DEMONSTRATIO MATHEMATICA
Vol. XLI No 4 2008

10.1515/dema-2013-0124

Włodzimierz M. Mikulski

ON NATURALITY OF THE LEGENDRE OPERATOR

Abstract. We deduce that all natural operators of the type of the Legendre operator from the variational calculus in fibred manifolds are the constant multiples of the Legendre operator.

Given fibred manifolds $Z_1 \to M$ and $Z_2 \to M$ over the same base M, we denote by $\mathcal{C}^\infty_M(Z_1, Z_2)$ the space of all base preserving fibred manifold morphisms of Z_1 into Z_2. Let $Y \to M$ be a fibred manifold, $dim(M) = m$, $dim(Y) = m+n$. Let $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ be an s-th order Lagrangian on $Y \to M$. Let $\delta\lambda \in \mathcal{C}^\infty_{J^sY}(J^sY, V^*J^sY \otimes \bigwedge^m T^*M)$ be the vertical differential of λ (the composition of the restriction $\tilde{\delta}\lambda : VJ^sY \to V\bigwedge^m T^*M = \bigwedge^m T^*M \times_M \bigwedge^m T^*M$ of the differential $d\lambda : TJ^sY \to T\bigwedge^m T^*M$ to the vertical sub-bundles with the second factor (essential) projection $\bigwedge^m T^*M \times_M \bigwedge^m T^*M \to \bigwedge^m T^*M$). Let $\Lambda(\lambda) : S^sT^*M \otimes VY \to \bigwedge^m T^*M$ be the restriction of the vertical differential $\delta\lambda : VJ^sY \to \bigwedge^m T^*M$ to the vector sub-bundle $S^sT^*M \otimes VY \subset VJ^sY$, the kernel of $V\pi^s_{s-1} : VJ^sY \to VJ^{s-1}Y$, where $\pi^s_{s-1} : J^sY \to J^{s-1}Y$ is the projection. The corresponding transformation $\Lambda(\lambda) : J^sY \to (\pi^{s-1})^*S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$ covering the identity of $J^{s-1}Y$ (where the pull-back is given by the projection $\pi^{s-1} : J^{s-1}Y \to Y$) is called the Legendre transformation determined by λ, [1]. It plays an important role in analytical mechanics, especially in the case of such λ (regular Lagrangians) for which $\Lambda(\lambda) : J^sY \to (\pi^{s-1})^*S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$ is a diffeomorphism (then it joints the Lagrange and Hamilton formalisms in fibred manifolds). Thus we have the Legendre operator

$$\Lambda : \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M) \to \mathcal{C}^\infty_Y(J^sY, V^*Y \otimes S^sTM \otimes \bigwedge^m T^*M)$$

sending a Lagrangian $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ into its Legendre transforma-

Key words and phrases: natural operator, Legendre operator.
1991 *Mathematics Subject Classification*: 58A20.

tion $\Lambda(\lambda) \in \mathcal{C}^\infty_Y(J^sY, V^*Y \otimes S^sTM \otimes \bigwedge^m T^*M)$. In the present paper we prove the following theorem.

THEOREM 1. *Any local $\mathcal{FM}_{m,n}$-natural regular operator*

$$A : \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M) \to \mathcal{C}^\infty_Y(J^sY, V^*Y \otimes S^sTM \otimes \bigwedge^m T^*M)$$

is of the form $A = c\Lambda$, $c \in \mathbf{R}$.

REMARK 1. We recall that $\mathcal{FM}_{m,n}$ is the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred embeddings. The naturality means that for any $\mathcal{FM}_{m,n}$-morphism $f : Y \to Y'$ and any s-th order Lagrangians $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ and $\lambda' \in \mathcal{C}^\infty_{M'}(J^sY', \bigwedge^m T^*M')$ if λ and λ' are f-related then so are $A(\lambda)$ and $A(\lambda')$. The regularity means that A transforms smoothly parametrized family of Lagrangians into smoothly parametrized family of respective type morphisms. The locality means that $A(\lambda)_u$ depends on the germ of λ at u.

Proof of Theorem 1. From now on let (x^i, y^k), $i = 1, \dots, m$, $k = 1, \dots, n$ be the usual fibred coordinates on $\mathbf{R}^{m,n}$, the trivial bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$. We will modify the proofs of the main results from the papers [3] and [4].

