Zeta Functions of Pseudodifferential Operators and Fourier Integral Operators on Manifolds with Boundary

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Abstract

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The aim of the first part of the thesis is the study of the asymptotic behavior of the eigenvalue counting function of selfadjoint operators in three different settings: SG-operators on $\mathbb{R}^n$ and on manifolds with cylindrical ends; bisingular operators defined on $M_1 \times M_2$, product of two closed manifolds; bisingular operators on Euclidean spaces. A precise formula for the first term in the asymptotic expansion is given and, in a particular case, the second term is determined as well. The results are achieved by the study of the complex powers of operators and of the spectral ζ-function. This analysis, in the case of SG-operators, leads also to the definition of non-commutative residue both in the case of $\mathbb{R}^n$ and of manifolds with cylindrical ends. Moreover, in the case of $\mathbb{R}^n$, endowed with a suitable metric, by mean of a regularized integral, a connection between the non-commutative residue and the Einstein-Hilbert action is showed.

The second part of the thesis treats an extension of Fourier Integral Operators (FIO) in the half space $\mathbb{R}^n_+$. As in Boutet de Monvel's calculus, we define a matrix of operators

$$\begin{pmatrix} r^+\mathrm{Op}^{\psi}(a)e^+ G^{\psi_\partial} & K^{\psi_\partial} \\ T^{\psi_\partial} & S^{\psi_\partial} \end{pmatrix}$$

where $\mathrm{Op}^{\psi}(a)$ is a Fourier Integral Operator defined by a symplectomorphism $\chi : T^*\mathbb{R}^n_+ \setminus \{0\} \to T^*\mathbb{R}^n_+ \setminus \{0\}$, represented in a neighborhood of the boundary by the phase function ψ, and by a principal symbol a; r^+ is the restriction operator, e^+ is the extension operator. In order to have nice continuity results in the scale of the $H^s(\mathbb{R}^n_+)$ Sobolev spaces, we need to impose conditions on the symplectomorphism. Essentially, we require that the symplectomorphism preserves the boundary and that all the components satisfy the transmission property. If χ fulfills these properties, then it induces a symplectomorphism $\chi_\partial : T^*\mathbb{R}^{n-1} \to T^*\mathbb{R}^{n-1}$, represented by the phase function ψ_∂. The operator G^{ψ_∂} is a FIO with phase ψ_∂ on $\partial\mathbb{R}^n_+$ and a singular Green symbol. The operator K^{ψ_∂} is a FIO defined on $\partial\mathbb{R}^n_+$ with phase ψ_∂ and with a potential symbol, T^{ψ_∂} is a FIO defined on $\partial\mathbb{R}^n_+$ with a trace symbol, S^{ψ_∂} is a usual FIO defined on $\partial\mathbb{R}^n_+$ with phase ψ_∂.

It is the goal of the thesis to show continuity results for such operators in the scale of $H^s(\mathbb{R}^n_+)$ spaces and to establish results similar to those of Boutet de Monvel.

Keywords: Spectral zeta function, Fourier Integral Operators, Boutet de Monvel's Calculus

Zusammenfassung

Ziel des ersten Teils dieser Dissertation ist die Untersuchung des asymptotischen Verhaltens der Zählfunktion der Eigenwerte selbstadjungierter Operatoren in drei verschiedenen Situationen: Für *SG*-Operatoren auf $\mathbb{R}^n$ und Mannigfaltigkeiten mit zylindrischen Enden, für bisinguläre Operatoren, die auf dem kartesischen Produkt $M_1 \times M_2$ zweier geschlossener Mannigfaltigkeiten definiert sind, sowie für bisinguläre Operatoren auf dem euklidischen Raum.

Eine genaue Formel für den ersten Term in der asymptotischen Entwicklung wird angegeben; in Spezialfällen wird auch der zweite Term bestimmt. Die Ergebnisse werden mit Hilfe einer Untersuchung komplexer Operatorpotenzen und der spektralen Zetafunktion erzielt.

Im Fall von *SG*-Operatoren führt diese Analyse weiterhin zur Definition eines nichtkommutativen Residuums sowohl auf $\mathbb{R}^n$ als auch auf Mannigfaltigkeiten mit zylindrischen Enden. Darüber hinaus wird für den Fall, dass $\mathbb{R}^n$ mit einer geeigneten Metrik versehen ist, mit Hilfe eines regularisierten Integrals ein Zusammenhang zwischen dem nichtkommutativem Residuum und der Einstein-Hilbert-Wirkung gezeigt.

Im zweiten Teil der Dissertation wird eine Erweiterung des Kalküls der Fourierintegraloperatoren (FIO) auf den Halbraum $\mathbb{R}^n_+$ behandelt. Wie in dem Kalkül von Boutet de Monvel definieren wir eine Matrix von Operatoren

$$\begin{pmatrix} r^+ A^\psi e^+ G^{\psi_\partial} & K^{\psi_\partial} \\ T^{\psi_\partial} & S^{\psi_\partial} \end{pmatrix}. \tag{1}$$

Dabei ist A^ψ ein Fourierintegraloperator, gegeben durch einen Symplektomorphismus $\chi : T^*\mathbb{R}^n_+ \setminus \{0\} \to T^*\mathbb{R}^n_+ \setminus \{0\}$, der wiederum in einer Umgebung des Randes durch die Phasenfunktion ψ bestimmt ist, und ein Hauptsymbol a. Ferner ist r^+ der Restriktionsoperator und e^+ der Fortsetzungsoperator durch Null. Um gute Stetigkeitsresultate in der Skala der Sobolevräume $H^s(\mathbb{R}^n_+)$ zu erzielen, müssen wir Bedingungen an den Symplektomorphismus stellen. Im Wesentlichen fordern wir, dass der Symplektomorphismus den Rand erhält und alle Komponenten die Transmissionseigenschaft haben.

Besitzt χ diese Eigenschaften, so induziert es einen Symplektomorphismus $\chi_\partial : T^*\mathbb{R}^{n-1} \to T^*\mathbb{R}^{n-1}$, dargestellt durch die Phasenfunktion ψ_∂. Der Operator G^{ψ_∂} ist ein FIO mit Phase ψ_∂ auf ∂M und einem singulären Greenschen Symbol. Der Operator K^{ψ_∂} ist ein FIO auf ∂M mit Phase ψ_∂ und einem Potentialsymbol k, T^{ψ_∂} ist ein FIO definiert auf ∂M mit einem Spursymbol, S^{ψ_∂} ist ein üblicher FIO auf ∂M mit Phase ψ_∂.

Hier ist das Ziel, die Stetigkeit dieser Operatoren auf der Sobolevraum-Skala $H^s(\mathbb{R}^n_+)$ zu zeigen und Resultate analog zu denen von Boutet de Monvel zu beweisen.

Schlagwörter: Spektrale Zetafunktion, Fourierintegraloperatoren, Kalkül von Boutet de Monvel

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Contents

Introduction

It is the aim of this thesis to show how microlocal techniques can be applied successfully to problems in analysis, geometry and spectral theory. In the first part we are mainly interested in eigenvalue asymptotics and geometric invariants. Specifically, we consider four situations where the manifold is either $\mathbb{R}^n$, a manifold with cylindrical ends, the product $M_1 \times M_2$ of two closed manifolds, or $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For each case, we use a specific pseudodifferential calculus, clarify the notion of ellipticity, define the complex powers and then study their properties. This leads to information about the non-commutative residue on manifolds with cylindrical ends and to connections with the Einstein-Hilbert action. We find then new results for the counting functions of selfadjoint operators belonging to the calculi described above. In a particular case, we determine not only the leading term, but also the second one. The result is obtained combining the study of the meromorphic extension of the spectral ζ-function and a Tauberian Theorem, due to J. Aramaki, [7].

Let us briefly present the various pseudodifferential calculi involved; more details are given in Chapter 1. *SG*-calculus was first introduced on $\mathbb{R}^n$ by H. O. Cordes [25] and C. Parenti [82], see also R. Melrose [71]. For an introduction to *SG*-calculus and, more generally to global calculus on $\mathbb{R}^n$, see F. Nicola and L. Rodino [81]. An *SG*-operator $A = a(x,D) = \mathrm{Op}\,(a)$ acting on $\mathbb{R}^n$ can be defined via the usual left-quantization

$$Au(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

starting from symbols $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with the property that, for arbitrary multi-indices α, β, there exist constants $C_{\alpha\beta} \geq 0$ such that the estimates

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m_1-|\alpha|} \langle x\rangle^{m_2-|\beta|}$$

hold for fixed $m_1, m_2 \in \mathbb{R}$ and all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\langle y\rangle = \sqrt{|y|^2+1}$, $y \in \mathbb{R}^n$. The set of such symbols is denoted by $SG^{m_1,m_2}(\mathbb{R}^n)$. The model example of an *SG*-operator is $(-\Delta+1)(1+|x|)^2$. In 1987, E. Schrohe [91] introduced a class of non-compact manifolds, the so-called *SG*-manifolds, on which it is possible to transfer from $\mathbb{R}^n$ the whole *SG*-calculus: in short, these are manifolds which admit a finite atlas whose changes of coordinates behave like symbols of order $(0,1)$ (see [91] for details and additional technical hypotheses). The manifolds with cylindrical ends are a special case of *SG*-manifolds, on which also the concept of *SG*-classical operator makes sense. Moreover, the principal symbol of an *SG*-classical operator A on a manifold with cylindrical ends M, has an invariant meaning, see Yu. V. Egorov and B.-W. Schulze [29], T. Hirschmann [45], L. Maniccia and P. Panarese [64], R. Melrose [71].

In [90], L. Rodino introduced bisingular operators: pseudodifferential operators defined on the product of two closed manifolds $M_1 \times M_2$. A main motivation of that paper was the multiplicative property of the Atiyah-Singer index, see [11]. A simple example of an operator in this class is the tensorial product $A_1 \otimes A_2$, where $A_1 \in L^{m_1}(M_1)$, $A_2 \in L^{m_2}(M_2)$ are pseudodifferential operators. Another example, studied in [90], is the vector-tensor product $A_1 \boxtimes A_2$. It is easy to verify that, if A_1 and A_2 are both differential operators, then $A_1 \otimes A_2$ is a differential operator on $M_1 \times M_2$, but if A_1 or A_2 are pseudodifferential operators then $A_1 \otimes A_2$ is not, in general, a pseudodifferential operator on the closed manifold $M_1 \times M_2$. Nevertheless, operators of this type arise naturally in different contexts. Bisingular calculus embeds this example into a wider theory. The class of bisingular operators is denoted by $L^{m_1,m_2}(M_1, M_2)$. We will study, in particular, the subclass $L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$ of operators which have a principal symbol. This calculus belongs to the framework of pseudodifferential operators with operator-valued symbols (see, e.g., Yu. V. Egorov and B.-W. Schulze [29], S. Rempel and B.-W. Schulze [88], B.-W. Schulze [96, 97], E. Schrohe [94] and the references therein for other examples). Namely, the principal symbol of an operator $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$ is a triple $(\sigma_1^{m_1}(A), \sigma_2^{m_2}(A), \sigma^{m_1,m_2}(A))$, where $\sigma_1^{m_1}(A)$ is a function on $T^*M_1 \setminus 0$ which takes values in $L^{m_2}(M_2)$, $\sigma_2^{m_2}(A)$ is a function on $T^*M_2 \setminus 0$ which takes values in $L^{m_1}(M_1)$ and $\sigma^{m_1,m_2}(A)$ is a function defined on $(T^*M_1 \setminus 0) \times (T^*M_2 \setminus 0)$. In [80], F. Nicola and L. Rodino introduced classical bisingular operators and proved an index formula, see also V. S. Pilidi [85].

Bisingular calculus on Euclidean spaces, recently introduced by U. Battisti, T. Gramchev, S. Pilipović and L. Rodino in [16], is a variant of bisingular calculus adapted to Shubin's calculus on $\mathbb{R}^n$. The simplest example of a bisingular operator on $\mathbb{R}^{n_1+n_2}$ is $A_1 \otimes A_2$ where $A_1 \in G^{m_1}(\mathbb{R}^{n_1})$ and $A_2 \in G^{m_2}(\mathbb{R}^{n_2})$ are operators of Shubin type, see M. A. Shubin [100, 101] for more details. As in the case of bisingular operators, the principal symbols in this setting is a triple and is operator-valued.

In Chapter 2 we describe complex powers and spectral ζ-functions of elliptic operators belonging to the pseudodifferential calculi described above. Complex powers of elliptic operators, in the case of closed manifolds, were first introduced by R. Seeley in [98]. Then, the theory has been extended to different settings, see, e.g, B. Ammann, R. Lauter, V. Nistor and A. Vasy [6], P. Boggiatto and F. Nicola [20], G. Dore and A. Venni [27], G. Grubb [35, 36], G. Grubb and L. Hansen [37], P. Loya [62, 63], L. Maniccia, E. Schrohe and J. Seiler [66], E. Schrohe [92]. In [92], E. Schrohe first developed a theory of complex powers in a setting close to *SG*-calculus; then L. Maniccia, E. Schrohe and J. Seiler in [66] described precisely the symbol of complex powers of *SG*-operators and investigated the case of classical *SG*-operators. In Section 2.1, we study the spectral ζ-function in the setting of *SG*-calculus on $\mathbb{R}^n$ and on manifolds with cylindrical ends, following the construction of L. Maniccia, E. Schrohe and J. Seiler [66] and of U. Battisti and S. Coriasco [15]. We restrict ourselves to the case of classical *SG*-symbols. Dealing with classical operators, we can prove that the spectral ζ-function can be extended as a meromorphic function to the whole of $\mathbb{C}$. In this way, following the original approach of M. Wodzicki [109], we introduce the non-commutative residue in the *SG*-setting, via ζ-functions.

The non-commutative residue was first considered by M. Wodzicki in 1984 [109], in the setting of pseudodifferential operators on closed manifolds, while studying the meromorphic continuation of the zeta function of elliptic operators. The non-commutative residue turns out to be a trace on the algebra of classical operators modulo smoothing operators. Moreover, if the dimension of the closed manifold is larger than one, it is the unique trace on this algebra, up to multiplication by a constant (the situation in dimension one is different, as a consequence of the fact that, in this case, S^*M is not connected, cf. C. Kassel [55]). In 1985, V. Guillemin [41] independently defined the so-called symplectic residue, equivalent to the non-commutative residue, with the aim of "finding a *soft* proof of Weyl formula". The non-commutative residue, sometimes called Wodzicki residue, gained a growing interest in the years, also in view of the links with non-commutative geometry and the Dixmier trace, see, e.g., A. Connes [24], B. Ammann and C. Bär [5], W. Kalau and M. Walze [54], D. Kastler [56]. The concept has been extended to different situations: manifolds with boundary by B. V. Fedosov, F. Golse, E. Leichtnam and E. Schrohe [31] and Y. Wang [106, 107], conic manifolds by E. Schrohe [93] and J.B. Gil and P.A. Loya [32], anisotropic operators on $\mathbb{R}^n$ by P. Boggiatto and F. Nicola [17], *CR*-manifolds by R. Ponge [86], [87]. Notice that the non-commutative residue was already defined by F. Nicola in [79] for *SG*-calculus on $\mathbb{R}^n$, by means of holomorphic families. We follow here a different approach, which leads to an invariant definition in the case of manifolds with cylindrical ends.

In Section 2.2, we investigate ζ-functions of *SG*-elliptic operators in a different direction. Here, we do not aim at finding a trace on the algebra of *SG*-classical operators, rather the goal is a regularized version of the Kastler-Kalau-Walze Theorem on $\mathbb{R}^n$, linking Dirac operators and the Einstein-Hilbert action. The contents of this section essentially come from U. Battisti and S. Coriasco [14]. A. Connes conjectured that the non-commutative residue could connect Dirac operators and the Einstein-Hilbert action. In 1995, D. Kastler [56], W. Kalau and M. Walze [54] proved this conjecture. Namely, let $D\!\!\!/$ be the classical Atiyah-Singer Dirac operator defined on a closed spin manifold $M = (M, g)$ of even dimension $n \geq 4$. Then

$$\mathrm{wres}(D\!\!\!/^{-n+2}) = -\frac{(n-2)\,2^{[\frac{n}{2}]}}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \int_M \frac{1}{12} s(x)\,dx, \tag{2}$$

where $s(x)$ is the scalar curvature and dx the measure on M induced by the Riemannian metric g (see, e.g., [5] for an overview on non-commutative residue and non-commutative geometry). Y. Wang [105, 106, 107], suggested an extension of the result to a class of manifolds with boundary. T. Ackermann [1] gave a proof of (2), using the relationship between heat trace and ζ-function and the properties of the second term in the asymptotic expansion of the heat trace of a generalized Laplacian. In Section 2.2, we follow T. Ackermann's idea and give a regularized version of (2), using the regularized integral introduced by L. Maniccia, E. Schrohe and J. Seiler in [65].

Next, in Sections 2.3 and 2.4, we study the complex powers of bisingular operators on compact closed manifolds and on Euclidean spaces, respectively, following U. Battisti [13] and U. Battisti, T. Gramchev, S. Pilipović and L. Rodino[16].

In Chapter 3 we state the main result about the asymptotic behavior of the counting functions $N_A(\lambda)$. For each of the three settings described above, we will prove that the counting function of an elliptic selfadjoint positive operator A has the asymptotic behavior

$$N_A(\lambda) \sim \begin{cases} C_1 \lambda^l \log(\lambda) + C_1' \lambda^l + O(\lambda^{l-\delta_1}) & \text{for } \frac{v_1}{m_1} = \frac{v_2}{m_2} = l \\ C_2 \lambda^{\frac{v_2}{m_2}} + O(\lambda^{\frac{v_2}{m_2}-\delta_2}) & \text{for } \frac{v_2}{m_2} > \frac{v_1}{m_1} \\ C_3 \lambda^{\frac{v_1}{m_1}} + O(\lambda^{\frac{v_2}{m_2}-\delta_3}) & \text{for } \frac{v_2}{m_2} < \frac{v_1}{m_1} \end{cases}, \quad \lambda \to \infty, \tag{3}$$

where m_1, m_2 are the orders of the operator A, v_1, v_2 are integer numbers depending on the calculus we analyze, C_1, C_1', C_2, C_3 are constants depending on the principal symbol of A and $\delta_1, \delta_2, \delta_3$ are strictly positive real numbers. In [79], F. Nicola expresses the leading terms of the counting function, in the case of SG-calculus on $\mathbb{R}^n$, by means of the Laurent coefficients of a suitable holomorphic family associated to the operator. In [64], L. Maniccia and P. Panarese describe the leading term of $N_A(\lambda)$ in the setting of SG-calculus on manifolds with cylindrical ends using heat kernel methods. The asymptotic expansion (3) improves these results in the case of SG-calculus, giving also the second term when $v_1/m_1 = v_2/m_2$. For bisingular operators, (3) has been proved in [13], and for bisingular operators on Euclidean spaces in [16]. Notice that, when $v_1/m_1 = v_2/m_2$, the asymptotic expansion of the counting function contains a logarithmic term. This case corresponds to a pole of order 2 of the ζ-function at the point v_1/m_1. Such a behavior appears in others settings: for example manifolds with conical singularities, see P. Loya [32] and manifolds with cusps, see S. Moroianu [73]. The approach used here is similar to the one in [73]. The exposition is completed by the analysis of an example, together with some numerical experiments on the expected results.

In the second part, we develop a calculus of Fourier Integral Operators (FIOs) on the half-space $\mathbb{R}^n$. The basic idea is to consider, similarly as M. I. Višik and G. I. Eskin [104] and L. Boutet de Monvel [19], a class of operators with contains both the classical boundary value problems and their inverses, whenever these exist. The elements of the calculus are given by matrices of operators

$$\begin{pmatrix} r^+ \mathrm{Op}^{\psi}(a) e^+ + G^{\psi_\partial} & K^{\psi_\partial} \\ T^{\psi_\partial} & S^{\psi_\partial} \end{pmatrix},$$

where the entries are Fourier integral operators associated to a symplectomorphism $\chi : T^*\mathbb{R}^n_+ \setminus 0 \to T^*\mathbb{R}^n_+ \setminus 0$, positively homogeneous of order one in the fibers, satisfying suitable conditions. Essentially, χ must preserve the boundary of $T^*\mathbb{R}^n_+$ and all the components satisfy the transmission condition (related ideas, in a different setting, are deeply investigated in A. Hirschowitz, A. Piriou [46]). Such a symplectomorphism induces naturally a symplectomorphism χ_∂ at the boundary. The function ψ is a phase function that represents the symplectomorphism χ, while ψ_∂ represents the symplectomorphism induced at the boundary. These two conditions arise naturally if one requires that $r^+\mathrm{Op}(a)e^+$ acts continuously from $\mathscr{S}(\mathbb{R}^n_+)$ to itself and the same holds for $r^+\mathrm{Op}(a)^*e^+$. Here we limit ourselves to the case of the half-space $\mathbb{R}^n_+$: this is the first step in order to develop in detail a calculus of Fourier Integral Operators of Boutet de Monvel type. The theory of Fourier Integral Operators is well known in the case of manifolds without boundary, see the classical paper by L. Hörmander [47]; for an overview on the theory see L. Hörmander [48, 49], J. J. Duistermaat [28], C. Sogge [102].

See V. Guillemin and S. Sternberg [43] for Fourier Integral operators in the semi-classical setting. In the case of manifolds with conical singularities, R. B. Melrose [69] introduced a notion of Fourier Integral Operator which has been refined by V. E. Nazaĭkinskiĭ, B.-W. Schulze and B. Yu. Sternin in [77, 78], with the aim to get an index formula for such operators, see also V. E. Nazaĭkinskiĭ, A. Yu. Savin, B.-W Schulze and B. Yu. Sternin [75] and V. E. Nazaĭkinskiĭ and B. Yu. Sternin [76]. The theory of FIOs we present here uses as a main ingredient the concept of vector-valued symbols, see e.g. Yu. V. Egorov and B.-W. Schulze [29] and B.-W. Schulze [96, 97] about their general theory. The main difference with the theory of pseudodifferential operators of Boutet de Monvel type is that if $\mathrm{Op}^{\psi}(a)$ is a FIO of order m, then, in general, the operator $r^{+}\mathrm{Op}^{\psi}_{n}(a)e^{+}$ does not belong to $S^{m}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{s}(\mathbb{R}_{+}), H^{s-m}(\mathbb{R}_{+}))$, see (6.2.18). This is a consequence of the global nature of FIOs. We overcome this problem proving that $r^{+}\mathrm{Op}^{\psi}(a)e^{+}$ belongs to $S^{m}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}(\mathbb{R}_{+}), \mathscr{S}(\mathbb{R}_{+}))$. The continuity on Sobolev spaces, by means of a splitting of the amplitude a, will be essentially reduced to the analysis of $r^{+}\mathrm{Op}^{\psi}(a)\delta_{0}^{(j)}$, where $\delta_{0}^{(j)}$ is the j-th derivative of the Dirac's distribution at the boundary.

In Chapter 4 we recall the basic definitions and properties of manifolds with boundary, following closely the approaches of B. W. Boothby [18], J. M. Lee [59] and J. R. Munkres [74]. For an introduction to manifolds with boundary, aimed at the calculus on manifold with singularities, see also E. Schrohe and B.-W. Schulze [95] and R. B. Melrose [70]. Then, we briefly introduce symplectic vector spaces and the Maslov index of Lagrangian subspaces. In this part we have used as main references A. Cannas da Silva [21], D. McDuff and D. Salamon [67] and J. Robbin and D. Salamon [89]. Eventually, we give the definition of symplectic manifolds and of the Keller-Maslov bundle associated with a Lagrangian submanifold of the cotangent bundle of a smooth manifold. For more details on the Keller-Maslov bundle and its connection with the Maslov index we refer to J. J. Duistermaat [28], L. Hörmander [48, 49].

In Chapter 5 we introduce the function spaces on the half-space involved in the theory. Then, we recall the basic properties of vector-valued symbols following closely the treatment of E. Schrohe [94]. Next, we introduce wedge-Sobolev spaces, cf. B.-W Schulze [96, 97], and describe continuity properties of pseudodifferential operator with vector-valued symbols, cf., e.g., J. Seiler [99] and T. Hirschmann [45].

Chapter 6 is devoted to the definition of Fourier Integral Operators of Boutet de Monvel type. First, we motivate the assumptions on the symplectomorphism and, consequently, on the phase function. The aim is to prove that the operator $r^{+}\mathrm{Op}^{\psi}(a)e^{+} : H^{s}(\mathbb{R}^{n}_{+}) \to H^{s-m}(\mathbb{R}^{n}_{+})$ is continuous, where $\mathrm{Op}^{\psi}(a)$ is a FIO on $\mathbb{R}^{n}$ of order m with phase function ψ, describing an admissible symplectomorphism. Then, we introduce *trace symbols*, *potential symbols* and *singular Green symbols* in order to define Fourier Integral Operators of Boutet de Monvel type. Boutet de Monvel pseudodifferential operators are a particular case of FIO of Boutet de Monvel type, namely, those defined by the identity symplectomorphism. We conclude showing that FIOs of Boutet de Monvel type are stable with respect to right and left composition with a pseudodifferential operator of Boutet de Monvel type. We also prove a version of Egorov Theorem adapted to FIOs of Boutet de Monvel type.

We plan to complete our theory giving the global definition of FIOs of

Boutet de Monvel type on compact manifolds with boundary, and the associated notions of principal symbol and ellipticity. It would be interesting to study the connections between our calculus and the parametrix constructions for mixed hyperbolic problems, studied, e.g., by J. Chazarain in [22] and [23]. Finally, a natural development is the analysis of index theory for this class of operators, following the ideas of A. Weinstein [108], C. Epstein and R. B. Melrose [30], E. Leichtnam, R. Nest and B. Tsygan [60] and, in the case of manifolds with conical singularities, of V. E. Nazaĭkinskiĭ, B.-W. Schulze and B. Yu. Sternin [77, 78].

Part I

Zeta Functions of Pseudodifferential Operators

Chapter 1

Pseudodifferential Algebras

This chapter is a brief introduction to the pseudodifferential algebras we will treat. In Section 1.1 we analyze *SG*-calculus, in section 1.2 Bisingular operators and in Section 2.4 Bisingular operators on Euclidean spaces, a global version of Bisingular operators suited to Shubin's Global calculus on $\mathbb{R}^n$.

1.1 *SG*-Operators

In this section we recall the basic properties of *SG*-calculus, first on $\mathbb{R}^n$, and then on manifold with cylindrical ends. For the details of the calculus and the extension to *SG*-manifolds see, e.g., [29, 91].

1.1.1 *SG*-Pseudodifferential Operators on $\mathbb{R}^n$

Definition 1.1.1. A function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to $SG^{m_1,m_2}(\mathbb{R}^n)$ if, for all multiindices $\alpha, \beta \in \mathbb{N}^n$, there exists a constants $C_{\alpha\beta} \geq 0$ such that

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m_1-|\alpha|} \langle x\rangle^{m_2-|\beta|}, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.1}$$

$SG^{m_1,m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ is a Fréchet space, with seminorms given by $\{C_{\alpha\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$, the best constant in (1.1). Moreover, *SG*-symbols form a graded algebra:

$$a \in SG^{m_1,m_2}(\mathbb{R}^n \times \mathbb{R}^n), b \in SG^{m_1',m_2'}(\mathbb{R}^n \times \mathbb{R}^n) \quad \Rightarrow \quad ab \in SG^{m_1+m_1',m_2+m_2'}(\mathbb{R}^n \times \mathbb{R}^n).$$

The set

$$S^{-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n) = \bigcap_{(m_1,m_2)\mathbb{R}^2} SG^{m_1,m_2}(\mathbb{R}^n \times \mathbb{R}^n) = \mathscr{S}(\mathbb{R}^{2n})$$

is called the set of smoothing symbols. We define the class of *SG*-pseudodifferential operators via left quantization. A linear operator $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is an *SG*-operator if it can be written as

$$Au(x) = \mathrm{Op}(a) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi \quad u \in \mathcal{S}(\mathbb{R}^n),$$

with $a \in SG^{m_1,m_2}(\mathbb{R}^n)$. The corresponding operators constitute the class

$$L^{m_1,m_2}(\mathbb{R}^n) = \mathrm{Op}\left(SG^{m_1,m_2}(\mathbb{R}^n)\right).$$

In the sequel we will often simply write SG^{m_1,m_2} and L^{m_1,m_2}, respectively, fixing the dimension of the base space to n. Using the classical property of composition of pseudodifferential operators, one can prove that SG-pseudodifferential operators form a graded algebra, that is

$$L^{m_1,m_2} \circ L^{m_1',m_2'} \subseteq L^{m_1+m_1',m_2+m_2'}.$$

The residual elements are operators with symbols in $SG^{-\infty}$, that is, those having kernel in $\mathscr{S}(\mathbb{R}^{2n})$, continuously mapping $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Notice that the class of SG-smoothing operators coincides with the class of smoothing global Shubin's operators, see [101]. The notion of ellipticity in this setting involves not only the behavior of the symbol w.r.t. the ξ-variable, as in the classical case, but also its decay at infinity w.r.t. the x-variable:

Definition 1.1.2. An operator $A = \mathrm{Op}(a) \in SG^{m_1,m_2}$ is SG-elliptic if there exists a positive constant R such that

$$a(x,\xi)^{-1} = O(\langle \xi \rangle^{-m_1} \langle x \rangle^{-m_2}),$$

holds for $|x| + |\xi| \geq R$.

It is immediate to check that SG-operators act continuously on the $\mathscr{S}(\mathbb{R}^n)$ space and, by duality on the tempered distribution $\mathscr{S}'(\mathbb{R}^n)$. In order to obtain Sobolev continuity results one has to introduce the weighted Sobolev spaces

$$H^{t_1,t_2}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \colon \|u\|_{t_1,t_2} = \langle x \rangle^{t_2} \|\mathrm{Op}\left(\langle \xi \rangle^{t_1}\right) u\|_{L^2} < \infty\}.$$

This scale of Sobolev spaces satisfies immersion property

$$H^{s_1,s_2}(\mathbb{R}^n) \hookrightarrow H^{r_1,r_2}(\mathbb{R}^n), \quad s_1 \geq r_1, s_2 \geq r_2.$$

If both inequalities are strict, the immersion is compact. Notice moreover that

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{(s_1,s_2)\in\mathbb{R}^2} H^{s_1,s_2}(\mathbb{R}^n) \text{ and } \mathcal{S}'(\mathbb{R}^n) = \bigcup_{(s_1,s_2)\in\mathbb{R}^2} H^{s_1,s_2}(\mathbb{R}^n).$$

The next proposition followss by the composition properties of SG-operators

Proposition 1.1.1. *If $A \in L^{m_1,m_2}$, then it can be extended to a continuous operator*

$$A : H^{s_1,s_2}(\mathbb{R}^n) \to H^{s_1-m_1,s_s-m_2}(R^n).$$

In view of the calculus and of the Definition of ellipticity 1.1.2, the following holds:

Proposition 1.1.2. *Let $A \in L^{m_1,m_2}$ be an elliptic operator. Then, there exists an inverse B of A modulo smoothing operators. Hence, A is a Fredholm operator.*

It is possible to introduce the notion of classical symbol also in this setting.

Definition 1.1.3. i) A symbol $a(x,\xi)$ belongs to the class $SG^{m_1,m_2}_{\mathrm{cl}(\xi)}(\mathbb{R}^n)$ if there exist $a_{m_1-i,\cdot}(x,\xi) \in \widetilde{\mathscr{H}}^{m_1-i}_{\xi}(\mathbb{R}^n)$, $i = 0,1,\ldots$, homogeneous functions of order $m_1 - i$ with respect to the variable ξ, smooth with respect to the variable x, such that, for a fixed 0-excision function ω,

$$a(x,\xi) - \sum_{i=0}^{N-1} \omega(\xi)\, a_{m_1-i,\cdot}(x,\xi) \in SG^{m_1-N,m_2}(\mathbb{R}^n), \quad N = 1,2,\ldots;$$

ii) A symbol $a(x,\xi)$ belongs to the class $SG^{m_1,m_2}_{\mathrm{cl}(x)}(\mathbb{R}^n)$ if there exist $a_{\cdot,m_2-k}(x,\xi) \in \widetilde{\mathscr{H}}^{m_2-k}_x(\mathbb{R}^n)$, $k = 0, \ldots$, homogeneous functions of order m_2-k with respect to the variable x, smooth with respect to the variable ξ, such that, for a fixed 0-excision function ω,

$$a(x,\xi) - \sum_{k=0}^{N-1} \omega(x)\, a_{\cdot,m_2-k}(x,\xi) \in SG^{m_1,m_2-N}(\mathbb{R}^n), \quad N = 1, 2, \ldots$$

Definition 1.1.4. A symbol $a(x,\xi)$ is SG-classical, and we write $a \in SG^{m_1,m_2}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^n) = SG^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^n) = SG^{m_1,m_2}_{\mathrm{cl}}$, if

i) there exist $a_{m_1-j,\cdot}(x,\xi) \in \widetilde{\mathscr{H}}^{m_1-j}_\xi(\mathbb{R}^n)$ such that, for a fixed 0-excision function ω, $\omega(\xi)\, a_{m_1-j,\cdot}(x,\xi) \in SG^{m_1-j,m_2}_{\mathrm{cl}(x)}(\mathbb{R}^n)$ and

$$a(x,\xi) - \sum_{j=0}^{N-1} \omega(\xi)\, a_{m_1-j,\cdot}(x,\xi) \in SG^{m_1-N,m_2}(\mathbb{R}^n), \quad N = 1, 2, \ldots;$$

ii) there exist $a_{\cdot,m_2-k}(x,\xi) \in \widetilde{\mathscr{H}}^{m_2-k}_x(\mathbb{R}^n)$ such that, for a fixed 0-excision function ω, $\omega(x)\, a_{\cdot,m_2-k}(x,\xi) \in SG^{m_1,m_2-k}_{\mathrm{cl}(\xi)}(\mathbb{R}^n)$ and

$$a(x,\xi) - \sum_{k=0}^{N-1} \omega(x)\, a_{\cdot,m_2-k} \in SG^{m_1,m_2-N}(\mathbb{R}^n), \quad N = 1, 2, \ldots$$

We set $L^{m_1,m_2}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^n) = L^{m_1,m_2}_{\mathrm{cl}} = \mathrm{Op}\left(SG^{m_1,m_2}_{\mathrm{cl}}\right)$.

Remark 1.1.1. *The definition could be extended in a natural way from operators acting between scalars to operators acting between (distributional sections of) vector bundles: one should then use matrix-valued symbols whose entries satisfy the estimates* (1.1) *and modify accordingly the various statements below. To simplify the presentation, we omit everywhere any reference to vector bundles, assuming them to be trivial and one-dimensional.*

The next two results are especially useful when dealing with SG-classical symbols, see, e.g, see Yu. V. Egorov and B.-W Schulze [29].

Theorem 1.1.3. *Let $a_k \in SG^{m_1-k,m_2-k}_{\mathrm{cl}}$, $k = 0, 1, \ldots$, be a sequence of SG-classical symbols and $a \sim \sum_{k=0}^{\infty} a_k$ its asymptotic sum in the general SG-calculus. Then, $a \in SG^{m_1,m_2}_{\mathrm{cl}}$.*

Theorem 1.1.4. *Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \le 1\}$ and let χ be a diffeomorphism from the interior of $\mathbb{B}^n$ to $\mathbb{R}^n$ such that*

$$\chi(x) = \frac{x}{|x|(1-|x|)} \quad \text{for} \quad |x| > 2/3.$$

Choosing a smooth function $[x]$ on $\mathbb{R}^n$ such that $1-[x] \neq 0$ for all x in the interior of $\mathbb{B}^n$ and $|x| > 2/3 \Rightarrow [x] = |x|$, for any $a \in SG^{m_1,m_2}_{\mathrm{cl}}$ denote by $(D^{\mathbf{m}}a)(y,\eta)$, $\mathbf{m} = (m_1, m_2)$, the function

$$b(y,\eta) = (1-[\eta])^{m_1}(1-[y])^{m_2} a(\chi(y),\chi(\eta)). \tag{1.2}$$

Then, $D^{\mathbf{m}}$ extends to a homeomorphism from $SG^{m_1,m_2}_{\mathrm{cl}}$ to $C^\infty(\mathbb{B}^n \times \mathbb{B}^n)$.

Remark 1.1.2. *Theorem 1.1.4 can be stated in an equivalent way using the* radial compactification map*, see [68, 70]. We consider the manifold with boundary*

$$\mathbb{S}^n_+ = \{x = (x', x_{n+1}) \mid x \in \mathbb{S}^n, x_{n+1} \geq 0\},$$

and the function

$$\begin{aligned} RC : \mathbb{R}^n &\to \mathbb{S}^n_+ \\ x &\to \left(\frac{x}{\langle x \rangle}, \frac{1}{\langle x \rangle} \right). \end{aligned}$$

Then, $a \in SG^{m_1,m_2}_{\mathrm{cl}}$ if and only if

$$[y]^{-m_1}[\eta]^{-m_2} a(RC^{-1}(y), RC^{-1}(\eta))$$

can be extended as a smooth function to $C^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$, where $[\cdot]$ is a boundary defining function of $\mathbb{S}^n_+$ such that, in a neighborhood of $\partial\mathbb{S}^n_+$, it is equal to the coordinate function x_{n+1}.

Note that the definition of SG-classical symbol implies a condition of compatibility for the terms of the expansions with respect to x and ξ. In fact, defining $\sigma^{m_1-j}_\psi$ and $\sigma^{m_2-i}_e$ on $SG^{m_1,m_2}_{\mathrm{cl}(\xi)}$ and $SG^{m_1,m_2}_{\mathrm{cl}(x)}$, respectively, as

$$\begin{aligned} \sigma^{m_1-j}_\psi(a)(x,\xi) &= a_{m_1-j,\cdot}(x,\xi), \quad j = 0,1,\dots, \\ \sigma^{m_2-i}_e(a)(x,\xi) &= a_{\cdot,m_2-i}(x,\xi), \quad i = 0,1,\dots, \end{aligned}$$

it is possible to prove that

$$\begin{aligned} a_{m_1-j,m_2-i} = \sigma^{m_1-j,m_2-i}_{\psi e}(a) = \sigma^{m_1-j}_\psi(\sigma^{m_2-i}_e(a)) = \sigma^{m_2-i}_e(\sigma^{m_1-j}_\psi(a)), \\ j = 0,1,\dots,\ i = 0,1,\dots \end{aligned} \tag{1.3}$$

Moreover, the algebra property of SG-operators and Theorem 1.1.3 implies that the composition of two SG-classical operators is still classical. For $A = \mathrm{Op}\,(a) \in L^{m_1,m_2}_{\mathrm{cl}}$ the triple $\sigma(A) = (\sigma_\psi(A), \sigma_e(A), \sigma_{\psi e}(A)) = (a_{m_1,\cdot}\,, a_{\cdot,m_2}\,, a_{m_1,m_2})$ is called the *principal symbol of A*. This definition keeps the usual multiplicative behavior, that is, for any $A \in L^{r_1,r_2}_{\mathrm{cl}}$, $B \in L^{s_1,s_2}_{\mathrm{cl}}$, $(r_1,r_2), (s_1,s_2) \in \mathbb{R}^2$, $\sigma(AB) = \sigma(A)\,\sigma(B)$, with componentwise product in the right-hand side. We also set

$$\begin{aligned} \mathrm{Sym_p}\,(A)\,(x,\xi) = {} & \mathrm{Sym_p}\,(a)\,(x,\xi) = \\ = {} & a_{\mathbf{m}}(x,\xi) = \omega(\xi) a_{m_1,\cdot}(x,\xi) + \omega(x)(a_{\cdot,m_2}(x,\xi) - \omega(\xi) a_{m_1,m_2}(x,\xi)) \end{aligned}$$

for a fixed 0-excision function ω. Theorem 1.1.5 below allows to express the ellipticity of SG-classical operators in terms of their principal symbol:

Theorem 1.1.5. *An operator $A \in L^{m_1,m_2}_{\mathrm{cl}}$ is elliptic if and only if each element of the triple $\sigma(A)$ is non-vanishing on its domain of definition; that is*

i) $a_{m_1,\cdot}(\omega,\xi) \neq 0$ *for all* $\omega \in \mathbb{S}^{n-1}$, $\xi \in \mathbb{R}^n$;

ii) $a_{\cdot,m_2}(x,\omega') \neq 0$ *for all* $x \in \mathbb{R}^n$, $\omega' \in \mathbb{S}^{n-1}$;

iii) $a_{m_1,m_2}(\omega,\omega') \neq 0$ *for all* $\omega \in \mathbb{S}^{n-1}$, $\omega' \in \mathbb{S}^{n-1}$.

1.1.2 *SG*-Operators on Manifolds with Cylindrical Ends

We analyze now an extension of *SG*-calculus for manifolds with cylindrical ends, a special case of *SG*-manifolds [91]. In this subsection we follow the idea of L. Maniccia and P. Panarese [64]. For simplicity, we restrict ourselves to the case of manifolds with one cylindrical end.

Definition 1.1.5. A manifold with a cylindrical end is a triple $(M, X, [f])$, where $M = \mathscr{M} \amalg_C \mathscr{C}$ is a n-dimensional smooth manifold and

i) $\mathscr{M}$ is a smooth manifold, given by $\mathscr{M} = (M_0 \setminus D) \cup C$ with a n-dimensional smooth compact manifold without boundary M_0, D a closed disc of M_0 and $C \subset D$ a collar neighbourhood of ∂D in M_0;

ii) $\mathscr{C}$ is a smooth manifold with boundary $\partial\mathscr{C} = X$, with X diffeomorphic to ∂D;

iii) $f : [\delta_f, \infty) \times \mathbb{S}^{n-1} \to \mathscr{C}$, $\delta_f > 0$, is a diffeomorphism, $f(\{\delta_f\} \times \mathbb{S}^{n-1}) = X$ and $f(\{[\delta_f, \delta_f + \varepsilon_f)\} \times \mathbb{S}^{n-1})$, $\varepsilon_f > 0$, is diffeomorphic to C;

iv) the symbol $\amalg_C$ means that we are gluing $\mathscr{M}$ and $\mathscr{C}$, through the identification of C and $f(\{[\delta_f, \delta_f + \varepsilon_f)\} \times \mathbb{S}^{n-1})$;

v) the symbol $[f]$ represents an equivalence class in the set of functions

$$\begin{aligned}\{g : [\delta_g, \infty) \times \mathbb{S}^{n-1} \to \mathscr{C} :\ & g \text{ is a diffeomorphism,}\\ & g(\{\delta_g\} \times \mathbb{S}^{n-1}) = X \text{ and}\\ & g([\delta_g, \delta_g + \varepsilon_g) \times \mathbb{S}^{n-1}),\ \varepsilon_g > 0, \text{ is diffeomorphic to } C\}\end{aligned}$$

where $f \sim g$ if and only if there exists a diffeomorphism $\Theta \in \mathrm{Diff}(\mathbb{S}^{n-1})$ such that

$$(g^{-1} \circ f)(\rho, \omega) = (\rho, \Theta(\omega)) \tag{1.4}$$

for all $\rho \geq \max\{\delta_f, \delta_g\}$ and $\omega \in \mathbb{S}^{n-1}$.

We use the following notation:

- $U_{\delta_f} = \{x \in \mathbb{R}^n : |x| > \delta_f\}$;
- $\mathscr{C}_\tau = f([\tau, \infty) \times \mathbb{S}^{n-1})$, where $\tau \geq \delta_f$. The equivalence condition (1.4) implies that $\mathscr{C}_\tau$ is well defined;
- $\pi : \mathbb{R}^n \setminus \{0\} \to (0, \infty) \times \mathbb{S}^{n-1} : x \mapsto \pi(x) = \left(|x|, \dfrac{x}{|x|}\right)$;
- $f_\pi = f \circ \pi : \overline{U_{\delta_f}} \to \mathscr{C}$ is a parametrisation of the end. Let us notice that, setting $F = g_\pi^{-1} \circ f_\pi$, the equivalence condition (1.4) implies

$$F(x) = |x|\, \Theta\left(\frac{x}{|x|}\right). \tag{1.5}$$

We also denote the restriction of f_π mapping U_{δ_f} onto $\dot{\mathscr{C}} = \mathscr{C} \setminus X$ by $\dot{f}_\pi$.

The couple $(\dot{\mathscr{C}}, \dot{f}_\pi^{-1})$ is called the exit chart. If $\mathscr{A} = \{(\Omega_i, \psi_i)\}_{i=1}^N$ is such that the subset $\{(\Omega_i, \psi_i)\}_{i=1}^{N-1}$ is a finite atlas for $\mathscr{M}$ and $(\Omega_N, \psi_N) = (\dot{\mathscr{C}}, \dot{f}_\pi^{-1})$, then M, with the atlas $\mathscr{A}$, is an SG-manifold (see [91]): an atlas $\mathscr{A}$ of such a kind is called *admissible*. From now on, we restrict the choice of atlases on M to the class of admissible ones. We introduce the following spaces, endowed with their natural topologies:

$$\mathscr{S}(U_\delta) = \left\{ u \in C^\infty(U_\delta) \colon \forall \alpha, \beta \in \mathbb{N}^n \; \forall \delta' > \delta \; \sup_{x \in U_{\delta'}} |x^\alpha \partial^\beta u(x)| < \infty \right\},$$

$$\mathscr{S}_0(U_\delta) = \bigcap_{\delta' \searrow \delta} \{u \in \mathscr{S}(\mathbb{R}_n) \colon \operatorname{supp} u \subseteq \overline{U_{\delta'}}\},$$

$$\mathscr{S}(M) = \{u \in C^\infty(M) \colon u \circ \dot{f}_\pi \in \mathscr{S}(U_{\delta_f}) \text{ for any exit map } f_\pi\},$$

$\mathscr{S}'(M)$ denotes the dual space of $\mathscr{S}(M)$.

Definition 1.1.6. The set $SG^{m_1,m_2}(U_{\delta_f})$ consists of all the symbols $a \in C^\infty(U_{\delta_f})$ which fulfill (1.1) for $(x, \xi) \in U_{\delta_f} \times \mathbb{R}^n$ only. Moreover, the symbol a belongs to the subset $SG^{m_1,m_2}_{\mathrm{cl}}(U_{\delta_f})$ if it admits expansions in asymptotic sums of homogeneous symbols with respect to x and ξ as in Definitions 1.1.3 and 1.1.4, where the remainders are now given by SG-symbols of the required order on U_{δ_f}.

Note that, since U_{δ_f} is conical, the definition of homogeneous and classical symbol on U_{δ_f} makes sense. Moreover, the elements of the asymptotic expansions of the classical symbols can be extended by homogeneity to smooth functions on $\mathbb{R}^n \setminus \{0\}$, which will be denoted by the same symbols. It is a fact that, given an admissible atlas $\{(\Omega_i, \psi_i)\}_{i=1}^N$ on M, there exists a partition of unity $\{\varphi_i\}$ and a set of smooth functions $\{\chi_i\}$ which are compatible with the SG-structure of M, that is:

- $\operatorname{supp} \varphi_i \subset \Omega_i$, $\operatorname{supp} \chi_i \subset \Omega_i$, $\chi_i \, \varphi_i = \varphi_i$, $i = 1, \dots, N$;
- $|\partial^\alpha(\varphi_N \circ \dot{f}_\pi)(x)| \leq C_\alpha \, \langle x \rangle^{-|\alpha|}$ and $|\partial^\alpha(\chi_N \circ \dot{f}_\pi)(x)| \leq C_\alpha \, \langle x \rangle^{-|\alpha|}$ for all $x \in U_{\delta_f}$.

