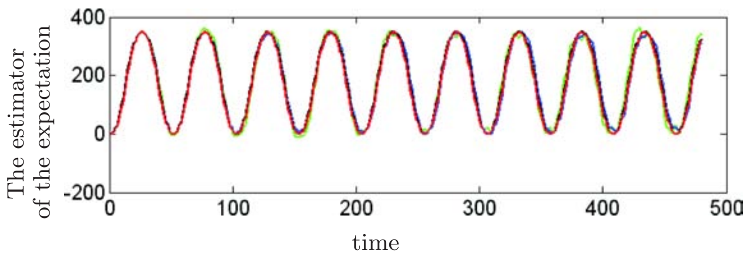

Fig. 5. Estimators of the expectation for the string without damping. Key: green line – 1 000 elements in a statistical sample, blue line – 10 000 elements in a statistical sample, turquoise line – 100 000 elements in a statistical sample, black line – 1 000 000 elements in a statistical sample, red line – the theoretical expectation.

In this paper we investigate statistical behaviour of the string with damping. Concerning the statistical behaviour of such a string we can derive similar conclusions as for a string without dumping. It suggests the behaviour of the systems of higher dimensions since their mathematical analysis is technically much more complicated.

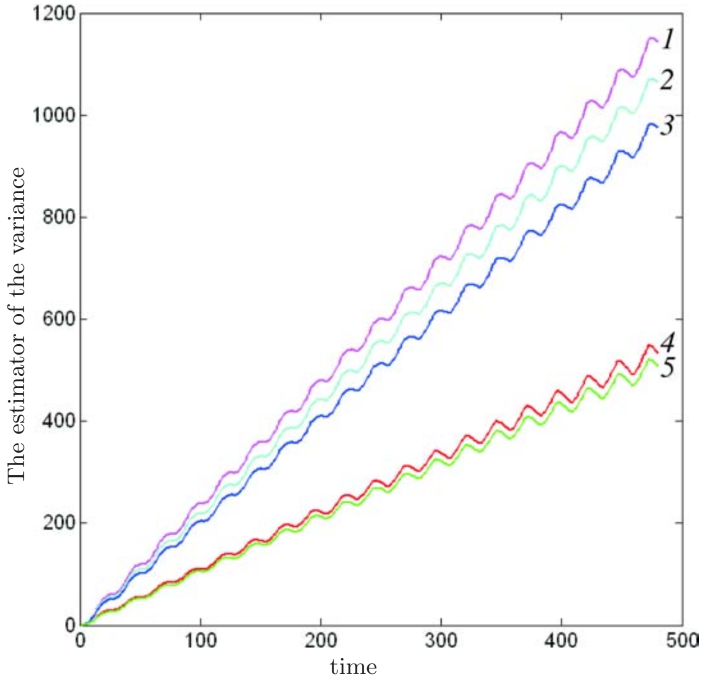

Fig. 6. Estimators of the variance for five different distributions η for the string without damping. Key: *1* – $\lambda E(\eta^2) = 397\,943.11$, *2* – $E(\eta^2) = 368\,588.00$, *3* – $E(\eta^2) = 339\,231.60$, *4* – $E(\eta^2) = 90\,723.67$, *5* – $E(\eta^2) = 86\,889.83$.

The obtained results will allow us to suggest possible ways of finding the statistical characteristics of forces influencing the system, having the trajectory of motion of the device we use to measure these forces.

The first partial theoretical results regarding vibration of oscillators excited by stochastic impulses can be found in the following papers [1, 2, 6, 8–10]. In the study [7], we can find the results that generalize the findings published in the previous papers.

2. Theoretical background

Now, for the convenience of the reader we quote the theorem from [3].

