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GEOMETRY OF SYMMETRIZED ELLIPSOIDS

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Abstract. We study the geometric properties of the symmetrized ellipsoids. In the paper we look for the differences and the similarities between the geometry of the symmetrized polydisc and symmetrized ellipsoids.

1. Introduction and results. The symmetrized polydisc has drawn quite a lot of attention recently. One of the most striking properties of that set is the one saying that in two-dimensional case the Lempert function, the Kobayashi distance and the Carathéodory distance coincide (see [**6**] and [**1**]) and, simultaneously, this domain cannot be exhausted by domains biholomorphic to convex ones (see [**7**] and [**8**]). Next interesting property of the symmetrized bidisc can be seen if we consider the question posed by Znamenskiĭ(see [**19**]): *Is any bounded* $\mathbb{C}$*-convex domain biholomorphic to a convex domain?* It turns out (see [**17**]) that the symmetrized bidisc gives a negative answer to that question.

Since the symmetrized polydisc can be exhausted by symmetrized ellipsoids, i.e. $\mathbb{G}_n = \bigcup_{p>0} \mathbb{E}_{p,n}$ (see the definition below), it seems reasonable to study the geometry of the symmetrized ellipsoid $\mathbb{E}_{p,n}$. This may be helpful in understanding whether the phenomena concerning the symmetrized polydisc are exceptional or not.

Let us start with some helpful notions and definitions.

For $p > 0$ let $\mathbb{B}_{p,n} := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^p + \dots + |z_n|^p < 1\}$. Moreover, put $\mathbb{B}_n := \mathbb{B}_{2,n}$, $\mathbb{D} := \mathbb{B}_1$, $\mathbb{B}(a, r) := a + r\mathbb{D}$, $\mathbb{B}(r) := \mathbb{B}(0, r)$, and $\mathbb{T} := \partial\mathbb{D}$.

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Let $\pi_n = (\pi_{n,1}, \dots, \pi_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows

$$\pi_{n,k}(z) = \sum_{1 \leqslant j_1 < \dots < j_k \leqslant n} z_{j_1} \dots z_{j_k}, \quad 1 \leqslant k \leqslant n, \ z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

The set $\mathbb{E}_{p,n} := \pi_n(\mathbb{B}_{p,n})$ is called the *symmetrized* (p,n)*-ellipsoid*. Moreover, for $p > 0$ put

$$\Delta_{p,n} := \{(z, \dots, z) \in \mathbb{C}^n : |z| < n^{-\frac{1}{p}}\}, \quad \Sigma_{p,n} := \pi_n(\Delta_{p,n}).$$

Note that π_n is a proper holomorphic mapping with multiplicity equal to $n!$, $\pi_n|_{\mathbb{B}_{p,n}} : \mathbb{B}_{p,n} \to \mathbb{E}_{p,n}$ is proper, and $\pi_n|_{\mathbb{B}_{p,n} \setminus \Delta_{p,n}} : \mathbb{B}_{p,n} \setminus \Delta_{p,n} \to \mathbb{E}_{p,n} \setminus \Sigma_{p,n}$ is a holomorphic covering.

In this note we deal not only with the geometric convexity but also with the notion of $\mathbb{C}$-convexity. Let us recall that a domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$*-convex* if $D \cap L$ is connected and simply connected for any complex affine line L such that $D \cap L$ is not empty.

Clearly, any convex domain is $\mathbb{C}$-convex, but the converse is not true. For the comprehensive information on the $\mathbb{C}$-convexity, see e.g. [**4**].

Below we present a number of results on the geometry of symmetrized ellipsoids.

Our first result concerns the convexity and $\mathbb{C}$-convexity of symmetrized ellipsoids and corresponds with Theorem 1 in [**17**].

PROPOSITION 1. *If* $p > 1$ *and* $n \geqslant k(p) := \min\{l \in \mathbb{N} : l \geqslant 3, \ \log_{l(l-1)} l^2 < p\}$, *then* $\mathbb{E}_{p,n}$ *is not* $\mathbb{C}$*-convex. In particular,* $\mathbb{E}_{p,n}$ *is not* $\mathbb{C}$*-convex for any* $p > \log_6 9$ *and* $n \geqslant 3$.

Since $\log_{n(n-1)} n^2 \searrow 1$ as $n \to +\infty$, we obtain the following

COROLLARY 2. *For any* $p > 1$ *there exists* $k(p) \in \mathbb{N}$ *such that* $\mathbb{E}_{p,n}$ *is not* $\mathbb{C}$*-convex for any* $n \geqslant k(p)$. *For example,* $k(\log_6 9) = 4$.

In general, as the following proposition shows, symmetrized ellipsoids are not convex. From that point of view, exceptional are the exponents $p = 1$ and $p = 2$, for which two-dimensional symmetrized ellipsoids are convex.

PROPOSITION 3. (i) *For any* $p \in (0, \log_2 \frac{5}{4}) \cup (2, +\infty)$ *and* $n \geqslant 2$, *the set* $\mathbb{E}_{p,n}$ *is not convex.*

(ii) *For any* $p > \log_3 \frac{9}{4}$ *and* $n \geqslant 3$, *the set* $\mathbb{E}_{p,n}$ *is not convex.*

(iii) *The sets* $\mathbb{E}_{2,2}$ *and* $\mathbb{E}_{1,2}$ *are convex.*

REMARK 4. It seems that in Proposition 3 (i), the number $\log_2 \frac{5}{4}$ may be replaced with 1. However, in such case we cannot give a formal proof. Using some technical method we are able to replace $\log_2 \frac{5}{4}$ with 0.648. However, we skip that proof since it does not solve the problem completely.

