6 BHLUMI: the path to 0.01% theoretical luminosity precision

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The state of the art of theoretical calculations for the low-angle Bhabha (LABH) process $e^-e^+ \to e^-e^+$ is critically examined and it is indicated how to attain a level comparable to the FCC-ee experimental precision of 0.01%, see Ref. [255]. The analysis starts from the context of the LEP experiments, through their current updates, up to prospects of their improvements for the sake of the FCC-ee. Possible upgrades of the Monte Carlo event generator BHLUMI and other similar MC programs for the LABH process are also discussed.

In the following, we are going to recall the main aspects of the theoretical precision in the LEP luminosity measurement and current important components of the corresponding error budget. Next, we will present improvements on some of these components that are already feasible. Finally, we will discuss, in detail, prospects for reaching the 0.01% theoretical precision for the FCC-ee luminometry and outline ways of upgrading the main Monte Carlo program for this purpose, BHLUMI, in this respect.

Luminosity measurements of all four LEP collaborations at CERN, and also of the SLD at the SLAC, relied on theoretical predictions for the low-angle Bhabha process obtained using the BHLUMI Monte Carlo multiphoton event generator, featuring a sophisticated QED matrix element with soft photon resummation. Version 2.01 of this event generator was published in 1992 (see Ref. [256]) and the upgraded version, 4.04, is published in Ref [257].

The theoretical uncertainty of the BHLUMI Bhabha prediction, initially rated at 0.25% [258], was re-evaluated in 1996 after extensive tests and debugging to be 0.16% [259]. From that time, the code of BHLUMI version 4.04 used by all LEP collaborations in their data analysis remains frozen. All following re-evaluation of its precision came from investigations using external calculations outside the BHLUMI code. For instance, the 0.11% estimate of Ref. [260], was based on better estimates of the QED corrections missing in BHLUMI and on improved knowledge of the vacuum polarization contribution. The detailed composition of the final estimate of the theoretical uncertainty $\delta\sigma/\sigma \simeq 0.061\%$ of the BHLUMI 4.04 prediction in 1999, based on published work, is shown in the second column of Table C.5, where we use $L_e = \ln(|t|/m_e^2)$, following Ref. [261]. This value was used in the final LEP1 data analysis in Ref. [16]. Conversely, at LEP2, the experimental error was substantially greater than the QED uncertainty of the Bhabha process; see Ref. [261] for more details.

All four LEP collaborations quoted experimental luminosity errors below 0.05% for LEP1 data; this is less than the theoretical error. The best experimental luminosity error, 0.034%, was quoted by the OPAL collaboration[28] – they also quoted a slightly smaller theory error, 0.054%, using the improved light fermion pair calculations of Refs. [263, 264]; see also the review article [265] and workshop presentations [266, 267].

From the end of the LEP until the present time, there has been limited progress on practical calculations for low-angle Bhabha scattering at energies around and above the Z resonance.[29] A new Monte Carlo generator, BabaYaga, based on the parton shower algorithm, was developed [230, 231, 233, 265]. It was intended mainly for low-energy electron–positron colliders with $\sqrt{s} \leqslant 10$ GeV, claiming precision at 0.1%, but was not validated for energies near the Z peak.

There was, however, a steady improvement in the precision of the vacuum polarization in the t-channel photon propagator; see the recent review in the FCC-ee workshop [268]. Using the uncertainty $\delta\Delta^{(5)}_{\text{had.}} = 0.63 \times 10^{-4}$ at $\sqrt{-t} = 2$ GeV quoted in Ref. [118], one obtains $\delta\sigma/\sigma = 1.3 \times 10^{-4}$. This is shown in the third column in Table C.5, marked 'Update 2018'. The improvement of the light-pair corrections of Refs. [263, 264] is also taken into account there.

[28] The OPAL collaboration has found all their experimental distributions for low-angle Bhabha data to be in striking agreement with the BHLUMI Monte Carlo simulation [262].

[29] This is in spite of considerable effort on the $\mathcal{O}(\alpha^2)$ so-called 'fixed-order' (without resummation) QED calculations for the Bhabha process; see further for more discussion.

Table C.5: Summary of total (physical+technical) theoretical uncertainty for typical calorimetric LEP luminosity detector within a generic angular range of 18–52 mrad. Total error is summed in quadrature.