Let g_i: $[0, \infty) \to R$, $i = 1, 2, 3, \ldots, m$ be a sequence of continuous functions, A being a bounded connected Borel subset of R^p for some $p \in N$, h_i: $A \to R$, $i = 1, 2, 3, \ldots, m$ denoting a sequence of bounded and continuous functions, $\{\tau_i\}_{i=1}^{\infty}$ – a sequence of independent and identically distributed random variables with exponential distribution $F(x) = 1 - \exp(-\lambda x)$ for $x > 0$ and $F(x) = 0$ for $x < 0$, $\{\eta_i\}_{i=1}^{\infty}$ – a sequence of independent and identically distributed random variables with finite expectation, $\{\zeta_i\}_{i=1}^{\infty}$ – a sequence of independent and identically distributed random variables with values in the set A, and finally let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Let us put

$$\begin{aligned} t_0 &= 0, \qquad t_i = \sum_{j=1}^{i} \tau_j, \qquad i = 1, 2, 3..., \\ \xi(t) &= \sum_{n=1}^{m} \alpha_n \sum_{0<t_j<t} \eta_j h_n(\zeta_j) g_n(t - t_j) \end{aligned} \tag{1}$$

and denote the distributions of ζ and η by ϕ_ζ and ϕ_η, respectively. Assume that $A_i \subset C$ and $B_i \subset [0, \infty)$ $i = 1, 2, \ldots, m$ denote the Borel sets and $k(i,j)$, for every fixed i, is an increasing sequence of all the natural numbers such that

$$\chi_{A_i}\left(\eta_{k(i,j)}\right) \chi_{B_i}\left(\xi_{k(i,j)}\right) = 1, \tag{2}$$

where $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Write $t_j^i = t_{k(i,j)}$ and $\tau_j^i = t_j^i - t_{j-1}^i$. We will say that $\xi(t)$ is decomposable if for every $n \in N$, all the Borel sets $A_i \subset A$ and $B_i \subset [0, \infty)$, $i = 1, 2, \ldots, n$ such that $A_i \times B_i$, are mutually disjoint

$$\bigcup_{i=1}^{\tau} A_i \times B_i = A \times B\,. \tag{3}$$

τ_j^i are independent and identically distributed random variables with exponential

$$\begin{aligned} F(x) &= 1 - \exp\left(-\lambda \Phi_\xi(A_i) \Phi_\eta(B_i)\right) \quad \text{for } x > 0 \text{ and} \\ F(x) &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{for } x < 0, \end{aligned} \tag{4}$$

$$\xi_i(t) = \sum_{n=1}^{m} \alpha_n \sum_{0<t_i^j<t} \eta_{k(i,j)} h_n(\zeta_{k(i,j)}) g_n(t - t_j^i) \tag{5}$$

are independent and

$$\xi(t) = \sum_{i=1}^{n} \xi_i \tag{6}$$

From the technical point of view, decomposability of the process $\xi(t)$ means that we can divide the acting stochastic forces in any way, and the space onto which they acted (a string, a membrane etc) can be divided into any areas. If we consider the processes corresponding to the acting forces, let us assume – a group number i acting on the area number j, we will obtain a series of processes. As regards these processes, we assume that they are independent.

THEOREM 1: *If the process defined in Eq. (6) is decomposable, then the characteristic function, expectation and variance of $\xi(t)$ are given by*

$$\varphi(s) = \exp\left\{\lambda t\left[\int_A \int_0^\infty \int_0^1 \exp\left(is\sum_{n=1}^{m} \alpha_n y h_n(z) g(ut)\right) \mathrm{d}u \phi_\eta(\mathrm{d}y) \phi_\zeta(\mathrm{d}z) - 1\right]\right\}, \tag{7}$$

$$E(\xi(t)) = \frac{\varphi'(0)}{i} = \lambda t E(\eta) \sum_{n=1}^{m} \alpha_n E(h_n(\zeta)) \int_0^1 g_n(tu)\,\mathrm{d}u, \tag{8}$$

$$\begin{aligned} D^2(\xi(t)) &= E(\xi^2(t)) - E^2(\xi(t)) = \frac{1}{i^2}\left(\varphi''(0) - (\varphi')^2(0)\right) \\ &= \lambda t E(\eta^2) \sum_{n=1}^{m}\sum_{j=1}^{m} \alpha_n \alpha_j E(h_n(\zeta) h_j(\zeta)) \int_0^1 g_n(tu) g_j(tu)\,\mathrm{d}u. \end{aligned} \tag{9}$$

This theorem allows to derive statistical properties of the motion of the excited and damped string.