For $p > 3$ even more than nonconvexity holds, namely the following is true (cf. [**1**] and [**17**] for similar results on the symmetrized polydiscs).

Proposition 5. (i) *The domain* $\mathbb{E}_{3,2}$ *is starlike with respect to the origin.*

(ii) *If* $\mathbb{E}_{p,2}$ *is starlike with respect to the origin then so is* $\mathbb{E}_{\frac{p}{2},2}$. *In particular,* $\mathbb{E}_{p,2}$ *is starlike for* $p \in \{\frac{l}{2^k} : l = 1, 3,\ k \in \mathbb{N}\}$.

(iii) *For* $p > 3$ *and* $n \geqslant 2$, *the domain* $\mathbb{E}_{p,n}$ *is not starlike with respect to the origin.*

It turns out that the two-dimensional symmetrized ellipsoid, just like the symmetrized bidisc, cannot be exhausted by domains biholomorphic to a convex ones, either. This property holds for $p > 2$, while $\mathbb{E}_{2,2}$ is even convex (cf. Proposition 3 (iii)).

Proposition 6. *The domain* $\mathbb{E}_{p,2}$, $p > 2$, *cannot be exhausted by domains biholomorphic to convex domains.*

Since $\mathbb{E}_{1,2}$ and $\mathbb{E}_{2,2}$ are convex bounded domains in $\mathbb{C}^2$, it was quite natural to ask whether these domains are Lu Qi-Keng. For $\mathbb{E}_{2,2}$ the answer is positive (see the proposition below). Moreover, we conjecture that $\mathbb{E}_{1,2}$ is Lu Qi-Keng, too.

Proposition 7. $\mathbb{E}_{2,2}$ *is the Lu Qi-Keng domain.*

Finally we want to discuss some partial results on automorphisms of symmetrized ellipsoids.

Recall that $\operatorname{Aut}(\mathbb{B}_n) = \{u \circ h_a : a \in \mathbb{B}_n,\ u \in \mathcal{U}(\mathbb{C}^n)\}$, where $\mathcal{U}(\mathbb{C}^n)$ denotes the class of unitary operators in $\mathbb{C}^n$ and

$$h_a(z) := \frac{\sqrt{1 - \|a\|^2}(\|a\|^2 z - \langle z, a\rangle a) - \|a\|^2 a + \langle z, a\rangle a}{\|a\|^2(1 - \langle z, a\rangle)}, \quad z, a \in \mathbb{B}_n,\ a \neq 0,$$

and $h_0 := \operatorname{id}_{\mathbb{B}_n}$.

Let $\mathfrak{S}_n$ denote the group of all permutations of the set $\{1, \dots, n\}$. For $\sigma \in \mathfrak{S}_n,\ z = (z_1, \dots, z_n) \in \mathbb{C}^n$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$.

For any domain $D \subset \mathbb{C}^n$ with $\sigma(D) = D,\ \sigma \in \mathfrak{S}_n$, let

$$\mathcal{O}_{\mathfrak{S}}(D) = \mathcal{O}_{\mathfrak{S}_n}(D) := \{f \in \mathcal{O}(D, D) : f_\sigma(z) = f(z_\sigma),\ z \in D,\ \sigma \in \mathfrak{S}_n\}.$$

Remark 8. (a) If $h \in \mathcal{O}_{\mathfrak{S}}(\mathbb{B}_{p,n})$ then the relation $H_h \circ \pi_n = \pi_n \circ h$ defines a holomorphic mapping $H_h : \mathbb{E}_{p,n} \to \mathbb{E}_{p,n}$ with $H_h(\Sigma_{p,n}) \subset \Sigma_{p,n}$. Moreover, if h is proper then H_h is proper, too.

(b) Observe that if $h \in \operatorname{Aut}(\mathbb{B}_{p,n}) \cap \mathcal{O}_{\mathfrak{S}}(\mathbb{B}_{p,n})$, then $H_h \in \operatorname{Aut}(\mathbb{E}_{p,n})$, $H_h^{-1} = H_{h^{-1}}$, and $H_h(\Sigma_{p,n}) = \Sigma_{p,n}$. In particular, if $u \in \mathcal{U}(\mathbb{C}^n) \cap \mathcal{O}_{\mathfrak{S}}(\mathbb{C}^n)$ and $a \in \Delta_{2,n}$, then $H_{u \circ h_a} \in \operatorname{Aut}(\mathbb{E}_{2,n})$.

(c) For any $u \in \mathcal{U}(\mathbb{C}^n) \cap \mathcal{O}_{\mathfrak{S}}(\mathbb{C}^n)$ and $z = \pi_n(a) \in \Sigma_{2,n}$, there is $H_{u \circ h_a}(z) = 0$. Consequently, the group $\operatorname{Aut}(\mathbb{E}_{2,n})$ acts transitively on $\Sigma_{2,n}$.

(d) Note that if $u \in \mathcal{U}(\mathbb{C}^2) \cap \mathcal{O}_{\mathfrak{S}}(\mathbb{C}^2)$ then

$$u(z_1, z_2) = u_\xi(z_1, z_2) := (\xi_1 z_1 + \xi_2 z_2, \xi_2 z_1 + \xi_1 z_2), \quad (z_1, z_2) \in \mathbb{C}^2,$$

where $\xi = (\xi_1, \xi_2) \in \partial\mathbb{B}_2$ is such that $\operatorname{Re}(\xi_1 \bar{\xi}_2) = 0$.