Moreover, φ_N and χ_N can be chosen so that $\varphi_N \circ \dot{f}_\pi$ and $\chi_N \circ \dot{f}_\pi$ are homogeneous of degree 0 on U_δ. We denote by u^* the composition of $u \colon \psi_i(\Omega_i) \subset \mathbb{R}^n \to \mathbb{C}$ with the coordinate patches ψ_i, and by v_* the composition of $v \colon \Omega_i \subset M \to \mathbb{C}$ with ψ_i^{-1}, $i = \dots, N$. It is now possible to give the definition of SG-pseudodifferential operator on M:

Definition 1.1.7. Let M be a manifold with a cylindrical end. A linear operator $A : \mathscr{S}'(M) \to \mathscr{S}'(M)$ is an SG-pseudodifferential operator of order (m_1, m_2) on M if, for any admissible atlas $\{(\Omega_i, \psi_i)\}_{i=1}^N$ on M with exit chart (Ω_N, ψ_N):

1) for all $i = 1, \dots, N-1$ and any $\varphi_i, \chi_i \in C_c^\infty(\Omega_i)$, there exist symbols $a^i(x, \xi) \in S^{m_1}(\psi_i(\Omega_i))$ such that

$$(\chi_i A \varphi_i \, u^*)_*(x) = \iint e^{i(x-y)\cdot\xi} a^i(x, \xi) u(y) dy dx, \quad u \in C^\infty(\psi_i(\Omega_i));$$

2) for any φ_N, χ_N of the type described above, there exists a symbol $a^N(x,\xi) \in SG^{m_1,m_2}(U_{\delta_f})$ such that

$$(\chi_N A \varphi_N u^*)_*(x) = \iint e^{i(x-y)\cdot\xi} a^N(x,\xi) u(y)\, dy\, dx, \quad u \in \mathscr{S}_0(U_{\delta_f});$$

3) K_A, the Schwartz kernel of A, is such that

$$K_A \in C^\infty\Big((M \times M) \setminus \Delta\Big) \bigcap \mathscr{S}\Big((\dot{\mathscr{C}} \times \dot{\mathscr{C}}) \setminus W\Big),$$

where Δ is the diagonal of $M \times M$ and $W = (\dot{f}_\pi \times \dot{f}_\pi)(V)$ with any conical neighbourhood V of the diagonal of $U_{\delta_f} \times U_{\delta_f}$.

The most important local symbol of A is a^N, which we will also denote a^f, to remind its dependence on the exit chart. Our definition of SG-classical operator on M differs slightly from the one in [64]:

Definition 1.1.8. Let $A \in L^{m_1,m_2}(M)$. A is an SG-classical operator on M, and we write $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$, if $a^f(x,\xi) \in SG^{m_1,m_2}_{\mathrm{cl}}(U_{\delta_f})$ and the operator A, restricted to the manifold $\mathscr{M}$, is classical in the usual sense.

The principal symbol $a_{m_1,\cdot}$ of an SG-classical operator $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$ is of course well-defined as a smooth function on $T^*M \setminus 0$. In order to give an invariant definition of principal symbol with respect to x of an operator $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$, the subbundle $T^*_X M = \{(x,\xi) \in T^*M \colon x \in X,\ \xi \in T^*_x M\}$ was introduced. The notion of ellipticity can be extended to operators on M as well:

Definition 1.1.9. Let $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$ and let us fix an exit map f_π. We can define local objects $a_{m_1-j,m_2-k}, a_{\cdot,m_2-k}$ as

$$a_{m_1-j,m_2-k}(\theta,\xi) = a^f_{m_1-j,m_2-k}(\theta,\xi), \quad \theta \in \mathbb{S}^{n-1},\ \xi \in \mathbb{R}^n \setminus \{0\},$$
$$a_{\cdot,m_2-k}(\theta,\xi) = a^f_{\cdot,m_2-k}(\theta,\xi), \quad \theta \in \mathbb{S}^{n-1},\ \xi \in \mathbb{R}^n.$$

Definition 1.1.10. An operator $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$ is SG-elliptic if the principal part of $a^f \in SG^{m_1,m_2}(U_{\delta_f})$ satisfies the SG-ellipticity conditions on $U_{\delta_f} \times \mathbb{R}^n$ and the operator A, restricted to the manifold $\mathscr{M}$, is elliptic in the usual sense.

Proposition 1.1.6. *The properties of $A \in L^{m_1,m_2}(M)$ and of $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$, as well as the notion of ellipticity, do not depend on the (admissible) atlas. Moreover, the local functions $a_{\cdot,m_2}$ and a_{m_1,m_2} give rise to invariantly defined elements of $C^\infty(T^*_X M)$ and $C^\infty(T^*_X M \setminus 0)$, respectively.*

Then, with any $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$, it is associated an invariantly defined principal symbol in three components $\sigma(A) = (a_{m_1,\cdot}, a_{\cdot,m_2}, a_{m_1,m_2})$. Finally, through local symbols given by $p^i(x,\xi) = \langle\xi\rangle^{s_1}$, $i = 1,\dots,N-1$, and $p^f(x,\xi) = \langle\xi\rangle^{s_1}\langle x\rangle^{s_2}$, $s_1, s_2 \in \mathbb{R}$, we get an SG-elliptic operator $\Pi_{s_1,s_2} \in L^{s_1,s_2}_{\mathrm{cl}}(M)$ and introduce the (invariantly defined) weighted Sobolev spaces $H^{s_1,s_2}(M)$ as

$$H^{s_1,s_2}(M) = \{u \in \mathscr{S}'(M) \colon \Pi_{s_1,s_2} u \in L^2(M)\}.$$

The properties of the spaces $H^{s_1,s_2}(\mathbb{R}^n)$ extend to $H^{s_1,s_2}(M)$ without any change, as well as the continuous action of the SG-operators.

1.2 Bisingular Operators

In this section we introduce the basic theory of bisingular operators. We refer to [90] and [80] for details and proofs. Here, Ω_i always denotes a bounded open domain of $\mathbb{R}^{n_i}$.

Definition 1.2.1. We define $S^{m_1,m_2}(\Omega_1,\Omega_2)$ as the set of functions belonging to $C^\infty(\Omega_1\times\Omega_2\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ such that, for all multi-indices α_i,β_i and for all compact subsets $K_i\subseteq\Omega_i$, $i=1,2$, there exists a positive constant $C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K_1,K_2}$ such that

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}a(x_1,x_2,\xi_1,\xi_2)|\leq C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K_1,K_2}\langle\xi_1\rangle^{m_1-|\alpha_1|}\langle\xi_2\rangle^{m_2-|\alpha_2|},$$

for all $x_i\in K_i$, $\xi_i\in\mathbb{R}^{n_i}$, $i=1,2$.

$S^{-\infty,-\infty}(\Omega_1,\Omega_2)$ is the set of smoothing symbols. Following [90], we introduce the subclass of bisingular operators with homogeneous principal symbol.

Definition 1.2.2. Let $a\in S^{m_1,m_2}(\Omega_1,\Omega_2)$; a has a homogeneous principal symbol if

i) there exists $a_{m_1,\cdot}(x_1,x_2,\xi_1,\xi_2)\in S^{m_1,m_2}(\Omega_1,\Omega_2)$ such that

$$a(x_1,x_2,t\xi_1,\xi_2)=t^{m_1}a(x_1,x_2,\xi_1,\xi_2),\quad \forall x_1,x_2,\xi_2,\quad \forall|\xi_1|>1,t>0,$$
$$a-\psi_1(\xi_1)a_{m_1,\cdot}\in S^{m_1-1,m_2}(\Omega_1,\Omega_2),\quad \psi_1\ \text{0-excision function.}$$

Moreover, $a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2)\in L^{m_2}_{\mathrm{cl}}(\Omega_2)$, so, being a classical symbol on Ω_2, it admits an asymptotic expansion w.r.t. the ξ_2 variable.

ii) there exists $a_{\cdot,m_2}(x_1,x_2,\xi_1,\xi_2)\in S^{m_1,m_2}(\Omega_1,\Omega_2)$ such that

$$a(x_1,x_2,\xi_1,t\xi_2)=t^{m_2}a(x_1,x_2,\xi_1,\xi_2),\quad \forall x_1,x_2,\xi_1,\quad \forall|\xi_2|>1,t>0,$$
$$a-\psi_2(\xi_2)a_{\cdot,m_2}\in S^{m_1,m_2-1}(\Omega_1,\Omega_2),\quad \psi_2\ \text{0-excision function.}$$

Moreover, $a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2)\in L^{m_1}_{\mathrm{cl}}(\Omega_1)$, so, being a classical symbol on Ω_1, it admits an asymptotic expansion w.r.t. the ξ_1 variable.

iii) The symbols $a_{m_1,\cdot}$ and $a_{\cdot,m_2}$ have the same leading term, so there exists a_{m_1,m_2} such that

$$a_{m_1,\cdot}-\psi_2(\xi_2)a_{m_1,m_2}\in S^{m_1,m_2-1}(\Omega_1,\Omega_2),$$
$$a_{\cdot,m_2}-\psi_1(\xi_1)a_{m_1,m_2}\in S^{m_1-1,m_2}(\Omega_1,\Omega_2),$$

and

$$a-\psi_1a_{m_1,\cdot}-\psi_2a_{\cdot,m_2}+\psi_1\psi_2a_{m_1,m_2}\in S^{m_1-1,m_2-1}(\Omega_1,\Omega_2).$$

The set of symbols with homogeneous principal symbol is denoted by $S^{m_1,m_2}_{\mathrm{pr}}$ (Ω_1,Ω_2). We will shortly write that the principal symbol of a is $\{a_{m_1,\cdot},a_{\cdot,m_2}\}$.

Remark 1.2.1. *In [80], classical bisingular operators were introduced using an approach very similar to the one of Remark 1.1.2. The authors consider the maps*

$$\widetilde{RC}_i: T^*\Omega_i\to\mathbb{S}^*_+\Omega_i=\Omega_1\times\mathbb{S}^{n_1}_+$$
$$(x,\xi)\to(x,RC(\xi)),$$

and the boundary defining functions ρ_1, ρ_2 of the two boundary hypersurfaces, $\partial\mathbb{S}^{n_1}_+ \times \mathbb{S}^{n_2}_+$ and $\mathbb{S}^{n_1}_+ \times \partial\mathbb{S}^{n_2}_+$, of the manifold with corners $\mathbb{S}^{n_1}_+ \times \mathbb{S}^{n_2}_+$, such that $\rho_1(RC^{-1}(\xi_1), \omega) = \langle\xi_1\rangle$, for all $\xi_1 \in \mathbb{R}^{n_1}$, and similarly for ρ_2. Then, setting $\tilde{\rho}_i = \pi_i^*\rho_i$, where $\pi_i : T^*\Omega_i \to \Omega_i$ is the canonical projection, $i = 1, 2$, one defines

$$S^{m_1,m_2}_{\mathrm{cl}}(\Omega_1 \times \Omega_2) = \left(\widetilde{RC}_1 \times \widetilde{RC}_2\right)^* \tilde{\rho}_1^{-m_1} \tilde{\rho}_2^{-m_2} C^\infty(\mathbb{S}^*_+\Omega_1 \times \mathbb{S}^*_+\Omega_2),$$

where $C^\infty(\mathbb{S}^*_+\Omega_1 \times \mathbb{S}^*_+\Omega_2)$ is the set of functions which admit smooth extension up to the boundary.

We define bisingular operators via their left quantization. A linear operator $A : C^\infty_c(\Omega_1 \times \Omega_2) \to C^\infty(\Omega_1 \times \Omega_2)$ is a bisingular operator if it can be written in the form

$$\begin{aligned} A(u)(x_1, x_2) =& \mathrm{Op}(a)(x_1, x_2) \\ =& \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} e^{ix_1\cdot\xi_1 + ix_2\cdot\xi_2} a(x_1, x_2, \xi_1, \xi_2)\hat{u}(\xi_1, \xi_2) d\xi_1 d\xi_2, \end{aligned}$$

with $a \in S^{m_1,m_2}(\Omega_1, \Omega_2)$ or $a \in S^{m_1,m_2}_{\mathrm{pr}}(\Omega_1, \Omega_2)$. Then, we write $A \in L^{m_1,m_2}(\Omega_1, \Omega_2)$ or $A \in L^{m_1,m_2}_{\mathrm{pr}}(\Omega_1, \Omega_2)$, respectively. The above definition can be extended to the product of closed manifolds; we refer to [90] for the details of the construction of global operators and the corresponding calculus.

Definition 1.2.2 implies that, for every operator $A \in L^{m_1,m_2}_{\mathrm{pr}}(\Omega_1, \Omega_2)$, we can define *principal symbol mappings*, $\sigma^{m_1}, \sigma^{m_2}, \sigma^{m_1,m_2}$, such that

$$\begin{aligned} \sigma_1^{m_1}(A) &: T^*\Omega_1 \setminus \{0\} \to L^{m_2}_{\mathrm{cl}}(\Omega_2) \\ &(x_1, \xi_1) \mapsto a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2), \\ \sigma_2^{m_2}(A) &: T^*\Omega_2 \setminus \{0\} \to L^{m_1}_{\mathrm{cl}}(\Omega_1) \\ &(x_2, \xi_2) \mapsto a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2), \\ \sigma^{m_1,m_2}(A) &: T^*\Omega_1 \setminus \{0\} \times T^*\Omega_2 \setminus \{0\} \to \mathbb{C} \\ &(x_1, x_2, \xi_1, \xi_2) \mapsto a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2). \end{aligned} \tag{1.6}$$

Moreover, denoting by $\sigma(P)(x, \xi)$ the principal symbol of a preudodifferential operator P on a closed manifold, the following *compatibility relation* holds

$$\begin{aligned} \sigma(\sigma_1^{m_1}(A)(x_1, \xi_1))(x_2, \xi_2) =& \sigma(\sigma_2^{m_2}(A)(x_2, \xi_2))(x_1, \xi_1) \\ =& \sigma^{m_1,m_2}(A)(x_1, x_2, \xi_1, \xi_2) = a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2). \end{aligned} \tag{1.7}$$

Comparing the compatibility condition (1.7) with (1.3), we observe a similarity, at least formal, between bisingular symbols with homogeneous principal symbol and SG-classical symbols.

Remark 1.2.2. *If we consider the product of closed manifolds $M_1 \times M_2$, then the whole symbol is a local object, in general. Nevertheless, similarly to the calculus on closed manifolds, it is possible to give an invariant meaning to the mappings* (1.6) *as functions defined on the cotangent bundle, see [90].*

As in the case of the calculus on closed manifolds, it is possible to define adapted Sobolev spaces and then prove some continuity results.

Definition 1.2.3. Let M_1, M_2 be two closed manifolds. The Sobolev space $H^{m_1,m_2}(M_1 \times M_2)$ is defined as

$$\{u \in \mathscr{D}'(M_1 \times M_2) \mid \|u\|_{H^{m_1,m_2}(M_1\times M_2)} = \|\mathrm{Op}(\langle \xi_1 \rangle^{m_1} \langle \xi_2 \rangle^{m_2})(u)\|_{L^2(M_1 \times M_2)} < \infty\}.$$

Using the formalism of tensor products, we can also write[1]

$$H^{m_1,m_2}(M_1 \times M_2) = H^{m_1}(M_1) \widehat{\otimes}_\pi H^{m_2}(M_2).$$

Similarly to Sobolev spaces $H^s(M)$, we have

i) $H^{m_1,m_2}(M_1 \times M_2) \hookrightarrow H^{m_1',m_2'}(M_1 \times M_2)$ is a continuous immersion if $m_i \geq m_i'$, $i = 1, 2$.

ii) $H^{m_1,m_2}(M_1 \times M_2) \hookrightarrow H^{m_1',m_2'}(M_1 \times M_2)$ is a compact immersion if $m_i > m_i'$, $i = 1, 2$.

Proposition 1.2.1. *A pseudodifferential operator $A \in L^{m_1,m_2}(M_1 \times M_2)$ can be extended to a continuous operator*

$$A : H^{s,t}(M_1 \times M_2) \to H^{s-m_1,t-m_2}(M_1 \times M_2).$$

Furthermore, the norm of the operator can be estimated using the seminorms of the symbol. It is also possible to prove the following proposition:

Proposition 1.2.2. *Let $A \in L^{m_1,m_2}(M_1 \times M_2)$ be a bisingular operator. If $m_i \leq 0$, $i = 1, 2$, then there exists $N \in \mathbb{N}$ such that $\|A\|_{0,0} \leq \sup \sum_{i \leq N} p_i(a(x_1, x_2, \xi_1, \xi_2))$, where $\{p_i(\cdot)\}_{i \in \mathbb{N}}$ are the seminorms of the Fréchet space $S^{m_1,m_2}(M_1, M_2)$.*

An operator $A \in L^{m_1,m_2}(M_1 \times M_2)$ is elliptic if $\sigma_1^{m_1}(A), \sigma_2^{m_2}(A), \sigma^{m_1,m_2}(A)$, the three components of its principal symbol, are invertible in their domain of definition. Explicitly:

Definition 1.2.4. Let $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$. A is bisingular elliptic if

i) $\sigma^{m_1,m_2}(A)(v_1, v_2) \neq 0$ for all $(v_1, v_2) \in T^*M_1 \setminus \{0\} \times T^*M_2 \setminus \{0\}$;

ii) $\sigma_1^{m_1}(A)(v_1) \in L^{m_2}_{\mathrm{cl}}(M_2)$ is invertible for all $v_1 \in T^*M_1 \setminus \{0\}$;

iii) $\sigma_2^{m_2}(A)(v_2) \in L^{m_1}_{\mathrm{cl}}(M_1)$ is invertible for all $v_2 \in T^*M_2 \setminus \{0\}$;

where $\sigma^{m_1,m_2}(A)$, $\sigma_1^{m_1}(A)$, $\sigma_2^{m_2}(A)$ are as in (1.6).

Remark 1.2.3. *If an operator $A \in L^{m_1,m_2}_{pr}$ satisfies condition* iii) *of Definition 1.2.4 then both the operators $\sigma^{m_1}(A)(v_1) \in L^{m_2}(M_2)$ and $\sigma^{m_2}(A)(v_2) \in L^{m_1}(M_1)$ are elliptic operators. Moreover, if A satisfies conditions* i) *and* ii)*, one can prove that both $\sigma^{m_1}(A)(v_1)$ and $\sigma^{m_2}(A)(v_2)$ are injective Fredholm operators with zero index, therefore invertible operators also in the scale of H^s spaces. Thus, in Definition 1.2.4, it is equivalent to require the invertibility of the operators on the spaces of smooth functions or on the Sobolev spaces H^s.*

In [90], it is proved that, if A satisfies Definition 1.2.4. Then, A is a Fredholm operator. This property is a corollary of the following theorem:

[1] For the definition of $\widehat{\otimes}_\pi$, see [103].

Theorem 1.2.3. *Let $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$ be bisingular elliptic. Then, there exists an operator $B \in L^{-m_1,-m_2}_{\mathrm{pr}}(M_1 \times M_2)$ such that*

$$AB = \mathrm{Id} + K_1,$$
$$BA = \mathrm{Id} + K_2,$$

where Id *is the identity map and K_1, K_2 are compact operators. Moreover,* $\mathrm{sym}(B) = b = \{\sigma_1^{m_1}(A)^{-1}, \sigma_2^{m_2}(A)^{-1}\}$.

The proof of Theorem 1.2.3 is a consequence of the global version of the following lemma:

Lemma 1.2.4. *Let $A \in L^{m_1,m_2}(\Omega_1 \times \Omega_2)$ and $B \in L^{m_1',m_2'}(\Omega_1 \times \Omega_2)$, then*

$$\{(a \circ b)_{m_1+m_1',\cdot}, (a \circ b)_{\cdot,m_2+m_2'}\} = \{a_{m_1,\cdot} \circ_{\xi_2} b_{m_1',\cdot}, a_{\cdot,m_2} \circ_{\xi_1} b_{\cdot,m_2'}\},$$

where

$$(a \circ_{\xi_1} b)(x_1, x_2, D_1, \xi_2)(u) = a(x_1, x_2, D_1, \xi_2) \circ b(x_1, x_2, D_1, \xi_2)(u) \quad \forall u \in C^\infty_c(\Omega_1),$$
$$(a \circ_{\xi_2} b)(x_1, x_2, \xi_1, D_2)(v) = a(x_1, x_2, \xi_1, D_2) \circ b(x_1, x_2, \xi_1, D_2)(v) \quad \forall v \in C^\infty_c(\Omega_2).$$

In the first row the composition is in $L^\infty(\Omega_1)$, the algebra of of pseudodifferential operators on Ω_1, in the second row, it is in $L^\infty(\Omega_2)$.

1.3 Bisingular Operators on Euclidean Spaces

In this section we illustrate a global version of bisingular operators, adapted to Shubin's calculus on $\mathbb{R}^n$, see [101]. This class of operators has been recently introduced in [16].

Definition 1.3.1. We define $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, $m_1 \in \mathbb{R}, m_2 \in \mathbb{R}$, as the subset of $C^\infty(\mathbb{R}^{2n_1+2n_2})$ functions such that for all multiindices α_i, β_i $(i = 1, 2)$ there exists a constant C so that

$$|\partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2} a(x_1, x_2, \xi_1, \xi_2)| \leq C\langle x_1, \xi_1\rangle^{m_1-|\alpha_1|-|\beta_1|}\langle x_2, \xi_2\rangle^{m_2-|\alpha_2|-|\beta_2|}, \tag{1.8}$$

for all x_1, ξ_1, x_2, ξ_2. We also define

$$\Gamma^{-\infty,-\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) = \bigcap_{m_1,m_2 \in \mathbb{R}^2} \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) = \mathscr{S}(\mathbb{R}^{2n_1+2n_2}),$$

the set of smoothing symbols in this context.

Definition 1.3.2. A linear operator $A : C^\infty_c(\mathbb{R}^{n_1+n_2}) \to C^\infty(\mathbb{R}^{n_1+n_2})$ is a globally bisingular operator if it can be written as[2]

$$A(u)(x_1, x_2) = \mathrm{Op}(a)(u)(x_1, x_2) = \iint e^{ix_1\cdot\xi_1+ix_2\cdot\xi_2} a(x_1, x_2, \xi_1, \xi_2)\hat{u}(\xi_1, \xi_2) đ\xi_1 đ\xi_2, \tag{1.9}$$

where $a \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. We define $G^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ as the set of operators (1.9) with symbol in $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$.

[2] $đ\xi_i = (2\pi)^{-n_i} d\xi_i$.

The $\mathcal{S}$-continuity of globally bisingular operators is immediate, we just have to check all seminorms. A continuity result on suitable Sobolev spaces can also be proved, as stated below.

Definition 1.3.3. For positive integers s_1, s_2, we define $Q^{s_1,s_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ as the space of all $u \in L^2(\mathbb{R}^{n_1+n_2})$ such that

$$\|u\|_{Q^{s_1,s_2}} = \sum_{\substack{|\alpha_1|+|\beta_1|\le s_1,\\ |\alpha_2|+|\beta_2|\le s_2}} \|x_1^{\beta_1} x_2^{\beta_2} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} u\|_{L^2}.$$

For general $s_1, s_2 \in \mathbb{R}$ we set

$$Q^{s_1,s_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) = \{u \in \mathscr{S}'(\mathbb{R}^{n_1+n_2}) \mid \|u\|_{Q^{s_1,s_2}} = \|\{\mathrm{Op}(\langle x_1, \xi_1\rangle^{s_1} \langle x_2, \xi_2\rangle^{s_2})\|_{L^2} < \infty\}.$$

Theorem 1.3.1. *An operator $A \in G^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ can be extended, for every $s_1, s_2 \in \mathbb{R}$, as a continuous operator*

$$A : Q^{s_1,s_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \to Q^{s_1-m_1,s_2-m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}).$$

The proof of Theorem 1.3.1 follows observing that $\Gamma^{0,0}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subseteq \Gamma^0_0(\mathbb{R}^{n_1+n_2})$, see [101] for the definition of $\Gamma^0_0(\mathbb{R}^n)$. Then, we use the well known results of L^2-continuity and the definition of $Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2})$. In order to make the notation simpler, in the sequel we will just write Γ^{m_1,m_2} and G^{m_1,m_2}, fixing the dimensions of the base spaces to n_1, n_2. We prove now that globally bisingular operators form an algebra.

Theorem 1.3.2. *Let $A \in G^{m_1,m_2}$ and $B \in G^{l_1,l_2}$. Then, $A \circ B \in G^{m_1+l_1,m_2+l_2}$.*

Proof. With a simple evaluation we obtain

$$(A \circ B)u(x_1, x_2) = \iint e^{ix_1\xi_1+ix_2\xi_2} c(x_1, x_2, \xi_1, \xi_2)\hat{u}(\xi_1, \xi_2) đ\xi_1 đ\xi_2,$$

where

$$c(x_1, x_2, \xi_1, \xi_2) = \int e^{-i(\mu_1+\mu_2)} a(x_1, x_2, \eta_1, \eta_2) b(y_1, y_2, \xi_1, \xi_2) dy_1 dy_2 đ\eta_1 đ\eta_2$$
$$\mu_1 = \langle y_1 - x_1, \eta_1 - \xi_1\rangle, \quad \mu_2 = \langle y_2 - x_2, \eta_2 - \xi_2\rangle. \tag{1.10}$$

We divide the product ab in (1.10) into four parts, for a fixed integer $N > 0$:

$$a(x_1, x_2, \eta_1, \eta_2) b(y_1, y_2, \xi_1, \xi_2) = (ab)_1^N + (ab)_2^N + (ab)_3^N + r_N,$$

where

$$(ab)_1^N = \sum_{|\beta_1|+|\alpha_1|<2N} \frac{1}{\beta_1!\alpha_1!} (y_1 - x_1)^{\beta_1} (\eta_1 - \xi_1)^{\alpha_1}$$
$$\partial_{\eta_1}^{\alpha_1} a(x_1, x_2, \xi_1, \eta_2) \partial_{y_1}^{\beta_1} b(x_1, y_2, \xi_1, \xi_2),$$

$$(ab)_2^N = \sum_{|\beta_2|+|\alpha_2|<2N} \frac{1}{\beta_2!\alpha_2!} (y_2 - x_2)^{\beta_2} (\eta_2 - \xi_2)^{\alpha_2}$$
$$\partial_{\eta_2}^{\alpha_2} a(x_1, x_2, \eta_1, \xi_2) \partial_{y_2}^{\beta_2} b(y_1, x_2, \xi_1, \xi_2),$$

$$(ab)_3^N = -\sum_{\substack{|\alpha_1|+|\beta_1|<2N \\ |\alpha_2|+|\beta_2|<2N}} \frac{1}{\beta_1!\beta_2!\alpha_1!\alpha_2!}(y_1-x_1)^{\beta_1}(y_2-x_2)^{\beta_2}(\eta_1-\xi_1)^{\alpha_1}(\eta_2-\xi_2)^{\beta_2}$$
$$\partial_{\eta_1}^{\alpha_1}\partial_{\eta_2}^{\alpha_2}a(x_1,x_2,\xi_1,\xi_2)\partial_{y_1}^{\beta_1}\partial_{y_2}^{\beta_2}b(x_1,x_2,\xi_1,\xi_2),$$
$$r_N = \sum_{\substack{|\alpha_1|+|\beta_1|<2N \\ |\alpha_2|+|\beta_2|<2N}} \frac{1}{\beta_1!\beta_2!\alpha_1!\alpha_2!}(y_1-x_1)^{\beta_1}(y_2-x_2)^{\beta_2}(\eta_1-\xi_1)^{\alpha_1}(\eta_2-\xi_2)^{\alpha_2}$$
$$\int_0^1\int_0^1 (1-t_1)^{N-1}(1-t_2)^{N-1}\partial_{\eta_1}^{\alpha_1}\partial_{\eta_2}^{\alpha_2}a(x_1,x_2,\xi_1+t_1(\eta_1-\xi_1),\xi_2+$$
$$t_2(\eta_2-\xi_2))\partial_{y_1}^{\beta_1}\partial_{y_2}^{\beta_2}b(x_1+t_1(y_1-x_1),x_2+t_2(y_2-x_2),\xi_1,\xi_2)dt_1dt_2.$$

Also, we define

$$c_i^N = \int e^{-i\mu_1-i\mu_2}(ab)_i^N dy_1dy_2đ\eta_1đ\eta_2, \quad R_N = \int e^{-i\mu_1-i\mu_2}r_N dy_1dy_2đ\eta_1đ\eta_2.$$

Let us focus on c_1^N. Notice that

$$(y_1-x_1)^{\beta_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle} = (-i)^{\beta_1}D_{\eta_1}^{\beta_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle}, \tag{1.11}$$
$$(\eta_1-\xi_1)^{\alpha_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle} = (-i)^{\alpha_1}D_{y_1}^{\alpha_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle}. \tag{1.12}$$

If $\alpha_1 \neq \beta_1$, there exists an index i such that, for example, $(\alpha_1)_i > (\beta_1)_i$. So, using relation (1.12) and integrating by parts, we derive $(\alpha_1)_i$ times w.r.t. y_1 the expression $(y_1-x_1)^{\beta_1}$, and, since $(\alpha_1)_i > (\beta_1)_i$, the derivative is zero. Clearly, the same scheme can be used if $(\alpha_1)_i < (\beta_1)_i$, by exchanging the role of the variable and the covariable, and by (1.11). This implies that we can restrict ourselves to consider the case $\alpha_1 = \beta_1$. Now, integrating by parts and using relation (1.12), we get

$$c_1^N = \frac{1}{\alpha!}\iint e^{-i\langle y_2-x_2,\eta_2-\xi_2\rangle}\sum_{|\alpha_1|<N}\partial_{\xi_1}^{\alpha_1}a(x_1,x_2,\xi_1,\eta_2)D_{x_1}^{\alpha_1}b(x_1,y_2,\xi_1,\xi_2)dy_2đ\eta_2. \tag{1.13}$$

The expression (1.13) can be written in the form

$$c_1^N = \sum_{|\alpha_1|<N}\frac{1}{\alpha_1!}\partial_{\xi_1}^{\alpha_1}a \circ_2 D_{x_1}^{\alpha_1}b,$$

where the symbol $\circ_2$ means the composition of the operators acting on $\mathbb{R}^{n_2}$. With the same scheme we can prove that

$$c_2^N = \sum_{|\alpha_2|<N}\frac{1}{\alpha_2!}\partial_{\xi_2}^{\alpha_2}a \circ_1 D_{x_2}^{\alpha_2}b.$$

Integrating by parts twice, we get

$$c_3^N = -\sum_{\substack{|\alpha_1|<N \\ |\alpha_2|<N}}\frac{1}{\alpha_1!\alpha_2!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}aD_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}b.$$

We have now to analyze the remainder. Consider the identity

$$\langle y_1, \eta_1\rangle^{2P}\langle y_2, \eta_2\rangle^{2P}(1-\Delta_{y_1}-\Delta_{\eta_1})^P(1-\Delta_{y_2}-\Delta_{\eta_2})^P e^{-i(\mu_1+\mu_2)} = e^{-i(\mu_1+i\mu_2)}. \quad (1.14)$$

By Peetre inequality, we have

$$\begin{aligned}|r_N| \leq &\langle x_1, \xi_1\rangle^{m_1+l_1-2P}\langle x_2, \xi_2\rangle^{m_2+l_2-2P}\langle y_1-x_1\rangle^{|l_1|+2P}\langle y_2-x_2\rangle^{|l_2|+2P}\\ &\langle \eta_1-\xi_1\rangle^{|m_1|+2P}\langle \eta_2-\xi_2\rangle^{|m_2|+2P}.\end{aligned}$$

Using (1.14) with P large enough and integrating by parts, we prove that $R_N \in \Gamma^{m_1+l_2-2N, m_2+l_2-2N}$. □

Remark 1.3.1. *It is useful to write c as*

$$c \sim \sum_{j=0}^{\infty} c_{m_1+l_1-2j, m_2+l_2-2j},$$

where

$$c_{m_1+l_1-2j,m_2+l_2-2j} = c^1_{m_1+l_1-2j,m_2+l_2-2j} + c^2_{m_1+l_1-2j,m_2+l_2-2j} + c^3_{m_1+l_1-2j,m_2+l_2-2j},$$

and

$$\begin{aligned}
c^1_{m_1+l_1-2j,m_2+l_2-2j} &= \sum_{|\alpha_1|=j}\frac{1}{\alpha_1!}\Big(\partial^{\alpha_1}_{\xi_1} a \circ_2 D^{\alpha_1}_{x_1} b - \sum_{|\alpha_2|\leq j}\frac{1}{\alpha_2!}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} a D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} b\Big),\\
c^2_{m_1+l_1-2j,m_2+l_2-2j} &= \sum_{|\alpha_2|=j}\frac{1}{\alpha_2!}\Big(\partial^{\alpha_2}_{\xi_2} a \circ_1 D^{\alpha_2}_{x_2} b - \sum_{|\alpha_1|\leq j}\frac{1}{\alpha_1!}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} a D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} b\Big),\\
c^3_{m_1+l_1-2j,m_2+l_2-2j} &= \sum_{|\alpha_1|=|\alpha_2|=j}\frac{1}{\alpha_1!\alpha_2!}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} a D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} b.
\end{aligned}$$

In the sequel, we will study a subclass of globally bisingular operators, namely operators with homogeneous principal part.

Definition 1.3.4. A symbol $a \in \Gamma^{m_1,m_2}$ has homogeneous principal part if

i) there exists a function $a_{m_1,\cdot}(x_1, x_2, \xi_1, \xi_2)$, homogeneous w.r.t. (x_1, ξ_1) of order m_1, such that

$$a - \psi_1(x_1, \xi_1)a_{m_1,\cdot} \in \Gamma^{m_1-1,m_2}(\mathbb{R}^{n_1+n_2}),$$

ψ_1 fixed 0-excision function, and the operator $a(x_1, x_2, \xi_1, D_2)$, with (x_1, ξ_1) *frozen*, is a classical global operator in $\mathbb{R}^{n_2}$;

ii) there exists $a_{\cdot,m_2}$, homogeneous w.r.t. (x_2, ξ_2) of order m_2, such that

$$a - \psi_2(x_2, \xi_2)a_{\cdot,m_2} \in \Gamma^{m_1,m_2-1}(\mathbb{R}^{n_1,n_2}),$$

ψ_2 fixed 0-excision function, and the operator $a(x_1, x_2, D_1, \xi_2)$, with (x_2, ξ_2) *frozen*, is a classical global operator in $\mathbb{R}^{n_1}$;

iii) there exists a function $a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2)$ homogeneous w.r.t. (x_1, ξ_1) of order m_1 and w.r.t. (x_2, ξ_2) of order m_2, such that a_{m_1,m_2} is equal to the principal symbol of $a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2)$ and of $a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2)$ and

$$a - \psi_1(x_1, \xi_1)a_{m_1,\cdot} - \psi_2(x_2, \xi_2)(a_{\cdot,m_2}) + \psi_1(x_1, \xi_1)\psi_2(x_2, \xi_2)a_{m_1,m_2}$$

belongs to $\Gamma^{m_1-1,m_2-1}(\mathbb{R}^{n_1+n_2})$.

In the rest of the section, the class of globally bisingular symbols with homogeneous principal part is denoted by $\Gamma^{m_1,m_2}_{\text{pr}}$, and the corresponding operators with homogeneous principal symbol by $G^{m_1,m_2}_{\text{pr}}(\mathbb{R}^{n_1+n_2})$. We introduce three functions, associated with an operator $A \in G^{m_1,m_2}_{\text{pr}}$: [3]

$$\begin{aligned}
\sigma_1^{m_1}(A) : T^*(\mathbb{R}^{n_1}) \setminus \{0\} &\to G^{m_2}_{cl}(\mathbb{R}^{n_2}) \\
(x_1, \xi_1) &\mapsto a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2), \\
\sigma_2^{m_2}(A) : T^*(\mathbb{R}^{n_2}) \setminus \{0\} &\to G^{m_1}_{cl}(\mathbb{R}^{n_1}) \\
(x_2, \xi_2) &\mapsto a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2), \\
\sigma^{m_1,m_2}(A) : T^*(\mathbb{R}^{n_1}) \setminus \{0\} \times T^*(\mathbb{R}^{n_2}) \setminus \{0\} &\to \mathcal{H}^{m_1,m_2}_{\xi_1,\xi_2} \\
(x_1, x_2, \xi_1, \xi_2) &\mapsto a_{m_1,m_2}(x_1, x_2, \xi, \xi_2).
\end{aligned}$$

Remark 1.3.2. *Analogously to the case of bisingular operators, one can introduce classical globally bisingular operator: this is achieved through the construction of Remark 1.2.1 in this setting. That is, one can set*

$$\Gamma^{m_1,m_2}_{\text{cl}} = (RC_1 \times RC_2)^* \, \rho_1^{-m_1} \rho_2^{-m_2} C^\infty(\mathbb{S}^{2n_1}_+ \times \mathbb{S}^{2n_2}_+),$$

where $\rho_1^{-1}(RC_1(x_1, \xi_1), \omega_2) = \langle x_1, \xi_1 \rangle$ *for all* $x_1, \xi_1 \in \mathbb{R}^{2n_1}$ *and* $\omega_2 \in \mathbb{S}^{2n_2}_+$*, and similarly for* ρ_2*. Here we will treat globally bisingular symbols with homogeneous principal symbols, therefore we do not detail the construction of classical globally bisingular operators.*

Now, we introduce the notion of ellipticity. As in the case of bisingular operators on the product of closed manifolds, we restrict ourselves to symbols with homogeneous principal symbol.

Definition 1.3.5. Let $A \in G^{m_1,m_2}_{\text{pr}}(\mathbb{R}^{n_1+n_2})$. A is an elliptic globally bisingular operator if there exist constants R_1, R_2 such that

i) the operator $\sigma_1^{m_1}(A)(x_1, \xi_1)$ is invertible for every $(x_1, \xi_1) \in T^*\mathbb{R}^{n_1} \setminus \{0\}$;

ii) the operator $\sigma_2^{m_2}(A)(x_2, \xi_2)$ is invertible for every $(x_2, \xi_2) \in T^*\mathbb{R}^{n_2} \setminus \{0\}$;

iii) for $(x_1, \xi_1) \in T^*\mathbb{R}^{n_1} \setminus \{0\}$, $(x_2, \xi_2) \in T^*\mathbb{R}^{n_2} \setminus \{0\}$

$$|\sigma^{m_1,m_2}(A)(x_1, x_x, \xi_1, \xi_2)| \neq 0. \tag{1.15}$$

Remark 1.3.3. *As in Remark 1.2.3, we notice that, if an operator* $A \in G^{m_1,m_2}_{\text{pr}}$ *satisfies condition* iii) *of Definition 1.3.5, then both the operators* $\sigma_1^{m_1}(x_1, \xi_1) \in G^{m_2}(\mathbb{R}^{n_2})$ *and* $\sigma_2^{m_2}(x_2, \xi_2) \in G^{m_1}(\mathbb{R}^{n_1})$ *are elliptic Shubin-type operators. Furthermore, if A satisfies conditions* i) *and* ii)*, one can prove that both* $\sigma_1^{m_1}(A)(x_2, \xi_2)$ *and* $\sigma_2^{m_2}(A)(x_1, \xi_1)$ *are injective Fredholm operator with zero index, therefore invertible operators also in the scale of* Q^s *spaces. Thus, in Definition 1.3.5, it is equivalent to require the invertibility of the operators on the Schwartz spaces or on the Sobolev spaces* Q^s*.*

[3] $\mathcal{H}^{m_1,m_2}_{\xi_1,\xi_2}$ is the set of homogeneous function of order m_i w.r.t. ξ_i, $i = 1, 2$.

Theorem 1.3.3. *If A is an elliptic globally bisingular operator then it is a Fredholm operator.*

Proof. It is a consequence of Theorem 1.3.1. From Remark 1.3.1, if A is elliptic one can define B as the operator with symbol

$$\begin{aligned} b =& \psi_1(x_1, \xi_1)\mathrm{sym}(\sigma_1^{m_1}(A)^{-1}) + \psi_2(x_2, \xi_2)\mathrm{sym}(\sigma_2^{m_2}(A)^{-1}) \\ &- \psi_1(x_1, \xi_1)\psi_2(x_2, \xi_2)\sigma^{m_1,m_2}(A)^{-1}. \end{aligned}$$

The calculus implies that B is an inverse of A modulo compact operator. □

Using a Neumann series procedure, by Theorem 1.3.3 we prove that, if a globally bisingular operator is elliptic, then it admits an inverse modulo smoothing operators. So we have this immediate corollary:

Corollary 1.3.4. *Let $A \in G^{m_1,m_2}_{\mathrm{pr}}$ be elliptic. Then*

i) if $Au \in Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2})$, then $u \in Q^{s_1+m_1,s_2+m_2}$;

ii) if $Au \in \mathcal{S}$, then $u \in \mathcal{S}$.

Chapter 2

Complex Powers and ζ-Function

In this chapter we define the complex powers and the spectral ζ-function of operators in the classes defined in Chapter 1. In Section 2.1, we study *SG*-classical operators, starting with the case of operators on $\mathbb{R}^n$ and then switching to the case of manifolds with cylindrical ends. In this part we follow the construction of L. Maniccia, E. Schrohe and J. Seiler [66]. Eventually, we will introduce the non-commutative residue via the corresponding ζ-function. The non-commutative residue in *SG*-calculus on $\mathbb{R}^n$ was already introduced by F. Nicola in [79] by means of the theory of holomorphic families. We compare the two constructions on $\mathbb{R}^n$, and show that the approach by means of the ζ-function is convenient for the extension of this concept to manifolds with cylindrical ends. In Section 2.2, we define a regularized version of the non-commutative residue, in order to prove a (regularized version) of the Kastler-Kalau-Walze Theorem on $\mathbb{R}^n$ endowed with a suitable metric. In Sections 2.3 and 2.4 we introduce complex powers of suitable operators with homogeneous principal symbols in the setting of bisingular operators and bisingular operators on Euclidean spaces, respectively. In both cases, we analyze the continuation of the spectral ζ-function and we give a precise description of the corresponding Laurent coefficients.

2.1 Complex Powers and ζ-Function of SG-Operators

In this section we prove, in particular, that the complex powers of suitable *SG*-classical operators are again *SG*-classical. Then, we study the corresponding ζ-function. The material in this section comes mainly from [15].

Theorem 2.1.1. *Given an elliptic operator $A \in L^{m_1,m_2}$ with $m_1, m_2 > 0$, only one of the following properties holds:*

i) the spectrum of A is the whole complex plane $\mathbb{C}$;

ii) the spectrum of A is a countable set, without any limit point.

Proof. If i) does not hold, there exists $\mu \in \mathbb{C}$ such that $(A - \mu I)$ is invertible. Without loss of generality, we can assume $\mu = 0$, so that

$$(A - \lambda I) = A(I - \lambda A^{-1}),$$

showing that $(A - \lambda I)$ is not invertible if and only if $\lambda \neq 0$ and $\frac{1}{\lambda}$ belongs to the spectrum of A^{-1}. From the properties of elliptic operators, we have that $A^{-1} \in L^{-m_1,-m_2}$. Moreover, in view of the hypothesis $m_1, m_2 > 0$, of the continuity of A^{-1} from $L^2 \equiv H^{0,0}$ to H^{m_1,m_2} and of the compact embeddings between the weighted Sobolev spaces stated in Section 1.1, $A^{-1}: L^2 \to H^{m_1,m_2} \hookrightarrow L^2$ is a compact operator, thus it has a countable spectrum with, at most, the origin as a limit point. □

Remark 2.1.1. *The proof of Theorem 2.1.1 also shows that the eigenfunctions of A are the same of A^{-1}.*

For fixed θ_0, θ, let $\Lambda = \{z \in \mathbb{C}: \theta_0 - \theta \leq \arg(z) \leq \theta_0 + \theta\}$ be a closed sector of the complex plane with vertex at the origin. We now recall the definition of SG-ellipticity with respect to Λ:

Definition 2.1.1. Let Λ be a closed sector of the complex plane with vertex at the origin. A symbol $a(x, \xi) \in SG^{m_1,m_2}$ and the corresponding operator $A = \mathrm{Op}\,(a)$ are called Λ-elliptic if there exist constants $C, R > 0$ such that

i) $a(x, \xi) - \lambda \neq 0$, for any $\lambda \in \Lambda$ and (x, ξ) satisfying $|x| + |\xi| \geq R$;

ii) $|(a(x, \xi) - \lambda)^{-1}| \leq C\langle\xi\rangle^{-m_1}\langle x\rangle^{-m_2}$ for any $\lambda \in \Lambda$ and (x, ξ) satisfying $|x| + |\xi| \geq R$.

Remark 2.1.2. *When matrix-valued symbols are involved, condition i) above is modified, asking that the spectrum of the matrix $a(x, \xi)$ does not intersect the sector Λ for $|x| + |\xi| \geq R$.*

To define the complex powers of an elliptic operator A, we need that the resolvent $(A - \lambda I)^{-1}$ exists, at least, for $|\lambda|$ large enough. The following Theorem 2.1.2 shows that this is always the case when $m_1, m_2 > 0$ and that the resolvent can be well approximated by a parametrix of $A - \lambda I$.

Theorem 2.1.2. *Let $m_1, m_2 > 0$ and $A \in L^{m_1,m_2}$ be Λ-elliptic. Then, there exists a constant L such that the resolvent set $\rho(A)$ includes $\Lambda_L = \{\lambda \in \Lambda: |\lambda| > L\}$. Moreover, for suitable constants $C, C' > 0$, we have that*

$$\|(A - \lambda I)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{C}{\lambda}$$

and

$$\|(A - \lambda I)^{-1} - B(\lambda)\|_{\mathcal{L}(L^2)} \leq \frac{C'}{\lambda^2}$$

where $B(\lambda)$ is a parametrix of $A - \lambda I$.

The next two results give estimates for the position of the eigenvalues of a Λ-elliptic operator in the complex plane and the relation between Λ-ellipticity and the principal symbol of a classical SG-operator, similarly to Theorem 1.1.5.

Lemma 2.1.3. *If $A = \mathrm{Op}(a) \in L^{m_1,m_2}$, $m_1, m_2 > 0$, is Λ-elliptic, there is a constant $c_0 \geq 1$ such that, for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, the spectrum of $a(x,\xi)$ is included in the set*

$$\Omega_{\langle x\rangle,\langle\xi\rangle} = \left\{ z \in \mathbb{C} \setminus \Lambda \colon \frac{1}{c_0}\langle\xi\rangle^{m_1}\langle x\rangle^{m_2} \leq |z| \leq c_0\langle\xi\rangle^{m_1}\langle x\rangle^{m_2} \right\}$$

and

$$|(\lambda - a(x,\xi))^{-1}| \leq C(|\lambda| + \langle\xi\rangle^{m_1}\langle x\rangle^{m_2})^{-1}, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \lambda \in \mathbb{C} \setminus \Omega_{\langle x\rangle,\langle\xi\rangle}.$$

Proposition 2.1.4. *For $a \in SG^{m_1,m_2}_{\mathrm{cl}}$, the Λ-ellipticity property is equivalent to*

$$a_{m_1,\cdot}(x,\omega) - \lambda \neq 0, \text{ for all } x \in \mathbb{R}^n,\ \omega \in \mathbb{S}^{n-1},\ \lambda \in \Lambda,$$
$$a_{\cdot,m_2}(\omega',\xi) - \lambda \neq 0, \text{ for all } \xi \in \mathbb{R}^n,\ \omega' \in \mathbb{S}^{n-1},\ \lambda \in \Lambda,$$
$$a_{m_1,m_2}(\omega',\omega) - \lambda \neq 0, \text{ for all } \omega \in \mathbb{S}^{n-1},\ \omega' \in \mathbb{S}^{n-1},\ \lambda \in \Lambda,$$

where $\mathbb{S}^{n-1} = \{u \in \mathbb{R}^n \colon |u| = 1\}$.

Remark 2.1.3. *If a is matrix-valued, the conditions in Proposition 2.1.4 have to be expressed in terms of the spectra of the three involved matrices, analogously to Remark 2.1.2.*

We can now give the definition of A^z, $z \in \mathbb{C}$. The following assumptions on A are natural:

Assumptions 1.

1. *$A \in L^{m_1,m_2}_{\mathrm{cl}}$, with m_1 and m_2 positive integers;*
2. *A is Λ-elliptic with respect to a closed sector Λ of the complex plane with vertex at the origin, therefore A is invertible;*
3. *The spectrum of A does not intersect the real interval $(-\infty, 0)$.*

Theorem 2.1.1 implies that, if A satisfies Assumptions 1, it has a discrete spectrum. In view of this, it is possible to find $\theta \in (0,\pi)$ so that $(A-\lambda)^{-1}$ exists for all $\lambda \in \Lambda = \Lambda(\theta) = \{z \in \mathbb{C} \colon \pi - \theta \leq \arg(z) \leq \pi + \theta\}$.