An $\mathcal{FM}_{m,n}$-morphism $\varphi : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$, $\varphi((x^i), (y^k)) = ((x^i), (y^k - \sigma^k(x^{i'})))$ sends $j^s_0((x), (\sigma^k))$ into

$$\Theta = j^s_0((x^i), (0)) \in (J^s\mathbf{R}^{m,n})_{(0,0)} \ .$$

Then (because of the invariance) A is uniquely determined by the evaluations

$$\langle A(\lambda)_\Theta, v \otimes \odot^s w \otimes u \rangle \in \mathbf{R}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s\mathbf{R}^{m,n}, \bigwedge^m T^*\mathbf{R}^m)$, all $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$, all $w \in T^*_0\mathbf{R}^m$ and all $u \in \bigwedge^m T_0\mathbf{R}^m$.

Using the invariance of A with respect to $\mathcal{FM}_{m,n}$-morphism of the form $id_{\mathbf{R}^m} \times \psi$ for linear isomorphisms $\psi : \mathbf{R}^n \to \mathbf{R}^n$, we get that A is uniquely determined by the evaluations

$$\langle A(\lambda)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s w \otimes v \rangle \in \mathbf{R}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s\mathbf{R}^{m,n}, \bigwedge^m T^*\mathbf{R}^m)$, all $w \in T^*_0\mathbf{R}^m$ and all $v \in \bigwedge^m T_0\mathbf{R}^m$.

Consider an arbitrary non-vanishing $f : \mathbf{R}^m \to \mathbf{R}$. There is a local diffeomorphism $F : \mathbf{R}^m \to \mathbf{R}^m$ such that $\frac{\partial F}{\partial x^1} = f$ and $F(0) = 0$. Then $\mathcal{FM}_{m,n}$-map $(F, x^2, \dots, x^m, y^1, \dots, y^n)^{-1}$ preserves Θ, $\frac{\partial}{\partial y^1}_0$ and sends $germ_0(d^mx)$ into $germ_0(fd^mx)$, where $d^mx = dx^1 \wedge \dots \wedge dx^m$. Then by the invariance, regularity and density arguments, A is uniquely determined by the evaluations

$$\langle A(\lambda + bd^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s w \otimes v\rangle \in \mathbf{R}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s\mathbf{R}^{m,n}, \bigwedge^m T^*\mathbf{R}^m)$ with the condition $\lambda(j^s_{x_o}((x^i),(0))) = 0$ for any $x_o \in \mathbf{R}^m$, all $b \in \mathbf{R}$, all $w \in T^*_0\mathbf{R}^m$ and all $v \in \bigwedge^m T_0\mathbf{R}^m$.

Then using the invariance of A with respect to $\mathcal{FM}_{m,n}$-maps of the form $\varphi \times id$ with linear isomorphisms $\varphi : \mathbf{R}^m \to \mathbf{R}^m$, we see that A is uniquely determined by the evaluations

$$\left\langle A(\lambda + bd^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s d_0x^m \otimes u^m\right\rangle \in \mathbf{R}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s\mathbf{R}^{m,n}, \bigwedge T^*\mathbf{R}^m)$ with the condition $\lambda(j^s_{x_o}((x^i),(0))) = 0$ and all $b \in \mathbf{R}$, where $u^m = \frac{\partial}{\partial x^1}_0 \wedge \cdots \wedge \frac{\partial}{\partial x^m}_0 \in \bigwedge^m T_0\mathbf{R}^m$.

Let λ and b be arbitrary as above. Using the invariance of A with respect to $\mathcal{FM}_{m,n}$-maps $\psi_\tau = ((x^i),(\frac{1}{\tau^k}y^k))$ for $\tau^k > 0$, we get the homogeneity condition

$$A\left((\psi_\tau)_*(\lambda + bd^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s d_0x^m \otimes u^m\right\rangle =$$
$$= \tau^1\left\langle A(\lambda + bd^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s d_0x^m \otimes u^m\right\rangle.$$

By Corollary 19.8 in [2] of the non-linear Peetre theorem we can assume that λ is a polynomial. It is easily seen that the coordinates of the polynomial $(\psi_\tau)_*\lambda$ are the multiplication by monomials in τ^k. The regularity of A implies that

$$\left\langle A(\lambda + bd^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s d_0x^m \otimes u^m\right\rangle$$

is smooth in the coordinates of λ and b. Then by the homogeneity function theorem (and the above type of homogeneity) we deduce that

$$\left\langle A(\lambda + bd^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s d_0x^m \otimes u^m\right\rangle$$

is a linear combination of the coordinates of λ on all $x^{\tilde\beta}y^1_\beta d^m x$ with coefficients being smooth functions in b, where (x^i, y^k_β) is the induced coordinate system on $J^s\mathbf{R}^{m,n}$. (Here and from now on β are m-tuples on non-negative integers with $|\beta| \le s$ and $\tilde\beta$ are arbitrary m-tuples of non-negative integers.) In other words A is determined by the values

$$(*) \qquad \left\langle A((ax^{\tilde\beta}y^1_\beta + b)d^m x)_\Theta, \frac{\partial}{\partial y^1}_0 \otimes \odot^s d_0x^m \otimes u^m\right\rangle = af^{\tilde\beta}_\beta(b)$$

for all $a, b \in \mathbf{R}$, all m-tuples $\tilde{\beta}$ and all m-tuples β as above.