(e) Let $p \neq 2$. If $h \in \operatorname{Aut}(\mathbb{B}_{p,n}) \cap \mathcal{O}_{\mathfrak{S}}(\mathbb{B}_{p,n})$ then $H_h(0) = 0$. This follows from the fact that $h(0) = 0$ (see Corollary 8.5.5 in [**11**]).

We already know from Remark 8 (b) that there are automorphisms of $\mathbb{E}_{2,n}$ generated by some automorphisms of $\mathbb{B}_n$. Next result shows that in the case of $n = 2$ there is no other automorphism of $\mathbb{E}_{2,2}$ (see [**12**] for a similar result on the symmetrized bidisc).

Proposition 9. $\operatorname{Aut}(\mathbb{E}_{2,2}) = \{H_{u_\xi \circ h_a} : \xi \in \partial\mathbb{B}_2,\ \operatorname{Re}(\xi_1 \bar{\xi}_2) = 0,\ a \in \Delta_{2,2}\}$.

Moreover, similarly as in [**12**] we prove

Proposition 10. (i) $\operatorname{Aut}(\mathbb{E}_{2,n})$ *does not act transitively on* $\mathbb{E}_{2,n}$ *for* $n > 1$.
(ii) $F(\Sigma_{2,n}) = \Sigma_{2,n}$ *for every* $F \in \operatorname{Aut}(\mathbb{E}_{2,n})$.

Numerous questions concerning symmetrized ellipsoids remain open. Below, we list some of them.

(a) Prove that $\mathbb{E}_{p,n}$ is not convex for $\log_2 \frac{5}{4} \leqslant p < 1$ and $n \geqslant 2$. Using some iteration method we are able to show non-convexity of $\mathbb{E}_{p,n}$ for $p < 0.648$.
(b) Is $\mathbb{E}_{p,n}$ not $\mathbb{C}$-convex for $1 < p \leqslant \log_6 9$ and $n \leqslant 3$? What about $0 < p \leqslant 1$?
(c) Is $\mathbb{E}_{p,2}$ convex for $1 < p < 2$?
(d) Is $\mathbb{E}_{p,2}$ $\mathbb{C}$-convex for $p > 2$? What about $0 < p < 1$?
(e) Is $\mathbb{E}_{p,n}$ or, at least, $\mathbb{E}_{p,2}$ starlike with respect to the origin for $0 < p < 3$?
(f) Is Proposition 6 valid for $p < 1$?
(g) Is $c_{\mathbb{E}_{p,2}} \neq \tilde{k}_{\mathbb{E}_{p,2}}$ for $p > 2$ or $p < 1$?
(h) Is $\mathbb{E}_{1,2}$ the Lu Qi-Keng domain?
(i) Is $\operatorname{Aut}(\mathbb{E}_{2,n}) = \{H_{u \circ h_a} : u \in \mathcal{U}(\mathbb{C}^n) \cap \mathcal{O}_{\mathfrak{S}}(\mathbb{C}^n),\ a \in \Delta_{2,n}\}$ for $n > 2$? Does any similar result hold for the holomorphic proper self-mappings of $\mathbb{E}_{2,n}$?

2. Proofs.

Proof of Proposition 1. The proof follows from the one of Theorem 1 (ii) in [**17**]. For the reader's convenience, we repeat the reasoning.

Let $k = k(p)$. For $t \in (0, k^{-\frac{1}{p}})$ consider the points

$$a_t := \pi_n(\underbrace{t, \dots, t}_{k}, 0, \dots, 0) = \left(\tbinom{k}{1}t, \dots, \tbinom{k}{k}t^k, 0, \dots, 0\right),$$

$$b_t := \pi_n(\underbrace{-t, \dots, -t}_{k}, 0, \dots, 0) = \left(\tbinom{k}{1}(-t)^1, \dots, \tbinom{k}{k}(-t)^k, 0, \dots, 0\right).$$

Obviously, $a_t, b_t \in \mathbb{E}_{p,n}$. Denote by L_t the complex line passing through a_t and b_t, that is,

$$L_t = \left\{c_{t,\lambda} := \left(\tbinom{k}{1}t(1-2\lambda), \dots, \tbinom{k}{k}t^k(1-2\lambda)^{k-2[\frac{k}{2}]}, 0, \dots, 0\right) : \lambda \in \mathbb{C}\right\}.$$

Assume that the set $L_t \cap \mathbb{E}_{p,n}$ is connected. Since $a_t = c_{t,0}$ and $b_t = c_{t,1}$, then $c_{t,\lambda} \in \mathbb{E}_{p,n}$ for some $\lambda = \frac{1}{2} + i\tau$, where $\tau \in \mathbb{R}$. It follows that

$$c_{t,\lambda} = \left(\tbinom{k}{1}(-2i\tau t), \tbinom{k}{2}t^2, \dots, \tbinom{k}{k}t^k(-2i\tau)^{k-2[\frac{k}{2}]}, 0, \dots, 0\right).$$

We may choose $\mu \in \mathbb{B}_{p,n}$ such that $\mu_j = 0,\ j = k+1, \dots, n$, and $c_{t,\lambda} = \pi_n(\mu)$. Observe that

$$-4k^2\tau^2t^2 = \left(\sum_{j=1}^{k}\mu_j\right)^2 = \sum_{j=1}^{k}\mu_j^2 + k(k-1)t^2. \tag{1}$$

We consider two cases.