3. Applications

We shall apply the theorem presented above to a one-dimensional continuous system with damping, i.e., to a string for which the equation describing the vibrations induced by forces is as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) - 2b\frac{\partial u(x,t)}{\partial t}\,. \tag{10}$$

In our model we assume that the damping forces are proportional to the velocity of the string at the point x with the coefficient $b > 0$, small $\left(\left(\frac{\pi}{l}\right)^2 > \left(\frac{b}{a}\right)^2\right)$ and independent of x. The coefficient $a = \sqrt{\frac{T}{A\rho}}$, where T is the force of the string stretch, A – the cross-sectional area, ρ – the mass density.

The boundary and the initial conditions are as follows:

$$u(0,t) = 0 \quad \text{and} \quad u(l,t) = 0, \tag{11}$$

$$u(x,0) = 0 \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = 0, \tag{12}$$

for $t \geq 0$, $x \in [0,l]$, $f(x,t)$ is given by

$$f(x,t) = \sum_{i=1}^{\infty} \eta_i \delta_{t_i \varsigma_i}\,, \tag{13}$$

where $\delta_{t_i\varsigma_i}$ are the Dirac distributions at t_i and ς_i; then the solution of Eqs. (10)–(12) takes the following form:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{l}\,, \tag{14}$$

where

$$T_n(t) = \frac{2}{a\sqrt{\left(\frac{n\pi}{l}\right)^2 - \left(\frac{b}{a}\right)^2}} \cdot \sum_{t_i<t} \eta_i \sin\frac{\pi n \varsigma_i}{l} e^{-b(t-t_i)} \sin\left(a\sqrt{\left(\frac{n\pi}{l}\right)^2 - \left(\frac{b}{a}\right)^2}(t-t_i)\right). \tag{15}$$

If η_i and ζ_i, $i = 1, 2, \ldots$ are two sequences of independent and identically distributed random variables with distributions Φ_η and Φ_ς respectively, and $\tau_i = t_i - t_{i-1}$, $i = 1, 2, \ldots$, are also independent and identically distributed random variables with exponential distribution ($F(u) = 1 - \exp(-\lambda u)$ for $u > 0$ and for some λ and $F(u) = 0$ for $u < 0$), then for any fixed x

$$\xi_m(t,x) = u_m(x,t) = \sum_{n=1}^{m} T_n(t) \sin\frac{n\pi x}{l} \tag{16}$$

is a stochastic process satisfying the assumptions of Theorem 1 with any m and $h_i = \sin(i\pi x/l)$, $i = 1, \ldots, m$. Due to the conditions of Theorem 1 we cannot assume $m = \infty$. Instead of that we can assume the limit of $u_m(x,t)$ as m tends to infinity, to take the required results. Applying this theorem to $u_m(x,t)$ we obtain the result:

$$\begin{aligned}
E(u_m(t,x)) &= \lambda E(\eta) \sum_{n=1}^{m} \frac{\sin\frac{\pi n x}{l} E\Big(\sin\Big(\frac{\pi n \varsigma}{l}\Big)\Big)}{a\sqrt{\left(\frac{\pi n}{l}\right)^2 - \left(\frac{b}{a}\right)^2}} \int_0^t e^{-bv} \sin a\sqrt{\left(\frac{\pi n}{l}\right)^2 - \left(\frac{b}{a}\right)^2}\, v\, dv \\
&= \lambda E(\eta) e^{-bt} \sum_{n=1}^{m} \frac{\sin\frac{\pi n x}{l} E\Big(\sin\Big(\frac{\pi n \varsigma}{l}\Big)\Big) \sin a\sqrt{\left(\frac{\pi n}{l}\right)^2 - \left(\frac{b}{a}\right)^2}\, t}{a^3 \left(\frac{n\pi}{l}\right)^2 \sqrt{\left(\frac{\pi n}{l}\right)^2 - \left(\frac{b}{a}\right)^2}} \\
&- \lambda E(\eta) e^{-bt} \sum_{n=1}^{m} \frac{\sin\frac{\pi n x}{l} E\Big(\sin\Big(\frac{\pi n \varsigma}{l}\Big)\Big) \cos a\sqrt{\left(\frac{\pi n}{l}\right)^2 - \left(\frac{b}{a}\right)^2}\, t}{a^2 \left(\frac{n\pi}{l}\right)^2} \\
&+ \lambda E(\eta) \sum_{n=1}^{m} \frac{\sin\frac{\pi n x}{l} E\Big(\sin\Big(\frac{\pi n \varsigma}{l}\Big)\Big) \sin a\sqrt{\left(\frac{\pi n}{l}\right)^2 - \left(\frac{b}{a}\right)^2}\, t}{a^2 \left(\frac{n\pi}{l}\right)^2},
\end{aligned} \tag{17}$$