Definition 2.1.2. Let A be an SG-operator that satisfies Assumptions 1. Let us define A_z, $z \in \mathbb{C}$, $\mathrm{Re}\, z < 0$, as

$$A_z = \frac{1}{2\pi i}\int_\Gamma \lambda^z (A - \lambda I)^{-1} d\lambda, \tag{2.1}$$

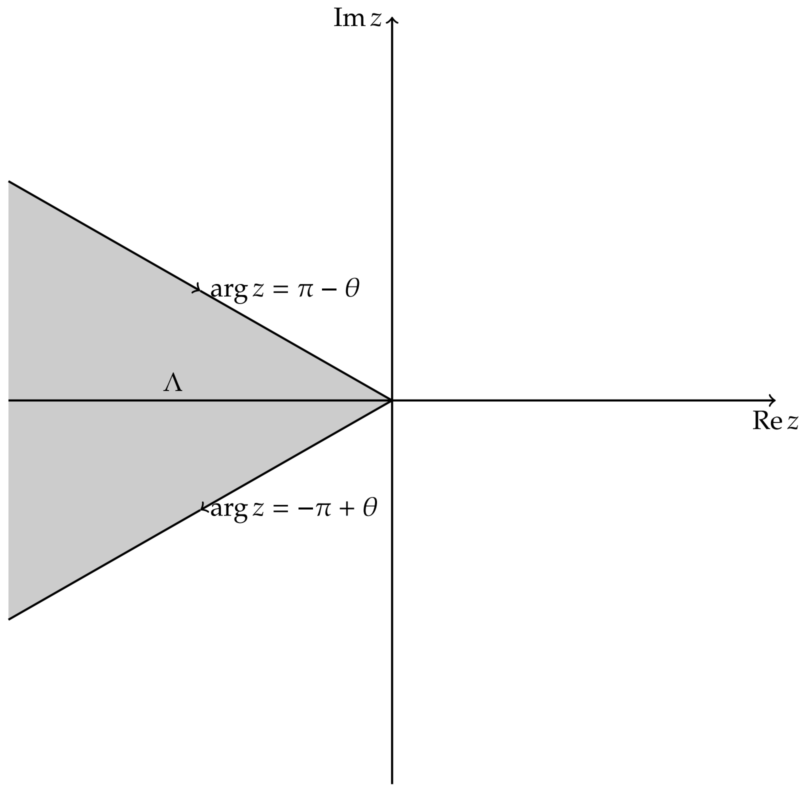
where $\Gamma = \partial^+\Lambda$ is the path in the Figure 2.1:

The operator A_z, $\mathrm{Re}\, z < 0$, is well defined since, from Theorem 2.1.2, $\|(A - \lambda I)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{1}{\lambda}$ and this gives the absolute convergence of the integral. The definition can be extended to arbitrary $z \in \mathbb{C}$:

Definition 2.1.3. Let A be an SG-operator that satisfies Assumptions 1. Define

$$A^z = \begin{cases} A_z & \text{for } \mathrm{Re}\, z < 0 \\ A_{z-l}A^l & \text{for } \mathrm{Re}\, z \geq 0, \text{ with } l = 1, 2, \dots,\ \mathrm{Re}\, z - l < 0. \end{cases}$$

Figure 2.1:



Proposition 2.1.5. *i) The Definition of A^z for $\operatorname{Re} z \geq 0$ does not depend on the integer l.*

ii) $A^z A^s = A^{z+s}$ *for all* $z, s \in \mathbb{C}$.

iii) $A^k = \underbrace{A \circ \ldots \circ A}_{k \text{ times}}$ *when z coincides with the positive integer k.*

iv) If $A \in L^{m_1,m_2}$ *then* $A^z \in L^{m_1 z, m_2 z}$.

The proof can be found, e.g., in [92] and [101]. Note that the definition and properties of *SG*-symbols and operators with complex double order (z_1, z_2) are analogous to those given above, with $\operatorname{Re} z_1, \operatorname{Re} z_2$ in place of m_1, m_2, respectively.

Remark 2.1.4. *An application of Lemma 2.1.3 implies that the symbol of the operator* A^z *has the form*

$$\operatorname{sym}(A^z) = \frac{1}{2\pi i} \int_{\partial^+ \Omega_{\langle x \rangle, \langle \xi \rangle}} \lambda^z \operatorname{sym}((A - \lambda I)^{-1}) d\lambda.$$

It is a fact that, given an *SG*-classical operator A satisfying Assumptions 1, A^z is still classical. L. Maniccia, E. Schrohe and J. Seiler proved this in [66] by direct computation, finding the *SG*-classical expansion of $\operatorname{sym}(A^z)$. We prove here the same result by a different technique, which makes use of the identification between SG-classical symbols and $C^\infty(\mathbb{B}^n \times \mathbb{B}^n)$ given in Theorem 1.1.4, see Yu. V. Egorov and B.-W Schulze [29].

Theorem 2.1.6. *Let $A \in L^{m_1,m_2}_{\mathrm{cl}}$ be an operator satisfying Assumptions 1. Then A^z, $\operatorname{Re} z < 0$, is SG-classical of order $(m_1 z, m_2 z)$.*

Proof. In this proof we use vector notation for the orders, setting $\mathbf{m} = (m_1, m_2)$, $\mathbf{e} = (1, 1)$. By Lemma 2.1.3 and Remark 2.1.4 we know that

$$a^z = \operatorname{sym}(A^z) = \frac{1}{2\pi i} \int_{\partial^+\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z \operatorname{sym}((A - \lambda I)^{-1}) d\lambda.$$

We have to prove that $a^z \in SG^{\mathbf{m}z}_{\mathrm{cl}}$. To begin, we claim that

$$b_{\mathbf{m}z}(x, \xi) = \frac{1}{2\pi i} \int_{\partial^+\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z (a_{\mathbf{m}}(x, \xi) - \lambda)^{-1} d\lambda = [a_{\mathbf{m}}(x, \xi)]^z \in SG^{\mathbf{m}z}_{\mathrm{cl}}, \ \operatorname{Re} z < 0.$$

In view of Theorem 1.1.4, it is enough to show that $(D^{\mathbf{m}z} b_{\mathbf{m}z})(y, \eta) \in C^\infty(\mathbb{B}^n \times \mathbb{B}^n)$. For $\mathbf{t} = (t_1, t_2)$, set $w_{\mathbf{t}}(y, \eta) = (1 - [\eta])^{t_1}(1 - [y])^{t_2}$. By the change of variable $\lambda = w_{-\mathbf{m}}(y, \eta)\mu$, we get

$$\begin{aligned} (D^{\mathbf{m}z} b_{\mathbf{m}z})(y, \eta) &= \frac{1}{2\pi i} \int_{\partial^+\Omega_{\langle\chi(y)\rangle,\langle\chi(\eta)\rangle}} \frac{\lambda^z \, w_{\mathbf{m}z}(y, \eta)}{a_{\mathbf{m}}(\chi(y), \chi(\eta)) - \lambda} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial^+\widetilde{\Omega}_{y,\eta}} \frac{\mu^z \, w_{-\mathbf{m}}(y, \eta)}{a_{\mathbf{m}}(\chi(y), \chi(\eta)) - \mu \, w_{-\mathbf{m}}(y, \eta)} d\mu \\ &= \frac{1}{2\pi i} \int_{\partial^+\widetilde{\Omega}_{y,\eta}} \frac{\mu^z}{(D^{\mathbf{m}} a_{\mathbf{m}})(y, \eta) - \mu} d\mu \end{aligned}$$

By Lemma 2.1.3, $|a_{\mathbf{m}}(x, \xi) - \lambda| \geq c(\langle\xi\rangle^{m_1} \langle x\rangle^{m_2} + |\lambda|)$, which implies $|(D^{\mathbf{m}} a_{\mathbf{m}})(y, \eta) - \mu| \geq c(1 + |\mu|)$, so that $D^{\mathbf{m}z} b_{\mathbf{m}z} \in C^\infty(\mathbb{B}^n \times \mathbb{B}^n)$, as claimed.
By the parametrix construction in the SG-calculus, and in view of the Λ-ellipticity of A, we have

$$(A - \lambda I)^{-1} = \operatorname{Op}((a - \lambda)^{-1}) + \operatorname{Op}(c) + \operatorname{Op}(q),$$

where $q \in SG^{-\infty}$, $c = \operatorname{sym}((A - \lambda I)^{-1} - \operatorname{Op}((a - \lambda)^{-1})) \sim \sum_{j=1}^{\infty} c_j$, $c_j = r_j \, (a - \lambda)^{-1}$, $r_j \in SG^{-j\mathbf{e}}_{\mathrm{cl}}$, $j \geq 1$, see [66]. We can then write

$$\begin{aligned} a^z =& \frac{1}{2\pi i} \int_{\partial^+\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z \operatorname{sym}((A - \lambda I)^{-1}) d\lambda \\ \sim& \frac{1}{2\pi i} \int_{\partial^+\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z (a - \lambda)^{-1} d\lambda + \frac{1}{2\pi i} \sum_{j=1}^{\infty} \int_{\partial^+\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z r_j (a - \lambda)^{-1} d\lambda \\ +& \frac{1}{2\pi i} \int_{\partial^+\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z q \, d\lambda. \end{aligned} \tag{2.2}$$

Let us consider the first term. The operator A is SG-classical so $a = a_{\mathbf{m}} + r$,

$r \in SG_{\rm cl}^{\mathbf{m}-\mathbf{e}}$. We have, for all $N \in \mathbb{N}$,

$$
\begin{aligned}
(a-\lambda)^{-1} &= (a_{\mathbf{m}} + r - \lambda)^{-1} \\
&= (a_{\mathbf{m}} - \lambda)^{-1}(1 + (a_{\mathbf{m}} - \lambda)^{-1} r)^{-1} \\
&= (a_{\mathbf{m}} - \lambda)^{-1} \left[\sum_{k=0}^{N} (-1)^k (a_{\mathbf{m}} - \lambda)^{-k} r^k \right. \\
&\qquad \left. + (-1)^{N+1}(1 + (a_{\mathbf{m}} - \lambda)^{-1} r)^{-1} (a_{\mathbf{m}} - \lambda)^{-(N+1)} r^{N+1} \right],
\end{aligned}
$$

and then

$$
\begin{aligned}
b &= \frac{1}{2\pi i} \int_{\partial\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z (a-\lambda)^{-1} d\lambda \\
&= \underbrace{\frac{1}{2\pi i} \int_{\partial\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z (a_{\mathbf{m}}-\lambda)^{-1} d\lambda}_{b_{\mathbf{m}z}} + \sum_{k=1}^{N} \underbrace{\frac{(-1)^k}{2\pi i} \int_{\partial\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z r^k (a_{\mathbf{m}}-\lambda)^{-k-1} d\lambda}_{b_k} + R_N,
\end{aligned}
$$

where

$$
R_N = \frac{(-1)^{N+1}}{2\pi i} \int_{\partial\Omega_{\langle x\rangle,\langle\xi\rangle}} \lambda^z (1 + (a_{\mathbf{m}} - \lambda)^{-1} r)^{-1} (a_{\mathbf{m}} - \lambda)^{-(N+2)} r^{N+1} d\lambda.
$$

By the calculus and the hypotheses, it turns out that $R_N \in SG^{\mathbf{m}-(N+1)\mathbf{e}}$. Moreover, $b_k \in SG_{\rm cl}^{\mathbf{m}z-k\mathbf{e}}$, $k \geq 1$. Indeed, as above,

$$
\begin{aligned}
(D^{\mathbf{m}z-k\mathbf{e}} b_k)(y,\eta) &= \frac{(-1)^k}{2\pi i} \int_{\partial^+\Omega_{\langle\chi(y)\rangle,\langle\chi(\eta)\rangle}} \frac{\lambda^z \, w_{\mathbf{m}z-k\mathbf{e}}(y,\eta)\, r^k(\chi(y),\chi(\eta))}{(a_{\mathbf{m}}(\chi(y),\chi(\eta)) - \lambda)^{k+1}} d\lambda \\
&= \frac{(-1)^k}{2\pi i} \int_{\partial^+\widetilde{\Omega}_{y,\eta}} \frac{\mu^z \, w_{-\mathbf{m}z}(y,\eta)\, w_{\mathbf{m}z-k\mathbf{e}}(y,\eta)\, r^k(\chi(y),\chi(\eta))}{w_{\mathbf{m}}(y,\eta)\,(a_{\mathbf{m}}(\chi(y),\chi(\eta)) - \mu\, w_{-\mathbf{m}}(y,\eta))^{k+1}} d\mu \qquad (2.3) \\
&= \frac{(-1)^k}{2\pi i} \int_{\partial^+\widetilde{\Omega}_{y,\eta}} \frac{\mu^z \, ((D^{\mathbf{m}-\mathbf{e}} r)(y,\eta))^k}{((D^{\mathbf{m}} a_{\mathbf{m}})(y,\eta) - \mu)^{k+1}} d\mu \in C^\infty(\mathbb{B}^n \times \mathbb{B}^n).
\end{aligned}
$$

Theorem 1.1.3 then gives $b \in SG_{\rm cl}^{\mathbf{m}z}$ with $b - b_{\mathbf{m}z} \in SG_{\rm cl}^{\mathbf{m}z-\mathbf{e}}$. In a completely similar fashion, it is possible to prove that the asymptotic sum in (2.2) gives a symbol in $SG_{\rm cl}^{\mathbf{m}z-\mathbf{e}}$, since $D^{-j\mathbf{e}} r_j$ is smooth and uniformly bounded, together with its derivatives, with respect to μ (see [66] for more details). Finally, it is easy to see that the third term in (2.2) gives a smoothing operator. Again by Theorem 1.1.3, $a^z \in SG_{\rm cl}^{\mathbf{m}z}$, with $a^z = [a_{\mathbf{m}}(x,\xi)]^z \mod SG_{\rm cl}^{\mathbf{m}z-\mathbf{e}}$. □

Remark 2.1.5. *By Definition 2.1.3,*

$$
A^z = A^l \circ A^{z-l}, \quad \mathrm{Re}(z-l) < 0,
$$

and, by Theorem 2.1.6, we obtain that A^z is an SG-classical operator for all $z \in \mathbb{C}$. So, denoting $a^l_{m_1 l - j, \cdot}(x,\xi)$, $j = 0, 1, \ldots$, the terms of the homogeneous expansion with respect to ξ of A^l, the SG-calculus implies

$$
a^z_{m_1 z - j, \cdot}(x,\xi) = \frac{1}{\alpha!} \sum_{|\alpha|+i+k=j} \partial^\alpha_\xi a^l_{m_1 l - i, \cdot} D^\alpha_x a^{z-l}_{m_1(z-l)-k, \cdot}. \qquad (2.4)
$$

The same holds for the x-expansion

$$a^z_{\cdot,m_2 z-k} = \frac{1}{\alpha!} \sum_{|\alpha|+i+j=k} \partial^\alpha_\xi a^l_{\cdot,m_2 l-i} D^\alpha_x a^{z-l}_{\cdot,m_2(z-l)-j}. \tag{2.5}$$

The following Proposition is immediate, in view of the proof of Theorem 2.1.6:

Proposition 2.1.7. *The top order terms in the expansions* (2.4), (2.5) *are such that*

$$\begin{aligned} a^z_{m_1 z,\cdot} &= (a_{m_1,\cdot})^z, \\ a^z_{\cdot,m_2 z} &= (a_{\cdot,m_2})^z, \\ a^z_{m_1 z,m_2 z} &= (a_{m_1,m_2})^z. \end{aligned} \tag{2.6}$$

Remark 2.1.6. *In order to define A^z we do not need m_1, m_2 integer numbers. Anyway, this hypothesis is essential in the definition of the non-commutative residue given below, so we included it from the very beginning in Assumptions 1.*

In [92], E. Schrohe noticed that, for $A \in L^{m_1,m_2}$ such that $\operatorname{Re} z\, m_1 < -n$ and $\operatorname{Re} z\, m_2 < -n$, A^z is trace class, so he defined

$$\zeta(A,z) = Sp(A^z) = \int K_{A^z}(x,x)dx, \tag{2.7}$$

where Sp is the spur of A^z, i.e., a trace on the algebra of trace class operators. Assuming that A is SG-classical and elliptic, we want to study the meromorphic extension of $\zeta(A,z)$: this will allow to define trace operators, in connection with the residues of $\zeta(A,z)$. We first consider the kernel $K_{A^z}(x,y)$ of the operator A^z defined in 2.1.3. The information provided by the knowledge of the homogeneous expansions of the symbol of A^z allows to investigate in detail the properties of $K_{A^z}(x,y)$ on the diagonal (x,x).

Theorem 2.1.8. *Let A be an elliptic operator that satisfies Assumptions 1. Then, $K_{A^z}(x,x)$ is a holomorphic function for $\operatorname{Re} z < -\frac{n}{m_1}$ and admits, at most, simple poles at the points $z_j = \frac{j-n}{m_1}$, $j = 0, 1, \ldots$.*

Proof. Let us consider the kernel $K_{A^z}(x,y)$ on the diagonal (x,x), given by

$$\begin{aligned} K_{A^z}(x,x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \operatorname{sym}(A^z)(x,\xi)d\xi \\ &= \frac{1}{(2\pi)^n} \int_{|\xi|<1} a^z(x,\xi)d\xi + \frac{1}{(2\pi)^n} \int_{|\xi|\geq 1} a^z(x,\xi)d\xi. \end{aligned}$$

Clearly, the first integral converges, and the resulting function is holomorphic, so we can focus on

$$\begin{aligned} \int_{|\xi|\geq 1} a^z(x,\xi)d\xi = &\int_{|\xi|\geq 1} \sum_{j=0}^{p-1} a^z_{m_1 z-j,\cdot}\left(x, \frac{\xi}{|\xi|}\right) |\xi|^{m_1 z-j} d\xi \\ &+ \int_{|\xi|\geq 1} r_{m_1 z-p,\cdot}(x,\xi)d\xi. \end{aligned}$$

The number p can be chosen such that $m_1 \operatorname{Re} z - p < -n$: this means that we have to deal with the terms appearing in the sum for $j = 0, \dots, p-1$. Switching to polar coordinates $\xi = \rho\omega$, $\rho \in [1, \infty)$, $\omega \in \mathbb{S}^{n-1}$,

$$\int_{|\xi|\geq 1} a^z(x,\xi)d\xi = \sum_{j=0}^{p-1} \int_1^\infty \rho^{m_1 z - j + n - 1} d\rho \int_{\mathbb{S}^{n-1}} a^z_{m_1 z - j}(x,\theta)d\theta$$
$$+ \int_{|\xi|\geq 1} r_{m_1 z - p,\cdot}(x,\xi)d\xi.$$

To have convergence in the first integral, we must impose $m_1 \operatorname{Re} z + n < 0$, i.e., $\operatorname{Re} z < -\frac{n}{m_1}$. Evaluating the integral, we find

$$\int_{|\xi|\geq 1} a^z(x,\xi)d\xi = -\sum_{j=0}^{p-1} \frac{1}{m_1 z - j + n} \int_{\mathbb{S}^{n-1}} a^z_{m_1 z - j,\cdot}(x,\theta)d\theta$$
$$+ \int_{|\xi|\geq 1} r^z_{m_1 z - p}(x,\xi)d\xi.$$

This proves that $K_{A^z}(x,x)$ is holomorphic for $\operatorname{Re} z < -\frac{n}{m_1}$, and that it can be extended as a meromorphic function on the whole complex plane with, at most, simple poles at the points $z_j = \frac{j-n}{m_1}$, $j = 0, 1, \dots$ □

Remark 2.1.7. *As in the case of a compact manifold, see [98], we can prove that the kernel $K_{A^z}(x,x)$ is regular for $z = 0$ and, if A is a differential operator, $K_{A^z}(x,x)$ is also regular for all integer.*

Now we proceed to examine the properties of $\zeta(A,z)$:

Theorem 2.1.9. *Let A be an elliptic operator that satisfies Assumptions 1, and define*

$$\zeta(A,z) = \int_{\mathbb{R}^n} K_{A^z}(x,x)dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \operatorname{sym}(A^z)(x,\xi)d\xi dx. \tag{2.8}$$

The function $\zeta(A,z)$ is holomorphic for $\operatorname{Re} z < \min\{-\frac{n}{m_1}, -\frac{n}{m_2}\}$. Moreover, it can be extended as a meromorphic function with, at most, poles at the points

$$z^1_j = \frac{j-n}{m_1},\ j = 0, 1, \dots, \quad z^2_k = \frac{k-n}{m_2},\ k = 0, 1, \dots$$

Such poles can be of order two if and only if there exist integers j, k such that

$$z^1_j = \frac{j-n}{m_1} = \frac{k-n}{m_2} = z^2_k. \tag{2.9}$$

Proof. We divide $\mathbb{R}^{2n}$ into the four regions

$$\{(x,\xi) \colon |x| \leq 1, |\xi| \leq 1\}, \quad \{(x,\xi) \colon |x| < 1, |\xi| > 1\},$$
$$\{(x,\xi) \colon |x| > 1, |\xi| < 1\}, \quad \{(x,\xi) \colon |x| \geq 1, |\xi| \geq 1\}.$$

Setting, as above, $a^z = \mathrm{sym}(A^z)$, we can write

$$\begin{aligned}
\zeta_1(A,z) &= \frac{1}{(2\pi)^n}\int_{|x|\le 1}\int_{|\xi|\le 1} a^z(x,\xi)d\xi dx,\\
\zeta_2(A,z) &= \frac{1}{(2\pi)^n}\int_{|x|<1}\int_{|\xi|>1} a^z(x,\xi)d\xi dx,\\
\zeta_3(A,z) &= \frac{1}{(2\pi)^n}\int_{|x|>1}\int_{|\xi|<1} a^z(x,\xi)d\xi dx,\\
\zeta_4(A,z) &= \frac{1}{(2\pi)^n}\int_{|x|\ge 1}\int_{|\xi|\ge 1} a^z(x,\xi)d\xi dx,\\
\zeta(A,z) &= \sum_{i=1}^{4}\zeta_i(A,z),
\end{aligned}$$

and examine each term of the sum separately.

1) The analysis of this term is straightforward. Since we integrate a^z, holomorphic function in z and smooth with respect to (x,ξ), on a bounded set with respect to (x,ξ), $\zeta_1(A,z)$ is holomorphic.

2) Using the asymptotic expansion of a^z with respect to ξ, we can write

$$\begin{aligned}
\zeta_2(A,z) = &-\frac{1}{(2\pi)^n}\sum_{j=0}^{p-1}\frac{1}{m_1 z+n-j}\int_{|x|<1}\int_{\mathbb{S}^{n-1}} a^z_{m_1 z-j,\cdot}(x,\theta)d\theta dx\\
&+\frac{1}{(2\pi)^n}\int_{|x|<1}\int_{\mathbb{R}^n} r^z_{m_1 z-p,\cdot}(x,\xi)d\xi dx.
\end{aligned}$$

Choosing $p > m_1\,\mathrm{Re}\,z+n$, the last integral is convergent. For the sum, we can argue as in the proof of the Theorem 2.1.8. So $\zeta_2(A,z)$ is holomorphic for $\mathrm{Re}\,z < -\frac{n}{m_1}$ and has, at most, poles at the points $z_j^1 = \frac{j-n}{m_1}$.

3) To discuss this term we need the asymptotic expansion of a^z with respect to x. Using Proposition 2.1.3, we can write

$$a^z(x,\xi) = \sum_{k=0}^{q-1} a^z_{\cdot,m_2 z-k}(x,\xi) + t^z_{\cdot,m_2 z-q}(x,\xi),$$

which implies

$$\begin{aligned}
\zeta_3(A,z) = &\frac{1}{(2\pi)^n}\int_{|x|>1}\int_{|\xi|<1}\sum_{k=0}^{q-1} a^z_{\cdot,m_2 z-k}\left(\frac{x}{|x|},\xi\right)|x|^{m_2 z-k}d\xi dx\\
&+\frac{1}{(2\pi)^n}\int_{|x|>1}\int_{|\xi|<1} t^z_{\cdot,m_2 z-q}(x,\xi)d\xi dx.
\end{aligned}$$

Now, switching to polar coordinates, we can write

$$\begin{aligned}
\zeta_3(A,z) = &\frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\int_1^{\infty}\rho^{m_2 z+n-1-k}\int_{\mathbb{S}^{n-1}}\int_{|\xi|<1} a^z_{\cdot,m_2 z-k}(\theta,\xi)d\xi d\theta d\rho\\
&+\frac{1}{(2\pi)^n}\int_{|x|>1}\int_{|\xi|<1} t^z_{\cdot,m_2 z-q}(x,\xi)d\xi dx.
\end{aligned}$$

Arguing as in point (2), it turns out that $\zeta_3(A,z)$ is holomorphic for $\operatorname{Re} z < -\frac{n}{m_2}$ and can be extended as a meromorphic function on the whole complex plane with, at most, poles at the points $z_k^2 = \frac{k-n}{m_2}$.

4) To treat the last term we need to use both the expansions with respect to x and with respect to ξ. We first expand a^z with respect to ξ

$$\zeta_4(A,z) = \frac{1}{(2\pi)^n}\sum_{j=0}^{p-1}\int_{|x|\geq 1}\int_{|\xi|\geq 1} a^z_{m_1 z - j,\cdot}(x,\xi)d\xi dx + \frac{1}{(2\pi)^n}\int_{|x|\geq 1}\int_{|\xi|\geq 1} r^z_{m_1 z - j,\cdot}(x,\xi)d\xi dx.$$

In order to integrate over $|\xi| \geq 1$, we assume $\operatorname{Re} z < -\frac{n}{m_1}$. Now, switching to polar coordinates and integrating the radial part, we can write

$$\zeta_4(A,z) = -\frac{1}{(2\pi)^n}\sum_{j=0}^{p-1}\int_{|x|\geq 1}\frac{1}{m_1 z + n - j}\int_{\mathbb{S}^{n-1}} a^z_{m_1 z - j,\cdot}(x,\theta)d\theta dx + \frac{1}{(2\pi)^n}\int_{|x|\geq 1}\int_{|\xi|\geq 1} r^z_{m_1 z - p,\cdot}(x,\xi)d\xi dx.$$

Now, in order to integrate over $|x| \geq 1$, we expand with respect to x

$$\begin{aligned}\zeta_4(A,z) = &-\frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\sum_{j=0}^{p-1}\int_{|x|\geq 1}\frac{1}{m_1 z + n - j}\int_{\mathbb{S}^{n-1}} a^z_{m_1 z - j, m_2 z - k}(x,\theta)d\theta dx\\ &- \frac{1}{(2\pi)^n}\sum_{j=0}^{p-1}\frac{1}{m_1 z + n - j}\int_{|x|\geq 1}\int_{\mathbb{S}^{n-1}} t^z_{m_1 z - j, m_2 z - q}(x,\theta)dx d\theta\\ &+ \frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\int_{|x|\geq 1}\int_{|\xi|\geq 1} r^z_{m_1 z - p, m_2 z - k}(x,\xi)d\xi d\theta\\ &+ \frac{1}{(2\pi)^n}\int_{|x|\geq 1}\int_{|\xi|\geq 1} r^z_{m_1 z - p, m_2 z - q}(x,\xi)dx d\xi.\end{aligned}$$

Imposing $\operatorname{Re} z < -\frac{n}{m_2}$, and integrating the radial part with respect to the x, we obtain

$$\begin{aligned}\zeta_4(A,z) = &\frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\sum_{j=0}^{p-1}\frac{1}{m_2 z + n - k}\frac{1}{m_1 z + n - j} I^{m_2 z - k}_{m_1 z - j}\\ &- \frac{1}{(2\pi)^n}\sum_{j=0}^{p-1}\frac{1}{m_1 z + n - j} R_{j,q}(z)\\ &- \frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\frac{1}{m_2 z + n - k} R_{p,k}(z) + R_{p,q}(z)\end{aligned}$$

where

$$I^{m_2 z - k}_{m_1 z - j} = \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a^z_{m_1 z - j, m_2 z - k}(\theta',\theta)d\theta d\theta'. \tag{2.10}$$

$R_{j,k}, R_{p,k}, R_{p,q}$ are holomorphic for $\operatorname{Re} z < \min\{-\frac{n}{m_1}, -\frac{n}{m_2}\}$, since p, q are arbitrary. Therefore, we obtain that $\zeta_4(A,z)$ is holomorphic for $\operatorname{Re} z < \min\{-\frac{n}{m_1}, -\frac{n}{m_2}\}$ and can be extended as a meromorphic function on the whole complex plane with, at most, poles at the points $z^1_j = \frac{j-n}{m_1}, z^2_k = \frac{k-n}{m_2}$. Clearly these poles can be of order two when the conditions (2.9) in the statement are fulfilled.

The proof is complete. □

We can now prove two Theorems which show the relation between $\zeta(A,z)$ and the functionals introduced by F. Nicola [79], namely

$$\begin{aligned}
\operatorname{Tr}_{\psi,e}(A) &= \frac{1}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_{-n,-n}(\theta,\theta')d\theta' d\theta = \frac{1}{(2\pi)^n} I^{-n}_{-n},\\
\widehat{\operatorname{Tr}}_{\psi}(A) &= \frac{1}{(2\pi)^n}\lim_{\tau\to\infty}\Big[\int_{|x|\le\tau}\int_{\mathbb{S}^{n-1}} a_{-n,\cdot}(x,\theta)d\theta dx\\
&\qquad - (\log\tau)\, I^{-n}_{-n} - \sum_{k=0}^{m_2+n-1}\frac{\tau^{m_2-k}}{(m_2-k)} I^{m_2-k}_{-n}\Big],\\
\widehat{\operatorname{Tr}}_{e}(A) &= \frac{1}{(2\pi)^n}\lim_{\tau\to\infty}\Big[\int_{\mathbb{S}^{n-1}}\int_{|\xi|\le\tau} a_{\cdot,-n}(\theta,\xi)d\xi d\theta\\
&\qquad - (\log\tau)\, I^{-n}_{-n} - \sum_{j=0}^{m_1+n-1}\frac{\tau^{m_1-j}}{(m_1-j)} I^{-n}_{m_1-j}\Big],
\end{aligned} \tag{2.11}$$

where $I^{-n}_{m_1-j}, I^{m_2-k}_{-n}$ are integrals of the form (2.10) with $a_{m_1-j,-n}$ and a_{-n,m_2-k} in place of $a^z_{m_1 z-j, m_2 z-k}$, respectively. We define the following new functional, that we call the *angular term*

$$\widehat{TR}_{\theta}(A) = \frac{1}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \frac{d}{dz}(a^z_{m_1 z-n-m_1, m_2 z-n-m_2})\Big|_{z=1}(\theta,\theta')d\theta' d\theta. \tag{2.12}$$

Remark 2.1.8. *In general, it is rather cumbersome to evaluate the angular term defined in* (2.12). *In the case $m_1 = m_2 = -n$, the computation is easier: by Proposition 2.1.7,*

$$\frac{d}{dz}(a^z_{-nz,-nz})\Big|_{z=1} = \lim_{z\to1}\frac{a^z_{-nz,-nz} - a_{-n,-n}}{z-1} = a_{-n,-n}\cdot\log(a_{-n,-n}).$$

Theorem 2.1.10. *Let A be an operator satisfying Assumptions 1. Then, defining*

$$TR(A) = m_1 m_2 \operatorname{Res}^2_{z=1}(\zeta(A,z)) = m_1 m_2 \lim_{z\to1}(z-1)^2\zeta(A,z), \tag{2.13}$$

we have

$$TR(A) = \operatorname{Tr}_{\psi,e}(A). \tag{2.14}$$

Proof. To evaluate the limit we split again $\zeta(A,z)$ into the four terms already examined in the proof of Theorem 2.1.9. We get:

1) $\lim_{z\to1}(z-1)^2\zeta_1(A,z) = 0$, since $\zeta_1(A,z)$ is holomorphic;

2) $\lim_{z\to 1}(z-1)^2\zeta_2(A,z) = 0$, since $\zeta_2(A,z)$ has a pole of order one at $z=1$;

3) Similarly, $\lim_{z\to 1}(z-1)^2\zeta_3(A,z) = 0$;

4) Finally,

$$\lim_{z\to 1}(z-1)^2\zeta_4(A,z) = \frac{1}{m_1 m_2 (2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a^1_{-n,-n}(\theta,\theta')d\theta' d\theta.$$

Now the theorem follows from Proposition 2.1.5, which gives $A^1 = A$, so that $a^1_{-n,-n} = a_{-n,-n}$.

□

Theorem 2.1.11. *Let A be an operator that satisfies Assumptions 1. Then, defining*

$$\widehat{TR}_{x,\xi}(A) = \lim_{z\to 1}(z-1)\left[\zeta(A,z) - \frac{\operatorname{Res}^2_{z=1}(\zeta(A,z))}{(z-1)^2}\right], \tag{2.15}$$

we have

$$\widehat{TR}_{x,\xi}(A) = -\frac{1}{m_1}\widehat{\operatorname{Tr}}_\psi(A) - \frac{1}{m_2}\widehat{\operatorname{Tr}}_e(A) + \frac{1}{m_1 m_2}\widehat{TR}_\theta(A). \tag{2.16}$$

Proof. We notice that the function

$$\zeta(A,z) - \frac{\operatorname{Res}^2_{z=1}(\zeta(A,z))}{(z-1)^2}$$

is meromorphic with a simple pole at the point $z=1$, so the limit (2.15) exists and is finite. In order to prove the assertion, we use a decomposition of $\mathbb{R}^{2n}$ into four sets defined by means of a parameter $\tau > 1$,

$$D_1 = \{(x,\xi)\colon |x|\le\tau, |\xi|\le\tau\}, \quad D_2 = \{(x,\xi)\colon |x|<\tau, |\xi|>\tau\},$$
$$D_3 = \{(x,\xi)\colon |x|>\tau, |\xi|<\tau\}, \quad D_4 = \{(x,\xi)\colon |x|\ge\tau, |\xi|\ge\tau\}.$$

and set

$$\zeta_i(A,z) = \iint_{D_i} a^z(x,\xi)d\xi dx, \quad i = 1,\dots,4.$$

1) D_1 is a compact set: $\zeta_1(A,z)$ is then holomorphic, so that, for any $\tau\ge 1$, $L_1 = \lim_{z\to 1}(z-1)\zeta_1(A,z) = 0$.

2) Expanding a^z with respect to ξ, we find

$$\zeta_2(A,z) = -\frac{1}{(2\pi)^n}\int_{|x|<\tau}\sum_{j=0}^{p-1}\frac{\tau^{m_1 z+n-j}}{m_1 z+n-j}\int_{\mathbb{S}^{n-1}} a^z_{m_1 z-j,\cdot}(x,\theta)d\theta dx$$
$$+\frac{1}{(2\pi)^n}\int_{|x|<\tau}\int_{\mathbb{R}^n} r_{m_1 z-p,\cdot}(x,\xi)d\xi dx.$$

For p big enough, $r_{m_1 z-p,\cdot}$ is absolutely integrable with respect to ξ, So we have, for q big enough, and any $\tau \geq 1$,

$$L_2 = \lim_{z\to 1}(z-1)\zeta_2(A,z) = -\frac{1}{m_1(2\pi)^n}\int_{|x|<\tau}\int_{\mathbb{S}^{n-1}} a^1_{-n,\cdot}(x,\theta)d\theta dx,$$

since any term in the limit goes to zero, apart the one corresponding to $j = n + m_1$.

3) Similarly, using the expansion of a^z with respect to x,

$$\begin{aligned}\zeta_3(A,z) = &-\frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\frac{\tau^{m_2 z+n-k}}{m_2 z+n-k}\int_{\mathbb{S}^{n-1}}\int_{|\xi|<\tau} a^z_{\cdot,m_2 z-k}(\theta,\xi)d\xi d\theta \\ &+\frac{1}{(2\pi)^n}\int_{x\geq\tau}\int_{\xi\leq\tau} t^z_{\cdot,m_2 z-q}(x,\xi)d\xi dx,\end{aligned}$$

so that, for q big enough and any $\tau \geq 1$,

$$L_3 = \lim_{z\to 1}(z-1)\zeta_3(A,z) = -\frac{1}{m_2(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{|\xi|\leq\tau} a^1_{\cdot,-n}(\theta,\xi)d\xi d\theta.$$

4) Expanding with respect to both the variables x and ξ,

$$\begin{aligned}\zeta_4(A,z) = &\frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\sum_{j=0}^{p-1}\frac{\tau^{m_1 z+n-j}}{m_1 z+n-j}\frac{\tau^{m_2 z+n-k}}{m_2 z+n-k} I^{m_2 z-k}_{m_1 z-j} \\ &-\frac{1}{(2\pi)^n}\sum_{j=0}^{p-1}\frac{\tau^{m_1 z+n-j}}{m_1 z+n-j}\int_{|x|\geq\tau}\int_{\mathbb{S}^{n-1}} t^z_{m_1 z-j,m_2 z-q}(x,\theta)d\theta dx \\ &-\frac{1}{(2\pi)^n}\sum_{k=0}^{q-1}\frac{\tau^{m_2 z+n-k}}{m_2 z+n-k}\int_{\mathbb{S}^{n-1}}\int_{|\xi|\geq\tau} r^z_{m_1 z-p,m_2 z-k}(\theta,\xi)d\xi d\theta \\ &+\frac{1}{(2\pi)^n}\int_{|x|\geq\tau}\int_{|\xi|\geq\tau} r^z_{m_1 z-p,m_2 z-q}(x,\xi)dx d\xi\end{aligned}$$

So, for p and q big enough, and any $\tau \geq 1$, we have

$$
\begin{aligned}
L_4 = \lim_{z\to 1}(z-1)\left(\zeta_4(A,z) - \frac{\mathrm{Res}^2_{z=1}(\zeta(A,z))}{(z-1)^2}\right) = \\
\lim_{z\to 1}\frac{(z-1)}{m_1m_2(2\pi)^n}\frac{\tau^{(m_1+m_2)(z-1)}-1}{(z-1)^2}I^{m_2z-n-m_2}_{m_1z-n-m_1} \\
+\lim_{z\to 1}\frac{(z-1)}{m_1m_2(2\pi)^n}\frac{I^{m_2z-n-m_2}_{m_1z-n-m_1}-I^{-n}_{-n}}{(z-1)^2} \\
+\frac{1}{m_2(2\pi)^n}\sum_{j=0,j\neq m_1+n}^{p-1}\frac{\tau^{m_1+n-j}}{m_1+n-j}I^{-n}_{m_1-j} \\
+\frac{1}{m_1(2\pi)^n}\sum_{k=0,k\neq m_2+k}^{q-1}\frac{\tau^{m_2+n-k}}{m_2+n-k}I^{m_2-k}_{-n} \\
-\frac{1}{m_1(2\pi)^n}\int_{|x|\geq\tau}\int_{\mathbb{S}^{n-1}} t^1_{-n,m_2-q}(x,\theta)d\theta dx \\
-\frac{1}{m_2(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{|\xi|\geq\tau} r^1_{m_1-p,-n}(\theta,\xi)d\xi d\theta
\end{aligned}
$$

The coefficients $I^{-n}_{m_1-j}$, $I^{m_2-k}_{-n}$, limits of corresponding integrals of the form (2.10), are as in (2.11), while the second limit coincides with the angular term $\widehat{TR}_\theta(A)$, defined in (2.12). Moreover, the first limit goes to

$$
\begin{aligned}
\frac{1}{(2\pi)^n}I^{-n}_{-n}\lim_{z\to 1}\frac{\tau^{(m_1+m_2)(z-1)}-1}{m_1m_2(z-1)} = \\
= \frac{1}{(2\pi)^n}I^{-n}_{-n}\frac{m_1+m_2}{m_1m_2}\log\tau = \frac{1}{(2\pi)^n}I^{-n}_{-n}\left(\frac{1}{m_2}\log\tau+\frac{1}{m_1}\log\tau\right).
\end{aligned}
$$

Clearly, $\widehat{TR}_{x,\xi}(A) = \lim_{\tau\to+\infty}(L_1+L_2+L_3+L_4) = \lim_{\tau\to+\infty}(L_2+L_3+L_4)$. The two terms

$$
\int_{|x|\geq\tau}\int_{\mathbb{S}^{n-1}} t^1_{-n,m_2-q}(x,\theta)d\theta dx \quad\text{and}\quad \int_{\mathbb{S}^{n-1}}\int_{|\xi|\geq\tau} r^1_{m_1-p,-n}(\theta,\xi)d\xi d\theta
$$

in L_4 vanish for $\tau \to +\infty$, by the uniform continuity of the integral. Moreover, the terms in L_4 involving $I^{-n}_{m_1-j}$ and $I^{m_2-k}_{-n}$ are relevant only for $m_1+n-j>0$ and

$m_2 + n - k > 0$, respectively. Then, finally,

$$
\begin{aligned}
&\lim_{z\to 1}(z-1)\left[\zeta(A,z) - \frac{\mathrm{Res}^2_{z=1}(\zeta(A,z))}{(z-1)^2}\right] = \\
&= \frac{1}{(2\pi)^n}\lim_{\tau\to\infty}\left[-\frac{1}{m_1}\int_{|x|\le\tau}\int_{\mathbb{S}^{n-1}} a_{-n,\cdot}(x,\theta)dxd\theta\right. \\
&+ \frac{1}{m_1}\sum_{k=0}^{m_2+n-1}\frac{\tau^{m_2-k}}{m_2-k}I^{m_2-k}_{-n} + \frac{1}{m_1}(\log\tau)I^{-n}_{-n} \\
&- \frac{1}{m_2}\int_{\mathbb{S}^{n-1}}\int_{|\xi|\le\tau} a_{\cdot,-n}(x,\theta)dxd\theta \\
&\left.+ \frac{1}{m_2}\sum_{j=0}^{m_1+n-1}\frac{\tau^{m_1-j}}{m_1-j}I^{-n}_{m_1-j} + \frac{1}{m_2}(\log\tau)I^{-n}_{-n}\right] \\
&+ \frac{1}{m_1m_2}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\frac{d}{dz}(a^z_{m_1z-n-m_1,m_2z-n-m_2})\Big|_{z=1}(\theta,\theta')d\theta' d\theta.
\end{aligned}
$$

which, by (2.11), coincides with (2.16). The proof is complete. □

The functional TR can be extended to all SG-classical operators with integer order in a standard way, cfr. [55]. Explicitely, let $A \in L^{m_1,m_2}_{\mathrm{cl}}$, m_1, m_2 integers, and choose an elliptic operator B of order (m_1', m_2'), satisfying Assumptions 1 and $m_1' > m_1$, $m_2' > m_2$. We can define $\zeta(B + sA, z)$, $s \in (-1,1)$, and then set

$$TR(A) = m_1'm_2'\,\frac{d}{ds}\mathrm{Res}^2_{z=1}(\zeta(B+sA,z))\Big|_{s=0} \tag{2.17}$$

Using the expression of TR given in Theorem 2.1.10, it is possible to prove that these definitions do not depend on the operator B. Moreover, with this approach it is also possibile to prove that TR is a trace on the algebra $\mathscr{A}$ of all SG-classical operators with integer order modulo operators in $L^{-\infty}$, see [55].

Now, we switch to the case of a manifold with a cylindrical end M, as defined in Subsection 1.1.2. First, we restate the notion of Λ-elliptic operator in this case.

Definition 2.1.4. Let $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$; A is Λ-elliptic if the principal part of a^f is Λ-elliptic on $U_{\delta_f} \times \mathbb{R}^n$ and A, restricted to $\mathscr{M}$, is Λ-elliptic in the standard sense.

We can now formulate a set of hypotheses, analogous to Assumptions 1, that imply the existence of A^z, $z \in \mathbb{C}$, for $A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$:

Assumptions 2.

1. *$A \in L^{m_1,m_2}_{\mathrm{cl}}(M)$, with m_1 and m_2 positive integers;*
2. *A is Λ-elliptic with respect to a closed sector Λ of the complex plane with vertex at the origin, therefore it is invertible;*
3. *The spectrum of A does not intersect the real interval $(-\infty, 0)$.*

The definitions of A^z and $\zeta(A,z)$ for such an operator on M follow by the known results on a closed manifold, see [98, 101], combined, via the SG-compatible partition of unity, with similar constructions on the end $\mathscr{C}$: through the exit chart, the latter are achieved by the same techniques used before in the case of $\mathbb{R}^n$. Note that, in view of the SG-structure on M given by the admissible atlases and the hypotheses, A^z and $\zeta(A,z)$ are invariantly defined on M. It is then easy to prove that the properties of $\zeta(A,z)$ extend from $\mathbb{R}^n$ to a general manifold with cylindrical ends. The next Theorem 2.1.12 is the global version of Theorem 2.1.9 on M:

Theorem 2.1.12. *Let $A \in L^{m_1,m_2}(M)$ satisfy Assumptions 2. Then $\zeta(A,z)$ is holomorphic for $\operatorname{Re} z < \min\{-\frac{n}{m_1}, -\frac{n}{m_2}\}$ and can be extended as a meromorphic function with, at most, poles at the points*

$$z_j^1 = \frac{j-n}{m_1},\ j = 0,1,\dots, \quad z_k^2 = \frac{k-n}{m_2},\ k = 0,1,\dots$$

Such poles can be of order two if and only if there exist j and k such that $z_j^1 = z_k^2$.

Proof. We have

$$\zeta(A,z) = \int_M K_{A^z}(y,y)dy = \int_{\mathscr{M}} K_{A^z}(y,y)dy + \int_{\mathscr{C}\setminus C} K_{A^z}(y,y)dy. \tag{2.18}$$

Since $K_{A^z}(y,y)dy$ has an invariant meaning on M, we can perform the computations through an arbitrary admissible atlas $\mathscr{A} = \{(\Omega_i, \psi_i)\}_{i=1}^N$. By the assumptions above, we know that $\{(\Omega_i, \psi_i)\}_{i=1}^{N-1}$ is an atlas on $\mathscr{M}$: then, by considerations completely similar to those that hold for compact manifolds without boundary, see, e.g., [101], Ch. 2, we can prove that the first integral in (2.18) is a complex function of z with the properties stated above and, at most, poles of the type z_j^1, $j = 0,1,\dots$ To handle the contribution on $\mathscr{C} \setminus C$, we fix an exit map f_π and compute the second integral, modulo holomorphic functions of z, as

$$\int_{\overline{U_{\delta_f+\varepsilon_f}}} K_{(\operatorname{Op}(a^f))^z}(x,x)dx.$$

We can then show that the remaining assertions on $\zeta(A,z)$ hold true by repeating the same steps of the proof of Theorem 2.1.9. □

We now extend the definition of the non-commutative residue for SG-operators on $\mathbb{R}^n$ to SG-operators on M in terms of the zeta function. First of all, choose an admissible atlas and introduce the following functionals on

$L^{m_1,m_2}_{\mathrm{cl}}(M)$, analogous to those defined in (2.11):

$$\begin{aligned}
TR(A) &= \frac{1}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_{-n,-n}(\theta,\theta')d\theta' d\theta,\\
\widehat{\mathrm{Tr}}^c_\psi(A) &= \frac{1}{(2\pi)^n}\lim_{\tau\to\infty}\Big[\int_{M\setminus\mathscr{C}_\tau}\int_{\mathbb{S}^{n-1}} a_{-n,\cdot}(x,\theta')d\theta' dx\\
&\qquad - (\log\tau)\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_{-n,-n}(\theta,\theta')d\theta' d\theta\\
&\qquad - \sum_{k=0}^{m_2+n-1}\frac{\tau^{m_2-k}}{m_2-k}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_{-n,m_2-k}(\theta,\theta')d\theta' d\theta\Big] \qquad (2.19)\\
\widehat{\mathrm{Tr}}^c_e(A) &= \frac{1}{(2\pi)^n}\lim_{\tau\to\infty}\Big[\int_{\mathbb{S}^{n-1}}\int_{|\xi|\le\tau} a_{\cdot,-n}(\theta,\xi)d\xi d\theta\\
&\qquad - (\log\tau)\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_{-n,-n}(\theta,\theta')d\theta' d\theta\\
&\qquad - \sum_{j=0}^{m_1+n-1}\frac{\tau^{m_1-j}}{(m_1-j)}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_{m_1-j,-n}(\theta,\theta')d\theta' d\theta\Big].
\end{aligned}$$

The angular term, analogous to (2.12), is defined as

$$\widehat{TR}^c_\theta(A) = \frac{1}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\frac{d}{dz}(a_{m_1z-n-m_1,m_2z-n-m_2})\Big|_{z=1}(\theta,\theta')d\theta' d\theta. \qquad (2.20)$$

Then, by arguments similar to those in the proofs of Theorems 2.1.10 and 2.1.11, we can prove:

Theorem 2.1.13. *Let A be an operator that satisfies Assumptions 2 and set*

$$\widehat{TR}_{x,\xi}(A) = -\frac{1}{m_1}\widehat{\mathrm{Tr}}^c_\psi(A) - \frac{1}{m_2}\widehat{\mathrm{Tr}}^c_e(A) + \frac{1}{m_1m_2}\widehat{TR}^c_\theta(A). \qquad (2.21)$$

The functionals $\dfrac{TR(A)}{m_1m_2}$ and $\widehat{TR}_{x,\xi}(A)$ are the coefficients of the polar parts of order two and of order one of $\zeta(A,z)$ evaluated at $z=1$, respectively.