Case 1. Let $p \geqslant 2$. Then (1) yields (if $p > 2$ we use the Hölder inequality):

$$t^2 = \frac{|\sum_{j=1}^{k}\mu_j^2|}{4k^2\tau^2 + k(k-1)} \leqslant \frac{\sum_{j=1}^{k}|\mu_j|^2}{k(k-1)} \leqslant \frac{k^{\frac{p-2}{p}}(\sum_{j=1}^{k}|\mu_j|^p)^{\frac{2}{p}}}{k(k-1)} \leqslant \frac{k^{-\frac{2}{p}}}{k-1}.$$

Therefore, $L_t \cap \mathbb{E}_{p,n}$ is not connected if $t \in [\frac{1}{\sqrt{k-1}}k^{-\frac{1}{p}}, k^{-\frac{1}{p}})$ (note that $k \geqslant 3$) and so $\mathbb{E}_{p,n}$ is not a $\mathbb{C}$-convex domain.

Case 2. Now let $p < 2$. Then (1) implies:

$$t^2 = \frac{|\sum_{j=1}^{k}\mu_j^2|}{4k^2\tau^2 + k(k-1)} \leqslant \frac{\sum_{j=1}^{k}|\mu_j|^p}{k(k-1)} < \frac{1}{k(k-1)}.$$

Moreover, since $\log_{k(k-1)} k^2 < p$, there follows $(k(k-1))^{-\frac{1}{2}} < k^{-\frac{1}{p}}$. Therefore, $L_t \cap \mathbb{E}_{p,n}$ is not connected if $t \in [(k(k-1))^{-\frac{1}{2}}, k^{-\frac{1}{p}})$ and so $\mathbb{E}_{p,n}$ is not a $\mathbb{C}$-convex domain. □

Before we continue, let us make the following very useful remark.

REMARK 11. Observe that

$$(s, t, 0, \dots, 0) \in \mathbb{E}_{p,n} \Leftrightarrow |s + \xi_1|^p + |s + \xi_2|^p < 2^p, \tag{2}$$

where $\{\xi_1, \xi_2\} = \sqrt{s^2 - 4t}$. If we consider the closure $\overline{E}_{p,n}$ then the "$\leqslant$" sign appears on the right hand side.

In the proof of Proposition 3 (iii), we will use the following simple result.

LEMMA 12. *Let* $a_j, b_j \in \mathbb{C},\ r_j > 0,\ j = 1, 2,$ *be such that* $|a_j^2| + |a_j^2 - b_j| < r_j,\ j = 1, 2$. *Then*

$$\left|\left(\frac{a_1 + a_2}{2}\right)^2\right| + \left|\left(\frac{a_1 + a_2}{2}\right)^2 - \frac{b_1 + b_2}{2}\right| < \frac{r_1 + r_2}{2}.$$

PROOF OF LEMMA 12. Since $b_j \in B(a_j^2, r_j - |a_j^2|),\ j = 1, 2$, then $\frac{b_1+b_2}{2} \in B(a_3, r_3)$, where $a_3 := \frac{a_1^2+a_2^2}{2}$ and $r_3 := \frac{r_1+r_2}{2} - \frac{|a_1^2|+|a_2^2|}{2}$. In our case it suffices to show that $\frac{b_1+b_2}{2} \in B(a_0, r_0)$, where $a_0 := (\frac{a_1+a_2}{2})^2$ and $r_0 := \frac{r_1+r_2}{2} - |(\frac{a_1+a_2}{2})^2|$. In other words, it is enough that $B(a_3, r_3) \subset B(a_0, r_0)$. We show that $r_0 = |a_0 - a_3| + r_3$. Indeed,

$$\begin{aligned} r_0 - |a_0 - a_3| - r_3 &= \frac{|a_1^2| + |a_2^2|}{2} - \left|\left(\frac{a_1 + a_2}{2}\right)^2\right| - \left|\left(\frac{a_1 + a_2}{2}\right)^2 - \frac{a_1^2 + a_2^2}{2}\right| \\ &= \frac{1}{4}\left(2(|a_1|^2 + |a_2|^2) - |a_1 + a_2|^2 - |a_1 - a_2|^2\right) = 0. \end{aligned}$$

□

PROOF OF PROPOSITION 3. *Re* (i). We consider two cases.

Case 1. Let $p < \log_2 \frac{5}{4},\ x := 2^{-\frac{1}{p}}$. Then $(1, 0, \ldots, 0), (2x, x^2, 0, \ldots, 0) \in \overline{\mathbb{E}}_{p,n}$ but $(\frac{1+2x}{2}, \frac{x^2}{2}, 0, \ldots, 0) \notin \overline{\mathbb{E}}_{p,n}$ since (use (2))

$$L := \left(1 + 2x + \sqrt{1 + 4x - 4x^2}\right)^p + \left(1 + 2x - \sqrt{1 + 4x - 4x^2}\right)^p > 4^p.$$

Indeed, using the estimates $1 < \sqrt{1 + 4x - 4x^2} < 1 + 2x - 2x^2$, we obtain

$$L > (2 + 2x)^p + (2x^2)^p = 2^p\left((1 + x)^p + \frac{1}{4}\right) > \frac{5}{4}2^p > 4^p.$$

Case 2. Let $p > 2,\ x := 2^{-\frac{1}{p}}$. Then $(2x, x^2, 0, \ldots, 0), (2xi, -x^2, 0, \ldots, 0) \in \overline{\mathbb{E}}_{p,n}$. On the other hand, $(x(1 + i), 0, \ldots, 0) \notin \overline{\mathbb{E}}_{p,n}$. Indeed,

$$|x(1 + i) - x(1 + i)|^p + |x(1 + i) + x(1 + i)|^p = (2\sqrt{2}x)^p = 2^{\frac{3}{2}p-1} > 2^p,$$

which contradicts (2).