$$D^2(u_m(x,t)) = \lambda E(\eta^2) \sum_{n=1}^{m}\sum_{j=1}^{m} \frac{\sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi j x}{l}\right)}{a^2\sqrt{\left(\frac{\pi n}{l}\right)^2-\left(\frac{b}{a}\right)^2}\sqrt{\left(\frac{\pi j}{l}\right)^2-\left(\frac{b}{a}\right)^2}}$$

$$\times E\left(\sin\frac{\pi n\varsigma}{l}\sin\frac{\pi j\varsigma}{l}\right)\int_0^t e^{-2bv}\sin\left(va\sqrt{\left(\frac{\pi n}{l}\right)^2-\left(\frac{b}{a}\right)^2}\right)$$

$$\times \sin\left(va\sqrt{\left(\frac{\pi j}{l}\right)^2-\left(\frac{b}{a}\right)^2}\right)\,\mathrm{d}v, \qquad (18)$$

being the expectation and variance of $u_m(x,t)$, respectively.

4. Numerical simulation

To verify numerically that the theoretical formulas are correct, we need a statistical sample T. To obtain one with n elements, we repeat the following procedure n times. First we choose randomly τ_i in accordance with the exponential distribution. Remember that $t_m = \sum_{i=1}^{m}\tau_i$. Then, we choose randomly the values of η_i (magnitude of the force exetred by the particles striking the string) with discrete distribution, and finally, we select randomly ς_i (the point of the action on the string). We substitute these data into Eq. (16) and thus we obtain an element of our statistical sample. The elements of the sample are denoted by $u_m^k(x,t)$.

Figure 7 presents estimators

$$\widetilde{E}_n(u_m(x,t)) = \frac{1}{n}\sum_{i=1}^{n} u_m^i(x,t)$$

for $m = 10$, $\lambda = 10$, $a = 20$, $b = 2$, $x = l/2$, statistical samples of 1 thousand, 10 thousand, 100 thousand and 1 million elements, distribution of the variable η such that $\eta \in \{728,\ 214\}$, $P(\eta = 728) = 2/3$, $P(\eta = 214) = 1/3$, and two different distributions of the variable ς: continuous (b), and uniform (a), on the points from the set $A = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$, ($P(\zeta = x_i) = 1/5$ for $x_i \in A$).

Figure 8 shows the estimators

$$\widetilde{D}_n^2(u_m(x,t)) = \frac{1}{n}\sum_{i=1}^{n}\left(u_m^i(x,t) - E(u_m(x,t))\right)^2$$

for $m = 10$, $\lambda = 10$, $a = 2$ and $b = 0.2$, and $a = 20$, $b = 2$, $x = l/2$, statistical samples equal to 1 thousand, 10 thousand, 100 thousand and one million and continuous distributions of the variable ς.