Type of correction or error	1999	Update 2018
(a) Photonic $\mathcal{O}(L_e\alpha^2)$	0.027% [269]	0.027%
(b) Photonic $\mathcal{O}(L_e^3\alpha^3)$	0.015% [270]	0.015%
(c) Vacuum polarization	0.040% [251, 271]	0.013% [268]
(d) Light pairs	0.030% [272]	0.010% [263, 264]
(e) Z and s-channel γ exchange	0.015% [260, 273]	0.015%
(f) Up–down interference	0.0014% [274]	0.0014%
(f) Technical precision	–	(0.027)%
Total	0.061% [261]	0.038%

Table C.6: Anticipated total (physical + technical) theoretical uncertainty for a FCC-ee luminosity calorimetric detector with an angular range of 64–86 mrad (narrow) near the Z peak. Description in brackets of photonic contributions is related to the 3rd column. Total error is summed in quadrature.

Type of correction or error	Update 2018	FCC-ee forecast
(a) Photonic $[\mathcal{O}(L_e\alpha^2)]$ $\mathcal{O}(L_e^2\alpha^3)$	0.027%	0.1×10^{-4}
(b) Photonic $[\mathcal{O}(L_e^3\alpha^3)]$ $\mathcal{O}(L_e^4\alpha^4)$	0.015%	0.6×10^{-5}
(c) Vacuum polarization	0.014% [268]	0.6×10^{-4}
(d) Light pairs	0.010% [263, 264]	0.5×10^{-4}
(e) Z and s-channel γ exchange	0.090% [273]	0.1×10^{-4}
(f) Up–down interference	0.009% [274]	0.1×10^{-4}
(f) Technical precision	(0.027)%	0.1×10^{-4}
Total	0.097%	1.0×10^{-4}

It should be emphasized that the technical precision, which is marked in parentheses as 0.027% in Table C.5, is not included in the total sum because, according to Ref. [260], it is included in the uncertainty of the photonic corrections. Future reduction of the photonic correction error will require a clear separation of the technical precision, which may turn out to be a dominant error, from other uncertainties.

Let us now address the following important question: *what steps are needed on the path to the* $\leq$0.01% *precision required for the low-angle Bhabha (LABH) luminometry at the FCC-ee experiments?* The last column in Table C.6 summarizes this goal component-by-component in the precision forecast for the FCC-ee luminometry. All necessary improvements in the next version or BHLUMI, which could bring us to the FCC-ee precision level, will also be specified.

In the following discussion, we must keep in mind certain basic features of the QED perturbative calculations for FCC-ee and LABH detectors. First of all, the largest photonic QED effects due to multiple real and virtual photon emission are strongly cut-off dependent. Monte Carlo implementation of QED perturbative results is mandatory, because the event acceptance of the LABH luminometer is quite complicated, and cannot be dealt with analytically. The LABH detector at the FCC-ee will be similar to that of the LEP, with calorimetric detection of electrons and photons (not distinguishing them) within the angular range $(\theta_{\rm min}, \theta_{\rm max})$ on opposite sides of the collision point [254]. The detection rings are divided into small cells; the angular range on both sides is slightly different, in order to minimize QED effects. The angular range at the FCC-ee is planned to be 64–86 mrads (narrow) [254], while at the LEP it was typically 28–50 mrads (narrow range, ALEPH/OPAL silicon detector); see Fig. 2 in Ref. [259] (also Fig. 16 in Ref. [275]) for an idealized detection algorithm of the generic LEP silicon detector. The average t-channel transfer near the Z resonance will be $|\bar{t}|^{1/2} = \langle|t|\rangle^{1/2} \simeq 3.25\,\mathrm{GeV}$ at the FCC-ee, instead of $1.75\,\mathrm{GeV}$ at the LEP.[30] The important scale factor controlling photonic QED effects, $\gamma = \frac{\alpha}{\pi} \ln \frac{|\bar{t}|}{m_e^2} = 0.042$ for the FCC-ee, is only slightly greater than 0.039 for the LEP. Conversely, the factor $x = |t|/s$ suppressing s-channel contributions will be 1.27×10^{-3}, significantly larger than 0.37×10^{-3} for the LEP.

The *photonic higher-order and subleading corrections* components in Tables C.5 and C.6 are large, but this is mainly due to collinear and soft singularities. Fortunately, they are known in QED at any perturbative order, hence can be resummed to infinite order. Why is the cross-section of the LABH luminometer highly sensitive to the emission of real soft and collinear photons? This is because the emission of even very soft or collinear photons in the initial state (ISR) may pull final electrons outside the acceptance angular range, while final-state photons can easily change the shape of the final-state 'calorimetric cluster' in the detector. This is why resummation of the multiphoton effects is of paramount importance, and must be implemented in an exclusive way, using the method of exclusive exponentiation (EEX), as in BHLUMI [257], or the parton shower (PS) method, as in BabaYaga [230]. It is well-known, see, for instance, Ref. [276], that the so-called 'fixed-order' $\mathcal{O}(\alpha^2)$ calculations without resummation[31] are completely inadequate for the LABH luminometry.