Re (ii). Consider the points

$$a_t := \pi_n(t, t, t, 0, \ldots, 0) = (3t, 3t^2, t^3, 0, \ldots, 0),$$

$$b_t := \pi_n(-t, -t, -t, 0, \ldots, 0) = (-3t, 3t^2, -t^3, 0, \ldots, 0), \quad t = 3^{-\frac{1}{p}}.$$

Obviously, $a_t, b_t \in \overline{\mathbb{E}}_{p,n}$. We show that $c_t := \frac{1}{2}(a_t + b_t) \notin \overline{\mathbb{E}}_{p,n}$. Suppose that $c_t \in \overline{\mathbb{E}}_{p,n}$. Then there exists $\mu \in \overline{\mathbb{B}}_{p,n}$ such that $\pi_n(\mu) = c_t$. Since $c_t = (0, 3t^2, 0, \ldots, 0)$, we may assume that $\mu = (\sqrt{3}ti, -\sqrt{3}ti, 0, \ldots, 0)$. A contradiction, since

$$\sum_{j=1}^{n} |\mu_j|^p = 2(\sqrt{3}t)^p = \frac{2}{3}3^{\frac{p}{2}} > 1.$$

Re (iii). First observe that for $n = 2$ we may rewrite condition (2) as

$$(s,t) \in \mathbb{E}_{2,2} \Leftrightarrow |s^2| + |s^2 - 4t| < 2, \quad s, t \in \mathbb{C}, \quad \text{for } p = 2,$$

$$(s,t) \in \mathbb{E}_{1,2} \Leftrightarrow |s^2| + |4t| + |s^2 - 4t| < 2, \quad s, t \in \mathbb{C}, \quad \text{for } p = 1.$$

Since $\mathbb{E}_{p,2}$ is open, to prove its convexity it suffices to show that $(\frac{s_1+s_2}{2}, \frac{t_1+t_2}{2}) \in \mathbb{E}_{p,2}$ whenever $(s_1, t_1),\ (s_2, t_2) \in \mathbb{E}_{p,2}$ for $p = 1, 2$.

If $p = 2$, use Lemma 12 with $a_j = s_j,\ b_j = 4t_j$, and $r_j = 2,\ j = 1, 2$.

If $p = 1$, then fix $(s_j, t_j) \in \mathbb{E}_{1,2},\ j = 1, 2$, and use Lemma 12 with $a_j = s_j,\ b_j = 4t_j$, and $r_j = 2 - |4t_j|,\ j = 1, 2$. ☐

PROOF OF PROPOSITION 5. *Re* (i). Fix $(s,t) \in \mathbb{E}_{3,2}$ and $u \in (0,1)$. Observe that (2) yields

$$(|s + \xi_1| + |s + \xi_2|)(|s^2| + |s^2 - 4t| - 2|t|) < 4,$$

where $\{\xi_1, \xi_2\} = \sqrt{s^2 - 4t}$. Hence,

$$(|s + \xi_1| + |s + \xi_2|) < \frac{4}{(|s^2| + |s^2 - 4t| - 2|t|)} =: 2c(s,t) = 2c,$$

i.e. $(\frac{s}{c}, \frac{t}{c^2}) \in \mathbb{E}_{1,2}$. Since $\mathbb{E}_{1,2}$ is convex, then $(u\frac{s}{c}, u\frac{t}{c^2}) \in \mathbb{E}_{1,2}$, i.e.

$$(|us + \xi_{1,u}| + |us + \xi_{2,u}|)(|s^2| + |s^2 - 4t| - 2|t|) < 4, \tag{3}$$

where $\{\xi_{1,u}, \xi_{2,u}\} = \sqrt{(us)^2 - 4ut}$.

Now we show that

$$|(us)^2| + |(us)^2 - 4ut| - 2|ut| < |s^2| + |s^2 - 4t| - 2|t|. \tag{4}$$

Since $|(us)^2| + |(us)^2 - 4ut| - 2|ut| < |us^2| + |us^2 - 4t| - 2|t|$, to prove (4) it suffices to show that

$$|us^2| + |us^2 - 4t| \leqslant |s^2| + |s^2 - 4t| =: r.$$

The above inequality holds true, since $B(s^2, r - |s^2|) \subset B(us^2, r - |us^2|)$.

Consequently, (3) and (4) imply that $(us, ut) \in \mathbb{E}_{3,2}$, which ends the proof of part (i).

Re (ii). Fix $(s,t) \in \mathbb{E}_{\frac{p}{2},2}$ and $u \in (0,1)$. Then from (2) there follows

$$|s + \xi_1|^p + |s + \xi_2|^p < 2^p(1 - 2^{-p}|4t|^{\frac{p}{2}}) =: 2^p c^p,$$

i.e. $(\frac{s}{c}, \frac{t}{c^2}) \in \mathbb{E}_{p,2}$. Since $\mathbb{E}_{p,2}$ is starlike with respect to the origin, $(u\frac{s}{c}, u\frac{t}{c^2}) \in \mathbb{E}_{p,2}$, i.e.

$$|us + \xi_{1,u}|^p + |s + \xi_{2,u}|^p < 2^p c^p.$$

Moreover, note that $c(u) := (1 - 2^{-p}|4ut|^{\frac{p}{2}})^{\frac{1}{p}} > c$, which gives

$$|us + \xi_{1,u}|^p + |s + \xi_{2,u}|^p < 2^p (c(u))^p.$$

Hence, using (2) again, $(us, ut) \in \mathbb{E}_{\frac{p}{2},2}$, which ends the proof of part (ii).