Let $\tilde{\beta}_{i_o} \neq 0$ for some $i_o = 1, \dots, m$. We are going to use the invariance of A with respect to the locally defined $\mathcal{FM}_{m,n}$-map

$$\psi^{i_o} = (x^i, y^1 + x^{i_o} y^1, y^2, \dots, y^n)^{-1}$$

preserving x^i, Θ, $\frac{\partial}{\partial y^1}{}_0$, $d_0 x^m$ and u^m and sending y^1_β into

$$y^1_\beta + x^{i_o} y^1_\beta + y^1_{\beta - 1_{i_o}} \text{ (if } \beta_{i_o} = 0 \text{ then the third term do not occur)}$$

(because we have

$$\begin{aligned} y^1_\beta \circ J^s((\psi^{i_o})^{-1})(j^s_{(x^i_o)}(x^i, \sigma^k)) &= \partial_\beta(\sigma^1 + x^{i_o}\sigma^1)(x^i_o) \\ &= \partial_\beta \sigma^1(x^i_o) + x^{i_o} \partial_\beta \sigma^1(x^i_o) + \partial_{\beta - 1_{i_o}} \sigma^1(x^i_o) \\ &= (y^1_\beta + x^{i_o} y^1_\beta + y^1_{\beta - 1_{i_o}})(j^s_{(x^i_o)}(x^i, \sigma^k)) \ , \end{aligned}$$

where ∂_β is the iterated partial derivative as indicated multiplied by $\frac{1}{\beta!}$). Then applying ψ^{i_o} to the left hand side of (*) for $\tilde{\beta} - 1_{i_o}$ instead of $\tilde{\beta}$, we see that the value (*) is determined by the values (*) for $\tilde{\beta} - 1_{i_o}$ instead of $\tilde{\beta}$.

Continuing this procedure we see that the values (*) are determined by the values (*) for $\tilde{\beta} = (0)$. In other hand, A is determined by the values (*) for $\tilde{\beta} = (0)$ and all m-tuples β of non-negative integers with $|\beta| \leq s$ and all reals a, b.

By the invariance of A with respect to $\mathcal{FM}_{m,n}$-maps $((\tau^i x^i), (y^k))$ for $\tau^i \neq 0$ we get the homogeneity conditions

$$\tau^{-\beta + (1,\dots,1)} f^{(0)}_\beta(\tau^{(1,\dots,1)} b) = \tau^{(1,\dots,1,1-s)} f^{(0)}_\beta(b) \ .$$

If $m \geq 2$, then by the homogeneous function theorem this type of homogeneity gives that $f^{(0)}_\beta$ are zero if it is not $\beta_1 = \dots = \beta_{m-1} = \beta_m - s$. Then if $m \geq 2$, we give that $f^{(0)}_\beta = 0$ for all $\beta \neq (0, \dots, 0, s)$. Moreover, for $\beta = (0, \dots, 0, s)$, $f^{(0)}_\beta$ is constant. If $m = 1$ this type of homogeneity gives that $f^{(0)}_\beta = 0$ if $\beta \neq (s)$. Moreover $f^{(0)}_{(s)}$ is constant. Then A is determined by the value

$$\left\langle A(y^1_{(0,\dots,0,s)} d^m x)_\Theta, \frac{\partial}{\partial y^1} \otimes \odot^s d_0 x^m \otimes u^m \right\rangle \in \mathbf{R}$$

if $m \geq 2$, and by the value

$$\left\langle A(y^1_{(s)} d^1 x)_\Theta, \frac{\partial}{\partial y^1} \otimes \odot^s d_0 x^1 \otimes u^1 \right\rangle \in \mathbf{R}$$

if $m = 1$. Therefore the vector space of all A in question is 1-dimensional. This ends the proof of Theorem 1. ∎

References

[1] I. Kolář, *A geometrical version of higher order Hamilton formalism in fibred manifolds*, J. Geom. Phys. **1** (2)(1984), 127–137.
[2] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer, Berlin 1993.
[3] W. M. Mikulski, *On naturality of the Helmholtz operator*, Archivum Math. Brno **41** (2) (2005), 145–149.
[4] W. M. Mikulski, *On naturality of the formal Euler oprator*, Demonstratio Math. **38** (1) (2005), 235–238.

INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
Reymonta 4
KRAKÓW, POLAND
e-mail: mikulski@im.uj.edu.pl

Received October 26, 2007.