Remark 2.1.9. *The functional TR extends to all SG-classical operators on M with integer order. The scheme is the same of* (2.17). *In this way, TR turns out to be a trace on the algebra $\mathcal{A}$ of SG-classical operators on M with integer order modulo smoothing operators.*

2.2 Kastler-Kalau-Walze type Theorem and Regularized ζ-Function

Theorem 2.1.8 shows that the kernel $K_{A^z}(x,y)$ of the complex power of suitable SG-elliptic operators behaves essentially as the kernel of the the complex power of elliptic operators on closed manifolds first studied in [98]. Namely, it admits a meromorphic extension for $\mathrm{Re}\, z < -\frac{n}{m_1}$ and has at most simple poles. Nevertheless, in Theorem 2.1.9, we have proved that the ζ-function is different from the case of closed manifolds, since it can have poles of order two. The different behavior is due to the non compactness of $\mathbb{R}^n$. In this section we introduce a regularized version of the ζ-function, following the idea of L. Maniccia, E. Schrohe and J. Seiler [65]. The regularized ζ-function looses the connections with the non-commutative residue. Indeed, we introduce a regularized non-commutative residue which is not a trace on the algebra of *SG*-classical operators, but has a deep link with the coefficients of the expansion of the heat trace. Since we are not interested in the *trace* property of the regularized non-commutative residue, we will define it on $L^{\infty,0}_{\mathrm{cl}}/L^{-\infty,0}$, rather then on $L^{\infty,\infty}_{\mathrm{cl}}/L^{-\infty,-\infty}$. In view of this different setting, we do not consider *SG*-ellipticity, but the usual notion of ellipticity. Clearly, elliptic operators are not Fredholm on the Sobolev spaces on $\mathbb{R}^n$, since they admit an inverse modulo $L^{-\infty,0}$, which, in general, is not compact. We call this almost-inverse *weak* parametrix. Notice that in this section, in order to make more transparent the link between ζ-function and heat trace, we use a slightly different convention to define ζ-function. The results presented in this subsection are published in [14].

2.2.1 Finite-Part Integral

The *finite-part integral*, introduced in [65], gives a meaning to the integral of a classical symbol a, and coincides with the usual integral when $a \in L^1(\mathbb{R}^n)$. dS denotes the usual measure on $|x| = 1$, induced by the Euclidean metric on $\mathbb{R}^n$, while, in this subsection, dx denotes the standard Lebesgue measure on $\mathbb{R}^n$.

Definition 2.2.1. Let a be an element of the classical Hörmander symbol class $S^m_{\mathrm{cl}}(\mathbb{R}^n)$, that is,

1. $a \in C^\infty(\mathbb{R}^n)$ and $\forall x \in \mathbb{R}^n$ $|D^\alpha a(x)| \leq C_\alpha (1 + |x|)^{m-|\alpha|}$;
2. a admits an asymptotic expansion in homogeneous terms a_{m-j} of order $m - j$: explicitly, for a fixed 0-excision function ω and all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} \omega\, a_{m-j} \in S^{m-N}(\mathbb{R}^n).$$

Then:

- if $m \in \mathbb{Z}$, set

$$\begin{aligned} ⨍ a(x)\,dx &:= \lim_{\rho\to\infty} \left[\int_{|x|\leq\rho} a(x)\,dx - \sum_{j=0}^{m-n} \int_{|x|\leq\rho} a_{m-j}(x)\,dx \right] \\ &= \lim_{\rho\to\infty} \left[\int_{|x|\leq\rho} a(x)\,dx - \sum_{j=0}^{m-n} \frac{\beta_j}{n+m-j}\rho^{n+m-j} - \beta_{n+m}\log\rho \right] \end{aligned}$$

  where

$$\beta_j := \int_{|x|=1} a_{m-j}\,dS; \tag{2.22}$$

- if $m \notin \mathbb{Z}$, set

$$\fint a(x)\,dx := \lim_{\rho\to\infty}\left[\int_{|x|\le\rho} a(x)\,dx - \sum_{j=0}^{[m]-n-1}\int_{|x|\le\rho} a_{m-j}(x)\,dx\right]. \tag{2.23}$$

From the above Definition it is clear that if $a \in L^1(\mathbb{R}^n)$ the finite part integral is equivalent to the standard integral. If $m \notin \mathbb{Z}$ the finite part integral coincides with the Kontsevich-Vishik density [57], [58].

Remark 2.2.1. *Now, we consider the radial compactification of $\mathbb{R}^n$ to $\mathbb{S}^n_+$ as in Remarks 1.1.2, 1.2.1:*

$$RC\colon \mathbb{R}^n \to S^n_+\colon x = (x_1,\dots,x_n) \mapsto y = \left[\frac{x_1}{(1+|x|^2)^{\frac12}},\dots,\frac{x_n}{(1+|x|^2)^{\frac12}},\frac{1}{(1+|x|^2)^{\frac12}}\right],$$

and choose y_{n+1} as boundary defining function on $\mathbb{S}^n_+$, such that its composition with RC coincides in the interior with $\frac{1}{\sqrt{(1+|x|^2)}}$, $x = RC^{-1}(y) \in \mathbb{R}^n$. Then

$$\fint a(x)\,dx = {}^R\!\!\int_{\mathbb{S}^n_+} a(rc^{-1}(y))\,dS(y)$$

where the right hand side is defined as the term of order ϵ^0 in the asymptotic expansion of

$$\int_{\mathbb{S}^n_+\cap\{y_{n+1}\ge\epsilon\}} a(rc^{-1}(y))\,dS(y), \quad \epsilon \searrow 0.$$

${}^R\!\!\int_{\mathbb{S}^n_+} f\,dS$ *is called* Renormalised integral, *see [3] and the references quoted therein for its precise definition, properties and applications.*

2.2.2 Regularised Trace and Regularised ζ-Function

We fix a closed sector of the complex plane Λ with vertex at the origin, as in Figure 2.1.

The definition of Λ-elliptic operator is the standard one, here given for operators defined through matrix-valued symbols, whose spectrum we denote by $\sigma(a(x,\xi))$:

Definition 2.2.2. The operator $A \in L^{\mu,0}(\mathbb{R}^n)$ is Λ-elliptic if there exists a constant $R > 0$ such that

$$\sigma(a(x,\xi)) \cap \Lambda = \emptyset \quad \forall|\xi| \ge R,\ \ \forall x \in \mathbb{R}^n \tag{2.24}$$

and

$$(a(x,\xi)-\lambda)^{-1} \in SG^{-\mu,0}(\mathbb{R}^n) \quad \forall|\xi| \ge R,\ \ \forall x \in \mathbb{R}^n,\ \ \forall\lambda \in \Lambda. \tag{2.25}$$

It is well known that, if an operator A is Λ-elliptic, we can build a *weak* parametrix $B(\lambda)$ such that

$$\begin{aligned} &B(\lambda)\circ(A-\lambda I) = \mathrm{Id} + R_1(\lambda),\\ &(A-\lambda I)\circ B(\lambda) = \mathrm{Id} + R_2(\lambda), \quad R_1, R_2 \in L^{-\infty,0}(\mathbb{R}^n). \end{aligned} \tag{2.26}$$

Moreover

$$\begin{aligned} &\lambda\, B(\lambda) \in L^{-\mu,0}(\mathbb{R}^n), \\ &\lambda^2 \left[(A-\lambda I)^{-1} - B(\lambda) \right] \in L^{-\infty,0}(\mathbb{R}^n), \quad \forall \lambda \in \Lambda \setminus \{0\}. \end{aligned} \tag{2.27}$$

From now on, $\mu > 0$ and A is considered as an unbounded operator with dense domain $D(A) = H^{\mu}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. To define the complex powers of a Λ-elliptic operator A, we assume that the following property holds for its spectrum $\sigma(A)$:

$$\sigma(A) \cap \{\Lambda \setminus \{0\}\} = \emptyset \text{ and the origin is at most an isolated point of } \sigma(A). \tag{A1}$$

Proposition 2.2.1. *Let $A \in L^{\mu,0}(\mathbb{R}^n)$, $\mu > 0$, be a Λ-elliptic operator that satisfies Assumption* (A1). *The complex power A^z,* $\operatorname{Re} z < 0$, *can be defined as*

$$A^z := \frac{1}{2\pi i} \int_{\partial^+ \Lambda_\epsilon} \lambda^z (A - \lambda I)^{-1} d\lambda, \tag{2.2.28}$$

where $\Lambda_\epsilon = \Lambda \cup \{z \in \mathbb{C} \mid |z| \leq \epsilon\}$, with $\epsilon > 0$ chosen such that $\sigma(A) \cap \{\Lambda_\epsilon \setminus \{0\}\} = \emptyset$ and $\partial^+ \Lambda_\epsilon$ is the (positively oriented) boundary of Λ_ϵ.

Proof. By the definition of Λ-elliptic operator, we know that $(A - \lambda I)^{-1}$ exists for all $\lambda \in \Lambda \setminus \{0\}$. Moreover, by (2.27) we have that A is sectorial, so the integral (2.2.28) converges in $\mathcal{L}(L^2(\mathbb{R}^n))$. □

Remark 2.2.2. *The definition of A^z is then extended to arbitrary $z \in \mathbb{C}$ in the standard way, that is $A^z := A^{z-j} \circ A^j$, where $j \in \mathbb{Z}_+$ is chosen so that* $\operatorname{Re} z - j < 0$, *as in Definition 2.1.3.*

Theorem 2.2.2. *Let $A \in L^{\mu,0}(\mathbb{R}^n)$, $\mu > 0$, be Λ-elliptic and satisfy Assumption* (A1). *Then, $A^z \in L^{\mu z,0}(\mathbb{R}^n)$. Moreover, if A is SG-classical then A^z is still SG-classical*

Remark 2.2.3. *In order to define the symbol of A^z, the resolvent $(A - \lambda I)^{-1}$ can be approximated with the weak parametrix $B(\lambda)$ defined in* (2.26). *In this way, a symbol for A^z can be computed, modulo smoothing operators w.r.t. the ξ-variable. A^z can then be considered as an element of the algebra $\mathscr{A}$ given by*

$$\mathscr{A} := \bigcup_{\mu \in \mathbb{Z}} L^{\mu,0}(\mathbb{R}^n)/L^{-\infty,0}(\mathbb{R}^n). \tag{2.2.29}$$

The proof of the Theorem 2.2.2 has been given in [66] and can be seen also as a slight modification of Theorem 2.1.6.

From here on, dx will denote the measure induced on $\mathbb{R}^n$ by a smooth Riemannian metric $g = (g_{jk})$. In order to obtain a result similar to (2) we have to impose some condition on g, namely[1]

$$g \text{ is a matrix-valued } SG\text{-classical symbol of order } (0,0). \tag{A2}$$

If $A \in L^{\mu,m}(\mathbb{R}^n)$ is trace class, that is $\mu < -n, m < -n$, we can define its trace

$$TR(A) := \int K_A(x,x)\, dx,$$

[1] In the $\flat$-calculus setting, this condition implies that the underlying metric is polyhomogeneous: this is used, for instance, in [4].

where $K_A(x,x)$ is the kernel of A restricted to the diagonal. The concept of regularised trace, valid for classical SG-operators under less restrictive hypotheses on the order, has been introduced in [65], using the finite part integral defined in the previous section:

Definition 2.2.3. Let $A \in L^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n)$ be such that $\mu < -n$. We define the regularised trace of A as

$$\widehat{TR}(A) := ⨍ K_A(x,x)\,dx. \tag{2.2.30}$$

Remark 2.2.4. *Note that the condition $\mu < -n$ implies that $K_A(x,x)$ is indeed a function and that the finite part integral* (2.2.30) *is well defined.*

Now, using the regularised integral, we can give the definition of regularised ζ-function:

Definition 2.2.4. Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be a Λ-elliptic operator that satisfies (A1); then we define

$$\hat{\zeta}(A,z) := \widehat{TR}(A^{-z}) = ⨍ K_{A^{-z}}(x,x)\,dx, \quad \operatorname{Re} z > \frac{n}{\mu}, \tag{2.2.31}$$

where $K_{A^{-z}}(x,x)$ is the kernel of the operator A^z.

It is simple to prove that $\hat{\zeta}(A,z)$ is holomorphic for $\operatorname{Re} z > \frac{n}{\mu}$, in view of the fact that the hypotheses imply that the kernel $K_{A^z}(x,x)$ is a function. As in the case treated in [98], we can look for meromorphic extensions of $\hat{\zeta}(A,z)$.

Theorem 2.2.3. *Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be an SG-operator that admits complex powers. Then the function $\hat{\zeta}(A,z)$ can be extended as a meromorphic function with, at most, poles at the points $z_j = \frac{n-j}{\mu}$, $j \in \mathbb{N}$.*

Following the idea of M. Wodzicki [109], see also [55], we can now introduce a regularised version of the non-commutative residue.

Definition 2.2.5. Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be a Λ-elliptic operator that satisfies (A1). We define the regularised non-commutative residue of A as

$$\widehat{\mathrm{wres}}(A) := \mu \operatorname{res}_{z=-1}\hat{\zeta}(A,z).$$

In the case $\mu \in \mathbb{N}$, using the explicit expression of the regularised integral and of the residues of $\hat{\zeta}(A,z)$, we get

$$\widehat{\mathrm{wres}}(A) = \frac{1}{(2\pi)^n}\lim_{\rho\to\infty}\left[\int_{|x|\le\rho}\int_{|\xi|=1} a_{-n,\cdot}(x,\xi)\,dS(\xi)dx - \sum_{j=0}^{\mu+n-1}\frac{\beta_j}{n-j}\rho^{n-j} - \beta_{\mu+n}\log\rho\right] \tag{2.2.32}$$

where

$$\beta_j = \int_{|x|=1}\int_{|\xi|=1} a_{n-j,\cdot}\,dS(\xi)\widetilde{dS}(x),$$

$\widetilde{dS}(x)$ the metric induced by g on $|x| = 1$. The case $\mu \notin \mathbb{Z}$ is not very interesting, since then $\widehat{\mathrm{wres}}(A)$ always vanishes, due to the fact that, in this case, the kernel $K_{A^{-z}}(x,x)$ has no poles at $z = -1$. The residue $\widehat{\mathrm{wres}}(\cdot)$ also vanishes on smoothing operators w.r.t. the ξ-variable, so it is well defined on the algebra $\mathscr{A}$. Incidentally, let us notice that the expression (2.2.32) is analogous to the functional $\operatorname{res}_\psi(A)$ defined by F. Nicola in [79], by means of holomorphic families.

2.2.3 A Kastler-Kalau-Walze type Theorem on $\mathbb{R}^n$

First, we restrict to the case of $\mathbb{R}^4$ and consider the classical Atiyah-Singer Dirac operator $\not{D}$ acting on the spinor bundle $\Sigma\mathbb{R}^4$. If the metric on $\mathbb{R}^4$ satisfies Assumption (A2), it is immediate to verify that $\not{D} \in L^{1,0}_{\mathrm{cl}}$. Let $\not{D}^{-2}$ denote a *weak* parametrix of the square of the Dirac operator, that is $\not{D}^2 \circ \not{D}^{-2} = I + R$, $R \in L^{-\infty,0}$. The calculus implies that $\not{D}^{-2} \in L^{-2,0}_{\mathrm{cl}}$. Via direct computation, following the idea of D. Kastler [56], it is possible to compute $a_{-4,\cdot}(x,\xi)$, the term of of order -4 in the asymptotic expansion w.r.t. the ξ-variable of the symbol of $\not{D}^{-2}$. Evaluating the integral on the sphere w.r.t. the ξ variable one gets

$$\int_{|\xi|=1} a_{-4,\cdot}(x,\xi)\, dS(\xi) = -\frac{1}{24\pi^2} s(x).$$

So we have that

$$\widehat{\mathrm{wres}}(\not{D}^{-2}) = -\frac{1}{24\pi^2} ⨍ s(x)\, dx. \tag{2.2.33}$$

The proof of (2.2.33) is contained in [12]. Let us notice the slight abuse of notation in (2.2.33), due to the fact that, in general, $\not{D}^{-2}$ does not satisfy Assumption (A1): anyway, we can use (2.2.32) as a definition of $\widehat{\mathrm{wres}}(\not{D}^{-2})$ in this case.

In order to obtain a generalisation of (2.2.33) to higher dimensions and to more general operators, the direct approach seems to be rather cumbersome. For this reason, we follow an idea of T. Ackermann [1] and take advantage of the properties of the asymptotic expansion of the heat kernel of generalised Laplacians.

As explained in the previous Section, if $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, is Λ-elliptic and satisfies Assumption (A1), we can define the complex powers of A and the heat semigroup e^{-tA} as well:

$$e^{-tA} := \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} e^{-t\lambda}(A - \lambda I)^{-1} d\lambda.$$

In [65] it has been proved that e^{-tA} is an SG-operator belonging to $L^{-\infty,0}(\mathbb{R}^n)$, so we can also consider the regularised heat trace $\widehat{TR}(e^{-tA})$. There is a deep link between regularised heat trace and $\hat{\zeta}$-function:

Theorem 2.2.4. *Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be an operator that admits complex powers. Then, for suitable constants $c_{kl} = c_{kl}(A)$, the following two asymptotic expansions hold:*

$$\Gamma(z)\, \hat{\zeta}(A,z) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{1} c_{kl} \left(z - \frac{n-k}{\mu}\right)^{-l-1}, \tag{2.2.34}$$

$$\widehat{TR}(e^{-tA}) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{1} (-1)^l c_{kl} t^{-\frac{n-k}{\mu}} log^l t, \quad t \searrow 0. \tag{2.2.35}$$

Proof. The statement follows by adapting the arguments given in [65] to the present situation. A main role in the proof is played by an abstract theorem by G. Grubb and R. Seeley [39], connecting ζ-functions and heat traces. □

Remark 2.2.5. *Notice that in* (2.2.34) *poles of order two arise just for negative integers* $-n$, $n \in \mathbb{N}$, *the points where the* Γ *function has poles of order one.*

Let us now consider a generalised positive Laplacian $\Delta \in L^{-2,0}_{\mathrm{cl}}(E)$, where E is a Hermitian vector bundle on $\mathbb{R}^n$ with connection ∇, that is

$$\Delta = \nabla^*\nabla + \mathscr{K}, \quad \mathscr{K} \in C^\infty(\mathrm{End}(E)) \text{ symmetric endomorphism field.}$$

We require that Δ satisfies Assumption (A1): in this way, we can define $e^{-t\Delta}$ as above. In the case of closed manifolds, it is well known (see, e.g., [5]) that

$$K_{e^{-t\Delta}}(x,x) = k_t(x,x) \sim (4\pi t)^{-\frac{n}{2}} \left[1 \cdot \mathrm{Id}_E + \Big(\frac{1}{6} s(x) \mathrm{Id}_E - \mathscr{K}_x \Big) t + O(t^2) \right], \quad t \searrow 0. \tag{2.2.36}$$

where $s(x)$ is the scalar curvature of the underlying manifold and the remainder term depends only on the connection and on the endomorphism field. The asymptotic expansion (2.2.36) also holds in the case of manifolds with cylindrical ends, since the computations are completely analogous and purely local, see [3]. The evaluation of the first term of the asymptotic expansion can be found in [64]: the expression of the second term then follows, using the properties of generalised Laplacians. In view of our hypotheses, the right hand side of (2.2.36) is a classical SG-symbol: then, we obtain

$$\begin{aligned} &\widehat{TR}(e^{-t\Delta}) \sim \\ &\quad (4\pi t)^{-\frac{n}{2}} \left\{ \fint \mathrm{Rk}(E)\, dx + t \fint \left[\frac{\mathrm{Rk}(E)}{6} s(x) - \mathrm{Trace}(\mathscr{K}_x) \right] dx + O(t^2) \right\},\ t \searrow 0. \end{aligned} \tag{2.2.37}$$

Since, trivially, when h is a meromorphic function with a simple pole in z_0, the function $\tilde{h}(z) = h(cz)$, $c \in \mathbb{R}$, is a meromorphic function with a simple pole in $\frac{z_0}{c}$ and

$$\mathrm{res}_{w=\frac{z_0}{c}} \tilde{h} = \frac{1}{c} \mathrm{res}_{z=z_0} h,$$

we also have that

$$\begin{aligned} \widehat{\mathrm{wres}}(\Delta^{-\frac{n}{2}+1}) &= (2-n)\, \mathrm{res}_{z=-1} \hat{\zeta}(\Delta^{-\frac{n}{2}+1}, z) = 2\, \mathrm{res}_{z=\frac{n-2}{2}} \hat{\zeta}(\Delta, z) \\ &= 2\, \Gamma\Big(\frac{n-2}{2}\Big)^{-1} c_{2,0}(\Delta), \end{aligned} \tag{2.2.38}$$

where $c_{2,0}(\Delta)$ is the coefficient of the term of order $t^{-\frac{n-2}{2}}$ in the asymptotic expansion (2.2.35). Finally, by (2.2.37) and the properties of $\Gamma(z)$,

$$\widehat{\mathrm{wres}}(\Delta^{-\frac{n}{2}+1}) = \frac{n-2}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \fint \left[\frac{\mathrm{Rk}(E)}{6} s(x) - \mathrm{Trace}(\mathscr{K}_x) \right] dx. \tag{2.2.39}$$

Remark 2.2.6. *Assumption* (A1) *does not imply that* Δ *is invertible, since we allow the origin to be an isolated point of* $\sigma(\Delta)$. *In view of this, the operator* $\Delta^{-\frac{n}{2}+1}$ *has to be interpreted in the sense of the complex powers defined above.*

If we consider a generalised Laplacian Δ, then its principal homogeneous symbol is $g^{jk}(x)\xi_j\xi_k = |\xi|^2 > 0$, $\xi \neq 0$. Δ turns out to be always Λ-elliptic with respect to a suitable sector of the complex plane, while $\sigma(\Delta)$ can admit the origin as an accumulation point. For example, it is well known that the classical Atiyah-Singer Dirac operator on $\mathbb{R}^n$, endowed with the canonical Euclidean metric, has no point spectrum, but the essential spectrum is the whole real line. In this case Assumption (A1) of course fails to be true[2]. A simple example such that Assumption (A1) is satisfied can be built in the following way. Let us consider a general Dirac operator D, defined on a Clifford bundle E over $\mathbb{R}^n$: D^2 is then a generalised Laplacian and a non-negative operator. If we consider $D^2_\epsilon = D^2 + \epsilon I$, we obtain an invertible generalised Laplacian, that clearly satisfies (A1). If we consider the classical Atyiah-Singer Dirac operator $D\!\!\!/$, formula (2.2.39) turns to

$$\widehat{\mathrm{wres}}((D\!\!\!/^2_\epsilon)^{-\frac{n}{2}+1}) = \frac{(n-2)2^{[\frac{n}{2}]}}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}}\left(-\frac{1}{12}⨍ s(x)\,dx - \epsilon ⨍ dx\right). \tag{2.2.40}$$

On the other hand, a natural example of a metric on $\mathbb{R}^n$ which can satisfy Assumption (A2) is an asymptotically flat one. In General Relativity, such an hypothesis on the metric is commonly assumed (e.g., in order to define the ADM-mass). Explicitly, we can consider a metric g such that, for a constant $\alpha > 0$,

$$g_{jk}(x) - \delta_{jk} = O(|x|^{-\alpha}) \text{ outside a compact set } K \subset \mathbb{R}^n.$$

Moreover, restricting ourselves to $\mathbb{R}^4$, if $\alpha > 2$ the scalar curvature $s(x)$ is integrable: in this case, (2.2.33) becomes

$$\widehat{\mathrm{wres}}(D\!\!\!/^{-2}) = -\frac{1}{24\pi^2}\int s(x)\,dx.$$

The method above can be used to treat also the case of manifolds with cylindrical ends, using the contents of [15]: one defines in this setting a regularised non-commutative residue and exploits its connection with the zeta function. The asymptotic expansion of the heat kernel as $t \searrow 0$ is locally defined, so, using suitable regularised integrals the results can be generalised to those manifolds in this class which admit a spin structure. We omit here any further detail.

2.3 Complex Powers and ζ-Function of Bisingular Operators

Here we define the complex powers of a subclass of elliptic bisingular operators. The contents of this section come from [13]. The first step is to give a suitable definition Λ-ellipticity for bisingular operators.

Definition 2.3.1. Let Λ be a sector of $\mathbb{C}$; we say that $a \in S^{m_1,m_2}_{\mathrm{pr}}(M_1, M_2)$ is Λ-elliptic if there exists a positive constant R such that

i)
$$\left(\sigma^{m_1,m_2}(A)(v_1, v_2) - \lambda\right)^{-1} \in S^{-m_1,-m_2}(M_1, M_2),$$
for all $|v_i| > R$, $i = 1, 2$, uniformly for all $\lambda \in \Lambda$.

[2]For further properties of the Dirac spectrum on open manifolds, the reader can refer, for instance, to the monograph by N. Ginoux [33].

ii)
$$\sigma_1^{m_1}(A)(v_1) - \lambda \,\mathrm{Id}_{M_2} \in L^{m_2}_{\mathrm{cl}}(M_2),$$
is invertible for all $|v_1| > R$, uniformly for all $\lambda \in \Lambda$.

iii)
$$\sigma_2^{m_2}(A)(v_2) - \lambda \,\mathrm{Id}_{M_1} \in L^{m_1}_{\mathrm{cl}}(M_1),$$
is invertible for all $|v_2| > R$, uniformly for all $\lambda \in \Lambda$.

In the following, in order to define the complex power of A, we assume that Λ is a sector of the complex plane with vertex at the origin, that is

$$\Lambda = \{z \in \mathbb{C} \mid \arg(z) \in [\pi - \theta, -\pi + \theta]\},$$

as in Figure 2.1.

Lemma 2.3.1. *Let $a \in S^{m_1,m_2}(\Omega_1, \Omega_2)$ be Λ-elliptic. For all $K_i \subseteq \Omega_i$, $i = 1, 2$, there exist $c_0 > 1$ and a set*

$$\Omega_{\xi_1,\xi_2} := \{z \in \mathbb{C} \setminus \Lambda \mid \frac{1}{c_0}\langle \xi_1\rangle^{m_1}\langle \xi_2\rangle^{m_2} < |z| < c_0\langle \xi_1\rangle^{m_1}\langle \xi_2\rangle^{m_2}\} \tag{2.3.41}$$

such that

$$\mathrm{spec}(a(x_1, x_2, \xi_1, \xi_2)) = \{\lambda \in \mathbb{C} \mid a(x_1, x_2, \xi_1, \xi_2) - \lambda = 0\} \subseteq \Omega_{\xi_1,\xi_2}, \quad \forall x_i \in \Omega_i, \xi_i \in \mathbb{R}^{n_i}.$$

Moreover, $\forall x_i \in K_i, \xi_i \in \mathbb{R}^{n_i}, \lambda \in \mathbb{C} \setminus \Omega_{\xi_1,\xi_2}, i = 1, 2,$

$$\begin{aligned}
&\left|\left(\lambda - a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2)\right)^{-1}\right| \leq C(|\lambda| + \langle \xi_1\rangle^{m_1}\langle \xi_2\rangle^{m_2})^{-1},\\
&\left|\mathrm{sym}\left(\sigma_1^{m_1}(a) - \lambda\right)^{-1}\right| \leq C(|\lambda| + \langle \xi_1\rangle^{m_1}\langle \xi_2\rangle^{m_2})^{-1},\\
&\left|\mathrm{sym}\left(\sigma_2^{m_2}(a) - \lambda\right)^{-1}\right| \leq C(|\lambda| + \langle \xi_1\rangle^{m_1}\langle \xi_2\rangle^{m_2})^{-1}.
\end{aligned}$$

The proof of Lemma 2.3.1 is essentially the same of Lemma 3.5 in [66].

Next, we prove that, if A Λ-elliptic, then we can define a parametrix of $(A - \lambda\,\mathrm{Id})$. Actually, we prove that, for $|\lambda|$ large enough, the resolvent $(A - \lambda\,\mathrm{Id})^{-1}$ exists. Restricting ourselves to differential operators, we could follow formally the idea of Shubin ([91], ch. II) of parameter depending operators. For general pseudodifferential operators, it is well know that this idea does not work, see [40].

Theorem 2.3.2. *Let $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$ be Λ-elliptic. Then there exists $R \in \mathbb{R}^+$, such that the resolvent $(A - \lambda I)^{-1}$ exists for $\lambda \in \Lambda_R = \{\lambda \in \Lambda \mid |\lambda| \geq R\}$. Moreover,*

$$\|(A - \lambda I)^{-1}\| = O(|\lambda|^{-1}), \quad \lambda \in \Lambda_R.$$

Proof. First, we look for an inverse of $(A - \lambda\,\mathrm{Id})$ modulo compact operators, that is an operator $B(\lambda)$ such that:

$$\begin{aligned}
(A - \lambda) \circ B(\lambda) &= \mathrm{Id} + R_1(\lambda), \quad \lambda R_1(\lambda) \in L^{-1,-1}(M_1 \times M_2),\\
B(\lambda) \circ (A - \lambda) &= \mathrm{Id} + R_2(\lambda), \quad \lambda R_2(\lambda) \in L^{-1,-1}(M_1 \times M_2),
\end{aligned} \tag{2.3.42}$$

for all $\lambda \in \Lambda$. In order to find such an operator, we make the principal symbol explicit

$$a - \lambda = \mathrm{psym}(a) - \lambda + c, \quad c \in S^{m_1-1, m_2-1}(M_1, M_2).$$

As we have noticed in Theorem 1.2.3, we can write the symbol of the inverse (modulo compact operators) of an elliptic operator. In this case we need to be more careful, because of the parameter λ. Following the same construction as in Theorem 1.2.3, we obtain

$$b(\lambda) = \Big\{ \big((\sigma_1^{m_1}(A) - \lambda \, \mathrm{Id}_{M_2})^{-1}, (\sigma_2^{m_2}(A) - \lambda \, \mathrm{Id}_{M_1})^{-1} \Big\}. \tag{2.3.43}$$

The above definition (2.3.43) is consistent in view of the Λ-ellipticity and of the relations

$$\begin{aligned}
\sigma\Big((\sigma_1^{m_1}(A) - \lambda \, \mathrm{Id}_{M_2})^{-1}(x_1, \xi_1) \Big)(x_2, \xi_2) &= (a_{m_1, m_2} - \lambda)^{-1}(x_1, x_2, \xi_1, \xi_2), \\
\sigma\Big((\sigma_2^{m_2}(A) - \lambda \, \mathrm{Id}_{M_1})^{-1}(x_2, \xi_2) \Big)(x_1, \xi_1) &= (a_{m_1, m_2} - \lambda)^{-1}(x_1, x_2, \xi_1, \xi_2).
\end{aligned}$$

Using the calculus and Lemma 2.3.1, we can check that $B(\lambda)$ satisfies conditions (2.3.42). By parameter-ellipticity, we get that $R_1(\lambda)$ and $R_2(\lambda)$ are compact operators for $\lambda \in \Lambda$, namely

$$\begin{aligned}
(A - \lambda \, \mathrm{Id}) \circ B(\lambda) &= \mathrm{Id} + R_1(\lambda), \\
(A - \lambda \, \mathrm{Id}) \circ B(\lambda) &= \mathrm{Id} + R_2(\lambda),
\end{aligned} \tag{2.3.44}$$

$\lambda R_1(\lambda), \lambda R_2(\lambda) \in S^{-1,-1}(M_1 \times M_2)$ uniformly w.r.t. $\lambda \in \Lambda$. So, $B(\lambda)$ is a parametrix and its symbol $b(\lambda)$ has the form

$$\begin{aligned}
b(\lambda) = &- (a_{m_1, m_2}(x_1, x_2, \xi_1, \xi_2) - \lambda)^{-1} \psi_1(\xi_2) \psi_2(\xi_1) \\
&+ \mathrm{sym}\{ (a_{m_1, \cdot}(x_1, x_2, \xi_1, D_2) - \lambda \, I_{M_2})^{-1} \} \psi_1(\xi_1) \\
&+ \mathrm{sym}\{ (a_{\cdot, m_2}(x_1, x_2, D_1, \xi_2) - \lambda \, I_{M_1})^{-1} \} \psi_2(\xi_2).
\end{aligned}$$

Moreover, we easily obtain

$$r_1(\lambda) = \mathrm{sym}(R_1(\lambda)) = (a - \mathrm{psym}(a)) \circ b(\lambda) + (\mathrm{psym}(a) \circ b(\lambda)) - 1, \tag{2.3.45}$$

hence $r_1(\lambda) \in S^{-1,-1}(M_1, M_2)$ is the asymptotic sum of terms of the type

$$\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} g D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} b(\lambda) \quad g \in S^{m_1, m_2}(M_1, M_2).$$

Clearly $(a_{m_1, m_2}(x_1, x_2, \xi_1, \xi_2) - \lambda)^{-1} = O(|\lambda|^{-1})$. By the theory of pseudodifferential operators on closed manifolds, the same property holds for

$$\mathrm{sym}(a_{m_1, \cdot}(x_1, x_2, \xi_1, D_2) - \lambda \, \mathrm{Id}_{M_2})^{-1}$$

and

$$\mathrm{sym}(a_{\cdot, m_2}(x_1, x_2, D_1, \xi_2) - \lambda \, \mathrm{Id}_{M_1})^{-1}$$

and their derivatives. Thus $r_1(\lambda) = O(|\lambda|^{-1})$, as a consequence of the calculus. By Proposition 1.2.2, this implies $\|R_1\|_{L^2} = O(|\lambda|^{-1})$, and the same is true for the operator R_2. So we can choose λ large enough such that R_1, R_2 have norm less than 1. In this way, using Neumann series, we prove that $(A - \lambda \, \mathrm{Id})$ is one to one and onto, therefore invertible with bounded inverse, by the Open

Map Theorem. Again, by Neumann series, we obtain $\tilde{B}(\lambda)$ such that (2.3.44) is fulfilled with $\tilde{R}_1, \tilde{R}_2$ smoothing and still with norm $O(\lambda^{-1})$. Now notice that $\lambda\Big[B(\lambda) - \tilde{B}(\lambda)\Big] \in S^{-m_1-1,-m_2-1}$ for all $\lambda \in \Lambda$. Furthermore, if we multiply both equations in (2.3.42) by $(A - \lambda I)^{-1}$ we obtain

$$(A - \lambda\,\mathrm{Id})^{-1} = \tilde{B}(\lambda) + \tilde{B}(\lambda)R_1(\lambda) + R_2(\lambda)(\lambda - A)^{-1}R_1(\lambda).$$

Hence $\|(A - \lambda I)^{-1}\| = O(|\lambda|^{-1})$ and $\lambda^2\Big[(A - \lambda)^{-1} - \tilde{B}(\lambda)\Big]$ is a smoothing operator in $L^{-\infty,-\infty}(M_1 \times M_2)$, uniformly w.r.t. λ. □

In order to define complex powers of an elliptic bisingular operator, we introduce some natural assumptions, similar to those assumed in Section 2.1.

Assumptions 3.

1. *$A \in L^{m_1,m_2}(M_1, M_2)$ is Λ-elliptic.*
2. *$\sigma(A) \cap \Lambda = \emptyset$ (in particular, A is invertible).*
3. *A has homogeneous principal symbols.*

Remark 2.3.1. *If we consider a Λ-elliptic operator $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$ with $m_i > 0$ ($i = 1, 2$), then $\sigma(A)$ is either discrete or the whole of $\mathbb{C}$, because the resolvent is a compact operator ([101], Ch. I). Since, by Theorem 2.3.2, we know that for large λ the resolvent is well defined, it turns out that the spectrum $\sigma(A)$ is discrete. Then, modulo a shift of the operator, we can find a suitable sector Λ such that Assumptions 3 is fulfilled.*

Definition 2.3.2. Let A be an operator fulfilling Assumptions 3. Then, we can define

$$A_z := \frac{i}{2\pi}\int_{\partial\Lambda_\epsilon^+} \lambda^z (A - \lambda\,\mathrm{Id})^{-1} d\lambda, \quad \operatorname{Re} z < 0, \tag{2.3.46}$$

where $\Lambda_\epsilon = \Lambda \cup \{z \in \mathbb{C} \mid |z| \le \epsilon\}$. The Dunford integral in (2.3.46) is convergent, since $\|(A - \lambda\,\mathrm{Id})^{-1}\| = O(|\lambda|^{-1})$ for λ large enough. As usual, we next define

$$A^z := A_{z-k} \circ A^k, \quad \operatorname{Re} z - k < 0.$$

Remark 2.3.2. *In Assumptions 3 we require $\Lambda \cap \sigma(A) = \emptyset$, that is, in particular, the operator must be invertible. It is possible to define complex powers of non invertible operator as well, provided the origin is an isolated point of the spectrum, see, e.g., [26]. For example, one can define the complex powers of $A = (-\Delta) \otimes (-\Delta)$ on the torus $\mathbb{S}^1 \times \mathbb{S}^1$, even if A has an infinite dimensional kernel.*

Theorem 2.3.3. *If the operator $A \in L^{m_1,m_2}(M_1, M_2)$ satisfies Assumptions 3, then $A^z \in L^{m_1 z, m_2 z}(M_1 \times M_2)$ and it admits a homogeneous principal symbol. Moreover, by Cauchy Theorem*[3]

$$\begin{aligned} a^z_{m_1 z, m_2 z} &= (a_{m_1,m_2})^z, \\ a^z_{m_1 z,\cdot} &= (a_{m_1,\cdot})^z, \\ a^z_{\cdot,m_2 z} &= (a_{\cdot,m_2})^z. \end{aligned} \tag{2.3.47}$$

[3] In equation (2.3.47), $a^z_{m_1 z,\cdot}, a^z_{\cdot,m_2 z}, a^z_{m_1 z,m_2 z}$ represent, respectively, $\sigma_1^{m_1 z}(A^z), \sigma_2^{m_2 z}(A^z), \sigma^{m_1 z, m_2 z}(A^z)$, while $(a_{m_1,\cdot})^z, (a_{\cdot,m_2})^z$ are the complex powers of the operators $\sigma_2^{m_1}(A), \sigma_2^{m_2}(A)$, and $(a_{m_1,m_2})^z$ is the complex power of the function $\sigma^{m_1,m_2}(A)$.

Proof. As a consequence of a general version of Fubini's Theorem we obtain

$$\mathrm{sym}(A^z) = \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} \lambda^z \mathrm{sym}((A - \lambda I)^{-1}) d\lambda, \quad \mathrm{Re}\, z < 0.$$

By Theorem 2.3.2, we know that $\lambda^2\Big[(A - \lambda I)^{-1} - B(\lambda)\Big] \in L^{-\infty,-\infty}(M_1 \times M_2)$ so, up to smoothing symbols, we have

$$\begin{aligned} \mathrm{sym}(A^z) =& \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} \lambda^z \mathrm{sym}(\tilde{B}(\lambda)) d\lambda \\ =& \frac{i}{2\pi} \int_{\Omega_{\xi_1,\xi_2}} \lambda^z \mathrm{sym}(\tilde{B}(\lambda)) d\lambda, \end{aligned} \tag{2.3.48}$$

where Ω_{ξ_1,ξ_2} is as in Lemma 2.3.1 and the second equality in (2.3.48) follows by Cauchy integral formula. Now, by Lemma 2.3.1 and by the explicit form of $\mathrm{sym}(\tilde{B}(\lambda))$, we get $A^z \in L^{m_1 z, m_2 z}(M_1 \times M_2)$. In order to show that A^z has homogeneous principal symbol, we write

$$\begin{aligned} \mathrm{sym}(\tilde{B}(\lambda)) =& \psi_1(\sigma^{m_1}(A) - \lambda I_{M_2})^{-1} + \psi_2(\sigma^{m_2}(A) - \lambda I_{M_1})^{-1} \\ & - \psi_1\psi_2(\sigma^{m_1,m_2}(A) - \lambda)^{-1} + c(\lambda), \end{aligned}$$

where $\lambda c(\lambda) \in m_1, m_2 - m_1 - 1, -m_2 - 1(M_1, M_2)$, $\forall \lambda \in \Lambda$. We split integral in (2.3.48) so that

$$\mathrm{sym}(A^z) = \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} \lambda^z \psi_1(\sigma^{m_1}(A) - \lambda I_{M_2})^{-1} \tag{2.3.49}$$

$$+ \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} \lambda^z \psi_2(\sigma^{m_2}(A) - \lambda I_{M_1})^{-1} d\lambda \tag{2.3.50}$$

$$- \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} \lambda^z \psi_1\psi_2(\sigma^{m_1,m_2}(A) - \lambda)^{-1} d\lambda \tag{2.3.51}$$

$$+ \frac{i}{2\pi} \int_{\partial^+\Lambda_\epsilon} \lambda^z c(\lambda) d\lambda. \tag{2.3.52}$$

The theorem follows from theory of complex powers on closed manifolds for the integrals (2.3.49) and (2.3.50), and from Cauchy Theorem for integral (2.3.51). Finally, we notice that the integral (2.3.52) gives a symbol of order $(m_1 z - 1, m_2 z - 1)$. □

We now introduce the function $\zeta(A, z)$ of an elliptic operator that satisfies Assumptions 3. The proof of the following property is similar to the case of compact manifolds (see [101], ch. II).

Proposition 2.3.4. *Let $A \in L^{m_1,m_2}(M_1 \times M_2)$, $m_i > 0$, $i = 1, 2$, be a selfadjoint operator satisfying Assumptions 3. Then we have*

$$A^z(u) = \sum_{i \in \mathbb{N}} \lambda_j^z (f_i, u),$$

where $\{\lambda_j\}_{j \in \mathbb{N}}$ is the sequence of the eigenvalues of A, and $\{f_j\}_{j \in \mathbb{N}}$ are the corresponding orthonormal eigenfunctions. We define

$$\zeta(A, z) := \sum_{j \in \mathbb{N}} \lambda_j^z, \quad \mathrm{Re}\, z < \min\Big\{ -\frac{n_1}{m_1}, -\frac{n_2}{m_2} \Big\}.$$

The definition of $\zeta(A,z)$ in the general case is the following:

Definition 2.3.3. Let $A \in L^{m_1,m_2}(M_1 \times M_2)$ be an operator satisfying Assumptions 3 then

$$\zeta(A,z) := \int_{M_1\times M_2} K_{A^z}(x_1,x_2,x_1,x_2)dx_1dx_2, \quad \operatorname{Re} z m_1 < -n_1, \operatorname{Re} z\, m_2 < -n_2,$$

where K_{A^z} is the kernel of A^z. The integral is well defined if $\operatorname{Re} z\, m_1 < -n_1$ and $\operatorname{Re} z\, m_2 < -n_2$ since, in this case, A^z is trace class.

Theorem 2.3.5. *$K_{A^z}(x_1,x_2,y_1,y_2)$ is a smooth function outside the diagonal. Furthermore, its restriction to the diagonal $K_{A^z}(x_1,x_2,x_1,x_2)$ can be extended as a meromorphic function on the half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z < \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\} + \epsilon\}$ with, at most, poles at the point $z_{\text{pole}} = \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$. The pole can be of order two if $\frac{n_1}{m_1} = \frac{n_2}{m_2}$, otherwise it is a simple pole.*

Proof. By definition, the kernel of A^z has the form

$$K_{A^z}(x_1,x_2,x_1,x_2) = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} a^z(x_1,x_2,\xi_1,\xi_2)d\xi_1 d\xi_2. \tag{2.3.53}$$

First, let us consider the case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$. Then, if $\operatorname{Re} z < -\frac{n_1}{m_1}$, $A^z \in L^{m_1 z, m_2 z}(M_1 \times M_2) \subseteq L^{-n_1-\epsilon,-n_2-\epsilon}(M_1 \times M_2)$; hence it is trace class and the integral of the kernel is finite. We can write $a^z = a^z_{m_1 z,\cdot} + a^z_r$, $a^z_r \in S^{m_1 z-1, m_2 z}(M_1, M_2)$ and we have then

$$\begin{aligned} K_{A^z}(x,x) =& \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_2}} \int_{|\xi_1|\geq 1} \left(a^z_{m_1 z,\cdot} + a^z_{r,\cdot}\right) d\xi_1 d\xi_2 \\ &+ \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_2}} \int_{|\xi_1|\leq 1} \left(a^z_{m_1 z,\cdot} + a^z_{r,\cdot}\right) d\xi_1 d\xi_2. \end{aligned} \tag{2.3.54}$$

The second integral in (2.3.54) is an holomorphic function for $\operatorname{Re} z \leq -\frac{n_1}{m_1} + \epsilon$ since we integrate w.r.t. the ξ_1 variable on a compact set. The same conclusion holds for the integral of $a^z_{r,\cdot}$ on the set $\{(\xi_1,\xi_2) \mid |\xi_1| \geq 1, \xi_2 \in \mathbb{R}^{n_2}\}$ because it has order $(m_1 z - 1, m_2 z)$. In order to analyze the integral of $a^z_{m_1,\cdot}$, we switch to polar coordinates and obtain

$$\int_{\mathbb{R}^{n_2}} \int_{|\xi_1|\geq 1} a^z_{m_1 z,\cdot} d\xi_1 d\xi_2 = -\frac{1}{m_1 z + n_1} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{S}^{n_1-1}} a_{m_1 z,\cdot} d\theta_1 d\xi_2. \tag{2.3.55}$$

Clearly, (2.3.55) can be extended as a meromorphic function on $\{z \in \mathbb{C} \mid \operatorname{Re} z < -\frac{n_1}{m_1} + \epsilon\}$, and, moreover, by (2.3.47), we get

$$\lim_{z\to -\frac{n_1}{m_1}} \left(z + \frac{n_1}{m_1}\right) K_{A^z}(x_1,x_2) = -\frac{1}{(2\pi)^{n_1+n_2} m_1} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{S}^{n_1-1}} \operatorname{sym}(\sigma^m_1(A)^{-\frac{n_1}{m_1}}) d\theta_1 d\xi_2.$$

The case $\frac{n_1}{m_1} < \frac{n_2}{m_2}$ is equivalent, by exchanging the role of x_1 and x_2. The case $\frac{n_1}{m_1} = \frac{n_2}{m_2}$ is a bit more delicate, since we have to analyze the whole principal symbol. First we write

$$\begin{aligned} K_{A^z}(x,x) = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} &\left(a^z_{m_1 z,\cdot} + a^z_{\cdot,m_2 z} - a^z_{m_1 z, m_2 z}\right) + \\ &\left(a^z - a^z_{m_1 z,\cdot} - a^z_{\cdot,m_2 z} + a^z_{m_1 z,m_2 z}\right) d\xi_1 d\xi_2. \end{aligned} \tag{2.3.56}$$

The definition of principal symbol implies that the second term in (2.3.56) belongs to $S^{m_1 z-1, m_2 z-1}(M_1, M_2)$, hence the second integral is well defined for $\operatorname{Re} z < -\frac{n_1}{m_1} + \epsilon$ and holomorphic for $\operatorname{Re} z < -\frac{n_1}{m_1} + \epsilon$. Now we have to analyze the integral of the principal symbol. Splitting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ into the following four regions,

$$\{(\xi_1, \xi_2) \mid |\xi_1| < \tau, |\xi_2| < \tau\}, \quad \{(\xi_2, \xi_2) \mid |\xi_1| \le \tau, |\xi_2| \ge \tau\},$$
$$\{(\xi_1, \xi_2) \mid |\xi_1| \ge \tau, |\xi_2| \le \tau\}, \quad \{(\xi_2, \xi_2) \mid |\xi_1| > \tau, |\xi_2| > \tau\},$$

one gets

$$\begin{aligned}
&\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \left(a^z_{m_1 z, \cdot} + a^z_{\cdot, m_2 z} - a^z_{m_1 z, m_2 z}\right) d\xi_1 d\xi_2 = \\
&\frac{\tau^{(m_1+m_2)z+n_1+n_2}}{(m_1 z + n_1)(m_2 z + n_2)} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} a^z_{m_1 z, m_2 z} d\theta_1 d\theta_2 \\
&- \frac{\tau^{m_1 z + n_1}}{(m_1 z + n_1)} \int_{|\xi_2| \le \tau} \int_{\mathbb{S}^{n_1-1}} a^z_{m_1 z, \cdot} d\theta_1 d\xi_2 \\
&- \frac{\tau^{m_2 z + n_2}}{(m_2 z + n_2)} \int_{|\xi_1| \le \tau} \int_{\mathbb{S}^{n_2-1}} a^z_{\cdot, m_2 z} d\theta_1 d\xi_1 \\
&- \frac{\tau^{m_1 z + n_1}}{(m_1 z + n_1)} \int_{|\xi_2| > \tau} \int_{\mathbb{S}^{n_1-1}} \left(a^z_{m_1 z, \cdot} - a^z_{m_1 z, m_2 z}\right) d\theta_1 d\xi_1 \\
&- \frac{\tau^{m_2 z + n_2}}{(m_2 z + n_2)} \int_{|\xi_1| > \tau} \int_{\mathbb{S}^{n_2-1}} \left(a^z_{\cdot, m_2 z} - a^z_{m_1 z, m_2 z}\right) d\theta_1 d\xi_1 \\
&+ h(z),
\end{aligned} \tag{2.3.57}$$

where $h(z)$ is an holomorphic function for $\operatorname{Re} z \le z_{\text{pole}} + \epsilon$. The evaluation of the integrals in (2.3.57) are similar to Proposition 3.3 in [79], and Theorem 2.2 in [15]. This concludes the proof. □

Since M_1, M_2 are closed manifolds, Theorem 2.3.5 implies the following:

Corollary 2.3.6. *Let $A \in L^{m_1, m_2}(M_1 \times M_2)$ be an operator satisfying Assumptions 3. Then, $\zeta(A, z)$ is holomorphic for $\operatorname{Re} z < \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$ and can be extended as a meromorphic function on the half plane $\operatorname{Re} z < \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\} + \epsilon$. Moreover, the Laurent coefficients of $\zeta(A, z)$ at $z = z_{\text{pole}} = \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$ are*

$$\lim_{z \to -\frac{n_1}{m_1}} \left(z + \frac{n_1}{m_1}\right) \zeta(A, z) = -\frac{1}{(2\pi)^{n_1+n_2} m_1} \iint_{M_1 \times M_2} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{S}^{n_1-1}} a_{m_1, \cdot}^{-\frac{n_1}{m_1}} d\theta_1 d\xi_2, \tag{2.3.58}$$

in the case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$.

$$\lim_{z \to -\frac{n_2}{m_2}} \left(z + \frac{n_2}{m_2}\right) \zeta(A, z) = -\frac{1}{(2\pi)^{n_1+n_2} m_2} \iint_{M_1 \times M_2} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{S}^{n_2-1}} a_{\cdot, m_2}^{-\frac{n_2}{m_2}} d\theta_2 d\xi_1, \tag{2.3.59}$$

in the case $\frac{n_2}{m_2} > \frac{n_1}{m_1}$.

$$\begin{aligned}
res^2(A) = \lim_{z \to -l} (z + l)^2 \zeta(A, z) = \\
\frac{1}{(2\pi)^{n_1+n_2}(m_1 m_2)} \iint_{M_1 \times M_2} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} (a_{m_1, m_2})^{-l} d\theta d\theta',
\end{aligned} \tag{2.3.60}$$

$$\lim_{z\to -l}(z+l)\Big(\zeta(A,z)-\frac{res^2(A)}{(z+l)^2}\Big) = -TR_{1,2}(A)+TR_\theta(A), \tag{2.3.61}$$

where

$$\begin{aligned} &TR_{1,2}(A) := \\ &\frac{1}{(2\pi)^{n_1+n_2}}\lim_{\tau\to\infty}\Big(\frac{1}{m_1}\iint_{M_1\times M_2}\int_{|\xi_2|\le\tau}\int_{\mathbb{S}^{n_1-1}}(a_{m_1,\cdot})^{-l}-\mathrm{res}^2(A)\log\tau\Big) \\ &+\frac{1}{(2\pi)^{n_1+n_2}}\lim_{\tau\to\infty}\Big(\frac{1}{m_2}\iint_{M_1\times M_2}\int_{|\xi_1|\le\tau}\int_{\mathbb{S}^{n_2-1}}(a_{\cdot,m_2})^{-l}-\mathrm{res}^2(A)\log\tau\Big) \end{aligned} \tag{2.3.62}$$

and

$$TR_\theta(A) := \frac{1}{(2\pi)^{n_1+n_2}(m_1m_2)}\int_{M_1\times M_2}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}} a^{-l}_{m_1,m_2}\log a_{m_1,m_2}d\theta_1 d\theta_2, \tag{2.3.63}$$

in the case $\frac{n_1}{m_1}=\frac{n_2}{m_2}=l$.