Re (iii). For $x := 2^{-\frac{1}{p}}$, we conclude $(2x, x^2, 0, \ldots, 0) \in \overline{\mathbb{E}}_{p,n}$. Using (2), we obtain $(2xu, x^2u, 0, \ldots, 0) \in \overline{\mathbb{E}}_{p,n},\ u \in (0,1)$, iff

$$f(u) := \left(u + \sqrt{u - u^2}\right)^p + \left(u - \sqrt{u - u^2}\right)^p \leqslant 2,\ u \in (0,1).$$

We show that there is $u_0 \in (0,1)$ with $f(u_0) > 2$, which contradicts the starlikeness of $\mathbb{E}_{p,n}$. First observe that f is differentiable and $f(1) = 2$. Therefore, we are done if we show that $\lim_{u\to 1^-} f'(u) < 0$. Simple calculation gives

$$\lim_{u\to 1^-} f'(u) = p(3-p) < 0,$$

which completes the proof. $\square$

Before we give the proof of Proposition 6, let us make the following

Remark 13. For $p \geqslant 1$, let

$$\rho(z) := \max\Big\{\sum_{j=1}^{n} |\lambda_j|^p : (\lambda_1, \ldots, \lambda_n) \in \pi_n^{-1}(z)\Big\}, \quad z \in \mathbb{C}^n.$$

Then ρ is a continuous plurisubharmonic function such that

$$\rho(\lambda z_1, \ldots, \lambda^n z_n) := |\lambda|^p \rho(z_1, \ldots, z_n), \quad (z_1, \ldots, z_n) \in \mathbb{C}^n,\ \lambda \in \mathbb{C},$$

and

$$\mathbb{E}_{p,n} = \{z \in \mathbb{C}^n : \rho(z) < 1\}, \quad \overline{\mathbb{E}}_{p,n} = \{z \in \mathbb{C}^n : \rho(z) \leqslant 1\}.$$

In particular, $\mathbb{E}_{p,n}$ is hyperconvex.

In the proof of Proposition 6, we will use the following

Lemma 14. *Let $p > 2$ and $\delta > 0$. Then there exist $x, y > 0$ such that $x^p + y^p = 1$ and*

$$A := x + \sqrt{x^2 + 4\delta y^2} > 2.$$

Proof of Lemma 14. Note that the condition $A > 2$ is equivalent to

$$x > 1 - \delta y^2.$$

Therefore, if we show that there exists $y \in (0,1)$ such that

$$y^p + (1 - \delta y^2)^p < 1, \tag{5}$$

then, taking $x := (1 - y^p)^{\frac{1}{p}}$, we are done.

Put $f(t) := t^p + (1-\delta t^2)^p,\ t \in [0,1]$. Since $f(0) = 1$, it suffices to show that f is a decreasing function on an interval $(0,\varepsilon)$ for some $\varepsilon > 0$. Fortunately, $f'(0) = 0$ and

$$f''(0) = -2p\delta < 0.$$

Hence, we are able to choose $y \in (0,1)$ satisfying (5). □

Proof of Proposition 6. This is a modification of the proof given in the case of the symmetrized bidisc by A. Edigarian [**8**] (see also Lemma 1.4.10 in [**13**]).

Fix $p > 2$. First observe that $\mathbb{E}_{p,2}$ is not convex (Proposition 3 (i)).

Suppose that $\mathbb{E}_{p,2} = \bigcup_{i \in I} G_i$, where each domain G_i is biholomorphic to a convex domain and for any compact $K \subset\subset \mathbb{E}_{p,2}$ there exists an $i_0 \in I$ with $K \subset G_{i_0}$. For any $0 < \varepsilon < 1$ take an $i = i(\varepsilon) \in I$ such that $\{(s,t) \in \mathbb{C}^2 : \rho(s,t) \leqslant 1-\varepsilon\} \subset G_{i(\varepsilon)}$ and let $f_\varepsilon = (g_\varepsilon, h_\varepsilon) : G_{i(\varepsilon)} \to D_\varepsilon$ be a biholomorphic mapping onto a convex domain $D_\varepsilon \subset \mathbb{C}_2$ with $f_\varepsilon(0,0) = (0,0)$ and $f'_\varepsilon(0,0) = \mathrm{id}_{\mathbb{C}^2}$.

Take arbitrary two points $(s_j, t_j) \in \mathbb{C}^2,\ j = 1,2$, and put

$$C := \max\{\rho(s_1,t_1), \rho(s_2,t_2)\}.$$

Our aim is to prove that $\rho(x(s_1,t_1) + (1-x)(s_2,t_2)) \leqslant C,\ x \in [0,1]$, which in particular shows that $\mathbb{E}_{p,2}$ is convex, a contradiction.

Observe that for $|\lambda| < (\frac{1-\varepsilon}{C})^{\frac{1}{p}}$, there is $\rho(\lambda s_j, \lambda^2 t_j) = |\lambda|^p \rho(s_j,t_j) < 1 - \varepsilon,\ j = 1,2$. Consequently, for any $x \in [0,1]$, the mapping $\varphi_{\varepsilon,x} : \mathbb{B}((\frac{1-\varepsilon}{C})^{\frac{1}{p}}) \to \mathbb{E}_{p,2}$,

$$\varphi_{\varepsilon,x}(\lambda) = (\psi_{\varepsilon,x}(\lambda), \chi_{\varepsilon,x}(\lambda)) := f_\varepsilon^{-1}(x f_\varepsilon(\lambda s_1, \lambda^2 t_1) + (1-x) f_\varepsilon(\lambda s_2, \lambda^2 t_2)),$$

is well defined. There holds $\varphi_{\varepsilon,x}(0) = (0,0),\ \varphi'_{\varepsilon,x}(0) = (xs_1 + (1-x)s_2, 0)$, and