Photonic uncertainty of $\mathcal{O}(L_e\alpha^2)$ in item (a) in Table C.5, neglected in the matrix element in BHLUMI version 4.04, scales as $L_e = \ln(|t|/m_e^2)$, where t is the relevant squared momentum transfer. Assuming that the technical precision is dealt with separately (see the later discussion), this item will disappear from the error budget completely once the EEX matrix element of BHLUMI is upgraded to include missing $\mathcal{O}(L_e\alpha^2)$ contributions, which are already known and published. In fact, these $\mathcal{O}(L_e\alpha^2)$ corrections consist of two real-photon contributions, one-loop corrections to one-real emission, and two-loop corrections.

Efficient numerical and analytic methods of calculating the exact $\mathcal{O}(\alpha^2)$ matrix element (spin amplitudes) for two real photons, keeping fermion masses, have been known for decades; see Refs. [278, 279]. In Ref. [280], exact two-photon amplitudes were compared with the matrix element of BHLUMI. The pioneering work on $\mathcal{O}(L_e\alpha^2, L_e^0\alpha^2)$ virtual corrections to one-photon distributions was reported in Ref. [281]. These were calculated by neglecting interference terms between e^+ and e^- lines, which, near the Z peak, are of the order of $\left(\frac{\alpha}{\pi}\right)^2 \frac{|t|}{s} L_e \sim 10^{-7}$ multiplied by some logarithm of the cut-off. Note that, from the s-channel analogue in Ref. [282], we know that the pure $\mathcal{O}(L_e\alpha^2)$ correction of this class (neglecting the $\mathcal{O}(L_e^0\alpha^2)$ term) is amazingly compact – it consists of merely a three-line formula at the amplitude level.[32] Finally, in Ref. [261], the two-loop $\mathcal{O}(L_e\alpha^2)$ t-channel photon form factor relevant for the LABH process (keeping in mind $|t|/s$ suppression) was continued analytically from the known s-channel result of Ref. [284], thus completing the entire photonic $\mathcal{O}(L_e\alpha^2)$ matrix element. This is known but not yet included in the MC BHLUMI version 4.04. Once the well-known photonic $\mathcal{O}(L_e\alpha^2)$ part is added in the future upgrade of the EEX matrix element in BHLUMI, the corresponding item will disappear from the list of projected FCC-ee luminometry uncertainties, as in Table C.6.

In view of this discussion, it is clear that the major effort of calculating the complete $\mathcal{O}(\alpha^2)$ QED correction to low and wide-angle Bhabha processes in Refs. [146,243,277,285], see also Refs. [186,191,192,286], is of rather limited practical importance for the LABH luminometry at the FCC-ee. It is more relevant for the wide-angle Bhabha processes, provided they are included in the MC with soft photon resummation. However, this is rather problematic, because, in all these works, soft real-photon contributions are added to loop corrections à la Bloch–Nordsieck, instead of subtracting the well-known virtual form factor from virtual loop results already at the amplitude level, before squaring them. All these works essentially add previously unknown $\mathcal{O}(L_e^0\alpha^2)$ corrections, which are of the order of $\sim 10^{-5}$.

Another important photonic correction, listed as item (b) in Table C.5 as an uncertainty of BHLUMI, is the $\mathcal{O}(\alpha^3 L_e^3)$ correction (third-order LO). It is already known from Ref. [270, 287] and is currently omitted in the EEX matrix element of version 4.04 of BHLUMI, although it is already included in the LUMLOG subgenerator

[30] At 350 GeV, the FCC-ee luminometer will have $|\bar{t}| = 12.5$ GeV.

[31] For instance, the calculations of Refs. [146, 277].

[32] Let us add for completeness that this correction was also calculated numerically [283].

of BHLUMI based on collinear kinematics. It would be relatively simple to add this class of corrections to the main EEX matrix element of BHLUMI, such that it would disappear from the uncertainty list. Once this is done, the uncertainty due to $\mathcal{O}(\alpha^4 L_e^4)$ and $\mathcal{O}(\alpha^3 L_e^2)$ should be estimated and included in the list of photonic uncertainties of the updated BHLUMI. Using scaling rules of thumb indicated in the previous discussion, one may estimate an error due to missing $\mathcal{O}(\alpha^4 L_e^4)$ as $0.015\% \times \gamma = 0.6 \times 10^{-5}$ near the Z peak and that of the missing $\mathcal{O}(\alpha^3 L_e^2)$ should also be of a similar order, $\gamma^2\alpha/\pi \simeq 10^{-5}$, although such estimates are rather uncertain.