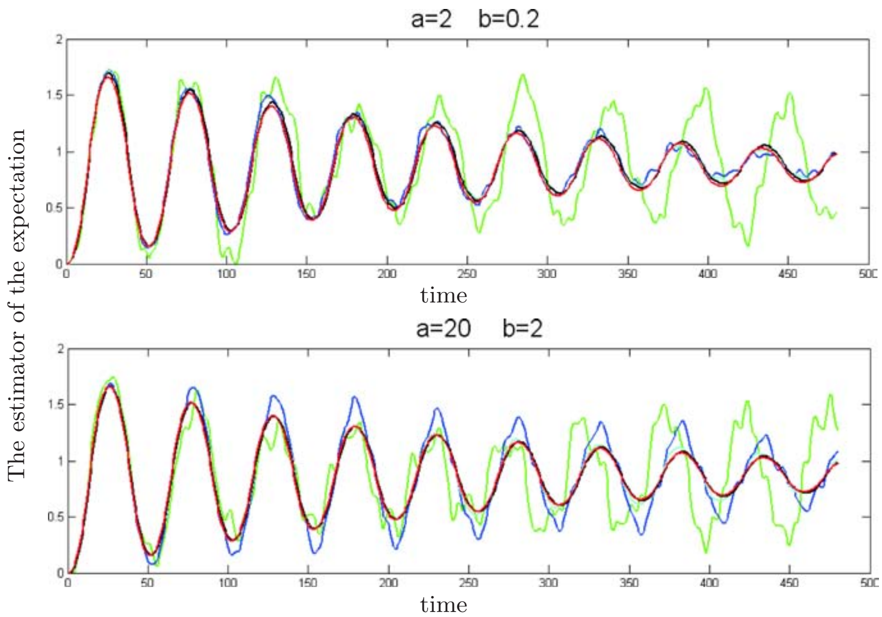

Fig. 7. Estimators of the expectation. Key: green line – 1000 elements in a statistical sample, blue line – 10 000 elements in a statistical sample, turquoise line – 100 000 elements in a statistical sample, black line – 1 000 000 elements in a statistical sample, red line – the theoretical expectation.

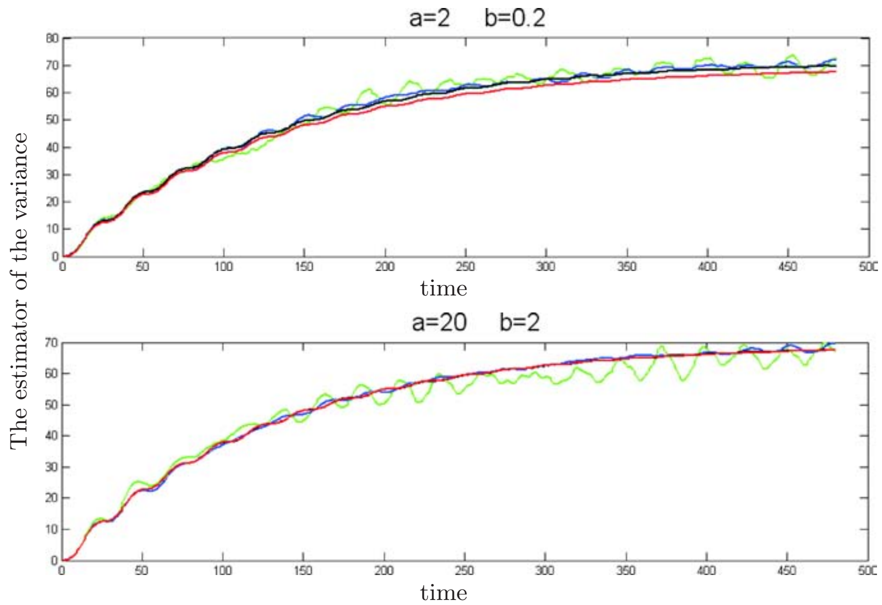

Fig. 8. Estimators of the variance. Key: green line – 1 000 elements in a statistical sample, blue line – 10 000 elements in a statistical sample, turquoise line – 100 000 elements in a statistical sample, black line – 1 000 000 elements in a statistical sample, red line – the theoretical variance expectation.

Figure 9 shows the estimators

$$\widetilde{D}_n^2(u_m(x,t)) = \frac{1}{n}\sum_{i=1}^{n}\left(u_m^i(x,t) - E(u_m(x,t))\right)^2$$

for $m = 10$, $\lambda = 10$, $a = 20$, $b = 2$, $x = l/2$, statistical samples of one million and continuous distributions of the variable ς, and the following cases for distributions of η:

1. $\eta \in \{728,\ 214\}$ and $P(\eta = 728) = 2/3$, $P(\eta = 214) = 1/3$,
2. $\eta \in \{728,\ 42.66\}$ – $P(\eta = 728) = 3/4$ and $P(\eta = 42.66) = 1/4$,
3. $\eta \in \{728,\ 385.33\}$ – $P(\eta = 728) = 1/2$ and $P(\eta = 385.33) = 1/2$,
4. $\eta \in \{352,\ 240,\ 120,\ 33\}$ – $P(\eta = 352) = 2/3$ and $P(\eta = 240) = 1/9$, $P(\eta = 120) = 1/9$, $P(\eta = 33) = 1/9$, $E(\eta^2) = 90\,723.67$,
5. $\eta \in \{330,\ 217,\ 120,\ 33\}$ – $P(\eta = 330) = 3/4$ and $P(\eta = 217) = 1/12$, $P(\eta = 120) = 1/12$, $P(\eta = 33) = 1/12$, $E(\eta^2) = 86\,889.83$.