In (2.3.62), $(a_{m_1,\cdot})^l$ and $(a_{\cdot,m_2})^l$ are the symbols of the complex powers of the operators $a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2)$ and $a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2)$. In order to obtain the terms in (2.3.61), (2.3.62), (2.3.63), we notice that the constant τ in (2.3.57) is arbitrary and the Laurent coefficients clearly do not change if we change the partition of $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$: therefore, we can let τ tend to infinity. In this way, both the fourth and fifth integral in (2.3.57) vanish, due to the continuity of the integral w.r.t. the domain of integration. The evaluation is similar to the proof of Theorem 2.9 in [15] and of Proposition 3.3 in [79].

2.4 Complex Powers and ζ-Function of Bisingular Operator on Euclidean Spaces

In this section we define complex powers of bisingular operators on Euclidean spaces. First we define parameter ellipticity:

Definition 2.4.1. Let Λ be a sector of the complex plane and a be a symbol belonging to $\Gamma^{m_1,m_2}_{\mathrm{pr}}$; a is called Λ-elliptic if there exists a constant R such that

i)
$$\sigma_1^{m_1}(A)(x_1,\xi_1)-\lambda I_{\mathbb{R}^{n_2}}\in G^{m_2}_{cl}(\mathbb{R}^{n_2})$$
is invertible for all $|x_1|+|\xi_1|\ge R$, uniformly w.r.t. $\lambda\in\Lambda$.

ii)
$$\sigma_2^{m_2}(A)(x_2,\xi_2)-\lambda I_{\mathbb{R}^{n_1}}\in G^{m_1}_{cl}(\mathbb{R}^{n_1})$$
is invertible for all $|x_2|+|\xi_2|\ge R$, uniformly w.r.t $\lambda\in\Lambda$.

iii)
$$\Big(\sigma^{m_1,m_2}(A)(x_1,x_2,\xi_1,\xi_2)-\lambda\Big)^{-1}\in\Gamma^{-m_1-m_2}$$
for all $|x_i|+|\xi_i|\ge R$, for all $\lambda\in\Lambda$.

In the remaining part of this Section, we consider sectors of the complex plane Λ with vertex at the origin as in the Figure 2.1.

It is an exercise to prove that, if $A \in G^{m_1,m_2}_{\mathrm{pr}}$ is Λ-elliptic, then the operator is sectorial: follow, for example, the scheme of Theorem 2.3.2. We make now some natural assumptions in order to perform the functional calculus.

Assumptions 4. *i) $A \in G^{m_1,m_2}_{\mathrm{pr}}$ is Λ-elliptic,*

ii) $\sigma(A) \cap \Lambda = \emptyset$; in particular, A is invertible.

Remark 2.4.1. *In condition* ii) *of Assumptions 4, we assume that the operator is invertible. We have made these assumptions in order to get a simpler theory. It is nevertheless possible to handle functional calculus of operators with non trivial kernel, even with infinite dimensional kernel: the crucial requirement is that the origin must be an isolated point of the spectrum, cf. [26].*

Definition 2.4.2. Let A be a globally bisingular operator that satisfies Assumptions 4. We can define

$$A_z := \frac{i}{2\pi} \int_{\partial\Lambda_\epsilon^+} \lambda^z (A - \lambda \mathrm{Id})^{-1} d\lambda, \quad \operatorname{Re} z < 0, \tag{2.4.64}$$

where $\Lambda_\epsilon = \Lambda \cup \{z \in \mathbb{C} \mid |z| < \epsilon\}$. The complex power of A is defined as

$$A^z = \begin{cases} A_z & \operatorname{Re} z < 0, \\ A_{z-k} \circ A^k & k \in \mathbb{N}, \operatorname{Re} z - k < 0. \end{cases}$$

Since the operator A is sectorial the Dunford integral in (2.4.64) converges. As usual, one can prove that the Definition 2.4.2 does not depend on k.

Theorem 2.4.1. *If $A \in G^{m_1,m_2}_{\mathrm{pr}}(\mathbb{R}^{n_1,n_2})$ fulfills Assumptions 4, then $A^z \in G^{m_1 z, m_2 z}$. Moreover,*[4]

$$\sigma_1^{m_1 z}(A^z)(x_1, \xi_1) = \left(\sigma_1^{m_1}(A)(x_1, \xi_1)\right)^z, \tag{2.4.65}$$

$$\sigma_2^{m_2 z}(A^z)(x_2, \xi_2) = \left(\sigma_2^{m_2}(A)(x_2, \xi_2)\right)^z, \tag{2.4.66}$$

$$\sigma^{m_1 z, m_2 z}(A^z)(x_1, x_2, \xi_1, \xi_2) = \left(\sigma^{m_1,m_2}(A)(x_1, x_2, \xi_1, \xi_2)\right)^z, \tag{2.4.67}$$

where the complex power in (2.4.65), (2.4.66) *is the complex power of operators, while in* (2.4.67) *is the standard complex power of a function.*

We now introduce the ζ-function of suitable globally bisingular operators. Then, we will study the meromorphic extension of ζ-function and we will analyze its first *left* pole. We do not write the proofs of the following statements, since they are similar to those of Theorem 2.1.9, 2.1.11, 2.3.5.

Definition 2.4.3. Let $A \in G^{m_1,m_2}$ be a globally bisingular operator that satisfies Assumptions 4. Then

$$\zeta(A, z) = \iint_{\mathbb{R}^{n_1+n_2}} K_{A^z}(x_1, x_2, x_1, x_2) dx_1 dx_2, \quad \operatorname{Re} z < 2 \min\left\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\right\},$$

where K_{A^z} is the kernel of A^z.

[4] We have just defined symbols Γ^{m_1,m_2} with $m_1, m_2 \in \mathbb{R}^2$. It is nevertheless possible to define the same class with complex numbers z_1, z_2, if, in the inequality (1.8), instead of m_i we use $\operatorname{Re} z_i$, $i = 1, 2$.

Theorem 2.4.2. *Let $A \in G^{m_1,m_2}$ satisfy Assumptions 4. Then $\zeta(A,z)$ can be extended as a meromorphic function on $\{z \in \mathbb{C} \mid \operatorname{Re} z < 2\min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\} + \epsilon\}$. Moreover, the Laurent coefficients at pole $z_{pole} = 2\min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$ depend on $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$.*
In the case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$:

$$\lim_{z \to -\frac{2n_1}{m_1}} \left(z + \frac{2n_1}{m_1}\right)\zeta(A,z) = \frac{(2\pi)^{-n_1-n_2}}{m_1} \int_{\mathbb{R}^{2n_2}} \int_{\mathbb{S}^{2n_1-1}} (a_{m_1,\cdot})^{-\frac{2n_1}{m_1}} d\theta_1 dx_2 d\xi_2. \quad (2.4.68)$$

In the case $\frac{n_2}{m_2} > \frac{n_1}{m_1}$:

$$\lim_{z \to -\frac{2n_2}{m_2}} \left(z + \frac{2n_2}{m_2}\right)\zeta(A,z) = \frac{(2\pi)^{-n_1-n_2}}{m_2} \int_{\mathbb{R}^{2n_1}} \int_{\mathbb{S}^{2n_2-1}} (a_{\cdot,m_2})^{-\frac{2n_2}{m_2}} d\theta_2 dx_1 d\xi_1. \quad (2.4.69)$$

In the case $\frac{n_1}{m_1} = \frac{n_2}{m_2} = l$:

$$res^2(A) = \lim_{z \to -l} (z+l)^2 \zeta(A,z) = \frac{(2\pi)^{-n_1-n_2}}{m_1 m_2} \int_{\mathbb{S}^{2n_2-1}} \int_{\mathbb{S}^{2n_1-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2, \quad (2.4.70)$$

$$\lim_{z \to -l} (z+l)\left(\zeta(A,z) - \frac{res^2(A)}{(z+l)^2}\right) = -TR_{1,2}(A) + TR_\theta(A), \quad (2.4.71)$$

where

$$TR_{1,2}(A) = (2\pi)^{-n_1-n_2}$$
$$\Big(\lim_{\tau\to\infty}\Big(\frac{1}{m_1}\int_{|x_2|+|\xi_2|<\tau}\int_{\mathbb{S}^{2n_1-1}}\Big((a_{m_1,\cdot})^{-l} d\theta_1 dx_2 d\xi_2 - res^2(A)\log\tau\Big)\Big)$$
$$+\lim_{\tau\to\infty}\Big(\frac{1}{m_2}\int_{|x_1|+|\xi_1|<\tau}\int_{\mathbb{S}^{2n_2-1}}\Big((a_{m_2,\cdot})^{-l} d\theta_2 dx_1 d\xi_1 - res^2(A)\log\tau\Big)\Big)\Big),$$

and

$$TR_\theta(A) = \frac{(2\pi)^{-n_1-n_2}}{m_1 m_2} \int_{\mathbb{S}^{2n_2-1}} \int_{\mathbb{S}^{2n_1-1}} (a_{m_1,m_2})^{-l} \log(a_{m_1,m_2}) d\theta_1 d\theta_2.$$

Chapter 3

Weyl Formulae

In this Chapter we study the asymptotic behavior of counting function of positive elliptic operators belonging to the classes introduced in Chapter 1.

3.1 Aramaki's Tauberian Theorem

Given a positive selfadjoint operator P with spectrum $\{\lambda_j\}_{j\in\mathbb{N}}$, one defines

$$N_P(\lambda) = \sum_{\lambda_j<\lambda} 1 = \sharp\{j \mid \lambda_j < \lambda\}.$$

We use Tauberian techniques to study the asymptotic behavior N_P, namely an extension of a classical Tauberian Theorem due to J. Aramaki [8].

Theorem 3.1.1. *Let P be a densely defined positive selfadjoint operator, if*

i) P^{-z} *is trace class for* $\operatorname{Re} z < N$, *and* $\zeta(P,z)$ *admits a meromorphic continuation in half plane* $\operatorname{Re} z < N + \epsilon$, *with at most poles on the real line;*

ii) $\zeta(P,z)$ *has the first left pole at the point* $-z_0$ *and*

$$\zeta(P,z) + \sum_{j=1}^{p} \frac{A_j}{(j-1)!}\left(\frac{d}{dz}\right)^{j-1} \frac{1}{z+z_0},$$

extends to an holomorphic function on the half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z < -z_0 + \epsilon\}$;

iii) $\Gamma(z)\zeta(P,z)$ *decays exponentially on vertical strips expect from neighborhoods of the poles;*

then,

$$N_P(\lambda) \sim \sum_{j=1}^{p} \frac{A_j}{(j-1)!}\left(\frac{d}{ds}\right)^{j-1}\left(\frac{\lambda^s}{s}\right)\Big|_{s=z_0} + O(\lambda^{z_0-\delta}), \quad \lambda \to \infty, \tag{3.1.1}$$

for a certain $\delta > 0$.

Remark 3.1.1. *In [8], J. Aramaki requires that $\zeta(P, z)$ has a polynomial growth on all vertical strips. Actually, in the proof he uses condition* iii) *in order to shift an integral in the complex plane. We have changed this condition with a weaker one, since in our setting this one is more easy to be verified. In [8], the authors requires that the $\zeta(P, z)$ admits an extension to the whole of $\mathbb{C}$, but actually condition* i) *in Theorem 3.1.1 is sufficient. Furthermore, notice that we have chosen a different orientation of ζ-function.*

Property 3.1.2. *If we consider a positive selfadjoint operator fulfilling Assumptions 1 or Assumptions 2 or Assumptions 3 or Assumptions 3, then it satisfies item* iii) *of Theorem 3.1.1.*

Proof. First, one has to notice that in these cases the sector used to define the complex powers can be chosen with an angle $\theta > \frac{\pi}{2}$.Then the proof follows from a general result proved by G. Grubb and R. Seeley [39], Corollary 2.10. □

Notice that J. Aramaki suggested others extension of Tauberian Theorem in order the get a better bound of the rest in the formula (3.1.1), see [9, 10].

3.2 Weyl Formulae for SG-Operators, Bisingular Operators and Bisingular Operators on Euclidean Spaces

In view of Theorems of 2.1.10, 2.1.11, Theorems 2.1.12, 2.1.13 and Property 3.1.2 a direct application of Theorem 3.1.1 implies the following theorems

Theorem 3.2.1. *Let A be a classical SG-operator, selfadjoint and positive, on $\mathbb{R}^n$ or a manifold with cylindrical ends, satisfying Assumptions 1 or 2, respectively. Then, for certain $\delta_i > 0$, $i = 0, 1, 2$, the counting function $N_A(\lambda)$ associated with A has the following asymptotic behavior for $\lambda \to +\infty$:*

$$N_A(\lambda) \sim \begin{cases} C_0^1 \lambda^{\frac{n}{m}} \log \lambda + C_0^2 \lambda^{\frac{n}{m}} + O(\lambda^{\frac{n}{m} - \delta_0}) & \text{for } m_1 = m_2 = m \\ C_1 \lambda^{\frac{n}{m_1}} + O(\lambda^{\frac{n}{m_1} - \delta_1}) & \text{for } m_1 < m_2 \\ C_2 \lambda^{\frac{n}{m_2}} + O(\lambda^{\frac{n}{m_2} - \delta_2}) & \text{for } m_1 > m_2. \end{cases} \tag{3.2.2}$$

Moreover, the constants appearing in the above estimates are given by

$$C_0^1 = \frac{1}{mn} \operatorname{TR}(A^{-\frac{n}{m}}), \quad C_0^2 = \widehat{\operatorname{TR}}_{x,\xi}(A^{-\frac{n}{m}}) - \frac{1}{n^2} \operatorname{TR}(A^{-\frac{n}{m}}), \tag{3.2.3}$$

$$C_1 = \widehat{\operatorname{TR}}_{x,\xi}(A^{-\frac{n}{m_1}}), \tag{3.2.4}$$

$$C_2 = \widehat{\operatorname{TR}}_{x,\xi}(A^{-\frac{n}{m_2}}). \tag{3.2.5}$$

If we consider instead bisingular operators, then Corollary 2.3.6, Property 3.1.2 and Theorem 3.1.1 imply the following:

Theorem 3.2.2. *Let $A \in L^{m_1,m_2}(M_1 \times M_2)$ be a positive selfadjoint bisingular operator satisfying Assumptions 3. Then, for $\lambda \to \infty$,*

$$N_A(\lambda) \sim \begin{cases} C_1 \lambda^l \log(\lambda) + C_1' \lambda^l + O(\lambda^{l-\delta_1}) & \text{for } \frac{n_1}{m_1} = \frac{n_2}{m_2} = l \\ C_2 \lambda^{\frac{n_2}{m_2}} + O(\lambda^{\frac{n_2}{m_2} - \delta_2}) & \text{for } \frac{n_2}{m_2} > \frac{n_1}{m_1} \\ C_3 \lambda^{\frac{n_1}{m_1}} + O(\lambda^{\frac{n_2}{m_2} - \delta_2}) & \text{for } \frac{n_2}{m_2} < \frac{n_1}{m_1}, \end{cases} \tag{3.2.6}$$

for certain $\delta_i > 0$, $i = 1, 2, 3$. The constants C_1, C_1', C_2, C_3 depend only on the principal symbol of A, namely

$$
\begin{aligned}
C_1 =& \frac{1}{(2\pi)^{n_1+n_2}(n_1\, m_2)} \iint_{M_1\times M_2} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2 \\
=& \frac{1}{(2\pi)^{n_1+n_2}(n_2\, m_1)} \iint_{M_1\times M_2} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2; \\
C_1' =& \frac{TR_{1,2}(A) - TR_\theta(A)}{l} - \frac{1}{n_1 n_2} \iint_{M_1\times M_2} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2; \qquad (3.2.7) \\
C_2 =& \frac{1}{(2\pi)^{n_1+n_2} n_2} \iint_{M_1\times M_2} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{S}^{n_2-1}} (a_{\cdot,m_2})^{-\frac{m_2}{n_2}} d\theta_2 d\xi_1; \\
C_3 =& \frac{1}{(2\pi)^{n_1+n_2} n_1} \iint_{M_1\times M_2} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{S}^{n_1-1}} (a_{m_1,\cdot})^{-\frac{n_1}{m_1}} d\theta_1 d\xi_2.
\end{aligned}
$$

The result in the case of globally bisingular operators is analogous, in view of Theorem 2.4.2:

Theorem 3.2.3. *Let $A \in G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ be self-adjoint and positive. Moreover, suppose that A satisfies Assumptions 4. Then, for $\lambda \to \infty$,*

$$
N_A(\lambda) = \begin{cases} C_1 \lambda^l \log \lambda + C_1' \lambda^l + O(\lambda^{l-\delta_1}) & \frac{2n_1}{m_2} = \frac{2n_2}{m_2} = l \\ C_2 \lambda^{2\frac{n_2}{m_2}} + O(\lambda^{2\frac{n_2}{m_2}-\delta_2}) & \frac{2n_2}{m_2} > \frac{2n_1}{m_1} \\ C_3 \lambda^{2\frac{n_1}{m_2}} + O(\lambda^{2\frac{n_1}{m_1}-\delta_3}) & \frac{2n_1}{m_1} > \frac{2n_2}{m_2}, \end{cases}
$$

for certain $\delta_i > 0$. The constants appearing in the asymptotic formulae above can be expressed in terms of $\{a_{m_1,\cdot}, a_{\cdot,m_2}, a_{m_1,m_2}\}$, the principal symbol of A, as follows:

$$
\begin{aligned}
C_1 =& \frac{1}{(2\pi)^{n_1+n_2} 2 n_1 m_2} \int_{\mathbb{S}^{2n_2-1}} \int_{\mathbb{S}^{2n_1-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2, \\
C_1' =& \frac{TR_{1,2}(A) - TR_\theta(A)}{l} - \frac{1}{4 n_1 n_2} \int_{\mathbb{S}^{2n_2-1}} \int_{\mathbb{S}^{2n_1-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2, \\
C_2 =& \frac{1}{(2\pi)^{n_1+n_2} 2 n_2} \int_{\mathbb{R}^{2n_1}} \int_{\mathbb{S}^{2n_2-1}} (a_{\cdot,m_2})^{-\frac{2n_2}{m_2}} d\theta_2 dx_1 d\xi_1, \\
C_3 =& \frac{1}{(2\pi)^{n_1+n_2} 2 n_1} \int_{\mathbb{R}^{2n_2}} \int_{\mathbb{S}^{2n_1-1}} (a_{m_1,\cdot})^{-\frac{2n_1}{m_1}} d\theta_1 dx_2 d\xi_2.
\end{aligned}
$$

3.3 An Example

We consider the bisingular operator $A = (-\Delta) \otimes (-\Delta)$ on the torus $\mathbb{S}^1 \times \mathbb{S}^1$. We clearly have $\sigma(A) = \{n^2 m^2\}_{(n,m)\in\mathbb{N}^2}$. Hence, the spectrum is countable and consists only of eigenvalues. The eigenvalue $\{0\}$ has an infinite dimensional eigenspace, while all other eigenspaces have dimension four. Therefore we get

$$
N_A(\lambda) = \sum_{0 < n^2 m^2 \le \lambda} 4. \qquad (3.3.8)
$$

Let us define the function $d(h) : \mathbb{N} \to \mathbb{N}$ so that $d(h)$ is equal to the numbers of ways we can write $h = m \cdot n$, with m, n natural positive numbers or, equivalently, it is equal to the number of divisors of h. This function is usually called Dirichlet divisor function. Setting

$$D(\lambda) = \sum_{n\leq\lambda} d(n),$$

we obtain the so-called divisor summatory function. It is linked to the lattice problem of counting the points with integer coordinates in the first quadrant which are below the iperbola $x_1x_2 = \lambda$. By a simple computation, we obtain

$$N_A(\lambda^2) = 4\,D(\lambda) = 4\sum_{n\leq\lambda} d(n). \tag{3.3.9}$$

Noticing that $\zeta(A) = 4\zeta_R(2z)\zeta_R(2z)$, where $\zeta_R(z)$ is the Riemann zeta-function, we can easily find the coefficients of the asymptotic expansion. Namely, we have

$$D(\lambda) \sim \lambda\log(\lambda) + (2\gamma-1)\lambda + O(\lambda^{1-\delta}), \quad \lambda\to\infty, \tag{3.3.10}$$

where

$$\gamma := \lim_{\tau\to\infty}\left[\sum_{i=1}^{[\tau]}\frac{1}{i} - \log\tau\right] \tag{3.3.11}$$

is the well known Euler-Mascheroni constant. The asymptotic expansion (3.3.10) is well known (see [51] for an overview on Dirichlet divisor problem; see also [52, 61]). It is still an open question to understand the behavior of remainder. In [44], G. H. Hardy proved that $O(\lambda^{\frac{1}{4}})$ is a lower bound for the third term. The best approximation presently known, found by M. Huxley in [50], is $O(\lambda^c(\log\lambda)^d)$, where

$$c := \frac{131}{416} \sim 0,3149038462 \quad d := \frac{18627}{8320} + 1 \sim 3,238822115.$$

It is conjectured that the remainder is $O(\lambda^{\frac{1}{4}})$.

It is interesting to investigate the link between the Dirichlet divisor function and the above results on the spectral properties of a suitable operators. Let us notice that in (3.3.8) we have a slight abuse of notation, since $N(\lambda)$ was only defined for positive operators. In this case $A = (-\Delta)\otimes(-\Delta)$ is non-negative, but has a non trivial kernel. In other words we actually consider

$$N_A := N_{\tilde{A}}$$

where $\tilde{A}$ correspond to the operator A with domain restricted to the orthogonal complement of the kernel. The variant of our theory to such a setting, which is possible, will be not detailed here. Rather, let us now consider the operator $A_c := (-\Delta + c)\otimes(-\Delta + c)$, $c > 0$, defined on the torus $\mathbb{S}^1\times\mathbb{S}^1$. Clearly, A_c satisfies Assumptions 3; so that we can apply Theorem 3.2.2. It is easy to see that the eigenvalues of A_c are $\{(n^2+c)(m^2+c)\}_{(n,m)\in\mathbb{N}^2}$, each one with multiplicity four. Hence

$$N(A_c;\lambda^2) = 4\,\sharp\{\text{ real numbers of the form } (n^2+c)(m^2+c) \mid \\ (n^2+c)(m^2+c)\leq\lambda,\ n,m\in\mathbb{N}\} = 4\,D_c(\lambda).$$

By Theorem 2.3.3, we know that $\sigma^{-1,-1}(A_c^{-\frac{1}{2}}) = (\sigma^{2,2}(A_c))^{-\frac{1}{2}}$ so the constant C_1 in (3.2.7) can be easily evaluated to be

$$C_1 = \frac{1}{2}\frac{1}{(2\pi)^2}(2\pi)^2\, 4 = 2. \tag{3.3.12}$$

Since in this case we know the eigenvalues of the operator, $TR(A_c)$ turns into

$$\begin{aligned} TR_{1,2}(A_c) &= 2\lim_{\tau\to\infty}\left[\sum_{i=-[\tau]}^{[\tau]}\frac{1}{(c+i^2)^{\frac{1}{2}}} - 2\log\tau\right] \\ &= 4\lim_{\tau\to\infty}\left[\sum_{i=0}^{[\tau]}\frac{1}{(c+i^2)^{\frac{1}{2}}} - \log\tau\right] = 4\gamma_c. \end{aligned} \tag{3.3.13}$$

We have named this constant γ_c because of the link with the usual constant of Euler-Mascheroni γ in (3.3.11). Notice that, letting c tend to 0, γ_c goes to $+\infty$; while, if c tends to infinity, γ_c goes to $-\infty$. Finally, we obtain

$$\begin{aligned} D_c(\lambda) &= \frac{1}{4}N(A_c;\lambda^2) \\ &\sim \lambda\log(\lambda) + (2\gamma_c - 1)\lambda + O(\lambda^{1-\delta}), \quad \lambda\to\infty. \end{aligned} \tag{3.3.14}$$

In this case, knowing exactly the eigenvalues of the operator, we can check our estimate with a numerical experiment. We have checked (3.3.14) for $D_c(\lambda)$ with $\lambda = 10.000.000$. In the second column of the Table 3.1 there is the estimate of the coefficient of first term of the asymptotic expansion obtained with the software Maple 15, in the third the coefficient obtained by (3.3.14), and in the fourth the error. We can notice that the error increases with c. This is not surprising, since (3.3.12) does not depend on c. In order to make the error smaller, we should increase the number of digits at which we truncate the series $D_c(\lambda)$. In Table 3.2 we analyze the coefficient of the second term. In this case the error is essentially independent of c, this is due to the fact that (3.3.13) does depend on c.

Remark 3.3.1. *Let us consider the following limit in the operator topology*

$$\lim_{c\to 0^+} A_c = A. \tag{3.3.15}$$

From (3.3.15) *one could suppose that that the limit*

$$\lim_{c\to 0^+}\frac{1}{4}N(A_c;\lambda^2) = \lim_{c\to 0^+} A_c(\lambda) = D(\lambda), \quad \lambda\to\infty \tag{3.3.16}$$

holds as well. Anyway, the limit (3.3.16) *is not correct. Indeed, we have noticed that, if c tends to* 0*, γ_c goes to ∞, not to γ. Moreover, $D(\lambda)$ is not linked with A, rather with $\tilde{A}$. Nevertheless, if we define $\tilde{A}_c$ to be equal to the operator A defined on the orthogonal complement of the eigenspace of c^2, then*

$$\lim_{c\to 0^+}\tilde{A}_c = \tilde{A},$$

and so we obtain

$$\lim_{c\to 0^+}\frac{1}{4}N(\tilde{A}_c;\lambda^2) = D(\lambda), \quad \lambda\to\infty. \tag{3.3.17}$$

The limit (3.3.17) *could also be checked via another numerical experiment.*

Table 3.1: 1st. term approximation

c	1st. term with Maple	1st. term in (3.3.14)	error
2	1,024846785	1	0,024846785
3	0,9916281891	1	0,008371811
4	0,968979304	1	0,031020696
5	0,951859819	1	0,048140181
6	0,938130598	1	0,061869402
7	0,926687949	1	0,073312051
8	0,916888721	1	0,083111279
9	0,908326599	1	0,091673401
10	0,900728511	1	0,099271489
11	0,893902326	1	0,106097674
12	0,887707593	1	0,112292407
13	0,882038865	1	0,117961135
14	0,876815128	1	0,123184872
15	0,871972341	1	0,128027659
16	0,867459966	1	0,132540034
17	0,863235614	1	0,136764386
18	0,859265437	1	0,140734563
19	0,855520776	1	0,144479224
20	0,851977951	1	0,148022049

Table 3.2: 2nd. term approximation

c	2nd. term with Maple	2nd. term in (3.3.14)	error
2	0,40048285	0,401484386	0,001001536
3	-0,13493765	-0, 1339381238	0,000999526
4	-0,499994550	-0,498993281	0,001001269
5	-0,775928050	-0,774926584	0,001001466
6	-0,997216950	-0,996213733	0,001003217
7	-1,181650650	-1,180647904	0,001002746
8	-1,339595550	-1,3385899520	0,001005598
9	-1,477600650	-1,476592538	0,001008112
10	-1,600067350	-1,599058126	0,001009224
11	-1,710092450	-1,7090842470	0,001008203
12	-1,809939750	-1,808931287	0,001008463
13	-1,901308850	-1,9002985710	0,001010279
14	-1,985505550	-1,9844949070	0,001010643
15	-2,063562050	-2,0625496430	0,001012407
16	-2,136292950	-2,1352865400	0,001006410
17	-2,204381450	-2,2033750580	0,001006392
18	-2,268373150	-2,2673662890	0,001006861
19	-2,328729950	-2,3277195600	0,001010390
20	-2,385833550	-2,3848212840	0,001012266

Let us now consider the harmonic oscillator appearing in Quantum Mechanics,

$$-\Delta^2 + |x|^2,$$

restricted to the one dimensional case. See, e.g., [83], for more details on the spectral theory of non-commutative harmonic oscillator. Then, we know that the Hermite basis of $L^2(\mathbb{R})$,

$$h_j(x) = h_j(x), \quad j = 0, 1, ...,$$

is the set of eigenfunctions of $-\partial^2 + x^2$ and that $(2j+1)$, $j = 0, 1, ...$, are the corresponding eigenvalues, all with multiplicity one. Thus, we can consider the shifted operator $-\partial^2 + x^2 + 1$, which has spectrum $2\,(j+1)$, $j = 0, 1, ...$, and define the bisingular operator

$$B = (-\partial^2_{x_1} + x_1^2 + 1) \otimes (-\partial^2_{x_2} + x_2^2 + 1) \in G^{2,2}(\mathbb{R}, \mathbb{R}).$$

Then, the spectrum of B is $\{4\,(j \cdot i)\}$, $j, i = 1, 2, \ldots$. So, we have that

$$N_B(4\,\lambda) = D(\lambda).$$

It is clear that B satisfies the hypothesis of Theorem 3.2.3. Hence, we can state the asymptotic (3.3.10), but using the results on asymptotic expansion of the counting function in the context of globally binsingular operators. Notice that, in this case, the link with the Dirichlet divisor function is more transparent, since we do not need to worry about the kernel: B is a positive operator. Our spectral approach to Dirichlet Divisor function suggests that, maybe, other *Weyl's formula techniques* (e. g. given by Fourier Integral Operators) could be useful to attack the Dirichlet Divisor conjecture.

Part II

Fourier Integral Operators on Manifolds with Boundary

Chapter 4

Background in Geometry

In this chapter we recall the basic notions of manifolds with boundary and we analyze the construction of the double of a manifold with non trivial boundary. We also include a brief introduction to symplectic geometry. The contents of this chapter mainly come from [18, 21, 59, 67, 70, 72, 74].

4.1 Smooth Manifolds with Boundary

The model case of manifold with boundary is the closed half-space $\overline{\mathbb{R}^n_+} = \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, in the induced topology; the boundary of the closed half-space is, of course, $\partial\overline{\mathbb{R}^n_+} = \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \mid x_n = 0\}$. The definition of topological manifold with boundary is similar to that of a topological manifold without boundary, but neighborhoods of points are of the form $\psi^{-1}(U \cap \overline{\mathbb{R}^n_+})$, U open set of $\mathbb{R}^n$, ψ local chart map.

Definition 4.1.1. A Hausdorff topological space M is a n-dimensional topological manifold with boundary if it has a countable base of open sets and for all $p \in M$ there is an open neighborhood U_p of p such that U_p is homeomorphic to an open set U'_p of $\overline{\mathbb{R}^n_+}$.

The open sets U'_p can be of two different types: interior open sets such that $U'_p \cap \partial\overline{\mathbb{R}^n_+} = \emptyset$ - that is open sets of $\mathbb{R}^n$- and boundary open sets such that $U'_p \cap \partial\overline{\mathbb{R}^n_+} \neq \emptyset$. We say that $p \in M$ is an interior point if there exists an open neighborhood U_p homeomorphic to an interior open set of $\overline{\mathbb{R}^n_+}$. The interior points form a topological manifold we denote $\mathring{M}$. The complement of the interior points is called the boundary of M and it is denoted by ∂M. The boundary ∂M can also be defined as the set of points $p \in M$ such that for an open neighborhood U_p, $\psi^{U_p}(p) \in \partial\overline{\mathbb{R}^n_+}$, ψ^{U_p} local homeomorphism on U_p. It is a consequence of the invariance of domain that, if the condition $\psi^{U_p}(p) \in \partial\overline{\mathbb{R}^n_+}$ holds for an open neighborhood of p, then it is true for any open neighborhood of p. The set ∂M turns out to be a topological manifold of dimension $n-1$. By Definition 4.1.1, $\overline{\mathbb{R}^n_+}$ is a topological manifold with boundary; the interior manifold is $\mathbb{R}^n_+$ while the boundary manifold is $\partial\overline{\mathbb{R}^n_+}$.

In order to introduce smooth manifolds with boundary one has to clarify what smooth on $\overline{\mathbb{R}^n_+}$ means. Of course, this is needed only for points belonging to the boundary. A map $f : \overline{\mathbb{R}^n_+} \to \mathbb{R}$ is smooth if there exists a smooth map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in \overline{\mathbb{R}^n_+}$. In the same way, a bijection $\chi : \overline{\mathbb{R}^n_+} \to \overline{\mathbb{R}^n_+}$ that sends boundary points into boundary points is a diffeomorphism of $\overline{\mathbb{R}^n_+}$ into itself, if there exists a diffeomorphism $\tilde{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{\psi}(x) = \psi(x)$ for all $x \in \overline{\mathbb{R}^n_+}$. The definition of smooth manifold with boundary is now analogue to the boundaryless case:

Definition 4.1.2. A topological manifold with boundary M is a smooth manifold with boundary if it has a differentiable structure $\mathscr{U} = \{\mathscr{U}_\alpha, \psi_\alpha\}$ such that

i) $\{\mathscr{U}_\alpha\}$ is an open covering of M.

ii) For all $(\mathscr{U}_\alpha, \psi_\alpha)$, $(\mathscr{U}_\beta, \psi_\beta)$ the maps

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \psi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$$
$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \psi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$$

are diffeomorphisms of open sets of $\overline{\mathbb{R}^n_+}$.

iii) $\mathscr{U}$ is maximal.

Notice that, if M is a smooth manifold with boundary, then, for all $p \in M$, if $\psi_\alpha(p) \in \partial\overline{\mathbb{R}^n_+}$ for a local chart, then $\psi_\beta(p) \in \partial M$ for any other local chart, because diffeomorphisms of $\overline{\mathbb{R}^n_+}$ do preserve the boundary. If the boundary of M is empty, Definitions 4.1.1 and 4.1.2 turns into the usual definitions of topological manifold and smooth manifold without boundary.

Proposition 4.1.1. *If M is a smooth manifold with non empty boundary ∂M, then the differentiable structure of M induces a differentiable structure on ∂M.*

Proof. One has to notice that if $\{\mathscr{U}_\alpha, \psi_\alpha\}$ induces a differentiable structure on M, then $\{\mathscr{U}_\alpha \cap \partial M, \psi_\alpha|_{\partial M}\}$ induces a differentiable structure on ∂M. □

4.1.1 Tangent and Cotangent Bundles of Manifolds with Boundary

In order to define the tangent space of a smooth manifold with boundary, we notice that, given a diffeomorphism $\chi : \overline{\mathbb{R}^n_+} \to \overline{\mathbb{R}^n_+}$, it is possible to consider the differential (or Jacobian) of χ, which is the differential of an extension $\tilde{\chi}$ of χ to the whole of $\mathbb{R}^n$, restricted to $\overline{\mathbb{R}^n_+}$. It is important to observe that value of $J(\tilde{\chi})|_M$ does not depend on the extension $\tilde{\chi}$.

Let us consider a smooth manifold with boundary M of dimension n with a differentiable structure defined by the atlas $\mathscr{A} = \{\mathscr{U}_\alpha, \psi_\alpha\}$. For each point $p \in M$ we consider the set of triples $(p, (\mathscr{U}_\alpha, \psi_\alpha), v) \in \{p\} \times \mathscr{A} \times \mathbb{R}^n$ such that $p \in \mathscr{U}_\alpha$. On this set we introduce the equivalence relation

$$((\mathscr{U}_\alpha, \psi_\alpha), v) \sim ((\mathscr{U}_\beta, \psi_\beta), w) \Leftrightarrow J(\psi_\beta \circ \psi_\alpha^{-1})|_{\psi_\alpha(p)} \cdot v = w.$$

Notice that this implies that the dimension of the tangent space of each point of M, even at the boundary, is n. This can be easily explained if one defines the tangent space from the germs of functions at a point. With this method, it is clear that we can approach a boundary point also in the normal direction.

Remark 4.1.1. *It is possible to define the tangent space of M also in an indirect way, embedding the manifold with boundary M into a smooth manifold $\tilde{M}$ which has no boundary. Then, one can define the tangent space of $T_p\tilde{M}$ for each point $p \in M$ and set $T_pM = T_p\tilde{M}$. In Section 4.1.2 we will explain how to build such an extension of M.*

Having defined the tangent space at an arbitrary point $p \in M$, we can define the cotangent space T_p^*M just as the dual of T_pM. The next step is to introduce the tangent bundle TM and the cotangent bundle T^*M. This is done as in the boundaryless case, by

$$
\begin{aligned}
&TM = M \times \mathbb{R}^n / \sim \\
&(p, v) \sim (q, w) \Leftrightarrow p = q \text{ and } J(\psi_\beta \circ \psi_\alpha^{-1})|_{\psi_\alpha(p)} \cdot v = w,
\end{aligned}
$$

where $(\mathscr{U}_\alpha, \psi_\alpha)$, $(\mathscr{U}_\beta, \psi_\beta)$ are local coordinates at p. It turns out that the tangent space TM is a fiber bundle with fiber $\mathbb{R}^n$, and it is a $2n$-dimensional manifold with boundary. Clearly, $\partial TM = T_{\partial M}M$. In the same way, we can define T^*M, the cotangent bundle of M, which also turns out to be a smooth $2n$-dimensional manifold with boundary, with $\partial T^*M = T^*_{\partial M}M$.

As we have already noticed, ∂M is a smooth $(n-1)$-dimensional manifold, so it is possible to define the bundles $T\partial M$ and $T^*\partial M$. Let us observe that the injection $i : \partial M \to M$ is an embedding. We have that the push-forward of the injection i gives a subbundle of $T_{\partial M}M$, namely $i_*(T\partial M) \subseteq T_{\partial M}M$. Moreover, there is an injection $i^* : T^*_{\partial M}M \to T^*\partial M$, obtained by the pull-back of the injection i. Now, we can define the conormal bundle at the boundary,

$$
\begin{aligned}
N^*\partial M = \{w \in T^*_{\partial M}M \,|\langle w(p), X(p)\rangle_p = 0 \\
\text{for all } p \in \partial M, \text{for all sections } X : \partial M \to i_*(T\partial M)\},
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle_p$ expresses the duality between T_p^*M and T_pM. With this notation, one has the exact sequence

$$0 \to N^*\partial M \to T^*_{\partial M}M \to T^*\partial M \to 0.$$

Lemma 4.1.2. *Given a smooth manifold with boundary M, there exists a smooth function $f : M \to [0, \infty)$ such that $f^{-1}(0) = \partial M$ and $df \neq 0$ on ∂M: f is then a* boundary defining function *of M. Moreover, $df(p)$ has rank 1 at the boundary, and there exists a vector field X_f such that $X_f(f) > 0$ at the boundary.*

Proof. Let us consider an atlas $\{\mathscr{U}_\alpha, \psi_\alpha\}$ such that $\{\mathscr{U}_\alpha\}$ is a locally finite covering of M. Let χ_α a partition of unit subordinate to the covering $\{\mathscr{U}_\alpha\}$, and let n be the dimension of M. Then, for all points $x \in U_\alpha \cap \partial M$, we set $f(x) = \psi_\alpha^n(x)$, where ψ_α^n is the n-component of ψ_α in $\mathbb{R}^n$. Clearly, by definition of boundary, one has $f(x) = 0$ if x is a boundary point. We consider now other coordinates $\psi_\beta(x) = (y_1, \ldots, y_n)$ at x. We notice that, since $\psi_\alpha \circ \psi_\beta^{-1}$ is a diffeomorphism of $\overline{\mathbb{R}^n_+}$, $\psi_\alpha^n \circ \psi_\beta^{-1}(y_1, \ldots, y_{n-1}, 0) = 0$, therefore $\frac{\partial}{\partial y_j}\Big(\psi_\alpha^n \circ \psi_\beta^{-1}(y_1, \ldots, y_{n-1}, 0)\Big) = 0$ for all $j = 1, \ldots, n-1$. Since $J(\psi_\alpha \circ \psi_\beta^{-1})$ is non-singular, it turns out that $\frac{\partial}{\partial y_n}(\psi_\alpha^n \circ \psi_\beta^{-1})$

does not vanish in a neighborhood of $\partial\overline{\mathbb{R}^n_+}$. Furthermore, since $\psi^n_\alpha(x)$ is non negative, the derivative at the boundary must be positive. We set now

$$f(x) = \sum_\alpha \chi_\alpha(x)\psi^n_\alpha(x).$$

Clearly $f(x) = 0$ if and only if $x \in \partial M$. Moreover, if $\psi_\beta(x) = (y_1, \dots, y_n)$ is another local chart, we have

$$\frac{\partial}{\partial y_j}(f \circ \psi_\beta^{-1}) = \sum_\alpha \psi^n_\alpha \frac{\partial}{\partial y_j}(\chi_\alpha \circ \psi_\beta^{-1}) + \sum_\alpha \chi_\alpha \frac{\partial}{\partial y_j}(\psi^n_\alpha \circ \psi_\beta^{-1}). \tag{4.1.1}$$

By the considerations above, one has that, for boundary points, the first term in (4.1.1) vanishes for all $j = 1, \dots, n$, the second term in (4.1.1) vanishes for all $j = 1, \dots, n-1$ and is strictly positive for $j = n$. This proves that $df(x)$ has rank 1 and that there exists a vector field X_f, which is locally the pullback of the normal vector field at $\partial\overline{\mathbb{R}^n_+}$, such that $X_f(f) > 0$ at the boundary. □

From a smooth $(n-1)$-dimensional boundaryless manifold N, one can obtain a n-dimensional manifold with boundary by setting $M = N \times [0, 1)$. M is usually called the *cylinder* of N. The Collar Neighborhood Theorem states that, at least in a neighborhood of the boundary, any manifold with boundary M can be seen as a cylindrical manifolds, i.e., locally near the boundary, it is of the form $\partial M \times [0, 1)$.

Theorem 4.1.3. *Let M be a compact smooth manifolds with boundary ∂M, and assume it to be connected. Then, there exists a neighborhood U of ∂M in M diffeomorphic to $\partial M \times [0, 1)$. U is called collar neighborhood of ∂M.*

Proof. By Lemma 4.1.2 we know that there exists a smooth boundary defining function f on M. We now consider U, a small neighborhood of ∂M, and let $\epsilon = \min_{M\setminus U} f(x) > 0$, which is well-defined in view of the compactness of $M \setminus U$. Next, let us set $W = f^{-1}([0, \epsilon))$. Lemma 4.1.2 guaranties the existence of a vector field X_f such that $X_f(f) > 0$ at the boundary: it is not a restriction to suppose that $X_f(f) \neq 0$ in W. So we set $\tilde{X}_f = \frac{X_f}{X_f(f)}$ and consider the flow $\phi_{\tilde{X}_f}(t, x_0)$ generated by $\tilde{X}_f$. We easily obtain that

$$\frac{d}{dt}(f \circ \phi_{\tilde{X}_f}(t, x_0)) = \tilde{X}_f(f) = 1,$$

that is

$$f \circ \phi_{\tilde{X}_f}(t, x_0)) = t + a, \quad a \in \mathbb{R}.$$

Setting $\tilde{\phi}_{\tilde{X}_f}(s, x_0) = \phi_{\tilde{X}_f}((s-a), x_0)$ we have $f \circ \tilde{\phi}_{\tilde{X}_f}(t, x_0) = t$. Since W is compact, every solution can be extended to all W and, modulo a rescaling, we can suppose the maximal domain of $\tilde{\phi}_{\tilde{X}_f}(t, x_0)$ to be $[0, 1]$. Define now

$$\begin{aligned} \chi : \partial M \times [0, 1) &\to f^{-1}([0, \epsilon) \\ (x, t) &\to \tilde{\phi}_{\tilde{X}_f}(t, x_0). \end{aligned}$$

By the properties of integral curves of a smooth vector field one has that χ is a diffeomorphism. □

Corollary 4.1.4. *Suppose that M is a connected compact manifold, and N is a submanifold of M such that there exists a two sided neighborhood U of N in M, that is $U \setminus N$ has two distinct connected components. Then, there exists a bicollar neighborhood V of N such that V is diffeomorphic to $N \times (-1, 1)$ and N corresponds to $N \times \{0\}$.*

Proof. We consider an open neighborhood U of N such that $\overline{U}$ is compact. Then it is possible to find two submanifolds with boundary U_1, U_2 such that $U_1 \cap U_2 = N$. Now we have to repeat the construction of the boundary defining function of Lemma 4.1.2, in order to obtain a smooth function $f : U \to \mathbb{R}$ such that $f^{-1}(0) = N$, $f^{-1}((-\infty, 0]) = U_1$ and $f^{-1}([0, \infty)) = U_2$ and a smooth vector field X_f such that $X_f(f) > 0$ on Y. Then, repeating the construction of Theorem 4.1.3, one gets the diffeomorphism. □

Remark 4.1.2. *Theorem 4.1.3 and Corollary 4.1.4 can also be proved in the non compact case, see [74] Ch. 5. In the sequel, we will anyway focus only on compact manifolds.*

4.1.2 Gluing Manifolds with Boundary

Let us now consider two manifolds with boundary X, Z, set $Y = \partial X$, $W = \partial Z$ and suppose that there exists a diffeomorphism $\phi : Y \to W$. We can then define the set

$$X \sqcup_\phi Z = X \cup Z / \sim,$$

where

$$y \sim w \Leftrightarrow y \in Y, w \in W \text{ and } \phi(y) = w.$$

Clearly, in this way we have just defined a set, and we have no differential structure. We will now build one such that the inclusions

$$\begin{aligned} i_X &: X \to X \sqcup_\phi Z, \\ i_Z &: Z \to X \sqcup_\phi Z, \end{aligned}$$

are diffeomorphisms onto the image. By Theorem 4.1.3, there are neighborhoods U_Y of Y and U_W of W, and diffeomorphism g_Y, g_W such that

$$\begin{aligned} g_Y &: Y \times (0, 1] \to U_Y, \\ g_W &: W \times [1, 2) \to U_W. \end{aligned}$$

Let us consider V_Y, V_W, open sets of Y and W, respectively, such that $\phi(V_Y) = V_W$. We define the map

$$g : V_Y \times (0, 2) \to X \sqcup_\phi Z$$

setting

$$\begin{aligned} g(x, t) &= i_X(g_Y(x, t)), \quad 0 < t \leq 1 \\ g(x, t) &= i_Z(g_W(\phi(x), t)), \quad 0 \leq t < 2 \end{aligned}$$

The smooth structure on $X \sqcup_\phi Z$ is defined choosing as open covering $i_X(X \setminus V_Y)$, $i_Z(Z \setminus V_W)$ and $g(V_Y \times (0, 2))$. By the definition of the function g, we have that the differentiable structure defined on the collar neighborhood of the boundary is compatible with the differentiable structures on X and on Z.