The so-called up–down interference between photon emission from e^+ and e^- lines was calculated in Ref. [274] at $\mathcal{O}(\alpha^1)$ to be roughly $\delta\sigma/\sigma \simeq 0.07\ |t|/s$. At LEP1, this contribution was rated as negligible, see Table C.5, but at the FCC-ee luminometer it will be a factor of ten larger, owing to the wider angular range of the FCC-ee detector, and must be included in the matrix element of the upgraded BHLUMI. Once this is done, its uncertainty will be negligible, see Table C.6, where we used $2\gamma \times 0.07\ |t|/s$, as a crude estimator.

BHLUMI multiphoton distributions obey a clear separation into an exact Lorentz invariant phase space and a squared matrix element. The matrix element is an independent part of the program and is currently built according to exclusive exponentiation (EEX) based on the Yennie–Frautshi–Suura [196] (YFS) soft photon factorization and resummation performed on the spin-summed squared amplitude. It includes complete $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(L_e^2\alpha^2)$ corrections, neglecting interference terms between electron and positron lines, suppressed by a $|t|/s$ factor. It has not been changed in the upgrades since version 2.01 [256]. It would be desirable to introduce the results from Refs. [261, 270, 281, 287] into the EEX matrix element, that is $\mathcal{O}(\alpha^2 L_e)$ and $\mathcal{O}(\alpha^3 L_e^3)$, again neglecting some $\sim|t|/s$ terms.

Conversely, a preferable solution would be to keep the same underlying multiphoton phase space MC generator of BHLUMI and exploit the results from Refs. [261, 270, 281, 287] to implement a more sophisticated matrix element of the CEEX [149] type, where CEEX stands for coherent exclusive exponentiation. In the CEEX resummation methodology, soft photon factors are factorized at the amplitude level and matching with fixed-order results is also done at the amplitude level (before squaring and spin summing). The big advantage of CEEX over EEX is that the separation of the infrared (IR) parts and matching with the fixed-order result are much simpler and more transparent – all IR cancellations for complicated interferences are managed automatically and numerically. The inclusion of the s-channel Z and photon exchange and t-channel Z exchange including $\mathcal{O}(\alpha)$ corrections, soft photon interference between electron and positron lines, and so on, would be much easier to take into account for CEEX than in the case of EEX. However, the inclusion of $\mathcal{O}(\alpha^3 L_e^3)$ in CEEX will have to be worked out and implemented.

The uncertainty of the low-angle Bhabha cross-section due to *imprecise knowledge of the QED running coupling constant* of the t-channel photon exchange is simply $\delta_{VP}\sigma/\sigma = 2\delta\alpha_{eff}(\bar{t})/\alpha_{eff}(\bar{t})$, where $\bar{t}$ is the average transfer of the t-channel photon. For the FCC-ee luminometer, it will be $|\bar{t}|^{1/2} \simeq 3.5$ GeV near the Z peak and $|\bar{t}|^{1/2} \simeq 13$ GeV at 350 GeV. The uncertainty of $\alpha_{eff}(t)$ is mainly due to the use of the experimental cross-section σ_{had} for $e^-e^+ \to$ hadrons below 10 GeV as an input to the (subtracted) dispersion relations. A comprehensive review of the corresponding methodology and the latest update of the results can be found in Refs. [288, 289]; see also the FCC-ee workshop presentation [268]. The hadronic contribution to α_{eff} from the dispersion relation is encapsulated in $\Delta\alpha^{(5)}(-s_0)$, where 2 GeV $\leq s_0^{1/2} \leq$ 10 GeV in order to minimize the dependence on $\sigma_{had}(s)$, such that the main contribution comes from $s^{1/2} \leq 2$ GeV. [33]

The parameter range 2 GeV $\leq s_0^{1/2} \leq$ 10 GeV, which is incidentally of paramount interest for FCC-ee luminometry, is part of a wider strategy in Refs. [288, 289] of obtaining $\alpha_{eff}(M_Z^2)$ in two steps. First, $\Delta\alpha^{(5)}(-s_0)$ is obtained from dispersion relations and next the difference $\Delta\alpha^{(5)}(M_Z^2) - \Delta\alpha^{(5)}(-s_0)$ is calculated using the perturbative QCD technique of the Adler function [290]. Taking $s_0^{1/2} = 2.0$ GeV and the value $\Delta\alpha^{(5)}(-s_0) = (64.09 \pm 0.63) \times 10^{-4}$ of Ref. [118] as a benchmark, in Table C.5 we quote $(\delta_{VP}\sigma)/\sigma = 1.3 \times 10^{-4}$. With the anticipated improvements of data for $\sigma_{had}(s)$, $s^{1/2} \leq 2.5$ GeV, one may expect an improvement of a factor of two by the time of the FCC-ee experiments, resulting in $\delta_{VP}\sigma/\sigma = 0.65 \times 10^{-4}$