$$\frac{1}{2}\chi''_{\varepsilon,x}(0) = xt_1 + (1-x)t_2 + \mu_\varepsilon x(1-x)(s_1-s_2)^2,$$

where $\mu_\varepsilon := \frac{1}{2}\frac{\partial^2 h_\varepsilon}{\partial s^2}(0,0)$. Define $\phi_{\varepsilon,x} : \mathbb{B}((\frac{1-\varepsilon}{C})^{\frac{1}{p}}) \to \mathbb{C}^2$ by

$$\phi_{\varepsilon,x}(\lambda) := \begin{cases} (\lambda^{-1}\psi_{\varepsilon,x}(\lambda), \lambda^{-2}\chi_{\varepsilon,x}(\lambda)), & \lambda \neq 0 \\ (\psi'_{\varepsilon,x}(0), \frac{1}{2}\chi''_{\varepsilon,x}(0)), & \lambda = 0 \end{cases}.$$

Then $\phi_{\varepsilon,x}$ is holomorphic and, by the maximum principle, we get

$$\rho(\phi_{\varepsilon,x}(0)) \leqslant \limsup_{s \to \left(\frac{1-\varepsilon}{C}\right)^{\frac{1}{p}}} \max_{|\lambda|=s} \rho(\phi_{\varepsilon,x}(\lambda)) = \limsup_{s \to \left(\frac{1-\varepsilon}{C}\right)^{\frac{1}{p}}} \frac{1}{s^p} \max_{|\lambda|=s} \rho(\varphi_{\varepsilon,x}(\lambda)) \leqslant \frac{C}{1-\varepsilon},$$

that is,

$$\rho(xs_1 + (1-x)s_2, xt_1 + (1-x)t_2 + \mu_\varepsilon x(1-x)(s_1-s_2)^2) \leqslant \frac{C}{1-\varepsilon}.$$

We only need to prove that $\mu_\varepsilon \to 0$.

Taking $x = \frac{1}{2}$ we get

$$\rho\left(\frac{1}{2}(s_1+s_2), \frac{1}{2}(t_1+t_2) + \frac{1}{4}\mu_\varepsilon(s_1-s_2)^2\right) \leqslant \frac{C}{1-\varepsilon}.$$

For $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^p + |\beta|^p = 1$, take $(s_1, t_1) := \pi_2(\alpha, \beta)$ and $(s_2, t_2) := \pi_2(\alpha, -\beta)$. Then $C = 1$ and

$$\rho(\alpha, \mu_\varepsilon \beta^2) \leqslant \frac{1}{1-\varepsilon}.$$

Hence $((1-\varepsilon)^{\frac{1}{p}}\alpha, (1-\varepsilon)^{\frac{2}{p}}\mu_\varepsilon\beta^2) \in \overline{\mathbb{E}}_{p,2}$ and so, by (2),

$$\left|\alpha + \sqrt{\alpha^2 - 4\mu_\varepsilon\beta^2}\right|^p + \left|\alpha - \sqrt{\alpha^2 - 4\mu_\varepsilon\beta^2}\right|^p \leqslant \frac{2^p}{1-\varepsilon}. \tag{6}$$

Suppose $\mu_\varepsilon \not\to 0$ as $\varepsilon \to 0$. Thus there exists $\delta > 0$ such that for any $\eta > 0$ there is $\varepsilon \in (0, \eta)$ with $|\mu_\varepsilon| > \delta$. For such an ε, define $\alpha := x$ and $\beta := \xi y$, where x, y are the numbers from Lemma 14 and $\xi \in \mathbb{T}$ is such that $\mu_\varepsilon \beta^2 < 0$. Then

$$\left|\alpha + \sqrt{\alpha^2 - 4\mu_\varepsilon\beta^2}\right|^p > A^p > \frac{2^p}{1-\varepsilon}$$

for ε small enough, which contradicts (6). □

PROOF OF PROPOSITION 7. Note that, due to (2), $\mathbb{E}_{2,2}$ is biholomorphic to the set $D_2 := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w| < 1\}$. Since K_{D_2} has no zeros on $D_2 \times D_2$ (see [**13**], Example 3.1.6. (c)), $K_{\mathbb{E}_{2,2}}$ has no zeros on $\mathbb{E}_{2,2} \times \mathbb{E}_{2,2}$ either (use the formula for the behavior of the Bergman kernel under biholomorphic mappings; see e.g. [**11**], Proposition 6.1.7). □

In the proof of Proposition 9 we use following

LEMMA 15. $\mathbb{T}^2 = \{(\xi_1 + \xi_2, (\xi_1 - \xi_2)^2) : (\xi_1, \xi_2) \in \partial\mathbb{B}_2,\ \operatorname{Re}(\xi_1\bar{\xi_2}) = 0\}$.

PROOF OF LEMMA 15. Fix $(\zeta_1, \zeta_2) \in \mathbb{T}^2$. Put $\xi_1 := \frac{1}{2}(\zeta_1 + \sqrt{\zeta_2})$, $\xi_2 := \frac{1}{2}(\zeta_1 - \sqrt{\zeta_2})$, where $\sqrt{\zeta_2}$ is taken arbitrarily. It is easy to check that $(\xi_1, \xi_2) \in \partial\mathbb{B}_2$ and $\operatorname{Re}(\xi_1\bar{\xi_2}) = 0$.