Remark 4.1.3. *It is also possible to prove that the differentiable structure given above is unique up to diffeomorphisms preserving the boundary. Moreover, it is possible to choose a differentiable structure such that the boundary defining function f_Y and f_W piece together, giving a differentiable function on $X \sqcup_\phi Z$. There is a version of this construction also in the case of non compact manifolds, see [74].*

If we are given a compact manifold X with a non-trivial boundary Y, there is a canonical way to build, starting from X, a closed manifold called the double of X, and denoted by $2X$. Indeed, it is enough to consider the manifold $X \sqcup_\phi \overline{X}$, where $\overline{X}$ is the manifold X with reversed orientation and $\phi : Y \to Y$ is the identity. We reverse the orientation since, in this way, the normal vector field at the boundary can be continued as a smooth vector field on the double manifold. $2X$ is a smooth manifold, therefore it is possible to define the tangent bundle as well as the cotangent bundle. The tangent bundle of X can be defined also as $T2X$ restricted to X, the same holds for the cotangent bundle T^*X.

4.2 Symplectic Geometry

In this section we recall some basic tools of symplectic geometry and we introduce the Maslov index, which will be used in the construction of the Keller-Maslov bundle. The results mainly come from [21, 67]

Definition 4.2.1. Let E be a real vector space. E is a symplectic vector space if there exists a non-degenerate skew-symmetric linear 2-form ω, that is

$$\omega(u,v) = 0, \quad \forall v \in E \Rightarrow u = 0.$$

We denote the symplectic vector space as (E, ω).

It is an easy consequence of the definition that the dimension of a symplectic vector space (E, ω) must be even. Notice that, for each skew-symmetric linear 2-form ω, one can associate the linear map $\tilde{\omega} : E \to E^*$

$$\tilde{\omega} : u \mapsto (v \mapsto \omega(u,v)).$$

Definition 4.2.1 is equivalent to $\ker(\tilde{\omega}) = \{0\}$.

Definition 4.2.2. Let (E, ω), (E', ω') be two symplectic spaces. Then, an isomorphism $\Phi : E \to E'$ is a symplectomorphism if $\Phi^*(\omega') = \omega$ where, by definition,

$$\Phi^*(\omega')(u,v) = \omega'(\Phi(u), \Phi(v)), \quad \forall u, v \in E.$$

Let W be a subspace of the symplectic space (E, ω). The corresponding orthogonal space is

$$W^\omega = \{u \in E \mid \omega(u,v) = 0, \forall v \in W\}.$$

W is called

i) isotropic, if $W \subseteq W^\omega$.

ii) coisotropic, if $W^\omega \subseteq W$.

iii) symplectic, if $W^\omega \cap W = \{0\}$.

iv) lagrangian, if $W^\omega = W$.

It is immediate to prove that

$$W \subseteq M \Rightarrow M^\omega \subseteq W^\omega, \quad (W^\omega)^\omega = W, \tag{4.2.2}$$
$$(M \cap W)^\omega = M^\omega + W^\omega, \quad (W + M)^\omega = W^\omega \cap M^\omega. \tag{4.2.3}$$

Proposition 4.2.1. *A subspace W of the symplectic vector space (E, ω) is Lagrangian if and only if $\omega|_W = 0$ and $2 \dim W = \dim E$.*

Let us consider the vector space $\mathbb{R}^{2n}$ with base

$$x_1 = (1, 0, \dots, 0),\ x_i = (0, \dots, \overbrace{\ 1\ }^{i}, 0, \dots), \quad 0 \le i \le n,$$
$$\xi_1 = (0, \dots, \underbrace{\ 1\ }_{n+1}, 0, \dots),\ \xi_i = (0, \dots, \underbrace{\ 1\ }_{n+i}, 0, \dots), \quad 0 \le i \le n,$$

and the skew-symmetric 2-form ω_0 such that

$$\omega_0(x_i, x_j) = \omega_0(\xi_i, \xi_j) = 0, \quad \forall i, j = 1, \dots, n,$$
$$\omega_0(x_i, \xi_j) = \delta_{i,j}, \quad \forall i, j = 1, \dots, n.$$

The space $(\mathbb{R}^{2n}, \omega_0)$ is the basic example of symplectic vector space. Note that every symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega_0)$:

Theorem 4.2.2 (Darboux)**.** *Let (E, ω) be a symplectic space of dimension $2n$. Then, there exists a basis $q_1, \dots, q_n, p_1, \dots, p_n$ such that*

$$\omega(q_i, q_j) = \omega(p_i, p_j) = 0, \quad \omega(q_i, p_j) = \delta_{i,j}.$$

Such a base is called symplectic base. Moreover, this base induces an isomorphism $\Phi : \mathbb{R}^{2n} \to E$.

By Theorem 4.2.2, since all symplectic vector spaces of dimension $2n$ are isomorphic to $(\mathbb{R}^{2n}, \omega_0)$, it is enough to examine this symplectic vector space. We consider now the set of symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$, that is the subset of $\mathrm{GL}(2n, \mathbb{R})$ such that $\Psi^* \omega_0 = \omega_0$, or, equivalently,

$$\Psi^T J_0 \Psi = J_0, \tag{4.2.4}$$

where J_0 is the symplectic matrix defined as a matrix block

$$J_0 = \begin{pmatrix} \mathbf{0} & \mathrm{Id} \\ -\mathrm{Id} & \mathbf{0} \end{pmatrix}.$$

It is an exercise to prove that, if we write

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

condition (4.2.4) turns to

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = \mathrm{Id}. \tag{4.2.5}$$

We denote by $\mathrm{Sp}(2n)$ the group of symplectomorphisms of $\mathbb{R}^{2n}$. If we consider the identification of $\mathbb{R}^2$ with $\mathbb{C}$ given by

$$(x, y) \to x + iy,$$

we obtain that the multiplication by J_0 turns out to be the multiplication by i. With this identification, one has that $\mathrm{Sp}(2n)$ is identified with a subset of $\mathrm{GL}(n, \mathbb{C})$ and that then $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$. Let us denote by $\mathrm{O}(2n)$ the usual orthonormal group.

Lemma 4.2.3. $\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = O(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n)$.

Proof. Let Ψ be a $(2n \times 2n)$-matrix. By definition,

$$\begin{aligned} &\Psi \in \mathrm{GL}(n, \mathbb{C}) \Leftrightarrow \Psi J_0 = J_0 \Psi, \det(\Psi) \neq 0 \\ &\Psi \in \mathrm{Sp}(n, \mathbb{C}) \Leftrightarrow \Psi^T J_0 \Psi = J_0 \\ &\Psi \in \mathrm{O}(2n) \Leftrightarrow \Psi \Psi^T = \mathrm{Id} = \Psi^T \Psi. \end{aligned} \tag{4.2.6}$$

By direct computation, one can check that every two conditions of (4.2.6) imply the third, therefore we can focus on the case $\Psi \in \mathrm{Sp}(2n) \cap O(2n)$ and set

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Since Ψ is orthogonal and symplectic, $A = D$ and $-B = C$, so we have

$$\Psi = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Furthermore, (4.2.5) implies

$$A^T B = B^T A, \quad A^T A + B^T B = \mathrm{Id}.$$

Therefore $U = A + iB$ is unitary. □

Lemma 4.2.4. *Let $\Psi \in \mathrm{Sp}(2n)$. If $\lambda \in \sigma(\Psi)$ then $\lambda^{-1} \in \sigma(\Psi)$. Moreover, the multiplicity of λ and λ^{-1} are the same and, if -1 is an eigenvalue, then it has even multiplicity. Finally,*

$$\text{if } \Psi(z) = \lambda z, \ \Psi(z') = \lambda' z', \text{ and } \lambda\lambda' \neq 1, \text{ then } \omega_0(z, z') = 0.$$

Proof. Since Ψ is symplectic we have

$$\Psi^T J_0 \Psi = J_0 \Rightarrow \Psi^T = J_0 \Psi^{-1} J_0^{-1},$$

that is, Ψ^T and Ψ are similar and this proves the first part of the lemma. Moreover, notice that a symplectic matrix has determinant equal to one, so the multiplicity of the eigenvalue -1 must be even. To prove the second part we write

$$\lambda\lambda' \omega_0(z, J_0 z') = \omega_0(\Psi(z), J_0 \Psi(z')) = \omega(z, \Psi^T J_0 \Psi(z')) = \omega_0(z, z').$$

If $\lambda\lambda' \neq 0$ this clearly implies $\omega(z, z') = 0$. □

Lemma 4.2.5. *Let $P = P^T \in Sp(2n)$. Then*[1] $P^\alpha \in \mathrm{Sp}(2n)$ *for all positive real α.*

Proof. We need to check that

$$\omega_0(P^\alpha z, P^\alpha z') = \omega_0(z, z'), \quad \forall z, z' \in \mathbb{R}^{2n}. \tag{4.2.7}$$

Since P is symmetric we can decompose E in the direct sum of the eigenspaces of P. By Lemma 4.2.4, we know that

$$\omega(P^\alpha(z_\lambda), P^\alpha(z_{\lambda'})) = \lambda^\alpha \lambda'^\alpha \omega(z_\lambda, z_{\lambda'}) = 0, \quad \text{if } \lambda\lambda' \neq 1,$$

so (4.2.7) is fulfilled. If $\lambda\lambda' = 1$ clearly condition (4.2.7) holds. □

Proposition 4.2.6. *The quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible.*

Proof. Let $\Psi \in \mathrm{Sp}(2n)$. By Lemma 4.2.5 we know that $(\Psi\Psi^T)^\alpha$ is symplectic, so we can define the retraction

$$\beta(t) : \mathrm{Sp}(2n) \times [0,1] \to \mathrm{Sp}(2n)$$
$$\Psi \mapsto \beta(t) = (\Psi\Psi^T)^{-\frac{t}{2}}\Psi.$$

This is a path in $\mathrm{Sp}(2n)$ and $\beta(1) \in \mathrm{U}(n)$, so it gives a retraction. □

Remark 4.2.1. *It is possible to prove that* $\mathrm{U}(n)$ *is the maximal compact subgroup of* $\mathrm{Sp}(2n)$.

Proposition 4.2.7. *The fundamental group of* $\mathrm{Sp}(2n)$ *is* $\mathbb{Z}$.

Proof. By Proposition 4.2.6, it is enough to prove that $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$. This follows by the fibration $\det : \mathrm{U}(n) \to \mathbb{S}^1$, with fiber $\mathrm{SU}(n)$. So we have the exact sequence

$$\pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(\mathbb{S}^1) \to \pi_0(\mathrm{SU}(n)).$$

Since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, we have just to prove that $\mathrm{SU}(n)$ is simply connected. If $n = 1$ this is clear. For $n \geq 2$, consider the map $\mathrm{SU}(n) \to \mathbb{S}^{2n-1}$ that sends a matrix into its components of the first column. This is a fibration, with fiber $\mathrm{SU}(n-1)$, so one has the exact sequence

$$0 = \pi_2(\mathbb{S}^{2n-2}) \to \pi_1(\mathrm{SU}(n-1)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathbb{S}^{2n-1}) = 0,$$

that is, $\mathrm{SU}(n)$ is simply connected if $\mathrm{SU}(n-1)$ is simply connected. Then, by induction, this is true for all n. □

Theorem 4.2.8 (Maslov Index of symplectomorphisms)**.** *There exists a functor μ_S, called Maslov Index,*

$$\mu_S : C(\mathbb{R}/\mathbb{Z}, \mathrm{Sp}(2n)) \to \mathbb{Z}$$

that satisfies the following axioms:

i) (Homotopy) *Two loops $\Lambda(t)$ and $\Lambda'(t)$ are homotopic if and only if $\mu_S(\Lambda) = \mu_S(\Lambda')$.*

ii) (Product) *For all loops $\Lambda, \Lambda' : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$, we have $\mu_S(\Lambda \circ \Lambda') = \mu_S(\Lambda) - \mu_S(\Lambda')$.*

[1]Notice that in this case P is symmetric and non-singular, hence it is possible to define P^α for all all $\alpha \in \mathbb{R}$.

iii) *(*Direct sum*) If $n = n_1 + n_2$, then we can consider the space* $\mathrm{Sp}(2n_1) \oplus \mathrm{Sp}(2n_2)$ *which is a subspace of* $\mathrm{Sp}(2n)$. *If Ψ_1 and Ψ_2 are loops in* $\mathrm{Sp}(n_1)$ *and* $\mathrm{Sp}(n_2)$, *respectively, then*

$$\mu_S(\Psi_1 \oplus \Psi_2) = \mu_S(\Psi_1) + \mu_S(\Psi_2).$$

iv) *(*Normalisation*) The loop*

$$\begin{aligned} \Psi : \mathbb{R}/\mathbb{Z} &\to \mathrm{U}(1) \\ t &\to e^{2\pi i t} \end{aligned}$$

has Maslov index 1.

Sketch of the proof. Define $\rho : \mathrm{Sp}(2n) \to \mathbb{S}^1$ setting

$$\rho(\Psi) = \det(X + iY), \quad \begin{pmatrix} X & -Y \\ -X & Y \end{pmatrix} = (\Psi\Psi^T)^{-\frac{1}{2}}\Psi.$$

So, for every loop $\Lambda(t)$ of symplectomorphisms, we can define the loop on $\mathbb{S}^1$ given by $\rho \circ \Lambda$. We define

$$\mu_S(\Lambda) = \deg(\rho \circ \Lambda),$$

where deg is the winding number. For the proof that this map has the required properties, see, e.g., [67].

4.2.1 Lagrangian Subspaces of Symplectic Vector Spaces

By Theorem 4.2.2, we can restrict ourselves to consider Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$. We define $\mathscr{L}(n)$ as the space of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$.

Lemma 4.2.9. *Let X, Y be real $(n \times n)$-matrixes. We define $\Lambda \subseteq \mathbb{R}^{2n}$ as the range of the $(2n \times n)$-matrix*

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^{2n}. \tag{4.2.8}$$

Then, $\Lambda \in \mathscr{L}(n)$ if and only if Z has rank n and $X^T Y = Y^T X$. Moreover, the space $\Lambda = \{(x, A\,x) \mid x \in \mathbb{R}^n\}$ is Lagrangian if and only if A is a symmetric.

Proof. Since Lagrangian spaces have dimension n, for all $\Lambda \in \mathscr{L}(n)$ it is possible to find $n \times n$ matrixes X, Y such that $\Lambda = \mathrm{Range}(Z)$, with Z as in (4.2.8). Indeed, let us define $\Lambda_{\mathrm{hor}} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n} \mid y = 0\}$. Then, there exists a matrix $A \in \mathrm{GL}(\mathbb{R}^{2n})$ such that $A(\Lambda_{\mathrm{hor}}) = \Lambda$. Choosing as X the upper-left corner $(n \times n)$-matrix of A and as Y the lower-left corner $(n \times n)$-matrix of A, one has that the Z in (4.2.8) has the desired properties. In the other direction, let us consider $z = (X\,u, Y\,u)$ and $z' = (X\,u, Y\,u')$ in Λ. Then,

$$\omega_0(z, z') = u^T(X^T Y - Y^T X)u',$$

so, if $X^T Y = Y^T X$, Λ defined as in (4.2.8) is Lagrangian. Finally, if we consider $X = \mathrm{Id}$, we obtain that $\{(x, A\,x) \mid x \in \mathbb{R}^n\}$ is Lagrangian if and only if $A^T = A$, that is, A is symmetric. □

A matrix (4.2.8) with $X^T Y = Y^T X$ is called Lagrangian Frame.

Remark 4.2.2. *Notice that Lemma 4.2.9 implies that $\mathscr{L}(n)$ is an analytic manifold of dimension* $\dfrac{n(n+1)}{2}$.

Remark 4.2.3. *A neighborhood of Λ_{hor} in $\mathscr{L}(n)$ can be identified with an open subspace of the space of symmetric matrixes. To see this, let us consider a path of Lagrangian frames $(X(t), Y(t))$ such that $X(0) = \mathrm{Id}$ and $Y(0) = 0$. By Lemma 4.2.9, one has that $X(t)^T Y(t) = Y^T(t)X(t)$. Differentiating this relation and evaluating the differential at zero, one gets*

$$\dot{Y}^T(0) = \dot{Y}(0).$$

This means that the tangent space at Λ_{hor} is parametrized by the space of symmetric matrices. Then, considering geodesic coordinates (see, e.g., [53] for the definition), one gets that a base of neighborhoods of Λ_{hor} is obtained by $\{(x, A\,x) \mid A \in U)\}$ where U is an open set in the topological space of symmetric matrices.

Proposition 4.2.10. *If Λ and Λ' are Lagrangian subspaces of $\mathbb{R}^{2n}$, then there exists a symplectomorphism $\Psi \in \mathrm{U}(n)$ such that $\Psi(\Lambda') = \Lambda$. Moreover, there exists a natural homeomorphism between $\mathscr{L}(n)$ and $\mathrm{U}(n)/\mathrm{O}(n)$.*

Proof. First notice that any Lagrangian submanifold Λ can be seen as the image, via a symplectomorphism, of the horizontal Lagrangian Λ_{hor}. We consider a Lagrangian frame (X, Y) of Λ and build the matrix

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

It is clear that $\Psi(\Lambda_{\text{hor}}) = \Lambda$. Furthermore, since $X^T Y = Y^T X$, Ψ turns out to be orthogonal. This proves the first part of the statement. Now, notice that this Lagrangian frame is unique, up to multiplication on the right by $\mathrm{O}(n)$: this implies $\mathscr{L}(n) = \mathrm{U}(n)/\mathrm{O}(n)$.

□

Theorem 4.2.11 (Maslov Index of Lagrangian subspaces)**.** *There exists a functor μ_L, called Maslov Index, such that*

$$\mu_L : C(\mathbb{R}/\mathbb{Z}, \mathscr{L}(n)) \to \mathbb{Z},$$

and satisfies the following axioms:

i) (Homotopy) *If two loops $\Lambda(t)$ and $\Lambda'(t)$ are homotopic, then $\mu_L(\Lambda) = \mu_L(\Lambda')$.*

ii) (Product) *If $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathscr{L}(n)$ and $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$, then*

$$\mu_L(\Psi\Lambda) = \mu_L(\Lambda) + 2\mu_S(\Psi)$$

with μ_S defined in Theorem 4.2.8

iii) (Direct Sum) *Let us consider two loops $\Lambda_1 : \mathbb{R}/\mathbb{Z} \to \mathscr{L}(n_1)$ and $\Lambda_2 : \mathbb{R}/\mathbb{Z} \to \mathscr{L}(n_2)$. Then, $\Lambda_1 \oplus \Lambda_2$ is a loop in $\mathscr{L}(n_1 + n_2)$, and*

$$\mu_L(\Lambda_1 \oplus \Lambda_2) = \mu_L(\Lambda_1) + \mu_L(\Lambda_2).$$

iv) *(*Normalisation*) The loop*

$$\begin{aligned}\Lambda : \mathbb{R}/\mathbb{Z} &\to \mathscr{L}(1)\\ t &\to \{e^{2\pi i t}x \mid x \in \mathbb{R}\}\end{aligned}$$

has index 1.

Sketch of the proof In Lemma 4.2.9 we have noticed that $\mathscr{L}(n)$ is isomorphic to $\mathrm{U}(n)/\mathrm{O}(n)$, so we define

$$\begin{aligned}\rho : \mathscr{L}(n) &\to \mathbb{S}^1\\ \Lambda &\mapsto \det{}^2(X + iY), \quad \Lambda = \mathrm{range}\begin{pmatrix}X\\Y\end{pmatrix} X + iY \in \mathrm{U}(n).\end{aligned} \tag{4.2.9}$$

Since the matrix $X + iY$ is unique up to right multiplication by matrices in $\mathrm{O}(n)$, the Definition (4.2.9) is well-given. Incidentally, note that, since we use the square of the determinant, we consider non-oriented Lagrangian subspaces. For a loop of Lagrangian subspaces $\Lambda(t)$, we define

$$\mu_L(\Lambda) = \deg(\rho \circ \Lambda),$$

where deg is the winding number as in Theorem 4.2.8. We refer to [67] for the proof that, with this definition, μ_L has the required properties.

For sake of completeness, we give an equivalent definition of Maslov Index for Lagrangian subspaces via intersection theory. First we need some technical lemmas.

Lemma 4.2.12. *For any Lagrangian subspace $\Lambda \in \mathscr{L}(n)$ there exists a Lagrangian subspace W such that $\Lambda \oplus W = \mathbb{R}^{2n}$ and the map*

$$\begin{aligned}A : \mathbb{R}^{2n} &\to \Lambda \times \Lambda^*\\ (l, w) &\mapsto (l, \omega(w, \cdot)|_\Lambda)\end{aligned}$$

is a symplectomorphism with (Λ, Λ^*) equipped with the symplectic form

$$\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}a'\\b'\end{pmatrix}\right) \mapsto b(a') - b'(a).$$

Proof. Let us consider an arbitrary subspace W such that $\Lambda \cap W = \{0\}$ and $W \subsetneq W^{\omega_0}$. Notice that $\dim(\Lambda) = n$ and $\dim(W^{\omega_0}) > n$, so $\dim(\Lambda \cap W^{\omega_0}) > 0$. Now, if $W^{\omega_0} \subseteq \Lambda \oplus W$, by (4.2.3) we have $W \supseteq W^{\omega_0} \cap \Lambda^{\omega_0} = W^{\omega_0} \cap \Lambda$, hence $W^{\omega_0} \cap \Lambda = \{0\}$, which is impossible. So we conclude that $W^{\omega_0} \nsubseteq \Lambda \oplus W$ and we can choose an element $e \in W^{\omega_0}$ such that $e \notin \Lambda \oplus W$. Now we can repeat the same procedure for the space $W + [e]$ and, by induction, we stop when we find a Lagrangian subspace, i. e., when $W = W^{\omega_0}$. To prove the second part of the Lemma we have just to notice that, since Λ and W are Lagrangian,

$$\omega_0(l + m, l' + m') = \omega_0(l, m') + \omega_0(m, l') = \omega_0(m, l') - \omega_0(m', l).$$

□

Corollary 4.2.13. *Let us consider a Lagrangian subspace Λ and its Lagrangian orthogonal complement W, defined by Lemma 4.2.12. Let Z be an orthogonal complement of W: we have that $Z = \{\lambda + A\,\lambda \mid \lambda \in \Lambda\}$, where $A : \Lambda \to W$ is a linear map.*

Proof. By definition, $\mathbb{R}^{2n} = W \oplus Z = \Lambda \oplus W$. Let us fix $\{z_1, \ldots, z_n\}$ and $\{\lambda_1, \ldots, \lambda_n\}$, bases of Z and Λ, respectively. We define

$$\begin{aligned} A &: Z \to W \\ z_i &\mapsto w_i \in W \text{ such that } \lambda_i + \omega_i = z_i. \end{aligned}$$

The elements w_i are well-defined, in view of Lemma 4.2.12. □

Theorem 4.2.14. *Let $\Lambda(t)$ be a path of Lagrangian subspaces such that $\Lambda(0) = \Lambda_0$ and $\dot{\Lambda}(0) = \dot{\Lambda}_0$. Then, the following statements hold true*

i) Let W be a Lagrangian complement of Λ_0, $v \in \Lambda_0$ and t small, and define $w(t) \in W$ such that $v + w(t) \in \Lambda(t)$. Then,

$$Q(v) = \frac{d}{dt}\omega_0(v, w(t))|_{t=0} \tag{4.2.10}$$

is independent of W.

ii) If $Z(t) = (X(t), Z(t))$ is a Lagrangian frame of $\Lambda(t)$, then

$$Q(v) = \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)u, \dot{X}(0)u\rangle, \quad u = Z(0)u,$$

$\langle\ ,\ \rangle$ being the scalar product on $\mathbb{R}^{2n}$ and Q as in (4.2.10).

iii) Q is natural, that is

$$Q(\Psi \circ \Lambda_0, \Psi \circ \dot{\Lambda}_0) \circ \Psi = Q(\Lambda_0, \dot{\Lambda}_0), \quad \forall \Psi \in \mathrm{Sp}(2n).$$

Proof. i) We can suppose, without loss of generality, that $\Lambda_0 = \Lambda_{\text{hor}} = \mathbb{R}^n \times \{0\}$. Then, $\{0\} \times \mathbb{R}^n$ is an orthogonal complement of Λ_{hor}. By Corollary 4.2.13, all Lagrangian complements of Λ_{hor} can be written as $\{(Ax, x) \mid x \in \mathbb{R}^n\}$, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a suitable linear map. Moreover, by Remark 4.2.3, if we suppose t small enough, we have that $\Lambda(t) = \{(x, A(t)\,x) \mid x \in \mathbb{R}^n\}$, $A(t)$ symmetric matrix. Let us suppose that $v = (x, 0)$, $w(t) = (B\,y(t), y(t))$. In order to have $v + w(t) \in \Lambda(t)$, we require $y(t) = A(t)(x + B\,y(t))$. If we derive this relation and evaluate at $t = 0$, we get $\dot{y}(0) = \dot{A}(0)\,x$ and obtain

$$Q(v) = \langle x, \dot{A}(0)x\rangle,$$

so Q does not depend on W.

ii) Let us consider $W = \{0\} \times \mathbb{R}^n$. We have proved at point *i)* above that this is not a restriction. Then, let $Z(t) = (X(t), Y(t))$ be a Lagrangian frame of the curve such that $v = (X(0)u, Y(0)u)$, $w(t) = (0, y(t))$. We have $Y(0)u + y(t) = Y(t)X(t)^{-1}X(0)u$, so $\omega_0(v, w(t)) = \langle X(0)u, y(t)\rangle$. Differentiating once, we get

$$\begin{aligned} &Q(v) = \langle X(0)u, \dot{y}(0)\rangle = \\ &\langle X(0)u, \dot{Y}(0)X^{-1}(0)X(0)u\rangle - \langle X(0)u, Y(0)X^{-1}(0)\dot{X}(0)X^{-1}(0)X(0)u\rangle. \end{aligned}$$

Now, if we simplify and recall that $X^T Y = Y^T X$, we get

$$Q(v) = \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)u, \dot{X}(0)u\rangle.$$

iii) This follows by the definition of Q.

□

Theorem 4.2.14 gives a bijection from the tangent space $T_\Lambda\mathscr{L}(n)$ to the quadratic forms on Λ. Let us now consider a general Lagrangian subspace V. We have a natural filtration

$$\mathscr{L}(n) = \bigcup_{k=0}^{n} \Sigma_k(V),$$

where $\Sigma_k(V)$ is the space of Lagrangian space having k-dimensional intersection with V. We define the Maslov cycle as $\Sigma(V) = \bigcup_{k=1}^{n} \Sigma_k(V)$. This is an algebraic variety, $\Sigma_1(V)$ being its regular part. Let us consider a path of Lagrangian space $\Lambda(t) : [0,1] \to \mathscr{L}(n)$. The point t_0 is a crossing point if $\Lambda(t_0) \in \Sigma(V)$. The crossing form is defined as

$$\Gamma(\Lambda, V, t) = Q(\Lambda(t), \dot{\Lambda}(t))|_{\Lambda(t)\cap V},$$

with Q from Theorem 4.2.14. A path $\Lambda(t)$ of Lagrangian subspaces is tangent to $\Sigma(V)$ at a crossing point t_0 if $\Gamma(\Lambda, V, t_0) = 0$. The crossing point t_0 is called regular if $\Gamma(\Lambda, V, t_0)$ is non-degenerate. If t_0 is regular and $\Lambda_{t_0} \in \Sigma_1(V)$, the crossing is called simple. So a path of Lagrangian subspace has only simple crossing points if it has transversal intersection with $\Sigma(V)$. The Maslov index of a path $\Lambda(t) : [a,b] \to \mathscr{L}(n)$ with just simple intersection is defined as

$$\mu(\Lambda, V) = \frac{1}{2}\operatorname{sgn}\Gamma(\Lambda, V, a) + \sum_{a<t<b} \operatorname{sgn}\Gamma(\Lambda, V, t) + \frac{1}{2}\operatorname{sgn}\Gamma(\Gamma, V, b), \qquad (4.2.11)$$

with the sum running over all simple crossing points. Using homotopy arguments, this definition can be extended to all paths of Lagrangian subspaces, see [89] for the details. If we consider loops, formula (4.2.11) is simpler, because the contribution at the end points vanishes. If $\Lambda(t) : \mathbb{S}^1 \to \Lambda(t)$ is a loop with simple crossing points, we have

$$\mu(\Lambda, V) = \sum_{a<t<b} \operatorname{sgn}\Gamma(\Lambda, V, t).$$

If one restrict to loops, the definition of $\Gamma(\Lambda, V, t)$ is independent on V, because for all Lagrangian subspaces W one can consider a symplectomorphism Ψ such that $\Psi(V) = W$. Then, using the naturality property of Γ and the the fact that $\mathrm{Sp}(2n)$ is connected, one gets that the the index is invariantly defined. One can prove that this definition of Maslov index satisfies all the axioms of Theorem 4.2.11, so the two definitions are equivalent, see [89].

Definition 4.2.3. Let us consider four Lagrangian subspacess M_1, M_2, W_1, W_2 such that M_i is transversal to W_j, $i = 1,2, j = 1,2$. The Hörmander index is defined as

$$s(M_1, M_2; W_1, W_2) = \mu(\Lambda, V)$$

where $\Lambda = \Lambda_1 \circ \Lambda_2$, $\Lambda_1 : [0,1] \to \mathscr{L}(n)$ being an arc of Lagrangian subspacess transversal to M_1 such that $\Lambda_1(0) = W_1$, $\Lambda_2(0) = W_2$. Similarly, $\Lambda_2 : [0,1] \to \mathscr{L}(n)$ is an arc of Lagrangian subspaces transversal to M_2 connecting W_2 and W_1. The definition does not depend on the choice of the path.

We now give the definition of symplectic manifold and Lagrangian submanifold.

Definition 4.2.4. Let M be a smooth manifold (possibly with non empty boundary). M is symplectic if there exists a non degenerate closed two form ω such that $\omega_p(\cdot,\cdot)$ is a symplectic form for the vector space T_pM, for all $p \in M$. We denote by (M,ω) the symplectic manifold with symplectic form ω.

A first immediate property is that a symplectic manifold (M,ω) is even-dimensional.

Example 4.2.1. *The first example of symplectic manifold is $\mathbb{R}^{2n}$ with the form $\sum_{i=1}^n dx_i \wedge dy_i$, where $\{x_i\}_{i=1}^n$ represents the first n variables and $\{y_i\}_{i=1}^n$ the remaining variables. To check this property one can notice that, at every point $p \in R^{2n}$, one can choose as symplectic base of $T_p\mathbb{R}^{2n}$ the span of*

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_n}\right)_p.$$

Example 4.2.2. *The symplectic manifold we will use in the following is the the cotangent bundle of a smooth manifold M, possibly with boundary. Let us consider local coordinates $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ in a neighborhood of a point $(x_0,\xi_0) \in T^*M$. The symplectic form is locally defined as*

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

Actually one can prove that ω is globally well defined and, introducing the fundamental 1-form $\alpha = \sum_{i=1}^n \xi_i dx_i$, one has that $\omega = -d\alpha$. If one considers a manifold with boundary $(M,\partial M)$, then T^*M turns out to be a symplectic manifold with boundary.

One can introduce also in this setting the notion of symplectomorphism.

Definition 4.2.5. Let (M_1,ω_1), (M_2,ω_2) two symplectic manifolds. A diffeomorphism $\chi : M_1 \to M_2$ is a symplectomorphism if $\omega_1 = \chi^*\omega_2$.

Also in the case of symplectic manifolds there exists a Darboux Theorem, analogous to Theorem 4.2.2.

Theorem 4.2.15. *Let (M,ω) be a smooth manifold without boundary. Then, for every point $p \in M$, there exists a neighborhood U of p and local coordinates in U, $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ such that*

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

Definition 4.2.6. Let (M,ω) be a symplectic manifold. A submanifold $N \subseteq M$ is a Lagrangian submanifold if, for all $p \in N$, T_pN is a Lagrangian subspace of T_pM w.r.t. the symplectic form ω. That is, we require that $i^*\omega = 0$, where $i : N \hookrightarrow M$ is the immersion. This implies that $\dim(N) = \frac{n}{2}$, n being the dimension of M.

Example 4.2.3. *Let us consider the cotangent bundle T^*M of a smooth manifold M. Then one defines the zero section as the space*

$$T^*M_0 = \{(x, \xi) \in T^*M \mid \xi = 0\}.$$

Since on this subspace the fundamental 1*-form $\alpha = \sum \xi_i dx_i$ vanishes identically, the zero section is a Lagrangian submanifold of T^*M. The zero section is often just written as* 0*, when there is no confusion. In a similar way one can define the tangent space of a fiber at a point $\lambda = (x_0, \xi_0)$*

$$T^{*0}_{\lambda}M = \{(x, \xi) \mid x = x_0\}.$$

This is also a Lagrangian space, since dx vanishes identically on it.

Let us now consider a smooth manifold M and its cotangent bundle $T^*M \setminus 0$. Let us suppose that we are given a Lagrangian submanifold Λ. Let us fix a point $\lambda = (x_0, \xi_0)$, and the corresponding Lagrangian submanifold $T^{*0}_{\lambda}M$. Considering $T_\lambda(T^*M)$ as a symplectic vector space, given two arbitrary Lagrangian space Λ_1, Λ_2 of $T_\lambda(T^*M)$, one can define, according to Definition 4.2.3,

$$s(T^{*0}_{\lambda}M, T_\lambda(\Lambda), \Lambda_1, \Lambda_1).$$

Given a Lagrangian submanifold $\Lambda \subseteq T^*M$, we will consider a bundle L on Λ with fiber, at each point $\lambda \in \Lambda$, given by the set of functions $\mathscr{L}(T_\lambda(T^*M)) \to \mathbb{C}$ such that

$$f(\Lambda_1) = i^{s(T^{*0}_{\lambda}M, T_\lambda(\Lambda), \Lambda_1, \Lambda_1)} f(\Lambda_2),$$

for all $\Lambda_1, \Lambda_2 \in \mathscr{L}(T_\lambda(T^*M))$. Such a bundle is called the Keller-Maslov bundle and is the right tool to parametrize the principal symbol of FIOs.

Chapter 5

Functions Spaces and Symbols on the Half-Space

5.1 Function Spaces

In the sequel, the space $H^s(\mathbb{R}^n)$ will be the usual Sobolev space on $\mathbb{R}^n$, while $H^{s_1,s_2}(\mathbb{R}^n)$, $\mathbf{s} = (s_1, s_2) \in \mathbb{R}^2$ is defined as

$$H^{\mathbf{s}}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \mid \|\langle x\rangle^{s_2}\mathrm{Op}(\langle\xi\rangle^{s_1})(u)\|_{L^2} < \infty\},$$

where $\mathrm{Op}(a)$ represents the pseudodifferential operator with symbol a.

Given a Fréchet space E, it is possible to define the space $\mathscr{S}(\mathbb{R}^q; E)$ of rapidly decreasing vector-valued functions. It can be defined as the subset of $C^\infty(\mathbb{R}^q; E)$ such that $\partial^\alpha x^\beta f$ is a bounded set in E for all multi-indices α, β. If we deal with projective limits or inductive limits of Banach spaces, the same definition can be used. Actually, it is possible to define $\mathscr{S}(\mathbb{R}^q; E)$ for all locally convex topological vector spaces E, see [103], Ch. 44.

Let us now fix a quantization for the Fourier Transform. For every function in $\mathscr{S}(\mathbb{R}^q, E)$, we set

$$\mathscr{F}(u)(\xi) = \hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\cdot\xi} u(x)dx, \quad \xi \in \mathbb{R}^q.$$

We recall from Section 4.1 the notation

$$\begin{aligned}
\mathbb{R}^n_+ &= \{(x_1, \ldots, x_n) \mid (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\},\\
\overline{\mathbb{R}^n_+} &= \{(x_1, \ldots, x_n) \mid (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n \geq 0\},\\
\partial\overline{\mathbb{R}^n_+} &= \{(x_1, \ldots, x_{n-1}, 0) \mid (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}.
\end{aligned}$$

We consider the restriction operator r^+ associated to $\mathbb{R}^n_+$. Since $\mathbb{R}^n_+$ is an open set, the restriction of a distributions is well known and

$$\begin{aligned}
&H^{\mathbf{s}}(\mathbb{R}^n_+) = \{r^+(u) \mid u \in H^{\mathbf{s}}(\mathbb{R}^n)\}\\
&\|f\|_{H^{\mathbf{s}}(\mathbb{R}^n_+)} = \inf\{\|u\|_{H^{\mathbf{s}}(\mathbb{R}^n)} \mid r^+(u) = f\}.
\end{aligned}$$

Moreover, we define

$$H^{\mathbf{s}}_0(\overline{\mathbb{R}^n_+}) = \{u \in H^{\mathbf{s}}(\mathbb{R}^n) \mid \mathrm{supp}(u) \subseteq \overline{\mathbb{R}^n_+}\}.$$

Notice that the spaces $H^{\mathbf{s}}(\mathbb{R}^n_+)$ and $H^{\mathbf{s}}_0(\overline{\mathbb{R}^n_+})$ have different nature, because $H^{\mathbf{s}}(\mathbb{R}^n_+)$ is a subspace of distributions defined on $\mathbb{R}^n_+$, while $H^{\mathbf{s}}_0(\overline{\mathbb{R}^n_+})$ is a closed subspace of $H^{\mathbf{s}}(\mathbb{R}^n)$. We now focus on the space $\mathscr{S}(\mathbb{R}^n_+)$, defined as

$$\mathscr{S}(\mathbb{R}^n_+) = \{r^+ f \mid f \in \mathscr{S}(\mathbb{R}^n_+)\}.$$

First of all, notice that there is a natural injection of $\mathscr{S}(\mathbb{R}^n_+)$ in $L^2(\mathbb{R}^n)$ extending the functions by zero in the negative half-space. Sometimes we identify $\mathscr{S}(\mathbb{R}^n_+)$ with its extension in $L^2(\mathbb{R}^n)$. The space $\mathscr{S}(\mathbb{R}^n_+)$ can be given a Fréchet structure via the family of seminorms

$$p_{\alpha,\beta}(u) = \sup_{x \in \mathbb{R}^n_+} |x^\beta \partial^\alpha u|, \quad \alpha, \beta \in \mathbb{N}^n. \tag{5.1.1}$$

The Sobolev spaces we have defined in the half-space and the Schwartz spaces in the half-space are closely related:

$$\begin{aligned} \text{ind-lim}_{(s_1,s_2)\to-\infty} H^{\mathbf{s}}_0(\overline{\mathbb{R}^n_+}) &= [\mathscr{S}(\mathbb{R}^n_+)]^* \\ &= \mathscr{S}'(\mathbb{R}^n_+) = \{u \in \mathscr{S}'(\mathbb{R}^n) \mid \operatorname{supp}(u) \subseteq \overline{\mathbb{R}^n_+}\}, \\ \text{proj-lim}_{s_1,s_2\to\infty} H^{\mathbf{s}}(\mathbb{R}^n_+) &= \mathscr{S}(\mathbb{R}^n_+). \end{aligned}$$

Remark 5.1.1. *Notice that the topology on $\mathscr{S}(\mathbb{R}^n_+)$ induced by the projective limit topology and the one defined by the seminorms* (5.1.1) *are equivalent. The topology induced by the inductive limit on $\mathscr{S}'(\mathbb{R}^n_+)$ is the topology of convergence on bounded sets. That is, the topology of $\mathscr{S}'(\mathbb{R}^n_+)$ is given by a non-countable set of seminorms $\{p_{A_i}\}$, where A_i are bounded sets of $\mathscr{S}(\mathbb{R}^n_+)$, and*

$$p_{A_i}(u) = \sup_{f \in A_i} |\langle u, f \rangle|.$$

5.2 The Extension Operator

In order to define operators on the half-space we have to introduce an operator of extension, which, roughly speaking, is the *dual* of the restriction operator. Here we focus on the case $n = 1$, we will see later why this is not a restriction.

Definition 5.2.1. Let f be a function defined on the half-space $\mathbb{R}_+$. We define $e^+ f$ as

$$e^+ f(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases} \tag{5.2.2}$$

We analyze now the extension operator in the Sobolev spaces $H^{\mathbf{s}}(\mathbb{R}_+)$. If $s_1 > \frac{1}{2}$, then, one can define e^+ as in (5.2.2). This obviously implies a loss of regularity: in this case we have

$$e^+ : H^{\mathbf{s}}(\mathbb{R}_+) \to H^{t,s_2}(\mathbb{R}_+), \quad t < \frac{1}{2}.$$

If $s_1 \in (-\frac{1}{2}, \frac{1}{2})$, r^+ is a bijection, and so one can define e^+ as the inverse of r^+. The case $s_1 < -\frac{1}{2}$ is more delicate and is explained in the following proposition.

Proposition 5.2.1. *Let $s_1 < -\frac{1}{2}$. Then, there exists a continuous extension operator e^+*

$$e^+ : H^{\mathbf{s}}(\mathbb{R}_+) \to H_0^{\mathbf{s}}(\overline{\mathbb{R}_+})$$

such that $r^+(e^+(u)) = u$, for all $u \in H^{\mathbf{s}}(\mathbb{R}_+)$.

Proof. First we prove that, if $s_1 < -\frac{1}{2}$, the mapping

$$r^+ : H_0^{\mathbf{s}}(\overline{\mathbb{R}_+}) \to H^{\mathbf{s}}(\mathbb{R}_+)$$

turns out to be surjective. In order to prove surjectivity, we recall that the spaces $H_0^{\mathbf{s}}(\overline{\mathbb{R}_+})$ and $H^{\mathbf{s}}(\mathbb{R}_+)$ are related by duality w.r.t. the $L^2(\mathbb{R}_+)$ scalar product, namely

$$H^{\mathbf{s}}(\mathbb{R}_+) = [H_0^{-\mathbf{s}}(\overline{\mathbb{R}_+})]^*, \quad H_0^{\mathbf{s}}(\overline{\mathbb{R}_+}) = [H^{-\mathbf{s}}(\mathbb{R}_+)]^*.$$

The proof can be found in [48], Appendix B.2. Therefore, to prove surjectivity, it is enough to find, for each element in $[H_0^{-\mathbf{s}}(\overline{\mathbb{R}_+})]^*$, an extension defined on $H^{-\mathbf{s}}(\mathbb{R}_+)$, and this is possible by the Hahn-Banach Theorem. Notice that, since $H^{-\mathbf{s}}(\overline{\mathbb{R}_+})$ is not dense in $H^{-\mathbf{s}}(\mathbb{R}_+)$, the extension is not unique. For example, the restriction of the zero function equals the restriction of the Dirac's distribution δ at the origin.

Let $u \in H^{\mathbf{s}}(\mathbb{R}_+)$; since $r^+ : H_0^{\mathbf{s}}(\overline{\mathbb{R}_+}) \to H^{\mathbf{s}}(\mathbb{R}_+)$ is surjective, we can consider the non empty set $U = \{\tilde{u} \mid r^+\tilde{u} = u\}$. For $u_1, u_2 \in U$ we have $(u_1 - u_2)|_{\mathbb{R}^+} = 0$. hence

$$u = \tilde{u} + \operatorname{span}\{\delta_0, \dots, \delta_0^{([-s+\frac{1}{2}])}\}.$$

U is a nonempty, closed, convex subset of $H_0^{\mathbf{s}}(\mathbb{R}^+)$. Hence there is an element $\bar{u} \in U$ for which $\|\bar{u}\|_{H_0^{\mathbf{s}}(\overline{\mathbb{R}_+})}$ is minimal. By the convexity, $\bar{u}$ is unique. The map $u \to \bar{u}$ is continuous, since

$$\begin{aligned}\|\bar{u}\|_{H^{\mathbf{s}}(\mathbb{R})} &= \min\{\|v\|_{H_0^{\mathbf{s}}(\overline{\mathbb{R}})} \mid r^+v = u\} \\ &= \min\{\|v\|_{H^{\mathbf{s}}(\mathbb{R})} \mid r^+v = u\} = \|u\|_{H^{\mathbf{s}}(\mathbb{R}_+)}.\end{aligned}$$

Indeed, the first equality holds in view of the definition of $\bar{u}$, while the second follows by the fact that we have, for the projection $\pi_0 : H^{\mathbf{s}}(\mathbb{R}) \to H_0^{\mathbf{s}}(\mathbb{R})$, $\|\pi_0 v\|_{H^{\mathbf{s}}} \leq \|v\|_{H^{\mathbf{s}}}$: then, the minimum is attained at $H_0^{\mathbf{s}}(\overline{\mathbb{R}_+})$. □

Remark 5.2.1. *Theorem 5.2.1 gives a general definition of e^+.The disadvantage of such an approach for $s_1 < -\frac{1}{2}$ is that the extension depends on the chosen Sobolev space: since we determine e^+u as the extension of minimal value of the corresponding norm, by changing the space, the minima can change as well. Nevertheless, two different extensions can differ only by a sum of derivatives of the Dirac's distribution at the origin.*

5.3 Operator-Valued Symbols and Wedge Sobolev Spaces

In this section we recall the basic tool of the theory of operator-valued symbols, introduced by B.-W Schulze. The contents mainly come from [29, 94, 96].

Definition 5.3.1. A strongly continuous group action of a Banach space B is a family $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}^+}$ of isomorphisms of B such that

i) $\kappa_\lambda \circ \kappa_\mu = \kappa_\mu \circ \kappa_\lambda = \kappa_{\mu\lambda}$, in particular $\kappa_1 = \mathrm{Id}$,

ii) for all fixed $x \in B$ the map

$$\mathbb{R}^+ \to B : \quad \lambda \mapsto \kappa_\lambda(x)$$

is continuous.

Definition 5.3.1 has been given for a Banach space B. If B is a Fréchet space the definition is the same.

Lemma 5.3.1. *Let B be a Banach space with group action κ_λ; then there exist constants M, C such that*

$$\|\kappa_\lambda\|_{\mathscr{L}(B,B)} \leq C \max\{\lambda, \lambda^{-1}\}^M. \tag{5.3.3}$$

The previous lemma follows from the Banach-Steinhaus Theorem, see [84]. Since Banach-Steinhaus Theorem holds, in particular, for Fréchet spaces, inductive limits of Banach spaces and projective limits of Banach spaces (see [34]), one has that, analogously, (5.3.3) holds also in such cases with inequalities as (5.3.3) for each seminorm.

In the following, we deal with specific group actions. In the case of functions on $\mathbb{R}^n$, we will consider

$$\kappa_\lambda f(x) = \lambda^{\frac{n}{2}} u(\lambda\, x). \tag{5.3.4}$$

We will use this group action also for all Sobolev space $H^{\mathbf{s}}$ with $s_1 \geq 0$. In the case $s_1 < 0$, the group action on distribution is given by duality, that is

$$\langle \kappa_\lambda u, f\rangle = \langle u, \kappa_\lambda^{-1} f\rangle, \quad f \text{ test function.}$$

In the scalar cases $B = \mathbb{R}, \mathbb{C}$, the group action will be the trivial one, that is, for each λ, $\kappa_\lambda = \mathrm{Id}$. The reason of these choices will be explained in Subsection 5.3.2, when we will recall the definition of wedge Sobolev spaces.