[33] The main contribution to the error in the measurement of the muon $g-2$ is from the same cross-section range [288].

near the Z peak, see Table C.6. At the high-energy end of the FCC-ee, 350 GeV, owing to the increase in the average transfer $|\bar{t}| = 12.5$ GeV, one currently obtains, from the dispersion relation, $\delta\alpha_{\rm eff}/\alpha_{\rm eff} = 1.190 \times 10^{-4}$ and $\delta_{\rm VP}\sigma/\sigma \simeq 2.4 \times 10^{-4}$, and again with the possible improvement of a factor of two, so that the FCC-ee expectation[34] is $(\delta_{\rm VP}\sigma)/\sigma \simeq 1.2 \times 10^{-4}$.

There are also alternative proposals for the measurement of $\alpha_{\rm eff}(t)$, not relying (or relying less) on dispersion relations – in Ref. [249], it was proposed to measure $\alpha_{\rm eff}(t)$ directly, for $t \sim -1$ GeV^2, from the elastic scattering of energetic muons on atomic electrons; this sounds interesting, but requires more studies.

The *light fermion pair effect* is not currently included in BHLUMI; hence, all of it is accounted for in the error budget of Table C.6 (current state of the art).

Let us start by summarizing the LEP-era state of the art. The largest correction to the light fermion pair due to additional electron pair production is given in Ref. [263], where the process $\rm e^+e^- \to e^+e^-e^+e^-$ is calculated using the numerical approach of Ref. [291], combined with virtual or soft corrections of Refs. [33, 292, 293], resulting in a theoretical error of 0.01%.[35] In Refs. [294, 295], a semi-analytical method of obtaining NLO accuracy is exploited, omitting non-logarithmic corrections and taking virtual corrections from Refs. [292, 293]. The third-order LO corrections due to simultaneous emission of the additional $\rm e^+e^-$ pair (non-singlet and singlet) and additional photons are also evaluated there. The overall precision of the results in Refs. [294, 295] is estimated to be 0.006%, mainly owing to the omission of the heavier lepton pairs ($\mu^+\mu^-$, $\tau^+\tau^-$) and quark pairs (0.005%). In Ref. [296], the complete LO semi-analytical calculations of additional pair production based on the electron structure functions technique are given up to the third-order for the non-singlet[36] and singlet structure functions. Contrary to Ref. [294], results are also provided for the asymmetric acceptances. In the approach of Ref. [272], based on the extension of the YFS [196], soft $\rm e^+e^-$ pair emission are resummed similarly as soft photons (omitting up–down interference, multiperipheral graphs, etc.) – with relevant real and virtual soft ingredients calculated in Ref. [297]. Numerical implementation exists in the unpublished BHLUMI version 2.30 MC code. The accuracy of the results was estimated to be 0.02% for the asymmetric angular acceptance, i.e., 3.3°–6.3° and 2.7°–7.0°, with the energy cut $1 - s'/s < z_{\rm cut} = 0.5$. Reference [263] claims that this precision is even better, 6×10^{-5} for $z_{\rm cut} \leq 0.5$, while for hard emission, $z_{\rm cut} > 0.5$, owing to significant multiperipheral components, it deteriorates to 0.01%.

What should be done to consolidate or extend these, mostly LEP-era, calculations of the fermion pair contribution and to reach an even better precision level needed for the FCC-ee?