In the first three cases $\lambda = 10$ and $E(\eta) = 556.67$, in the last two cases $\lambda = 20$ and $E(\eta) = 278.33$. The values of $E(\eta^2)$ are equal to 368588.00, 397943.11, 339231.60, 90723.67, 86889.83 respectively. The quantity $\lambda E(\eta) = 5566.7$ represents the mass of the medium flowing through the pipe in the unit of the time.

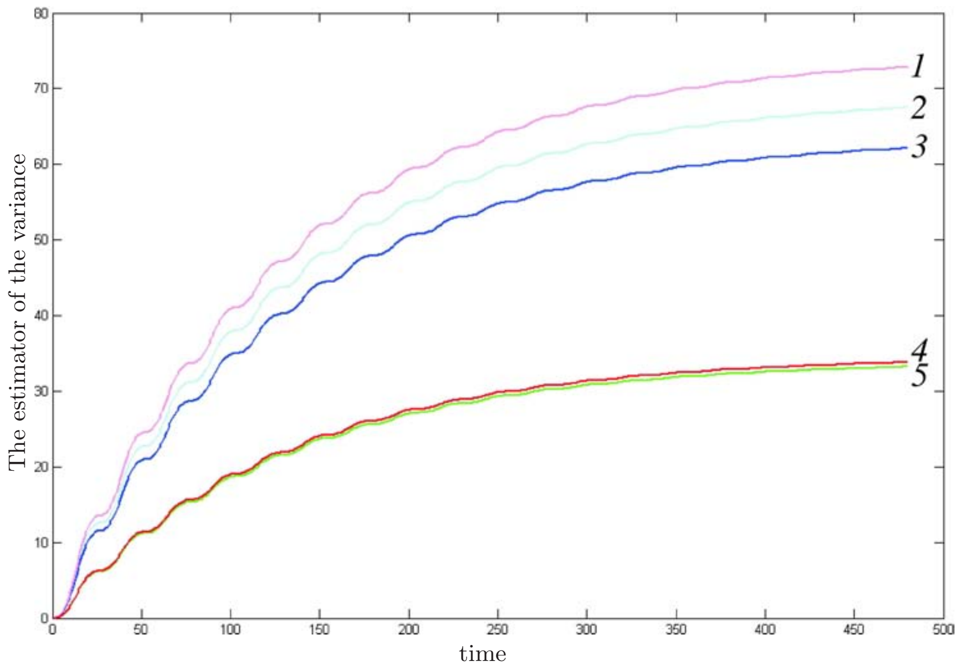

Fig. 9. The variance for five different distributions η. $\lambda E(\eta) = \text{const} = 5566.7$. Key: *1* – $\lambda E(\eta^2) = 397\,943.11$, *2* – $E(\eta^2) = 368\,588.00$, *3* – $E(\eta^2) = 339\,231.60$, *4* – $E(\eta^2) = 90\,723.67$, *5* – $E(\eta^2) = 86\,889.83$.

5. Conclusions

The occurrence of too large particles stimulating the motion of the measuring device, with too high probability at the same mean value of transported mass in a unit of time, is an unwanted effect. Calculation of variance allows for detection of this phenomenon. The diagrams of variance estimators presented above (Fig. 9) suggest how we should conclude about the distribution of stochastic impulses forcing a string, from the values of the variance estimators of the process $u(x,t)$ given by Eq. (14). As we see, greater impulses imply faster rise of variance in time.

If λ increases, then, at constant flow $\lambda E(\eta) = \text{const}$, the smaller is $D^2(u(x,t))$, the closer will be the size of a falling particle to the mean value, and the smaller – the probability of a large particle strike. The mean value of the distribution of particle sizes multiplied by the mean strike rate is constant, at least in certain time intervals. The value of variance that we will be able to measure with the methods similar to those used in simulation, will inform us about irregularities in the technological process.

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