Definition 5.3.2. Let us consider E, B Banach spaces with strongly continuous group actions $\kappa, \tilde{\kappa}$, respectively. A function $a(x, y, \eta) \in C^\infty(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; \mathscr{L}(E, B))$ is a symbol in the set $S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; \mathscr{L}(E, B)) = S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, B)$ if for all $\alpha, \beta, \gamma \in \mathbb{N}^q$ there exists a constant $C_{\alpha,\beta,\gamma}$ such that

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\Big(\partial_x^\beta \partial_y^\gamma \partial_\eta^\alpha a(x, y, \eta)\Big)\kappa_{\langle\eta\rangle}\Big)\|_{\mathscr{L}(E,B)} \leq C_{\alpha,\beta,\gamma}\langle\eta\rangle^{m-|\alpha|}, \quad \text{for all } x, y, \eta.$$

In the sequel, the Banach spaces we will consider will be mainly $H^{\mathbf{s}}(\mathbb{R}_+)$, $H^{\mathbf{s}}_0(\overline{\mathbb{R}^n_+})$ and the Fréchet spaces will be $\mathscr{S}(\mathbb{R})$ or $\mathscr{S}(\mathbb{R}_+)$.

As we have noticed, the spaces $\mathscr{S}(\mathbb{R}_+)$ and $\mathscr{S}'(\mathbb{R}_+)$ are related to $H^{\mathbf{s}}(\mathbb{R}_+)$ and $H^{\mathbf{s}}_0(\overline{\mathbb{R}_+})$ via projective limit and direct limit. We give the definition of operator-valued symbols in the case of projective limit and direct limit.

Definition 5.3.3. Let $E_1 \hookrightarrow E_2 \ldots$ be an inductive family of Banach spaces such that the group action on E_{i+1}, restricted to E_i, is the same of E_i, and $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ a projective family of Banach space such that the group action of F_i, restricted to F_{i+1}, is equal to the group action of F_{i+1}. Then, setting $E = \text{ind-lim}_i E_i$ and $F = \text{proj-lim}_i F_i$, we define

$$\begin{aligned}
S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;B,F) &= \text{proj-lim}_i S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;B,F_i),\\
S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,B) &= \text{proj-lim}_i S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E_i,B),\\
S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F) &= \text{proj-lim}_{(i,j)} S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E_j,F_i),
\end{aligned}$$

where B is a general Banach space with group action.

The case of of symbols with values in the vector space of linear operators from an inductive limit space to an inductive limit space or from a projective limit space to a projective limit space is more delicate.

Definition 5.3.4. Let $E_1 \hookrightarrow E_2 \dots$ and $F_1 \hookrightarrow F_2 \dots$ be inductive families of Banach spaces with group actions κ_i and $\tilde{\kappa}_j$, respectively, such that the group action of E_{i+1}, restricted to E_i, is the same of E_i, and similarly for the family F_j. We set $E = \text{ind-lim}_i E_i$ and $F = \text{ind-lim}_j F_j$. Then, a smooth function $a(x,y,\eta)$ which takes values in $\mathscr{L}(E,F)$ is a symbol in the class $S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F)$ if, for all i and for all α,β,γ, there exists j and a constant $C^j_{\alpha,\beta,\gamma}$ such that

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\Big(\partial^\beta_x\partial^\gamma_y\partial^\alpha_\eta a(x,y,\eta)\Big)\kappa_{\langle\eta\rangle}u\Big)\|_{F_j} \le C^j_{\alpha,\beta,\gamma}\|u\|_{E_i}\langle\eta\rangle^{m-|\alpha|}. \tag{5.3.5}$$

The following lemma follows from Definition 5.3.4.

Lemma 5.3.2. *Let $a \in S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F)$ and $b \in S^t(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;F,G)$, E,F,G Banach spaces, or projective limits of Banach spaces, or inductive limit of Banach spaces. Then*

i) $\partial^\alpha_\eta\partial^\beta_x\partial^\gamma_y a(x,y,\eta) \in S^{m-|\alpha|}(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F)$, for all multi-indices $\alpha,\beta,\gamma \in \mathbb{N}^q$;

ii) The point wise composition $(ba)(x,y,\eta)$ belongs to $S^{m+t}(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,G)$.

Theorem 5.3.3. *Let $a \in S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F)$, E,F as in Definition 5.3.4. Then,*

$$\begin{aligned}
\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^q;E) &\to \mathscr{S}(\mathbb{R}^q;F)\\
u &\mapsto (2\pi)^{-\frac{n}{2}}\int e^{i(x'-y')\cdot\xi'}a(x',y',\xi')u(y')dy'\,đ\xi'
\end{aligned}$$

is a continuous operator, where $đ\xi = (2\pi)^{-\frac{n}{2}}d\xi$.

Theorem 5.3.4. *Let us consider symbols $a \in S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F)$, $b \in S^t(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;F,G)$, and the associated pseudodifferential operators $A = \mathrm{Op}(a)$, $B = \mathrm{Op}(b)$. Then,*

i) there exist a right symbol a_R and a left symbol a_L, defining the same operator;

ii) the composition $B \circ A$ is again a pseudodifferential operator with symbol $c \in S^{m+t}(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,G)$ such that

$$c \sim \sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\partial^\alpha_\xi b_L D^\alpha a_L.$$

We give now a few examples of vector-valued operators.

Example 5.3.1 (Trace Operators). *Let us consider the trace operators $\gamma_j : \mathscr{S}(\mathbb{R}_+) \to \mathbb{C}$, $j = 0, 1, \ldots$ defined as*

$$\gamma_j f = \lim_{t\to 0^+} \partial_t^j f(t) = f^{(j)}(0).$$

In view of the trace theorem for Sobolev spaces [2], one can prove that the trace operators extends to $H^{\mathbf{s}}(\mathbb{R}_+)$ with $s_1 > \frac{1}{2} + j$. Actually, γ_j can be seen as an operator-valued symbol in $S^{j+\frac{1}{2}}(\mathbb{R}^q, \mathbb{R}^q; H^s(\mathbb{R}_+), \mathbb{C})$, with the canonical group action on the Sobolev space and the trivial one on $\mathbb{C}$. Indeed, we find

$$|\gamma_j(\kappa_{\langle\eta\rangle}(u))| = |\langle\eta\rangle^{\frac{1}{2}} \lim_{t\to 0^+} (\partial_t^{(j)} u(\langle\eta\rangle t))| = \langle\eta\rangle^{j+\frac{1}{2}} \partial_t^{(j)} u(0).$$

This implies that the required estimates in Definition 5.3.2 are fulfilled.

Example 5.3.2. *Another important example is a pseudodifferential operator acting in one variable only. Namely, let us consider $a \in S^m(\mathbb{R}^n, \mathbb{R}^n)$ and define the operator*

$$\mathrm{Op}_n(a)u(x_n) = \int e^{ix_n\cdot\xi_n} a(x', x_n, \xi', \xi_n)\hat{u}(\xi_n) đ\xi_n.$$

It is possible to prove that

$$u \mapsto \kappa_{\langle\xi'\rangle^{-1}} \circ \mathrm{Op}_n(a)u \circ \kappa_{\langle\xi'\rangle}$$

is a continuous operator from $\mathscr{S}(\mathbb{R})$ to itself. Moreover, one can extend it as a continuous operator from $H^{\mathbf{s}}(\mathbb{R})$ to $H^{\mathbf{s}-(m,0)}(\mathbb{R})$. Namely,

$$\kappa_{\langle\xi'\rangle^{-1}} \circ \mathrm{Op}_n(a) \circ \kappa_{\langle\xi'\rangle} u(x_n) = \\ \int e^{ix_n\cdot\frac{\xi_n}{\langle\xi'\rangle} - iy_n\cdot\xi_n} a\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\right) u(\langle\xi'\rangle y_n) dy_n đ\xi_n.$$

If we set $\eta_n\langle\xi'\rangle = \xi_n$ and $t_n = y_n\langle\xi'\rangle$ we get

$$\kappa_{\langle\xi'\rangle^{-1}} \circ \mathrm{Op}_n(a)u \circ \kappa_{\langle\xi'\rangle}(x_n) = \\ \int e^{ix_n\cdot\eta_n} a\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \eta_n\langle\xi'\rangle\right) \hat{u}(\eta_n) đ\eta_n.$$

Now, the $\mathscr{S}$-continuity follows because the function $a\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \eta_n\langle\xi'\rangle\right)$, for fixed $(x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, belongs to $S^m(\mathbb{R}, \mathbb{R})$. A simple observation, useful in the computations, is that $(1 + \xi^2 + \langle\xi'\rangle^2\eta_n^2) = \langle\xi'\rangle^2\langle\eta_n\rangle^2$. More precisely, one can prove that $\mathrm{Op}_n(a) \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{\mathbf{s}}(\mathbb{R}), H^{\mathbf{s}-(m,0)}(\mathbb{R}))$, since the operator norm of

$$\kappa_{\langle\xi'\rangle^{-1}} \circ \left(\partial_{x'}^\beta \partial_{\xi'}^\alpha \mathrm{Op}_n(a)u\right) \circ \kappa_{\langle\xi'\rangle}$$

comes from the seminorms of the symbol $\partial_{x'}^\beta \partial_\xi^\alpha \left[a\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \eta_n\langle\xi'\rangle\right)\right]$ seen as a symbol in $S^m(\mathbb{R}, \mathbb{R})$, with (x', ξ') fixed. Knowing that $a \in S^m(\mathbb{R}^n, \mathbb{R}^n)$, it follows that

$$\sup_{x\in\mathbb{R}} \left|\langle\eta_n\rangle^{-m+|q|} \partial_{x_n}^t \partial_{\eta_n}^q \left(\partial_{\xi'}^\alpha \partial_{x_n}^\beta a\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \eta_n\langle\xi_n\rangle\right)\right)\right| \le C\langle\xi_n\rangle^{m-|\alpha|}$$

where C depends on $\{p_{\gamma,\delta}\}_{\gamma,\delta\in\mathbb{N}^n}$, the set of seminorms of the Fréchet space $S^m(\mathbb{R}^n, \mathbb{R}^n)$.

Also in the case of operator-valued symbols it is possible to consider classical symbols, but first one has to give a meaning to asymptotic expansions. The following results are given without proofs: they are an extension of usual proofs in the scalar case.

Theorem 5.3.5. *Given $a_j \in S^{m-j}(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, F)$, $j \in \mathbb{N}$, E, F being Banach spaces or inductive or projective limit of Banach spaces, there exists a symbol $a \in S^m(\mathbb{R}^n, \mathbb{R}^n; E, F)$ such that $a \sim \sum a_j$, where $\sim$ means that, for all $N \in \mathbb{N}$,*

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N}(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, F).$$

Definition 5.3.5. A symbol $a \in S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, F)$, E, F being Banach spaces or inductive or projective limit of Banach spaces with group actions $\tilde{\kappa}, \kappa$, respectively, is classical if it has an asymptotic expansion $a \sim \sum_j a_j$ with $a_j \in S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, F)$ such that

$$a_j(x, y, \lambda\eta) = \lambda^{m-j} \tilde{\kappa}_\lambda a_j(x, y, \eta) \kappa_{\lambda^{-1}}$$

for all $\lambda \geq 1$, $|\eta| > R$. We write, in this case, $a \in S^m_{\mathrm{cl}}(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, F)$. Clearly, if E, F are equal to $\mathbb{R}$ or $\mathbb{C}$, the definition coincides with the usual one.

5.3.1 Dual and Transposed Symbol

It is well known that $[H^s(\mathbb{R})]^* = H^{-s}(\mathbb{R})$, where the duality is understood in the L^2-sense. Moreover, since κ is unitary on functions, $\kappa_\lambda \bar{u} = \overline{\kappa_\lambda u}$, we have that the definition of group action on distribution by transposition or by L^2-duality is the same. Now, we embed this example in a more abstract theory. Let us consider a triple of Hilbert vector spaces (E_-, E_0, E_+) such that there exists a topological vector space V such that $V \supseteq E_- \cup E_0 \cup E_+$. Furthermore, suppose that $E_0 \cap E_- \cap E_+$ is dense in E_+, E_0, E_- and that E_- is the dual of E_+ via the scalar product of E_0. Explicitly: there is a continuous, non-degenerate, sesquilinear form $(\cdot, \cdot)_E$ such that

$$(\cdot, \cdot) : E_- \times E_+ \to \mathbb{C},$$

and $(\cdot, \cdot)_E$ coincides with the scalar product of E_0 on $(E_- \cap E_0) \times (E_+ \cap E_0)$. We can then identify $E_\pm$ with the dual of $E_\mp$ with the norm

$$\|x\|_{E_-} = \sup_{\|y\|=1} |(x, y)|_E, \quad \|y\|_{E_+} = \sup_{\|x\|=1} |(x, y)|_E. \tag{5.3.6}$$

We assume that the group action defined on V is compatible with the sesquilinear form $(\cdot, \cdot)_E$, that is

$$(\kappa_\lambda u, v)_E = (u, \kappa_\lambda^{-1} v), \quad \forall u \in E_0, v \in E_+.$$

In the sequel, the triples of Hilbert spaces we use will be mainly

$$(H^{-\mathbf{s}}(\mathbb{R}), L^2(\mathbb{R}), H^{\mathbf{s}}(\mathbb{R})), \quad (H_0^{-s}(\overline{\mathbb{R}_+}), L^2(\mathbb{R}_+), H^{\mathbf{s}}(\mathbb{R}_+)).$$

The aim of this abstract construction is the following: given a triple (E_-, E_0, E_+), associate with each symbol $a(x, y, \eta) \in S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E_-, F_-)$ an adjoint symbol $a^*(x, y, \eta) \in S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; F_+, E_+)$ such that

$$(a(x, y, \eta)e, f)_F = (e, a^*(x, y, \eta)f)_E, \qquad \forall x, y, \eta;\ \forall e, f.$$

In order to check that a^* belongs to $S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; F_+, E_+)$ one can use (5.3.6). Moreover, one can verify that

$$(\mathrm{Op}(a)u, v)_{\mathscr{S}(\mathbb{R}^q;F)} = (u, \mathrm{Op}(a^*)v)_{\mathscr{S}(\mathbb{R}^q;E)},$$

where the scalar product on $\mathscr{S}(\mathbb{R}^q; E)$ is defined as

$$\begin{aligned}(\cdot,\cdot)_{\mathscr{S}(\mathbb{R}^q;E)} &: \mathscr{S}(\mathbb{R}^q; E) \times \mathscr{S}(\mathbb{R}^q; E) \to \mathbb{C} \\ &(u, v) \mapsto (u, v)_{\mathscr{S}(\mathbb{R}^q;E)} = \int (u(x), v(x))_E dx.\end{aligned} \tag{5.3.7}$$

In the case of general locally convex topological vector spaces with seminorms $\{p_\alpha\}_{\alpha\in I}$ one has $|I|$ semidefinite sesquilinear forms.

A similar argument can be used also for the transposed of a symbol $a \in S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; E, F)$. We want to define the transposed symbol $a^t \in S^m(\mathbb{R}^q, \mathbb{R}^q, \mathbb{R}^q; F', E')$ such that

$$\langle a^t(g), f\rangle = \langle u, a(f)\rangle, \quad \forall f \in E, g \in F'.$$

The dual spaces F', E' are endowed with the topology of convergence on bounded sets. Similarly to (5.3.7) one obtains

$$\langle a^t(u), v\rangle_{\mathscr{S}(\mathbb{R}^q,E)} = \langle u, a(v)\rangle_{\mathscr{S}(\mathbb{R}^q;F)}, \quad u \in \mathscr{S}(\mathbb{R}^q; F), v \in \mathscr{S}(\mathbb{R}^q; E).$$

where

$$\begin{aligned}\langle\cdot,\cdot\rangle_{\mathscr{S}(\mathbb{R}^q;E)} &: \mathscr{S}(\mathbb{R}^q; E) \times \mathscr{S}(\mathbb{R}^q; F) \to \mathbb{C} \\ &(u, v) \mapsto \langle u, v\rangle_{\mathscr{S}(\mathbb{R}^q;E)} = \int \langle u(x), v(x)\rangle dx.\end{aligned}$$

5.3.2 Wedge Sobolev Spaces

In Example 5.3.2 we have seen that

$$\mathrm{Op}_n(a) \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{\mathbf{s}}(\mathbb{R}), H^{\mathbf{s}-(m,0)}(\mathbb{R})).$$

Moreover, Theorem 5.3.3 states that

$$\mathrm{Op}_{x'}(\mathrm{Op}_n(a)(x', \xi')) : \mathscr{S}(\mathbb{R}^{n-1}, H^{\mathbf{s}}(\mathbb{R})) \to \mathscr{S}(\mathbb{R}^{n-1}, H^{\mathbf{s}-(m,0)}(\mathbb{R})) \tag{5.3.8}$$

is a continuous operator. Actually, by the general theory of pseudodifferential operators one knows that (5.3.8) has a continuous extension from $H^{\mathbf{s}}(\mathbb{R}^n)$ to $H^{\mathbf{s}-(m,0)}(\mathbb{R}^n)$. In order to capture this Sobolev continuity, we introduce adapted Sobolev spaces that, in the standard case, are equivalent to the usual Sobolev spaces. The results of this section mainly come from [94, 96]. About properties of wedge Sobolev spaces see also [45, 95].

Definition 5.3.6. Let E be a Banach space with group action κ. We define $\mathscr{W}^s(\mathbb{R}^q; E)$ as the completion of $\mathscr{S}(\mathbb{R}^q; E)$ with respect to the norm

$$\|u\|^2_{\mathscr{W}^s(\mathbb{R}^q;E)} = \int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}} \hat{u}(\eta)\|^2_E d\eta.$$

If we deal with general locally convex topological vector spaces F with seminorms $\{p_\alpha\}_{\alpha\in I}$, we define $\mathscr{W}^s(\mathbb{R}^q;F)$ as the completion of $\mathscr{S}(\mathbb{R}^q;F)$ w.r.t. the seminorms

$$p^s_\alpha(u)^2 = \int \langle\eta\rangle^{2s} p_\alpha(\kappa_{\langle\eta\rangle^{-1}}\hat{u}(\eta))^2 d\eta.$$

That is: if a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{W}^s(\mathbb{R}^q;F)$ is such that $\{p^s_\alpha(u_n)\}$ is a Cauchy sequence for all α, then there exists $u \in \mathscr{W}^s(\mathbb{R}^q;F)$ such that $p^s_\alpha(u_n) \to p^s_\alpha(u)$, for all α.

It is possible to introduce, as in the standard case, weighted wedge Sobolev spaces

$$\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;E) = \{\langle x\rangle^{-s_2}u \mid u \in \mathscr{W}^{s_2}(\mathbb{R}^q;E)\}.$$

The following properties can be proved by techniques similar to those used in [45, 99].

Proposition 5.3.6. *i) If $\mathbf{s}_1 \geq \mathbf{s}_2$ then $\mathscr{W}^{\mathbf{s}_1}(\mathbb{R}^q;E) \hookrightarrow \mathscr{W}^{\mathbf{s}_2}(\mathbb{R};E)$ is a continuous immersion, if $\mathbf{s}_1 > \mathbf{s}_2$ the immersion in compact. By $\mathbf{s}_1 \geq \mathbf{s}_2$ or $\mathbf{s}_1 > \mathbf{s}_2$ we mean that the inequalities hold for both the components of $\mathbf{s}_1$ and $\mathbf{s}_2$.*

ii) $[\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;E)]^* = \mathscr{W}^{-\mathbf{s}}(\mathbb{R}^q;E^*)$.

iii) If $E \hookrightarrow F$ continuously and $\kappa_E = \kappa_F$ on E then

$$\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;E) \hookrightarrow \mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;F), \quad \forall \mathbf{s} \in \mathbb{R}^2.$$

Let us notice that if $E = \text{ind-lim}_j E_j$ and $F = \text{proj-lim}_j F_j$ then

$$\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;E) = \text{ind-lim}_j \mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;E_j),$$
$$\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;F) = \text{proj-lim}_j \mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;F_j).$$

So, we can define the space

$$\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;\mathscr{S}'(\mathbb{R}^n_+)) = \text{ind-lim}_{(t_1,t_2)\to\infty} \mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;H^{t_1,t_2}_0(\overline{\mathbb{R}_+})),$$
$$\mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;\mathscr{S}(\mathbb{R}^n_+)) = \text{proj-lim}_{(t_1,t_2)\to\infty} \mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;H^{t_1,t_2}(\mathbb{R}_+)).$$

Moreover, notice that the following equality holds:

$$\mathscr{S}(\mathbb{R}^n_+) = \text{proj-lim}_{(s_1,s_2),(t_1,t_2)\to\infty} \mathscr{W}^{\mathbf{s}}(\mathbb{R}^{n-1};H^{\mathbf{t}}(\mathbb{R}_+)),$$
$$\mathscr{S}'(\mathbb{R}^n_+) = \text{ind-lim}_{(s_1,s_2),(t_1,t_2)\to\infty} \mathscr{W}^{\mathbf{s}}(\mathbb{R}^{n-1};H^{\mathbf{t}}_0(\overline{\mathbb{R}_+})).$$

The following Theorem is proved in [99].

Theorem 5.3.7. *Let $a \in S^m(\mathbb{R}^q,\mathbb{R}^q,\mathbb{R}^q;E,F)$, E,F Hilbert spaces. Then, the operator* $\text{Op}(a)$*, defined in Theorem 5.3.3, admits a continuous extension to wedge Sobolev spaces*

$$\text{Op}(a) : \mathscr{W}^{\mathbf{s}}(\mathbb{R}^q;E) \to \mathscr{W}^{s_1-m,s_2}(\mathbb{R}^q;F).$$

Chapter 6

Fourier Integral Operators of Boutet de Monvel Type

In this chapter we consider operators acting on the half-space $\overline{\mathbb{R}^n_+}$. In order to prove continuity properties in the scale of Sobolev spaces, we need some hypotheses on the phase of the FIOs and on the symbol. This local theory will be the first step in order to introduce a global definition of FIOs on manifolds with boundary, starting from a symplectomorphism fulfilling suitable conditions.

6.1 Transmission Condition and Admissible Phase Functions

In this section we will consider FIOs arising from symplectomorphisms of manifolds with boundary. As we have noticed in the Example 4.2.2, the cotangent bundle T^*M of a manifold with boundary $(M, \partial M)$ is a symplectic manifold with boundary $T^*_{\partial M}M$. Let us consider $(M, \partial M)$, $(Z, \partial Z)$, two compact manifolds with boundary and $\chi : T^*M \setminus 0 \to T^*Z \setminus 0$, a symplectomorphism positively homogeneous of order one in the fibers. It is natural to require that the the symplectomorphism preserves the boundary: $\chi(\partial T^*M \setminus 0) = \partial(T^*Z \setminus 0)$. The following lemma, which is proved in [69], analyzes symplectomorphisms of this type.

Lemma 6.1.1. *If $(M, \partial M)$ and $(Z, \partial Z)$ are compact manifolds with boundary and $\chi : T^*M \setminus 0 \to T^*Z \setminus 0$ is a symplectomorphism, positively homogeneous of order 1 in the fibers, such that $\chi(\partial T^*M \setminus 0) = \partial T^*Z \setminus 0$, then χ induces a symplectomorphism $\chi_\partial : T^*\partial M \setminus 0 \to T^*\partial Z \setminus 0$, positively homogeneous of order one in the fibers, such that the following diagram commutes:*

$$\begin{array}{ccc} T^*_{\partial M}M \setminus N^*\partial M & \xrightarrow{\ \chi\ } & T^*_{\partial Z}Z \setminus N^*\partial Z \\ \big\downarrow{\scriptstyle i^*_M} & & \big\downarrow{\scriptstyle i^*_Z} \\ T^*\partial M \setminus 0 & \xrightarrow{\ \chi_\partial\ } & T^*\partial Z \setminus 0 \end{array}$$

Proof. In 4.1.1 we have noticed that $i_M^* : T^*_{\partial M}M \to T^*\partial M$ has a kernel given by the normal bundle $N^*\partial M$. More precisely, it is a Hamiltonian foliation. The leaves are integral curves of a Hamiltonian vector field. In this case, locally, the Hamiltonian vector field is ∂_{ξ_n}, where ξ_n is the dual variable of the normal direction at the boundary. This structure is preserved by the symplectomorphism χ, which sends fibers into fibers. Since $(i^*)^{-1}(i^*(p))$ is connected for all $p \in T^*M_{\partial M} \setminus N^*\partial M$, the diffeomorphism χ_∂ is well defined and turns out to be positively homogeneous of order one in the fibers, since i_M^* and χ have this property. Notice that, if α_M is the fundamental 1-form on T^*M, then $(i_M^*)^*\alpha_M = \alpha_M|_{T^*\partial M}$ and the same property holds for α_Z. Then, we have

$$(i_M^*)^*(\chi_\partial)^*\alpha_{\partial Z} = \chi^*(i_Z^*)^*\alpha_{\partial Z} = \chi^*(\alpha_Z|_{\partial T^*Z}) = \alpha_M|_{\partial T^*M},$$

that is, χ_∂ preserves the fundamental 1-form, so it is a symplectomorphism. □

Remark 6.1.1. *In Lemma 6.1.1 we have considered the induced symplectomorphism χ_∂ outside the zero section. Actually, since χ is smooth on $\partial T^*M \setminus 0$, the induced symplectomorphism χ_∂ is also smooth on the zero section. Since χ_∂ is positively homogeneous of order one in the fibers, the smoothness at the zero section implies that χ_∂ is then trivial in the fibers.*

Property 6.1.2. *Let $(M, \partial M), (Z, \partial Z)$ be smooth manifolds with boundary and $\chi : T^*M \setminus 0 \to T^*Z \setminus 0$ be a symplectomorphism positively homogeneous of order one in the fibers, preserving the boundary. We can consider cylindrical coordinates at the boundaries ∂M and ∂Z: these coordinates induce cylindrical coordinates at ∂T^*M and ∂T^*Z. For all possible choices of such cylindrical coordinates, denoting by $(\chi_{x'}, \chi_{x_n}, \chi_{\xi'}, \chi_{\xi_n})$ the components of χ, we have that the Jacoban matrix of the symplectomorphism at the boundary has the form*

$$J_\chi = \begin{pmatrix} \partial_{x'}\chi_{x'} & \partial_{\xi'}\chi_{x'} & \partial_{x_n}\chi_{x'} & \partial_{\xi_n}\chi_{x'} \\ \partial_{x'}\chi_{\xi'} & \partial_{\xi'}\chi_{\xi'} & \partial_{x_n}\chi_{\xi'} & \partial_{\xi_n}\chi_{\xi'} \\ \partial_{x'}\chi_{x_n} & \partial_{\xi'}\chi_{x_n} & \partial_{x_n}\chi_{x_n} & \partial_{\xi_n}\chi_{x_n} \\ \partial_{x'}\chi_{\xi_n} & \partial_{\xi'}\chi_{\xi_n} & \partial_{x_n}\chi_{\xi_n} & \partial_{\xi_n}\chi_{\xi_n} \end{pmatrix},$$

where

i) $\partial_{x'}\chi_{x_n}, \partial_{\xi'}\chi_{x_n}, \partial_{\xi_n}\chi_{x_n}$ are null vectors,

ii) $\partial_{\xi_n}\chi_{x'}, \partial_{\xi_n}\chi_{\xi'}$ are null vectors.

Proof. Part *i)* follows because the symplectomorphism maps the boundary to the boundary, that is $\chi_{x_n}(x', \xi', 0, \xi_n) = 0$ for all (x', ξ', ξ_n). Part *ii)* follows by Lemma 6.1.1, which implies that the symplectomorphism induced at the boundary depends only on the coordinates at $T^*\partial M$. So the Jacobian has the form

$$J_\chi = \begin{pmatrix} \partial_{x'}\chi_{x'} & \partial_{\xi'}\chi_{x'} & \partial_{x_n}\chi_{x'} & \mathbf{0} \\ \partial_{x'}\chi_{\xi'} & \partial_{\xi'}\chi_{\xi'} & \partial_{x_n}\chi_{\xi'} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \partial_{x_n}\chi_{x_n} & 0 \\ \partial_{x'}\chi_{\xi_n} & \partial_{\xi'}\chi_{\xi_n} & \partial_{x_n}\chi_{\xi_n} & \partial_{\xi_n}\chi_{\xi_n} \end{pmatrix}$$

Let us now recall that the Jacobian matrix of a symplectomorphism always has determinant equal to one. Lemma 6.1.1 implies that the sub-matrix

$$\begin{pmatrix} \partial_{x'}\chi_{x'} & \partial_{\xi'}\chi_{x'} \\ \partial_{x'}\chi_{\xi'} & \partial_{\xi'}\chi_{\xi'} \end{pmatrix}$$

is equal to J_{χ_∂}. □

In order to define a suitable calculus for FIOs on manifolds with boundary, we need to introduce the notion of transmission property, see, e.g., [19, 35, 38, 88, 94]. Consider the function spaces:

$$H^+ = \{\mathscr{F}(e^+u) \mid u \in \mathscr{S}(\mathbb{R}_+)\} \quad \text{and} \quad H_0^- = \{\mathscr{F}(e^-u) \mid u \in \mathscr{S}(\mathbb{R}_+)\}.$$

It is possible to prove that H^+ and H_0^- are spaces of functions decaying of first order at infinity. Moreover, we denote by H' the set of all polynomials in one variable. Then, we define

$$H = H^+ \oplus H_0^- \oplus H', \quad H^- = H_0^- \oplus H', \quad H_0 = H^+ \oplus H_0^-.$$

Definition 6.1.1. Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Then, a has the transmission property at $x_n = y_n = 0$ provided that, for all k, l,

$$\partial_{y_n}^k \partial_{x_n}^l a(x', 0, y', 0, \xi', \langle \xi' \rangle \xi_n) \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes}_\pi H_{\xi_n}.$$

We denote by $S^m_{\mathrm{tr}}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ the subset of symbols with the transmission property.

For symbols positively homogeneous of order m w.r.t. the ξ variable, Definition 6.1.1 is equivalent to

$$\partial_{x_n}^k \partial_{y_n}^l \partial_{\xi'}^\alpha a(x', 0, y', 0, 0, 1) = (-1)^{m-|\alpha|} \partial_{x_n}^k \partial_{y_n}^l \partial_{\xi'}^\alpha a(x', 0, y', 0, 0, -1) \tag{6.1.1}$$

for all $k, l \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-1}$. The above condition is often called symmetry condition: the proof of the equivalence can be found in [88].

Definition 6.1.2 (Admissible symplectomorphism)**.** Let $(M, \partial M)$ and $(Z, \partial Z)$ be compact manifolds with boundary and $\chi : T^*M \setminus 0 \to T^*Z \setminus 0$ a symplectomorphism. We call this symplectomorphism admissible if all the components of χ locally satisfy the transmission condition. This definition has a global meaning, because a change of coordinates in the cotangent bundle is linear w.r.t. the fibers. A phase function that represents an admissible symplectomorphism will be called admissible.

Property 6.1.3. *Let $\psi(x', x_n, \xi', \xi_n)$ be an admissible phase function, locally representing a symplectomorphism χ close to the boundary of M. Then, the phase function at the boundary $x_n = 0$ is linear in the ξ' variable and does not depend on the ξ_n variable.*

Proof. The phase function, in a local chart, represents the symplectomorphism at the boundary, that is the graph of χ is described as

$$(x', x_n, \partial_{x'}\psi, \partial_{x_n}\psi, \partial_{\xi'}\psi, \partial_{\xi_n}\psi, \xi', \xi_n).$$

Since $\partial_{\xi_n}\psi(x', 0, \xi', \xi_n) = 0$ for all x', ξ', ξ_n, we can write

$$\psi_\partial(x', \xi') = \psi(x', 0, \xi', \xi_n), \quad |\xi'| \neq 0.$$

If $|\xi'| = 0$, in view of the non-continuity at $|\xi'| + |\xi_n| = 0$, we cannot, in general, define ψ_∂. Nevertheless, in Remark 6.1.1 we have noticed that the symplectomorphism χ_∂ induced at the boundary is smooth: this implies that the phase function is also smooth, and, since it is positively homogeneous of order one, it turns out to be linear in the fibers. □

6.2 Fourier Integral Operators on the Half-Space

In this section we analyze the continuity properties of FIOs on the half-space $\mathbb{R}^n_+$. We restrict ourselves to FIOs defined by Lagrangian submanifolds obtained from admissible symplectomorphisms as in Definition 6.1.2, and in this case the base manifold is $\mathbb{R}^n_+$. In the following, we will make use of a general statement on FIOs associated to a symplectomorphism, that allows us to consider left and right quantization of the symbol and of the phase, see [49], Ch. 25. Since we are concerned with the behavior of FIOs near the boundary, not with the behavior at infinity,

in the sequel we will always consider
symbols with compact support in the space variable.

The first step is to analyze the action of a FIO with these properties on Dirac's distribution at the origin.

Theorem 6.2.1. *Let $a \in S^m_{tr}(\mathbb{R}^n, \mathbb{R}^n)$ and ψ a phase function that represents, locally at the boundary, an admissible symplectomorphism χ. Then*

$$\begin{aligned} k_j(x', \xi') =& r^+ \mathrm{Op}^{\psi}_n(a)\delta_0^{(j)} \\ =& r^+ \int e^{i\psi(x',x_n,\xi',\xi_n) - i\psi_\partial(x',\xi')} a(x', x_n, \xi', \xi_n)\hat{\delta}_0^{(j)} đ\xi_n \end{aligned}$$

defines an operator-valued symbol in $S^{m+\frac{1}{2}+j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}(\mathbb{R}_+))$. Here, $\delta_0^{(j)}$ is the j-th derivative of the Dirac's distribution at 0, while ψ_∂ is the phase function that defines the symplectomorphism induced on the boundary as in Lemma 6.1.1.

Proof. First we consider the operator $\mathrm{Op}^\psi(a)$ acting on smooth functions defined on the whole of $\mathbb{R}^n$.

$$\begin{aligned} \mathrm{Op}^\psi(a) : C^\infty_c(\mathbb{R}^n) &\to C^\infty(\mathbb{R}^n) \\ u \mapsto & \int e^{i\psi(x',x_n,\xi',\xi_n)} a(x', x_n, \xi', \xi_n)\hat{u}(\xi', \xi_n) đ\xi' đ\xi_n = \\ & \int e^{i\psi_\partial(x',\xi')} \int e^{ir(x',x_n,\xi',\xi_n)} a(x', x_n, \xi', \xi_n)\hat{u}(\xi', \xi_n) đ\xi' đ\xi_n, \end{aligned}$$

where $r(x', x_n, \xi', \xi_n) = \psi(x', x_n, \xi', \xi_n) - \psi_\partial(x', \xi')$. Since ψ represents a canonical symplectomorphism, from the results in [49], Ch.25, it admits a right quantization. Therefore

$$\begin{aligned} &\mathrm{Op}^\psi(a)(\phi \otimes \delta_0)(x_n) \\ &= \int e^{i\psi_\partial(x',\xi')} r^+ \int e^{i\psi(x',x_n,\xi',\xi_n) - \psi_\partial(x',\xi')} a(x', x_n, \xi', \xi_n)\hat{\delta}_0^{(j)} đ\xi_n \hat{\phi}(\xi') đ\xi' \\ &= \int e^{i\psi_\partial(x',\xi')} k_j(x', \xi')\hat{\phi}(\xi') đ\xi' \\ &= \int e^{ix'\cdot\xi' + ix_n\cdot\xi_n - i\psi^{-1}(y',y_n,\xi',\xi_n)} a_R(y', y_n, \xi', \xi_n)\phi(y') \otimes \delta_0(y_n) dy' dy_n đ\xi' đ\xi_n \\ &= \int e^{ix'\cdot\xi' - i\psi_\partial^{-1}(y',\xi')} \int e^{ix_n\cdot\xi_n - ir^{-1}(y',y_n,\xi',\xi_n)} a_R(y', y_n, \xi', \xi_n) \\ &\phi(y') \otimes \delta_0(y_n) dy' dy_n đ\xi' đ\xi_n, \end{aligned} \tag{6.2.2}$$

where the equality is modulo operators with smooth kernel,

$$r^{-1}(y', y_n, \xi', \xi_n) = \psi^{-1}(y', y_n, \xi', \xi_n) - \psi_\partial^{-1}(x', \xi'),$$

and ψ^{-1} is the phase function representing χ^{-1}, inverse of the symplectomorphism χ. Now, we focus on the action in the normal direction, namely, the expression

$$B(y', \xi', x_n)(\delta_0) = \int e^{ix_n\cdot\xi_n} \int e^{-ir^{-1}(y', y_n\xi'\xi_n)} a_R(y', y_n, \xi', \xi_n)\delta_0 dy_n đ\xi_n.$$

By the definition of operators on distributions, we have, for all $u \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
&\langle \kappa_{\langle\xi'\rangle^{-1}} B\delta_0, u\rangle = \langle \delta_0, B^t \circ \left(\kappa_{\langle\xi'\rangle} u\right)\rangle = \\
&\langle \delta_0, \langle\xi'\rangle^{\frac{1}{2}} \int e^{-ir^{-1}(y', x_n, \xi', \xi_n) + iy_n\xi_n} a_R(y', x_n, \xi', \xi_n) u(\langle\xi'\rangle y_n) dy_n đ\xi_n\rangle = \\
&\langle\xi'\rangle^{\frac{1}{2}} \int a_R(y', 0, \xi', \xi_n\langle\xi'\rangle)\hat{u}(-\xi_n)đ\xi_n = \text{ (transmission property)} \\
&\langle\xi'\rangle^{\frac{1}{2}} \sum_{k=0}^{m} s_k^R(y', \xi') \int \xi_n^k \hat{u}(\xi_n)đ\xi_n + \langle\xi'\rangle^{\frac{1}{2}} \sum_{l=0}^{\infty} \lambda_l b_l^R(y', \xi') \int \hat{h}_l(\xi_n)\hat{u}(-\xi_n)đ\xi_n
\end{aligned}$$

where $s_k^R \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$, $\lambda_l \in l^1$, $\{b_l^R\} \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$, is a null sequence, $h_l \in \mathscr{S}(\mathbb{R}_+) \oplus \mathscr{S}(\mathbb{R}_-)$. Using the properties of the Fourier transform,

$$\begin{aligned}
\langle \kappa_{\langle\xi'\rangle^{-1}} B\delta_0, u\rangle = &\langle\xi'\rangle^{\frac{1}{2}} \sum_{k=0}^{m} s_k^R(y', \xi') i^k \delta_0^{(k)}(u) + \\
&\langle\xi'\rangle^{\frac{1}{2}} \sum_{l=0}^{\infty} \lambda_l b_l^R(x', \xi') \int h_l(x_n)u(x_n)dx_n.
\end{aligned} \tag{6.2.3}$$

Applying the restriction operator r^+, all terms that depend on $\delta_0^{(k)}$ vanish, so we get

$$\kappa_{\langle\xi'\rangle^{-1}} r^+ B\delta_0 = \langle\xi'\rangle^{\frac{1}{2}} \sum_{l=0}^{\infty} \lambda_l b_l^R(y', \xi') r^+ h_l(x_n) = B(y', \xi', x_n). \tag{6.2.4}$$

As derivatives w.r.t. (x', ξ') can be treated in the same way, we see that $r^+ B\delta_0\big(y', \xi') \in S^{m+\frac{1}{2}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}(\mathbb{R}_+))$. Now, since $\psi_\partial(x', \xi')$ is a symplectomorphism of the boundary, inserting (6.2.4) into (6.2.2), we obtain

$$\begin{aligned}
&\kappa_{\langle\xi'\rangle^{-1}} r^+ \mathrm{Op}^\psi(a)(\phi \otimes \delta_0) = \int e^{ix'\cdot\xi' - i\psi_\partial^{-1}(y', \xi')} \kappa_{\langle\xi'\rangle^{-1}} r^+ B(y', \xi', x_n)\phi(y')dy' đ\xi' \\
&= \langle\xi'\rangle^{\frac{1}{2}} \sum_{l=0}^{\infty} r^+ h_l(x_n) \int e^{ix'\cdot\xi' - i\psi_\partial^{-1}(y', \xi')} \lambda_l b_l^R(y', \xi')\phi(y')dy' đ\xi'.
\end{aligned}$$

Finally, switching back to the left quantization, we get, modulo smoothing operators,

$$\begin{aligned}
&\kappa_{\langle\xi'\rangle^{-1}} r^+ \int e^{i\psi(x', x_n, \xi', \xi_n) - i\psi_\partial(x', \xi')} a(x', x_n, \xi', \xi_n)\hat{\delta}_0 đ\xi_n \\
&= \langle\xi'\rangle^{\frac{1}{2}} \sum_{l=0}^{\infty} \lambda_l b_l(x', \xi') r^+ h_l(x_n).
\end{aligned}$$

This implies the assertion for $j = 0$, since $r^+h_k \in \mathscr{S}(\mathbb{R}_+)$ and $b_j \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$. The proof for $j > 0$ is similar: it is enough to notice that the phase function has the transmission property, since it is admissible, so we can follow the same steps, but with a symbol of order $m + j$. □

Remark 6.2.1. *We could prove, with a and ψ as in Theorem 6.2.1, that*

$$r^- \int e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')} a(x', x_n, \xi', \xi_n)\hat{\delta}_0^{(j)} đ\xi_n$$

is a symbol in $S^{m+\frac{1}{2}+j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}(\mathbb{R}_-))$.

Remark 6.2.2. *Theorem 6.2.1 gives a precise description of $r^+\mathrm{Op}^\psi(a)\delta_0^{(j)}$. Suppose that $a \in S^m_{tr}(\mathbb{R}^n, \mathbb{R}^n)$ and there exists N such that*

$$\partial^k_{x_n} a(x', 0, \xi', \langle\xi'\rangle\xi_n) \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\hat{\otimes}_\pi H_0,$$

for $k \le N$, that is, the expansion of $\partial^k_{x_n} a$, $k \le N$, at $x_n = 0$ has no polynomial part w.r.t. the ξ_n-variable. Then, $r^+\mathrm{Op}^\psi(a)\delta_0^{(j)}$ is a regular distribution and

$$r^+\mathrm{Op}^\psi(a)\delta_0^{(j)} = \chi_{\mathbb{R}_+}\mathrm{Op}^\psi(a)\delta_0^{(j)}, \quad j = 0, \dots N, \tag{6.2.5}$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}^+$. Relation (6.2.5) *holds also in the case the symbol a vanishes at $x_n = 0$ at least of order $m + j + 1$, because the multiplication makes vanish all the derivatives of the Dirac's distribution appearing in* (6.2.3).

Definition 6.2.1. A function $a \in C^\infty(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}_{x_n} \times \mathbb{R}_{\xi_n})$ belongs to the set $BS^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^l(\mathbb{R}))$ if, for all $\alpha, \beta, \gamma, \delta$, there exists a constant $C_{\alpha,\beta,\gamma,\delta}$ such that

$$|\partial^\alpha_{\xi'}\partial^\beta_{x'}\partial^\gamma_{x_n}\partial^\delta_{\xi_n} a(x', x_n, \xi', \xi_n)| \le C\langle\xi'\rangle^{m-|\alpha|-|\gamma|}\langle\xi_n\rangle^{l-|\delta|}.$$

Clearly, $BS^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^l(\mathbb{R})) \subseteq BS^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^{l'}(\mathbb{R}))$ if $l' \ge l$.

A direct computation implies the following statements:

i) the classes $BS^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^l(\mathbb{R}))$ have a multiplicative property, that is

$$\begin{aligned} BS^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^l(\mathbb{R}))\cdot BS^{m'}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^{l'}(\mathbb{R})) \\ \subseteq BS^{m+m'}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^{l+l'}(\mathbb{R})). \end{aligned} \tag{6.2.6}$$

ii) Let $a \in S^m(\mathbb{R}^n, \mathbb{R}^n)$, then

$$\tilde{a}(x', x_n, \xi', \xi_n) = a\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) \tag{6.2.7}$$

belongs to $BS^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, S^m(\mathbb{R}))$.

Lemma 6.2.2. *Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and ψ be chosen as in Theorem 6.2.1. Then*

$$\begin{aligned} &\partial^\alpha_{\xi'}\partial^\beta_{x'} e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')} a(x', x_n, \xi', \xi_n) \\ &= e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}\tilde{a}(x', x_n, \xi', \xi_n), \end{aligned} \tag{6.2.8}$$

where $\tilde{a}\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \langle\xi'\rangle\xi_n\right) \in BS^{m-|\alpha|}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^{m+|\beta|}(\mathbb{R}))$.

Proof. The assertion is proved by induction. It is trivially true if $|\alpha| = |\beta| = 0$ by (6.2.7). So, let us suppose that (6.2.8) is true for $|\alpha| + |\beta| < t$, $t \in \mathbb{N}$, and show that it holds true for $|\alpha| + |\beta| = t$. If $|\alpha| \neq 0$ we can write

$$\begin{aligned}
&\partial_{\xi'_j}\Big(\partial_{\xi'}^{\alpha-1_j} D_{x'}^{\beta} e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')} a(x',x_n,\xi',\xi_n)\Big) = \text{(by the inductive hypothesis)}\\
=&\partial_{\xi_j}\Big(e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}\tilde{a}(x',x_n,\xi',\xi_n)\Big) =\\
&e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi)}\Big(\partial_{\xi'_j}(i\psi(x',x_n,\xi',\xi_n) - i\psi_\partial(x',\xi'))\tilde{a}(x',x_n,\xi',\xi_n)\\
&+ \partial_{\xi'_j}\tilde{a}(x',x_n,\xi',\xi_n)\Big) =\\
=&e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}\Big(b(x',x_n,\xi',\xi_n)\tilde{a}(x',x_n,\xi',\xi_n)\\
&+ \partial_{\xi'_j}\tilde{a}(x',x_n,\xi',\xi_n)\Big),
\end{aligned} \tag{6.2.9}$$

where

$$b(x',x_n,\xi',\xi_n) = x_n \int_0^1 \partial_{\xi'_j}\partial_{x_n}\psi(x',\theta,\xi',\xi_n)d\theta. \tag{6.2.10}$$

In (6.2.9), we have used Taylor expansion around $x_n = 0$ and the condition $\psi(x',0,\xi',\xi_n) - \psi_\partial(x',\xi') = 0$. The function b in (6.2.10) is the integral remainder. Now, we have to verify that

$$b\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\tilde{a}\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) + (\partial_{\xi'_j}\tilde{a})\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)$$

belongs to $BS^{m-|\alpha|}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, S^{|\beta|}(\mathbb{R}))$. This is true since, by the inductive hypothesis, $\tilde{a}\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) \in BS^{m-|\alpha|-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, S^{|\beta|}(\mathbb{R}))$, $\frac{x_n}{\langle\xi'\rangle} \in BS^{-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, S^0(\mathbb{R}))$, and $\int_0^1 \partial_{\xi'_j}\partial_{x_n}\psi(x',\theta,\xi',\xi_n)d\theta$ is a symbol of order zero, so b in (6.2.10) belongs to $BS^{-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^0(\mathbb{R}))$ by (6.2.7); then, we just apply the multiplicative property (6.2.6). If $|\alpha| = 0$, then we have

$$\begin{aligned}
&\partial_{x'_j}\Big(\partial_{x'}^{\beta-1_j} e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')} a(x',x_n,\xi',\xi_n)\Big) =\\
&\partial_{x'_j}\Big(e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}\tilde{a}(x',x_n,\xi',\xi_n)\Big) = e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}\\
&\Big(c(x',x_n,\xi',\xi_n)\tilde{a}(x',x_n,\xi',\xi_n) + \partial_{x_j}\tilde{a}(x',x_n,\xi',\xi_n)\Big).
\end{aligned}$$

where

$$c(x',x_n,\xi',\xi_n) = x_n \int_0^1 \partial_{x'_j}\partial_{x_n}\psi(x',tx_n,\xi',\xi_n)dt,$$

is the integral remainder of Taylor expansion of $\partial_{x'_j}((\psi(x',x_n,\xi',\xi_n) - \psi_\partial(x',\xi'))$ at $x_n = 0$. Again, by the inductive hypothesis, $\tilde{a}(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle) \in BS^m$ $(\mathbb{R}^{n-1}$, $\mathbb{R}^{n-1}$, $S^{m+|\beta|-1}(\mathbb{R}))$, moreover $c \in BS^0(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; S^1(\mathbb{R}))$, thus, applying the multiplicative property (6.2.6), the assertion is proved. □

Theorem 6.2.3. *Let $a(x', x_n, \xi', \xi_n) \in S^m_{tr}(\mathbb{R}^n, \mathbb{R}^n)$ and ψ represent locally at the boundary an admissible symplectomorphism. Then,*

$$\begin{aligned} r^+\mathrm{Op}_n^\psi(a)e^+ : \mathscr{S}(\mathbb{R}_+) &\to \mathscr{S}(\mathbb{R}_+) \\ u &\mapsto r^+ \iint e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')-iy_n\xi_n} a(x', x_n, \xi', \xi_n) \\ &\quad e^+u(y_n)dy_n đ\xi_n \end{aligned}$$

is an operator-valued symbol in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$. The same property holds for $r^-\mathrm{Op}_n^\psi(a)e^+$ and $r^+\mathrm{Op}_n^\psi(a)e^-$.