As in Ref. [263], for the additional real $\rm e^+e^-$ pair radiation, the complete matrix element should be used, because non-photonic graphs can contribute as much as 0.01% for the cut-off $z_{\rm cut} \sim 0.7$. A number of MC generators for the $\rm e^+e^- \to 4f$ process, developed for the LEP2 physics, are available to be exploited for that purpose. To improve on the 0.005% uncertainty of Ref. [294], owing to the emission of the $\mu^+\mu^-$, $\tau^+\tau^-$, and quark pairs, one may use LO calculation of Ref. [296], incorporating lepton pair contributions by means of the modification of the running coupling. A naive rescaling of the electron logarithm (owing to the mass of the muon) $\ln^2 \frac{|t|}{m_\mu^2} / \ln^2 \frac{|t|}{m_{\rm e}^2} = 0.16$, applied to the $\rm e^+e^-$ pair contribution of 0.05% provides an estimate of the muon pair contribution of 0.008%.[37] A similar estimate for tau lepton pairs shows that this contribution can be neglected. Note that adding $\mu^+\mu^-$ pairs to the BHLUMI version 2.30 code of Ref. [297] would be straightforward. Moreover, in the approach of Refs. [263, 264] this should be possible.[38] The contribution of light quark pairs (π pairs, etc.) can be roughly estimated using the quantity $\rm R_{had} = \sigma_{had}/\sigma_\mu \simeq 3$ for the effective hadronic production threshold of the order of 1 GeV. One obtains $\rm R_{had} \ln^2 \frac{|t|}{0.5^2\ GeV^2} / \ln^2 \frac{|t|}{m_\mu^2} = 0.9$,

[34] We thank F. Jegerlehner for elucidating private communications on these predictions.

[35] The emission of a μ-pair is also discussed in Ref. [263].

[36] This is contrary to the incorrect statement in Ref. [294]: third-order non-singlet $\rm e^+e^-\gamma$ corrections are realized in Ref. [296] by second-order structure functions with running coupling.

[37] This is less optimistic than the estimate in Ref. [294].

[38] The other option is to use the previously described general-purpose LEP2 4f codes, including also the earlier-discussed corresponding virtual corrections.

i.e., this contribution is of the size of the muon pair contribution, that is, of the order of 0.008%. (This is less optimistic than the estimate in Ref. [294]. Adding quadrature errors due to muon and light quark pairs, one obtains 0.011%, rather than the 0.006% of Ref. [294]. However, it is consistent with the estimate of Ref. [263].)

Another important group of corrections to light fermion pair emission are the higher-order terms. The emission of two (or more) electron pairs is suppressed by another factor $(\frac{\alpha}{\pi}\ln\frac{|t|}{m_e^2})^2 \sim 10^{-3}$ and is negligible. The additional $e^+e^-+n\gamma$ correction is non-negligible. Its evaluation was based either on LO structure functions (Table 1 in Ref. [294], Fig. 8 in Ref. [263], [296]) or on the YFS [196] soft approximation (Fig. 4 in Ref. [272]), resulting in quite different results, and their comparison is rather inconclusive. They are, at most, of the order of 0.5–0.75 of the additional e^+e^- correction (without γ). The remaining non-leading, non-soft additional $e^+e^- + n\gamma$ corrections are suppressed by another $1/\ln\frac{|t|}{m_e^2} \sim 0.06$ and should be negligible ($\sim 0.003\%$). It would also be possible to calculate the additional $e^+e^- + \gamma$ real emission in a way similar to the existing code for LEP2 physics [298].

We summarize as follows on uncertainties due to light fermion pair emissions: (1) the contribution of light quark pairs must be calculated with an accuracy of 25%, i.e., 0.0027%; (2) the contribution of the muon pairs will be known to 10%, i.e., to 0.0008%; (3) the non-leading, non-soft additional $e^+e^- + n\gamma$ corrections will be treated as an error of 0.003%. Adding points (1)–(3) in quadrature, we obtain 0.004%. Applying a safety factor of 1.25, we end up with 0.005% for the possible pair production uncertainty forecast for the FCC-ee, quoted in Table C.6. These improvements can be implemented either directly in the upgraded `BHLUMI` or using a separate calculation, such as `BHLUMI` version 2.30 [272] code, or external MC programs, like those of Refs. [263, 264].

In addition to γ exchange in the t-channel γ_t, there are also, in the Bhabha process, contributions from γ exchange in the s-channel γ_s and Z exchange in both t- and s-channels, Z_t and Z_s. Once they are added at the amplitude level (relevant Feynman diagrams) and then squared to obtain the differential cross-section, $|\gamma_t+Z_t+\gamma_s+Z_s|^2$ gives rise to six interference and three squared contributions. Apart from the pure t-channel γ exchange, $\gamma_t\otimes\gamma_t$, numerically the most important are interferences of other contributions with the γ_t amplitude, owing to the enhancement factor $\sim s/|t|$. Near the Z peak, among them, the most sizeable is the interference $\gamma_t \otimes Z_s$. This is discussed in detail in Ref. [273] for two types of detector: SICAL, with an angular coverage of $\sim 1.5°$–$3°$, and LCAL, with an angular coverage of $\sim 3°$–$6°$. The implementation of the $\gamma_t \otimes Z_s$ contribution in `BHLUMI` version 4.02 and the assessment of its theoretical precision for the LEP luminosity measurement is based on this work. The following estimated theoretical errors of all other contributions beyond the dominant $\gamma_t \otimes \gamma_t$ will be also based on the study of Ref. [273]. Since the angular coverage of the planned FCC-ee luminometer [254] is close to the LCAL one, we shall exploit the corresponding results of Ref. [273].