To prove the opposite inclusion it suffices to observe that $1 = |\xi_1|^2 \pm 2\operatorname{Re}(\xi_1\bar{\xi_2}) + |\xi_2|^2 = |\xi_1 \pm \xi_2|^2$. □

PROOF OF PROPOSITION 9. Since $\mathbb{E}_{2,2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_1^2 - 4z_2| < 2\}$ is biholomorphic to $\mathbb{E}_{(1,\frac{1}{2})} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2| < 1\}$ and $\operatorname{Aut}(\mathbb{E}_{(1,\frac{1}{2})})$ is known (cf. [**14**], Theorem 2.3.4), we get $\operatorname{Aut}(\mathbb{E}_{2,2}) = \{\Phi_{c,\zeta} : c \in \mathbb{D},\ \zeta \in \mathbb{T}^2\}$, where

$$\Phi_{c,\zeta}(z_1, z_2) := \Big(\zeta_1\sqrt{2}h_c(\tfrac{z_1}{\sqrt{2}}), \frac{1}{2}\Big(\zeta_1^2 h_c^2(\tfrac{z_1}{\sqrt{2}}) - \frac{1}{2}\zeta_2(z_1^2 - 4z_2)\frac{1-|c|^2}{(1-\bar{c}\frac{z_1}{\sqrt{2}})^2}\Big)\Big),$$

with $c \in \mathbb{D}$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$.

Let $a = (a_0, a_0) \in \Delta_{2,2}$, i.e. $|a_0| < \frac{1}{\sqrt{2}}$. If $h_a = (h_1, h_2)$, then, for any $(\lambda_1, \lambda_2) \in \mathbb{B}_2$,

$$h_j(\lambda_1, \lambda_2) = \frac{\sqrt{1-2|a_0|^2}(2\lambda_j - \lambda_1 - \lambda_2) - 2a_0 + \lambda_1 + \lambda_2}{2(1 - \bar{a}_0(\lambda_1 + \lambda_2))}, \quad j = 1, 2,$$

and, consequently,

$$\begin{aligned} h_1(\lambda_1, \lambda_2) + h_2(\lambda_1, \lambda_2) &= \frac{\lambda_1 + \lambda_2 - 2a_0}{1 - \bar{a}_0(\lambda_1 + \lambda_2)}, \\ h_1(\lambda_1, \lambda_2)h_2(\lambda_1, \lambda_2) &= \frac{(\lambda_1 + \lambda_2 - 2a_0)^2 - (1 - 2|a_0|^2)(\lambda_1 - \lambda_2)^2}{4(1 - \bar{a}_0(\lambda_1 + \lambda_2))^2}, \\ h_1^2(\lambda_1, \lambda_2) + h_2^2(\lambda_1, \lambda_2) &= \frac{(\lambda_1 + \lambda_2 - 2a_0)^2 + (1 - 2|a_0|^2)(\lambda_1 - \lambda_2)^2}{2(1 - \bar{a}_0(\lambda_1 + \lambda_2))^2}. \end{aligned}$$

Next, if $\xi \in \partial\mathbb{B}_2$ with $\operatorname{Re}(\xi_1\bar{\xi}_2) = 0$ then, in virtue of Remark 8 (d),

$$\pi_2 \circ u_\xi \circ h_a = ((\xi_1 + \xi_2)(h_1 + h_2), (\xi_1^2 + \xi_2^2)h_1 h_2 + \xi_1\xi_2(h_1^2 + h_2^2)).$$

If we put $(z_1, z_2) = \pi_2(\lambda_1, \lambda_2)$ and use the fact that

$$(\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = z_1^2 - 4z_2,$$

then the relation $H_{u_\xi \circ h_a} \circ \pi_2 = \pi_2 \circ u_\xi \circ h_a$ and the equalities above give $H_{u_\xi \circ h_a} = \Phi_{c,\zeta}$, with $c = a_0\sqrt{2}$ and $\zeta = \zeta(\xi) = (\xi_1 + \xi_2, (\xi_1 - \xi_2)^2)$ which, together with Lemma 15, finishes the proof. $\square$

It remains to prove Proposition 10.

Proof of Proposition 10. *Re* (i). Suppose that $\operatorname{Aut}(\mathbb{E}_{2,n})$ acts transitively on $\mathbb{E}_{2,n}$. Then, by the Cartan classification theorem (cf. [**2**], [**10**]), $\mathbb{E}_{2,n}$ is biholomorphic to $\mathbb{B}_n$ or $\mathbb{D}^n$; a contradiction.

Indeed, in the case of $\mathbb{E}_{2,n} \simeq \mathbb{B}_n$, we use the characterization of proper holomorphic self-mappings of $\mathbb{B}_n$ due to H. Alexander (cf. [**3**] or [**18**], Theorem 15.4.2), saying that any such mapping is an automorphism. In the case of $\mathbb{E}_{2,n} \simeq \mathbb{D}^n$, we use the fact that there is no proper holomorphic mapping from $\mathbb{B}_n$ to $\mathbb{D}^n$ (cf. [**18**], Theorem 15.2.4).

Re (ii). Let $V := \{F(0) : F \in \operatorname{Aut}(\mathbb{E}_{2,n})\}$. By W. Kaups' theorem, V is a connected complex submanifold of $\mathbb{E}_{2,n}$ (cf. [**15**]). We already know that $\Sigma_{2,n} \subset V$ (Remark 8 (c)). Since $\operatorname{Aut}(\mathbb{E}_{2,n})$ does not act transitively (Proposition 10 (i)), then $V \subsetneq \mathbb{E}_{2,n}$. Thus $V = \Sigma_{2,n}$. Take a point $z = H_h(0) \in$

$\Sigma_{2,n}$ with $h \in \mathrm{Aut}(\mathbb{B}_n)$ (Remark 8 (c) again). Then for every $F \in \mathrm{Aut}(\mathbb{E}_{2,n})$ we get $F(z) = (F \circ H_h)(0) \in V = \Sigma_{2,n}$. □

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