The Born level $\gamma_t \otimes Z_s$ contribution is up to $\sim 1\%$ and changes from being positive below the Z peak to negative above it, reaching the maximal absolute value at about ± 1 GeV from the peak. Corrections due to additional photon emission are sizeable, up to $\sim 0.5\%$. `BHLUMI` includes the QED corrections and running-coupling effects for this contribution within the $\mathcal{O}(\alpha)$ YFS exclusive exponentiation. The theoretical uncertainty for $\gamma_t \otimes Z_s$ was estimated at 0.090% for the LCAL detector and is used as an initial estimate of the theoretical error in Table C.6.

The other contributions will be estimated by means of relating them to the $\gamma_t \otimes Z_s$ or $\gamma_t \otimes \gamma_t$, using rescaling factors, $|t|/s \approx 1.3 \times 10^{-3}$ and $\tilde{\gamma}_Z = \Gamma_Z/M_Z \approx 2.7 \times 10^{-2}$. The interference $\gamma_t \otimes \gamma_s$ (included in `BHLUMI`), the next-most-sizeable contribution near the Z peak, is smaller at the Born level than the $\gamma_t \otimes Z_s$ contribution by the factor[39] $\sim 4\,\tilde{\gamma}_Z \approx 0.1$. Taking $\sim 1\%$ for the Born level $\gamma_t \otimes Z_s$, we get $\sim 0.1\%$ for $\gamma_t \otimes \gamma_s$. For not-too-tight cuts on radiative photons, the missing photonic QED corrections are smooth near the Z peak and should stay within 10%, leading to an estimate of the theoretical precision of the $\gamma_t \otimes \gamma_s$ contribution in `BHLUMI` for the FCC-ee luminometry of $\sim 0.01\%$. At the Born level, the resonant $Z_s \otimes Z_s$, is smaller than the $\gamma_t \otimes Z_s$ term by the factor $\sim |t|/s \times 1/(4\,\tilde{\gamma}_Z) \approx 1.3 \times 10^{-2}$; thus, its size is $\sim 0.01\%$. This enters into the theoretical error as a whole, as it is omitted in the current version of `BHLUMI`. However, it can be included

[39] The factor of four comes from the ratio of the corresponding coupling constants.

rather easily, such that, finally, only missing radiative corrections will matter. They can reach $\sim 50\%$ of the Born level contribution, owing to the presence of the Z resonance; hence, the corresponding theoretical error estimate is $\sim 0.005\%$. The less important t-channel interference $\gamma_t \otimes Z_t$ is estimated by multiplying the $\gamma_t \otimes Z_s$ contribution by the $\sim |t|/s \times \tilde{\gamma}_{\rm Z} \approx 3.5 \times 10^{-5}$ factor. It can be easily implemented in BHLUMI, by leaving out only photonic corrections below 10^{-5}. The non-resonant s-channel $\gamma_s \otimes \gamma_s$ contribution is suppressed by the factor $\sim (4\,\tilde{\gamma}_{\rm Z})^2 \approx 0.01$ with respect to $Z_s \otimes Z_s$ (which is worth $\sim 0.01\%$), so is rated at an order of 10^{-6}. Finally, the $Z_t \otimes Z_t$ contribution is smaller than the dominant $\gamma_t \otimes \gamma_t$ one by the factor $\sim (|t|/s/4)^2 < 10^{-6}$; thus, it is completely negligible.

Combining all these theoretical errors in the quadrature, the total uncertainty (contributions omitted in BHLUMI) due to the Z exchanges and γ_s exchange for the FCC-ee luminometer near the Z peak is estimated at the level of 0.090% and quoted as the current state of the art (for the FCC-ee luminometer) in Table C.6.

It is worth emphasizing that this uncertainty is completely dominated by the uncertainty of the $\gamma_t \otimes Z_s$ contribution, which comes from a rather conservative estimate in Ref. [273] based on comparisons of BHLUMI with the MC generator BABAMC [299] and the semi-analytical program ALIBABA [300, 301], the latter including higher-order leading-log QED effects. Later on, the new MC event generator BHWIDE [235] was developed for wide-angle Bhabha scattering, including all Born level contributions for the Bhabha process and $\mathcal{O}(\alpha)$ YFS exponentiated EW radiative corrections. A comparison of BHLUMI with BHWIDE for the FCC-ee luminometer would help to reduce all these theoretical errors. In principle, not only the Born level but also the $\mathcal{O}(\alpha)$ QED matrix elements of BHWIDE, could be implemented in the EEX-style matrix elements of BHLUMI. This would reduce the theoretical error for this group of contributions below 0.01%, as indicated in Table C.6. However, the most efficient and elegant way to reduce the uncertainty of these contributions practically to zero would be to include these Z exchanges and s-channel photon exchange into the CEEX matrix element at $\mathcal{O}(\alpha^1)$ in BHLUMI. Concerning EW corrections, it would most probably be sufficient to add them in the form of effective couplings in the Born amplitudes. Conversely, such a CEEX matrix element with the Z exchanges in BHLUMI would serve as a starting point for a better wide-angle Bhabha MC generator, much as BHLUMI version 4.04 served as a starting point for BHWIDE [235].

The question of the *technical precision* due to programming bugs, numerical instabilities, technical cut-off parameters, etc. is quite non-trivial and most difficult. The evaluation of the technical precision of BHLUMI version 4.04 with YFS soft photon resummation and complete $\mathcal{O}(\alpha^1)$ relies on two pillars: the comparison with semi-analytical calculations given in Ref. [287] and comparisons with two hybrid MC programs LUMLOG+OLDBIS and SABSPV, reported in Ref. [275]. This precision was established to be 0.027% (together with missing photonic corrections). This was not an ideal solution, because the two hybrid MCs did not feature complete soft photon resummation and disagreed with BHLUMI by more than 0.17% for sharp cut-offs on the total photon energy.

Conversely, after the LEP era, another MC program, BabaYaga [230, 231, 233], with soft photon resummation, has been developed using a parton shower (PS) technique, and could, in principle, also be used for a better validation of the technical precision of both BHLUMI and BabaYaga. In fact, such a comparison with BHWIDE MC [235] was made for $s^{1/2} \leq 10$ GeV and 0.1% agreement was found. It is quite likely that such an agreement persists near $s^{1/2} \simeq M_{\rm Z}$. Note that the complete $\mathcal{O}(\alpha^1)$ was included in BabaYaga before the three technologies of matching fixed-order NLO calculations with a parton shower (PS) algorithm were unambiguously established: MC@NLO [302], POWHEG [303], and KrkNLO [304]. The algorithm of NLO matching in BabaYaga is quite similar to that of KrkNLO. (Single MC weight introduces NLO correction in both methods, but in KrkNLO it sums over real photons, while in BabaYaga it takes the product over them. However, it is the same when truncated to $\mathcal{O}(\alpha^1)$. We are grateful to the authors of BabaYaga for clarification on this point.)

Ideally, in the future validation of the upgraded BHLUMI, to establish its *technical precision* at a level of 10^{-5} for the total cross-section and of 10^{-4} for single differential distributions, one would need to compare it with another MC program developed independently, which properly implements the soft photon resummation, LO corrections up to $\mathcal{O}(\alpha^3 L_{\rm e}^3)$, and second-order corrections with the complete $\mathcal{O}(\alpha^2 L_{\rm e})$. In principle, an extension of a program like BabaYaga to the level of NNLO for the hard process, keeping the correct soft photon resummation, would be the best partner for the upgraded BHLUMI, to establish the technical precision of both programs at the 10^{-5} precision level.[40] In the meantime, a comparison between the upgraded BHLUMI with EEX and CEEX matrix elements would also offer a very good test of its technical precision, since the basic multiphoton phase space integration module of BHLUMI was already well tested [287] and such a test can be repeated at an even higher precision level.

Summarizing, we conclude that an upgraded new version of BHLUMI with the error budget of 0.01% shown in Table C.6 is perfectly feasible. With appropriate resources, such a version of BHLUMI with the $\mathcal{O}(\alpha^2)$ CEEX matrix element and a precision tag of 0.01%, necessary for FCC-ee physics, could be realized. Keeping in mind that the best experimental luminosity error achieved at the LEP was 0.034% [262], it would be interesting to study whether the systematic error of the designed FCC-ee luminosity detector [253] can match this anticipated theory precision.

Let us also remark that the process $\mathrm{e}^+\mathrm{e}^- \rightarrow 2\gamma$ is also considered for FCC-ee luminometry, see Refs. [234, 265] for further discussion on the QED radiative corrections to this process.

[40] The upgrade of the BHLUMI distributions will be relatively straightforward because its multiphoton phase space is exact [34] for any number of photons.