Proof. The following argument is a slight modification of the proof of the $\mathscr{S}$-continuity of FIOs with non homogeneous phase in [110], Sec. 1.5. We are interested in the behavior at the boundary, so it is no restriction to consider

$$r^+\omega(x_n) \iint e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')-iy_n\xi_n} a(x', x_n, \xi', \xi_n)\chi(y_n)e^+(u)(y_n)dy_n đ\xi_n, \tag{6.2.11}$$

where ω, χ are cut-off functions near the origin. We consider now another cut-off function ω' such that $\omega\omega' = \omega$: we can rewrite (6.2.11) as

$$r^+ \iint e^{i\psi'(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')-iy_n\xi_n} a'(x', x_n, \xi', \xi_n)\chi(y_n)e^+(u)(y_n)dy_n đ\xi_n,$$

where

$$\begin{aligned} \psi'(x', x_n, \xi', \xi_n) &= \omega'(x_n)\psi(x', x_n, \xi', \xi_n) - [1 - \omega'(x_n)]x_n \cdot \xi_n, \\ a'(x', x_n, \xi', \xi_n) &= a(x', x_n, \xi', \xi_n)\omega(x_n). \end{aligned}$$

We have to prove that, for each choice of l, s, α, β, there exists l', s' such that

$$p_{l,s}\{\kappa_{\langle\xi'\rangle^{-1}} D^\alpha_{\xi'} D^\beta_{x'} r^+\mathrm{Op}_n^\psi(a)e^+\kappa_{\langle\xi'\rangle}u\} \prec \langle\xi'\rangle^{m-|\alpha|} p_{l',s'}(u).$$

where $\{p_{l,s}\}$ are the seminorms of $\mathscr{S}(\mathbb{R}_+)$. Let us start with the case $|\alpha| = |\beta| = l = s = 0$, that is, let us estimate

$$\begin{aligned} \left(\kappa_{\langle\xi'\rangle^{-1}} r^+\mathrm{Op}_n^\psi(a)e^+\kappa_{\langle\xi'\rangle}\right)u(x', \xi') = & r^+ \iint e^{i\psi'(x',\frac{x_n}{\langle\xi'\rangle},\xi',\xi_n\langle\xi'\rangle)-i\psi_\partial(x',\xi')-iy_n\xi_n} \\ & a'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\chi(y_n)e^+(u)(y_n)dy_n đ\xi_n. \end{aligned} \tag{6.2.12}$$

We introduce the operators

$$L_0^t = \frac{1 - i\left(\partial_{\xi_n}\left[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\right] - y_n\right)\partial_{\xi_n}}{1 + |\partial_{\xi_n}\left[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\right] - y_n|^2}$$

and

$$L_1^t = \frac{1}{1 + |\xi_n|^2}\left(1 + i\xi_n\partial_{y_n}\right),$$

which satisfy

$$\begin{aligned} L_0^t e^{i\psi' - i\psi_\partial - iy_n\xi_n} &= e^{i\psi' - i\psi_\partial - iy_n\xi_n}, \\ L_1^t e^{i\psi' - i\psi_\partial - iy_n\xi_n} &= e^{i\psi' - i\psi_\partial - iy_n\xi_n}. \end{aligned}$$

Let us notice now that

$$\left|\partial_{\xi_n}\left[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) - i\psi_\partial(x',\xi')\right] - x_n\right| =$$
$$\left|\omega'\left(\frac{x_n}{\langle\xi'\rangle}\right)\left(\langle\xi'\rangle\partial_{\xi_n}\psi\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) - x_n\right)\right|.$$

By hypothesis, $\partial_{\xi_n}\psi(x',0,\xi',\xi_n) = 0$ for all x',ξ',ξ_n. Hence, we write

$$\left|\omega'\left(\frac{x_n}{\langle\xi'\rangle}\right)\left(\partial_{\xi_n}\psi\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) - x_n\right)\right| =$$
$$\left|\omega'\left(\frac{x_n}{\langle\xi'\rangle}\right)(x_n\partial_{x_n}\partial_{\xi_n}\psi\left(x', \eta, \xi', \xi_n\langle\xi'\rangle\right) - x_n)\right| =$$
$$\left|x_n\left(\left(\omega'(\frac{x_n}{\langle\xi'\rangle}\right)\partial_{x_n}\partial_{\xi_n}\psi(x', \eta, \xi', \xi_n\langle\xi'\rangle) - 1\right)\right|$$

where $\eta \in [0, \frac{x_n}{\langle\xi'\rangle}]$. Clearly $\omega'\left(\frac{x_n}{\langle\xi'\rangle}\right)\partial_{x_n}\partial_{\xi_n}\psi'(x',\eta,\xi',\xi_n\langle\xi'\rangle)$ is bounded, because $\psi \in S^1(\mathbb{R}^n,\mathbb{R}^n)$, and $\omega' \in C_c^\infty(\mathbb{R})$. So, choosing x_n small enough, we can assume that

$$\left|\partial_{\xi_n}\left[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) - \psi_\partial(x',\xi')\right] - x_n\right| \le \tau < 1.$$

Now let us examine $1 + |\partial_{\xi_n}[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)] - y_n|^2$. We have

$$\begin{aligned} &1 + \left|\partial_{\xi_n}\left[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\right] - y_n\right|^2 \\ &\ge \frac{1}{2}\left[1 + \left|\partial_{\xi_n}\left[\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\right] - y_n\right|\right]^2 \\ &\ge \frac{1}{2}\Big[1 + |x_n - y_n| - \tau\Big]^2 \end{aligned}$$

Now, using integration by parts, we can write (6.2.12) as

$$\begin{aligned} &r^+ \iint e^{i\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) - i\psi_\partial(x',\xi') - iy_n\xi_n} a'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) \\ &\chi(y_n)e^+(u)(y_n)dy_n đ\xi_n = \\ &r^+ \iint e^{i\psi'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right) - i\psi_\partial(x',\xi') - iy_n\xi_n} \\ &L_0^{l_0}\left(a'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\right) L_1^{l_1}(\chi(y_n)\, e^+(u)(y_n))dy_n đ\xi_n. \end{aligned} \tag{6.2.13}$$

Let us examine the terms in the last integral of (6.2.13). Since $a' \in S^m(\mathbb{R}^n,\mathbb{R}^n)$, we get

$$\left|L_0^{l_0}a'\left(x', \frac{x_n}{\langle\xi'\rangle}, \xi', \xi_n\langle\xi'\rangle\right)\right| \prec \Big((1-\tau) + |x_n - y_n|\Big)^{-2l_0}\langle\xi'\rangle^m\langle\xi_n\rangle^m \tag{6.2.14}$$

and

$$L_1^{l_1} e^+ u(y) = \langle \xi_n \rangle^{-2l_1} \left\{ \sum_{i=0}^{l_1} c_i \xi_n^i e^+ \partial_{y_n}^i (u) + \sum_{i=0}^{l_1-1} \sum_{k=0}^{i} d_i \xi_n^i \gamma_k(u) \delta_0^{(i-1)} \right\}, \tag{6.2.15}$$

where c_i, d_i are constants. We split (6.2.15) into two parts: $(L_1^{l_1} e^+ u)_s$, which contains the terms with no Dirac's distributions involved, and $(L_1^{l_1} e^+ u)_d$, which contains all the others. Now, (6.2.13) turns into

$$\begin{aligned} r^+ \iint & e^{i\psi'(x', \frac{x_n}{\langle \xi' \rangle}, \xi', \xi_n \langle \xi' \rangle) - i\psi_\partial(x', \xi') - iy_n \xi_n} \\ & L_0^{l_0} a' \left[(L_1^{l_1} e^+ u)_s + (L_1^{l_1} e^+ u))_d \right] dy_n đ\xi. \end{aligned} \tag{6.2.16}$$

In order to get the desired inequality for the integral containing $L_0^{l_0} a' (L_1^{l_1} e^+ u)_s$ we just impose l_0, l_1 large enough, so that one can evaluate the integral using (6.2.14) and (6.2.15), exactly as in [110], p. 66. For the term depending on Dirac's distribution, we notice that $L_0^{l_0} a'$ is still a symbol with the transmission property of order m and $\langle \xi_n \rangle^{-2}$ satisfy the transmission property: then, using the properties of trace operators in Example 5.3.1 and Theorem 6.2.1, we get the assertion for $|\alpha| = |\beta| = m = n = 0$.

To prove the general case it is enough to apply Lemma 6.2.2, that shows

$$\begin{aligned} & \partial_{x'}^\beta \partial_{\xi'}^\alpha e^{i\psi(x', x_n, \xi', \xi_n) - i\psi_\partial(x', \xi')} a(x', x_n, \xi', \xi_n) = \\ & e^{i\psi(x', x_n, \xi', \xi_n) - i\psi_\partial(x', \xi')} \tilde{a}(x', x_n, \xi', \xi_n), \end{aligned} \tag{6.2.17}$$

$\tilde{a}\left(x', \frac{x_n}{\langle \xi' \rangle}, \xi', \xi_n \langle \xi' \rangle\right) \in BS^{m-|\alpha|}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, S^{m+|\beta|}(\mathbb{R}))$. Now, by (6.2.17), we can prove Theorem 6.2.1 with $\tilde{a}$ instead of a and then repeat the same scheme we have used above to get the desired inequality. By Remark 6.2.1 it is clear that the result of the Theorem also holds for $r^+ \mathrm{Op}_n^\psi(a) e^-$ and $r^- \mathrm{Op}_n^\psi(a) e^+$. □

Note that, as it can be seen by the proof of Theorem 6.2.3, if we derive w.r.t. the x' variable, the operator in the normal direction can increase the order: this is the reason why, in this setting, it is not possible to prove that

$$\begin{aligned} & r^+ \mathrm{Op}_n^\psi(a) e^+ : \mathscr{S}(\mathbb{R}_+) \to \mathscr{S}(\mathbb{R}_+) \\ & u \to r^+ \iint e^{i\psi(x', x_n, \xi', \xi_n) - i\psi_\partial(x', \xi') - iy_n \xi_n} a(x', x_n, \xi', \xi_n) e^+ u(y_n) dy_n đ\xi_n \end{aligned}$$

is a symbol in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-m}(\mathbb{R}_+))$, a being a symbol of order m. This can be seen explicitly, through the following counterexample:

$$\begin{aligned} & A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) \\ & \quad u \mapsto \iint e^{i[(x'-y') \cdot \eta' + (f(x') x_n - y_n) \cdot \eta_n]} u(y', y_n) dy' dy_n đ\eta' đ\xi_n, \end{aligned} \tag{6.2.18}$$

where f is a strictly positive function. The phase function of the FIO A in (6.2.18) represents a symplectomorphism χ of $T^*\mathbb{R}_+^n = \mathbb{R}_+^{2n}$ in itself of the form

$$\begin{aligned} & \chi : \mathbb{R}_+^{2n} \to \mathbb{R}_+^{2n} \\ & \quad (x', x_n, \xi', \xi_n) \mapsto \left(x', f(x') x_n, \xi' - f'(x') x_n \frac{\xi_n}{f(x')}, \frac{\xi_n}{f(x')} \right). \end{aligned} \tag{6.2.19}$$

Indeed, setting $x = (x', x_n)$, $y = (y', y_n)$ and $\eta = (\eta', \eta_n)$ the phase function of A turns out to be

$$\phi(x, y, \eta) = (x' - y') \cdot \eta' + (f(x')x_n - y_n) \cdot \eta_n.$$

Notice that $\phi'_\eta(x, y, \eta) = 0$ implies $x' = y'$ and $f(x')x_n = y_n$, so ϕ parametrize the Lagrangian submanifold

$$C'_\phi = (x', x_n, \eta' + f'(x')x_n\eta_n, f(x')\eta_n, x', f(x')x_n, -\eta', -\eta_n).$$

Thus, C'_ϕ correspond to the canonical relation

$$\begin{aligned}
C_\phi =& (x', x_n, \eta' + f'(x')x_n\eta_n, f(x')\eta_n, x', f(x')x_n, \eta', \eta_n) \\
=& \mathrm{graph}(\chi) = \left(x', x_n, \xi', \xi_n, x', f(x')x_n, \xi' - f'(x')x_n\frac{\xi_n}{f(x')}, \frac{\xi_n}{f(x')}\right).
\end{aligned}$$

The symplectomorphism χ is admissible since preserves the boundary:

$$\chi(x', x_n, \xi', \xi_n) \in \partial\mathbb{R}^n_+ \Leftrightarrow (x', x_n, \xi', \xi_n) \in \mathbb{R}^n_+, \text{ that is } x_n = 0,$$

and it is linear in the fibers, therefore all components have the transmission property. Looking at the action along the normal direction, we see that (6.2.18) cannot be an operator-valued symbol in $S^0(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^s(\mathbb{R}_+))$. Indeed,

$$\begin{aligned}
&\kappa_{\langle\eta'\rangle^{-1}}\partial_{x'_j} \iint e^{i(f(x')x_n - y_n)\cdot\eta_n}\kappa_{\langle\eta'\rangle}u(y_n)dy_n đ\xi_n \\
&= \kappa_{\langle\eta'\rangle^{-1}}\left(\langle\eta'\rangle^{\frac{1}{2}} \iint e^{i(f(x')x_n - y_n)\cdot\eta_n} i(\partial_{x'_j} f)(x')x_n\eta_n u(\langle\eta'\rangle y_n)dy_n đ\eta_n\right) \\
&= \kappa_{\langle\eta'\rangle^{-1}}\left(\langle\eta'\rangle^{-\frac{1}{2}} \iint e^{i(\langle\eta'\rangle f(x')x_n - z_n)\cdot\frac{\eta_n}{\langle\eta'\rangle}} i(\partial_{x'_j} f)(x')x_n\eta_n u(z_n)dy_n đ\eta_n\right) \\
&= \kappa_{\langle\eta'\rangle^{-1}}\left(\langle\eta'\rangle^{-\frac{1}{2}} \iint e^{if(x')x_n\cdot\eta_n} i(\partial_{x'_j} f)(x')x_n\eta_n \hat{u}\left(\frac{\eta_n}{\langle\eta'\rangle}\right) đ\eta_n\right) \\
&= \kappa_{\langle\eta'\rangle^{-1}}\left(\langle\eta'\rangle^{\frac{3}{2}} \iint e^{i\langle\eta'\rangle f(x')x_n\cdot\theta_n} i(\partial_{x'_j} f)(x')x_n\theta_n \hat{u}(\theta_n)đ\theta_n\right) \\
&= \kappa_{\langle\eta'\rangle^{-1}}\left(\langle\eta'\rangle^{\frac{3}{2}} \iint e^{i\langle\eta'\rangle f(x')x_n\cdot\theta_n} (\partial_{x'_j} f)(x')x_n\widehat{\partial_{y_n} u}(\theta_n)đ\theta_n\right) \\
&= \kappa_{\langle\eta'\rangle^{-1}}\left(\langle\eta'\rangle^{\frac{3}{2}}(\partial_{x'_j} f)(x')x_n\partial_{x_n}u(\langle\eta'\rangle f(x')x_n)\right) \\
&= (\partial_{x'_j} f(x'))x_n\partial_{x_n}u(f(x')x_n).
\end{aligned}$$

Now, we recall a technical lemma, proved in [88], p. 122.

Lemma 6.2.4. *Let $a \in S^m(\mathbb{R}^n, \mathbb{R}^n)$ be a symbol with the transmission property. Then, there exists a symbol $a_1 \in S^m(\mathbb{R}^n, \mathbb{R}^n)$ having the transmission property for all hyperplanes $x_n = \epsilon$, $\epsilon \geq 0$, such that*

$$\partial^k_{x_n}(a(x, x_n, \xi', \xi_n) - a_1(x', x_n, \xi', \xi_n))|_{x_n=0} = 0$$

for all $k \in \mathbb{N}$ and for all x', ξ', ξ_n.

Proof. We set

$$a_1(x', x_n, \xi', \xi_n) = \sum_{j=0}^{\infty} \frac{x_n^j}{j!} \partial_{x_n}^j a(x', 0, \xi', \xi_n)\phi(t_j x_n) \tag{6.2.20}$$

where ϕ is a cut-off function at the origin and $\{t_j\}$ is a sequence such that the series in (6.2.20) converges in $S^m(\mathbb{R}^n, \mathbb{R}^n)$. Clearly, choosing a_1 as in (6.2.20), we get that $a - a_1$ vanishes of infinite order at $x_n = 0$. The symbol a_1 has then the transmission property for all hyperplanes $x_n = \epsilon$, $\epsilon > 0$, since a has it w.r.t. to $x_n = 0$. □

Proposition 6.2.5. *Let a and ψ be as in Theorem 6.2.1. Then, for all α and β it is possible to write*

$$r^+\partial_{x'}^\beta \partial_{\xi'}^\alpha \mathrm{Op}_n^\psi(a)e^+ = r^+\mathrm{Op}_n^\psi(a_d)e^+ + r^+\mathrm{Op}_n^\psi(a_0)e^+$$

with a_0 such that

$$(\partial_{x'}^\beta \partial_{\xi'}^\alpha \mathrm{Op}_n^\psi(a_0)e^+)f \in L^2(\mathbb{R}), \tag{6.2.21}$$

and $a_d \in S_{tr}^m(\mathbb{R}^n, \mathbb{R}^n)$ is a polynomial in ξ.

Proof. The proof follows from Remark 6.2.2 and an observation of the proof of Theorem 6.2.3. First we consider $|\alpha| = |\beta| = 0$. We prove that

$$(\partial_{x'}^\beta \partial_{\xi'}^\alpha \mathrm{Op}_n^\psi(a_0)e^+)\, f \in \mathscr{S}(\mathbb{R}_+) \quad f \in \mathscr{S}(\mathbb{R}_+), \tag{6.2.22}$$

where we consider $\mathscr{S}(\mathbb{R}_+)$ as a subset of $L^2(\mathbb{R}_+)$. Let us consider a, a_1 as in Lemma 6.2.4 and set $b = a - a_1$. In view of the transmission property of a, we can write

$$\partial_{x_n}^j a(x', 0, \xi', \xi_n) = \sum_{k=0}^{m} a_{k,j}(x', \xi')\xi_n^k + \sum_{k=0}^{\infty} \lambda_{k,j} b_{k,j}(x', \xi') h_{k,j}\left(\frac{\xi_n}{\langle \xi' \rangle}\right),$$

where the $a_{k,j} \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ are polynomials in ξ'. Then, we set

$$\begin{aligned}
a_d(x', x_n, \xi', \xi_n) &= \sum_{j=0}^{\infty} \sum_{k=0}^{m} x_n^j a_{k,j}(x', \xi')\xi_n^k \phi(t_j x_n), \\
a_0^1(x', x_n, \xi', \xi_n) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x_n^j b_{k,j}(x', \xi') h_{k,j}\left(\frac{\xi_n}{\langle \xi' \rangle}\right)\phi(t_j x_n), \\
a_0 &= a_0^1 + b.
\end{aligned}$$

By construction, it is clear that $a = a_d + a_0$. Notice that a_0^1 has no polynomial part w.r.t. the ξ_n variable, while b vanishes to infinite order at $x_n = 0$. In view of Remark 6.2.2, we have that

$$e^+r^+\mathrm{Op}^\psi(a_0^1)\delta_0 = \chi_{\mathbb{R}_+}\mathrm{Op}^\psi(a_0^1)\delta_0 \tag{6.2.23}$$

$$e^+r^+\mathrm{Op}^\psi(b)\delta_0^{(j)} = \chi_{\mathbb{R}_+}\mathrm{Op}^\psi(b)\delta_0^{(j)}, \quad \forall j \in \mathbb{N}, \tag{6.2.24}$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$. Let us consider relations (6.2.14) and (6.2.15). If we replace there a' witha_0^1, with the notation of (6.2.14), we get that $L_0^{l_0}(a_0^1)$ has no polynomial part w.r.t. the ξ_n. Similarly, using notation from (6.2.16), $(L_0^{l_0}(a_0^1)L_1^{l_1}e^+u)_d$ equals a sum of derivatives of Dirac's distribution up to the order $l_1 - 1$. Each $\delta_0^{(j)}$ is associated with a symbol of type $L_0^{l_0}(a_0^1)\frac{\xi_n^{j+1}}{\langle\xi_n\rangle^{2(j+1)}}$ (that is, a symbol vanishing at infinity at least of order $j+2$). (6.2.22) then follows by Remark 6.2.2, since the singular terms vanish, by the properties of a_0^1. Now, we have to check (6.2.22) for $|\alpha|+|\beta|>0$. Performing the derivatives, we obtain symbols $\tilde{a}_0^1$ and $\tilde{b}$ and $\tilde{b}$ still vanishes of infinite order at the origin. The symbol $\tilde{a}_0^1$ can have, in general, polynomial parts up to order $|\beta|-1$. Notice, nevertheless, that

$$\partial_{x'}(\psi(x',x_n,\xi',\xi_n)-\psi_\partial(x',\xi'))$$

vanishes of the first order at x_n. So $\tilde{a}_0^1$ vanishes at x_n of order $|\beta|$, and we can repeat the same scheme as above. □

Remark 6.2.3. *We have proved that $r^+\mathrm{Op}_n^\psi(a)e^+$ is a continuous operator from $\mathscr{S}(\mathbb{R}_+)$ to itself, so it is possible to define the transposed operator*

$$(r^+\mathrm{Op}_n^\psi(a)e^+)^t.$$

It is important to stress that, in general,

$$(r^+\mathrm{Op}_n^\psi(a)e^+)^t f \neq r^+(\mathrm{Op}_n^\psi(a))^t e^+ f, \quad f\in\mathscr{S}(\mathbb{R}_+). \tag{6.2.25}$$

A simple counterexample is the operator $(r^+\partial e^+)^t$. It is immediate that

$$(r^+\partial e^+)^t f = -r^+\partial e^+ + f(0)\delta_0.$$

Nevertheless, if (6.2.21) *is satisfied, then* (6.2.25) *is true. Indeed, we have*

$$\begin{aligned}&\langle (r^+\mathrm{Op}_n^\psi(a)e^+)^t f, u\rangle = \langle f, r^+\mathrm{Op}_n^\psi(a)e^+u\rangle = \\ &\langle f, \chi_{\mathbb{R}_+}\mathrm{Op}_n^\psi(a)e^+u\rangle = \langle \chi_{\mathbb{R}_+}f, \mathrm{Op}_n^\psi(a)e^+u\rangle = \\ &\langle \mathrm{Op}_n^\psi(a)^t e^+ f, e^+u\rangle.\end{aligned}$$

Moreover, if (6.2.21) *is satisfied, since $C_c^\infty(\mathbb{R}_+)$ is dense in $\mathscr{S}'(\mathbb{R}_+)$, we have*

$$(r^+\mathrm{Op}_n^\psi e^+)u = \lim_{k\to\infty} r^+\mathrm{Op}_n^\psi(a)e^+\phi_k,$$

$\phi_k \to u$ in $\mathscr{S}'(\mathbb{R}_+)$. As a consequence of the $\mathscr{S}(\mathbb{R}_+)$ continuity, we obtain as well that $r^+\mathrm{Op}_n^\psi(a_0)e^+$ is an element of $S^m(\mathbb{R}^{n-1},\mathbb{R}^{n-1};\mathscr{S}'(\mathbb{R}_+),\mathscr{S}'(\mathbb{R}_+))$.

Lemma 6.2.6. *Let $f\in\mathscr{S}'(\mathbb{R})$ be a distribution such that r^+f is a C^∞ function in the open set $\mathbb{R}_+$ with a $\mathscr{S}$ behavior at $+\infty$. Explicitly, $f\chi\in\mathscr{S}(\mathbb{R})$, where χ is a smooth function that vanishes in $(-\infty,\epsilon)$ and equals one in $(2\epsilon,+\infty)$. Then, the following statements are equivalent:*

i) for all $j\in\mathbb{N}$

$$\lim_{x\to 0^+}\partial^j f(x) = c^j, \quad c^j\in\mathbb{C},$$

that is, the function f can be extended as a smooth function in a neighborhood of zero.

ii) for all $j \in \mathbb{N}$ and for all sequences $\{\psi_m^j\}_{m\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}_+)$ such that

$$\psi_m^j \to (-1)^j \delta_0^{(j)} \quad \text{in } \mathscr{S}'(\mathbb{R}), \tag{6.2.26}$$

we have

$$\lim_{m\to\infty} \langle f, \psi_m^j \rangle = c^j, \quad c^j \in \mathbb{C}.$$

There is a trivial continuous inclusion $i : C_c^\infty(\mathbb{R}_+) \to C_c^\infty(\mathbb{R})$ given by the extension by zero, so the limit (6.2.26) is well defined.

Proof.

i)⇒ ii) If f can be extended as a smooth function in a neighborhood of the origin, we can choose an extension $\tilde{f} \in \mathscr{S}(\mathbb{R})$. Then we have, for all j and for all m, and for all $\{\psi_m^j\}_{m\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}_+)$,

$$\langle (\tilde{f} - f), \psi_m^j \rangle = 0 \tag{6.2.27}$$

since $\operatorname{supp}(\tilde{f} - f) \subseteq \overline{\mathbb{R}_-}$ and $\operatorname{supp} \psi_m^j \subset \mathbb{R}_+$. Equality (6.2.27) implies

$$\langle \tilde{f}, \psi_m^j \rangle = \langle f, \psi_m^j \rangle = c_m^j \quad \forall j, m, \tag{6.2.28}$$

so that

$$c^j = \lim_{m\to\infty} c_m^j = \lim_{m\to\infty} \langle \tilde{f}, \psi_m^j \rangle = \langle \tilde{f}, \lim_{m\to\infty} \psi_m^j \rangle = (-1)^j \delta_0^{(j)} \tilde{f}.$$

This gives the desired result, observing that

$$(-1)^j \delta_0^{(j)} \tilde{f} = \lim_{x\to 0^+} \partial^j \tilde{f}(x) = \lim_{x\to 0^+} \partial^j f(x).$$

ii) ⇒ i) Conversely, let us suppose that condition *ii)* holds but condition *i)* is not fulfilled. So, there exists a $j = 0, 1, \ldots,$ such that the limit

$$\lim_{x\to 0^+} \partial^j f(x)$$

is not c^j. This means that there exists $\bar{\epsilon}$ such that for all $m \in \mathbb{N}$ there exists $x_m \in (0, \frac{1}{m})$ such that

$$|c^j - \partial^j f(x_m)| > \bar{\epsilon}. \tag{6.2.29}$$

We can suppose that f is real valued: in fact, if it is not the case, then either its real or imaginary part satisfy (6.2.29). Since f is smooth in $\mathbb{R}_+$, there exists a neighborhood U_m of x_m such that (6.2.29) holds for all $x \in U_m$. We can suppose that U_m is balanced and we call r_m its radius. In order to simplify the notation set $r_0 = 1, x_0 = 0$. Now, let us consider the sequence

$$\psi_m = \begin{cases} r_m^{-1} a_m \exp\left(\left(1 - |\frac{x - x_m}{r_m}|^2\right)^{-1}\right) & x_m - r_m < x < x_m + r_m \\ 0 & \text{otherwise} \end{cases}$$

where the constant a_m is chosen so that $\int \psi_m = 1$. Then $\psi_m \to \delta_0$ in $\mathscr{S}'(\mathbb{R})$. We can write

$$\begin{aligned}
|\langle f, (-1)^j \psi_m^{(j)} \rangle - c^j| &= \left| \int_{U_m} \partial^j f(x) \psi_m(x) dx - c^j \right| \\
&= \left| \int_{U_m} (\partial^j f(x) - c^j) \psi_n dx \right| \geq \inf_{U_m} |\partial^j f(x) - c^j| \cdot \left| \int \psi_m dx \right| \\
&= \inf_{U_m} |\partial^j f(x) - c^j|.
\end{aligned}$$

By the definition of the sets U_m, we have $\inf_{U_m} |f^j(x) - c_j| > \bar{\epsilon}$, so finally we find

$$|\langle f, \psi_m^{(j)} \rangle - c^j| \geq \bar{\epsilon} \quad \forall m \in \mathbb{N},$$

that is

$$\lim_{n\to\infty} \langle f, \psi_m^{(j)} \rangle \neq c^j,$$

and we get the contradiction.

□

Theorem 6.2.7. *Let ψ and a be as in Theorem 6.2.3. By Proposition 6.2.5 we can write $a = a_d + a_0$. Then, $\mathrm{Op}_n^\psi(a_0)e^+$ maps $\mathscr{S}(\mathbb{R}_+)$ to $L^2(\mathbb{R})$. Hence $(e^+r^+ - 1)\mathrm{Op}_n^\psi(a_0)e^+ = -e^-r^-\mathrm{Op}_n^\psi(a_0)e^+$. Moreover, $r^-\mathrm{Op}_n^\psi(a_0)e^+$ extends to an operator*

$$r^-\mathrm{Op}_n^\psi(a_0)e^+ : \mathscr{S}'(\mathbb{R}_+) \to \mathscr{S}(\mathbb{R}_-)$$

and defines a symbol in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_-))$.

Proof. In Proposition 6.2.5 we have noticed that $e^+r^+\mathrm{Op}_n^\psi(a_0)e^+ = \chi_{\mathbb{R}_+}\mathrm{Op}_n^\psi(a_0)e^+$, $\chi_{\mathbb{R}_+}$ being the characteristic function of $\mathbb{R}_+$. So, we can write

$$(e^+r^+ - 1)\mathrm{Op}_n^\psi(a_0)e^+u = (\chi_{\mathbb{R}_+} - 1)\mathrm{Op}_n^\psi(a_0)e^+u = -e^-\mathrm{Op}_n^\psi(a_0)e^+u, \quad u \in \mathscr{S}(\mathbb{R}_+). \tag{6.2.30}$$

Since, for every $u \in \mathscr{S}'(\mathbb{R}_+)$, $(r^+\mathrm{Op}_n^\psi(a_0)e^+)u = \lim_{m\to\infty} r^+\mathrm{Op}_n^\psi(a_0)e^+\phi_m$, $\{\phi_m\} \subseteq C_c^\infty(\mathbb{R}_+)$, such that $\phi_m \to u$ in $\mathscr{S}'(\mathbb{R}_+)$, we restrict ourselves to functions in $C_c^\infty(\mathbb{R}_+)$. We want to prove that, for all α, β, there exist s_1, s_2 such that

$$p_{\alpha,\beta}(r^-\mathrm{Op}_n^\psi(a_0)e^+u) \prec \langle \xi' \rangle^m \|u\|_{H^{s_1,s_2}}, \quad u \in C_c^\infty(\mathbb{R}_+),$$

$\{p_{\alpha,\beta}\}$ being the seminorms of $\mathscr{S}(\mathbb{R}_-)$. Notice that $\partial_{\xi_n}\psi(x', x_n, \xi', \xi_n)$ is negative for all x_n negative. Hence, if $x_n < -\epsilon < 0$, the phase function has no critical points, an integration by parts arguments implies that singsupp $(r^-\mathrm{Op}_n^\psi e^+(a_0)u) \subseteq \{0\}$, and we get as well that

$$\sup_{x_n < -\epsilon} |(1 + x_n^2)^{\frac{\beta}{2}} \partial_{x_n}^\alpha (e^+r^+ - 1)\mathrm{Op}_n^\psi(a_0)u| \prec \langle \xi' \rangle^m \|u\|_{H^{s_1,s_2}}.$$

Notice that we have used the fact that the symbol has compact support w.r.t. the space variable x. Now, we need to consider only the behavior when x_n approaches the origin.

From Theorem 6.2.3 and Proposition 6.2.5, we notice that the following maps are continuous:

$$r^+\mathrm{Op}_n^{\psi}(a_0)e^+ : \mathscr{S}(\mathbb{R}_+) \to \mathscr{S}(\mathbb{R}_+) \tag{6.2.31}$$

and

$$r^-\mathrm{Op}_n^{\psi}(a_0)e^+ : \mathscr{S}'(\mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}_-). \tag{6.2.32}$$

In order to prove that $(e^+r^+ - 1)\mathrm{Op}_n^{\psi}(a_0)e^+$ belongs to the set $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_-))$, we have to analyze

$$\lim_{x_n \to 0-} \kappa_{\langle \xi' \rangle^{-1}}\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\partial_{x_n}^{k}(r^-\mathrm{Op}_n^{\psi}(a_0)e^+\Big)\kappa_{\langle \xi' \rangle}(u)(x_n). \tag{6.2.33}$$

We recall that $\mathscr{S}'(\mathbb{R}_-)$ is endowed with the topology of the inductive limit w.r.t. the inductive set $\{H_0^{s_1,s_2}(\overline{\mathbb{R}_-})\}$, so, by definition, we have to prove that, for all s_1, s_2,

$$|\lim_{x_n \to 0-} \kappa_{\langle \xi' \rangle^{-1}}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\partial_{x_n}^{k}(r^-\mathrm{Op}_n^{\psi}(a_0)e^+\kappa_{\langle \xi' \rangle}u)(x_n)| < \langle \xi' \rangle^{m-|\alpha|}\|u\|_{H^{s_1,s_2}(\mathbb{R})},$$

for u $\in C_c^{\infty}(\mathbb{R}_+)$. Using the idea of Lemma 6.2.6, we do not focus on

$$\lim_{x_n \to 0^-} \kappa_{\langle \xi' \rangle^{-1}}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\partial_{x_n}^{k}r^-\mathrm{Op}_n^{\psi}(a_0)e^+\kappa_{\langle \xi' \rangle}u(x_n),$$

but, rather on

$$\langle \kappa_{\langle \xi' \rangle^{-1}}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}r^-\mathrm{Op}_m^{\psi}(a_0)e^+\kappa_{\langle \xi' \rangle}u, (-1)^k\partial_{x_n}^{k}\psi_m\rangle,$$

where $\{\psi_m\}_{m\in\mathbb{N}} \in C_c^{\infty}(\mathbb{R}_-)$ is a sequence such that

$$e^-\psi_m \to \delta_0, \quad \text{in } \mathscr{S}'(\mathbb{R}). \tag{6.2.34}$$

Notice that (6.2.34) implies that $\kappa_{\langle \xi' \rangle}\psi_m$ converges to $\langle \xi' \rangle^{-\frac{1}{2}}\delta_0$. By definition we have

$$\begin{aligned}&\langle \psi_m, \kappa_{\langle \xi' \rangle^{-1}}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}r^-\mathrm{Op}_n^{\psi}(a_0)\kappa_{\langle \xi' \rangle}e^+u\rangle = \\ &\langle \kappa_{\langle \xi' \rangle^{-1}}r^+\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)e^-\psi_m, u\rangle\end{aligned}$$

This equality holds in view of Remark 6.2.3 and Proposition 6.2.5. By (6.2.34) and (6.2.32) we get

$$\begin{aligned}&\lim_{m\to\infty}\langle \kappa_{\langle \xi' \rangle^{-1}}r^+\Big(\partial_{\xi'}^{\beta}\partial_{x'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)e^-\psi_m, u\rangle = \\ &\quad\langle \xi' \rangle^{-\frac{1}{2}}\langle \kappa_{\langle \xi' \rangle^{-1}}r^-\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)\delta_0, u\rangle.\end{aligned}$$

By Theorem 6.2.1 we know that

$$\kappa_{\langle \xi' \rangle^{-1}}r^+\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)\delta_0 \in S^{m-|\alpha|}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}(\mathbb{R}_+)),$$

so, finally

$$\begin{aligned}&\lim_{m\to\infty}|\langle \kappa_{\langle \xi' \rangle^{-1}}r^+\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)e^-\psi_m, u\rangle| \\ &\le \|\kappa_{\langle \xi' \rangle^{-1}}r^+\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)\delta_0\|_{H^{s_1,s_2}(\mathbb{R}_+)}\|u\|_{H^{-s_1,-s_2}(\mathbb{R})} \\ &\le Cp_{\delta,\rho}(\kappa_{\langle \xi' \rangle^{-1}}r^+\Big(\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\mathrm{Op}_n^{\psi}(a_0)^t\kappa_{\langle \xi' \rangle}\Big)\delta_0)\|u\|_{H^{-s_1,-s_2}} \\ &\le C_a\langle \xi' \rangle^{m-|\alpha|}\|u\|_{H^{-s_1,-s_2}(\mathbb{R})}.\end{aligned}$$

where we have used the continuous immersion of $\mathscr{S}(\mathbb{R}_+)$ in $H^{s_1,s_2}(\mathbb{R}_+)$, and $p_{\delta,\rho}$ is a suitable seminorm of $\mathscr{S}(\mathbb{R}_+)$ such that $\|u\|_{H^{s,s_2}(\mathbb{R}_+)} \leq C p_{\delta,\rho}(u)$, for a certain $C \in \mathbb{R}$. $\square$

Theorem 6.2.8. *Let $a \in S^m_{tr}(\mathbb{R}^n, \mathbb{R}^n)$, and ψ an admissible phase function. Then,*

$$r^+ \mathrm{Op}^{\psi}_n(a) e^+ : H^s(\mathbb{R}^n_+) \to H^{s-m}(\mathbb{R}^n_+), \quad s > -\frac{1}{2},$$

continuously.

Proof. First notice that if $s \leq 0$ the result follows from the continuity of e^+ and the properties of FIOs with homogeneous phase. If $s > 0$, using interpolation we may assume $s \in \mathbb{N}$, we notice that

$$r^+ \mathrm{Op}^{\psi}(a) e^+ = r^+ \mathrm{Op}^{\psi}(a) e^+ \circ \Lambda^{-s}_+ \circ \Lambda^s_+,$$

where $\Lambda^s_+ = r^+ \Lambda^s e^+$ are pseudodifferential operators in the sense of Boutet de Monvel such that $\Lambda^s_+ : H^s(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+)$ is continuous and invertible. Denote by Λ^{-s}_+ the inverse of Λ^s_+. Since $\Lambda^s_+ : H^s(\mathbb{R}^n_+) \to L^2(\mathbb{R}_+)$ is continuous, we have just to prove that $r^+ \mathrm{Op}^{\psi}(a) e^+ \circ \Lambda^{-s}_+ : L^2(\mathbb{R}^n_+) \to H^{s-m}(\mathbb{R}^n_+)$ is continuous. We observe that

$$r^+ \mathrm{Op}^{\psi}(a) e^+ \circ r^+ \Lambda^{-s} e^+ = r^+ \mathrm{Op}^{\psi}(a) \circ \Lambda^{-s} e^+ - r^+ \mathrm{Op}^{\psi}(a)(e^+ r^+ - 1) \Lambda^{-s} e^+. \quad (6.2.35)$$

The operator $\mathrm{Op}^{\psi}(a) \circ \Lambda^{-s}$, by the properties of FIOs is, modulo operators with smoothing kernel, a FIO of order $m - s$ with phase ψ. Thus, $r^+ \mathrm{Op}^{\psi}(a) \circ \Lambda^{-s} e^+ : L^2(\mathbb{R}^n_+) \to H^{s-m}(\mathbb{R}^n_+)$ is continuous, since e^+ is continuous on L^2. We have now to analyze the second term of (6.2.35). We treat it as a FIO defined on the boundary with operator-valued symbol. Notice that Λ^{-s} is of negative order, and the differential part of the decomposition in 6.2.5 vanishes, so

$$r^+ \mathrm{Op}^{\psi}_n(a)(e^+ r^+ - 1) \Lambda^{-s} e^+ u = -r^+ \mathrm{Op}^{\psi}_n(a) e^- r^- \Lambda^{-s} e^+ u, \quad u \in C^{\infty}_c(\mathbb{R}_+).$$

Applying Theorem 6.2.7, or using the general theory of Boutet de Monvel calculus, we obtain that $r^- \Lambda^{-s}_+ e^+$ extends to a symbol belonging to $S^{-s}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_-))$; by Theorem 6.2.3, we know that

$$r^+ \mathrm{Op}^{\psi}_n(a) e^- \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{-1}; \mathscr{S}(\mathbb{R}_-), \mathscr{S}(\mathbb{R}_+)).$$

So, $r^+ \mathrm{Op}^{\psi}_n(a) e^- r^- \Lambda^{-s} e^-$ is a symbol in $S^{m-s}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$. We can therefore write $r^+ \mathrm{Op}^{\psi}_n(a)(e^+ r^+ - 1) \Lambda^{-s} e^+$ as an operator-valued FIO defined on the boundary with phase function ψ_∂ and an amplitude belonging to $S^{m-s}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$, so we get

$$\begin{array}{ccc} L^2(\mathbb{R}^n_+) & \hookrightarrow & \mathscr{W}^0(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+)) \\ & & \downarrow {\scriptstyle r^+ \mathrm{Op}^{\psi}(e^+ r^+ - 1) \Lambda^{-s} r^+} \\ H^{s-m}(\mathbb{R}^n_+) & \hookleftarrow & \mathscr{W}^{s-m}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}(\mathbb{R}_+)) \end{array} .$$

$\square$

Remark 6.2.4. *As we have seen in Property 5.2.1, we can define the extension operator for all $s \in \mathbb{R}$. Anyway, as noticed in Remark 5.2.1, this extension,in general, could depend on the Sobolev space: for this reason we have imposed in Theorem 6.2.8 that $s > -\frac{1}{2}$.*

6.3 Fourier Integral Operators of Boutet de Monvel Type on the Half-Space

In order to define FIOs of Boutet de Monvel type, we recall the definition of *potential symbols*, *trace symbols*, *singular Green symbols*. First we define ∂_+, the derivative in the normal direction:

$$\partial_+ = r^+\partial_{x_n}e^+ : H^s(\mathbb{R}_+) \to H^{s-1}(\mathbb{R}_+), \quad s > -\frac{1}{2}.$$

One can consider the operator ∂_+ as an operator-valued symbol belonging to $S^1(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-1}(\mathbb{R}_+))$. Recall that we write $\mathbf{s} = (s_1, s_2)$.

i) A *potential symbol* of order m is an element of

$$S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}'(\mathbb{R}_+)) = \text{proj-lim}_{\mathbf{s}} S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, H^{\mathbf{s}}(\mathbb{R}_+)).$$

ii) A *trace symbol* of order m and type zero is an element of the set

$$S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathbb{C}) = \text{proj-lim}_{\mathbf{s}} S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H_0^{\mathbf{s}}(\overline{\mathbb{R}}_+), \mathbb{C}).$$

Clearly, a trace symbol of order m and type zero defines also a symbol in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{s_1,s_2}(\mathbb{R}_+), \mathbb{C})$, if $s_1 > -\frac{1}{2}$. A *trace symbol* of type d is a sum of the form

$$t = \sum_{j=0}^{d} t_j \partial_+^j, \quad t_j \in S^{m-j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathbb{C}).$$

where t is in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{s_1,s_2}(\mathbb{R}_+), \mathbb{C})$, $s_1 > d - \frac{1}{2}$.

iii) A *singular Green symbol* of order m and type zero is an element of

$$S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}), \mathscr{S}(\mathbb{R}_+)) =$$
$$\text{proj-lim}_{\mathbf{s}} S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H_0^{\mathbf{s}}(\overline{\mathbb{R}_+}), H^{\mathbf{s}}(\mathbb{R}_+)).$$

A singular Green symbol of order m and type zero furnishes a symbol in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{s_1,s_2}(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$, provided $s_1 > -\frac{1}{2}$. A *singular Green symbol* of order m and type d is a sum of the form

$$g = \sum_{j=0}^{d} g_j \partial_+^j, \quad g_j \in S^{m-j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+)).$$

Obviously, g is in $S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; H^{s_1,s_2}(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$, $s_1 > d - \frac{1}{2}$.

Remark 6.3.1. *The trace operator γ_j is a trace symbol of order $j + \frac{1}{2}$ and type $j + 1$. In fact, one can write*

$$\gamma_0(f) = \int_0^\infty \langle\xi'\rangle e^{-y_n\langle\xi'\rangle} f(y_n)dy_n - \int_0^\infty e^{-y_n\langle\xi'\rangle}\partial_{y_n} f(y_n)dy_n,$$

$f \in \mathscr{S}(\mathbb{R}_+)$. That is $\gamma_0 = t_0 + t_1\partial_+$, where

$$t_0 f = \langle\xi'\rangle \int_0^\infty e^{-y_n\langle\xi'\rangle} f(y_n)dy_n, \quad t_1 f = -\int_0^\infty e^{-y_n\langle\xi'\rangle} f(y_n)dy_n.$$

One can check that t_0 and t_1 admit an extension to $\mathscr{S}'(\mathbb{R}_+)$, that t_0 belongs to $S^{\frac{1}{2}}(\mathbb{R}^{n-1},\mathbb{R}^{n-1};\mathscr{S}'(\mathbb{R}_+),\mathbb{C})$ and $t_1 \in S^{-\frac{1}{2}}(\mathbb{R}^{n-1},\mathbb{R}^{n-1};\mathscr{S}'(\mathbb{R}_+),\mathbb{C})$. Therefore, γ_0 is a trace symbol of order $\frac{1}{2}$ and type one. By iteration, one can prove the general result for γ_j.

Definition 6.3.1. Let ψ be an admissible phase function, describing an admissible symplectomorphism χ. Moreover, let ψ_∂ be the phase function induced by ψ on the boundary. Then, a FIO of Boutet de Monvel type of order m and type d is a matrix of the type

$$\mathcal{A} := \begin{pmatrix} r^+\mathrm{Op}^{\psi}(a)e^+ + G^{\psi_\partial} & K^{\psi_\partial} \\ T^{\psi_\partial} & S^{\psi_\partial} \end{pmatrix},$$

where: $\mathrm{Op}^{\psi}(a)$ is a FIO with phase function ψ and symbol $a \in S^m_{\mathrm{tr}}(\mathbb{R}^n,\mathbb{R}^n)$; G^{ψ_∂} is a FIO with phase function ψ_∂ and singular Green symbol g of order m and type d; K^{ψ_∂} is a FIO with phase function ψ_∂ and potential symbol k of order m and type d; T^{ψ_∂} is a FIO with phase function ψ_∂ and trace symbol t of order m and type d; S^{ψ_∂} is a FIO with phase function ψ_∂ and symbol $s \in S^m(\mathbb{R}^{n-1},\mathbb{R}^{n-1})$. The set of such operators is denoted by $\mathscr{B}^{m,d}_{\chi}(\overline{\mathbb{R}^n_+})$.

Pseudodifferential operators of Boutet de Monvel type [19, 36, 88, 94] are a particular case of FIOs of Boutet de Monvel type when we assume that the symplectomorphism χ is the identity: we denote this class by $\mathscr{B}^{m,d}(\overline{\mathbb{R}^n_+})$. As a consequence of Theorem 6.2.8 and of the Sobolev continuity of FIOs defined through operator-valued symbols, we get the following Theorem:

Theorem 6.3.1. *Every $\mathcal{A} \in \mathscr{B}^{m,d}_{\chi}(\overline{\mathbb{R}^n_+})$ induces a continuous operator*

$$\mathcal{A} : H^s(\mathbb{R}^n) \oplus H^s(\mathbb{R}^{n-1}) \to H^{s-m}(\mathbb{R}^n) \oplus H^{s-m}(\mathbb{R}^{n-1}),$$

provided $s > d - \frac{1}{2}$.

Now, we analyze the composition of a Boutet de Monvel pseudodifferential operator with a FIO of Boutet de Monvel type. Recall that we assume the involved symbols to have compact support w.r.t. the space variable. To this aim we introduce two lemmas.

Lemma 6.3.2. *Let ψ be a phase function which represents an admissible symplectomorphism and ψ_∂ be the corresponding phase function at the boundary. Then, if $a \in S^m(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\mathbb{R}^{n-1};E,F)$, E,F being Banach spaces or projective limit of Banach spaces or inductive limits of Banach spaces, there exist a left symbol $a_L \in S^m(\mathbb{R}^{n-1},\mathbb{R}^{n-1};E,F)$ and a right symbol $a_R \in S^m(\mathbb{R}^{n-1},\mathbb{R}^{n-1};E,F)$ such that*

$$\int e^{i\psi_\partial(x',\xi')-iy'\cdot\xi'}a(x',y',\xi)đ\xi' =$$
$$\int e^{i\psi_\partial(x',\xi')-iy_n\cdot\xi_n}a_L(x',\xi)đ\xi' = \int e^{ix'\cdot\xi'-i\psi_\partial(y',\xi')}a_R(y',\xi)đ\xi',$$

where the equality is modulo operators with kernel in $C^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathscr{L}(E,F))$.

Proof. Since the phase function represents a symplectomorphism at the boundary it is linear in the fibers, that is $\psi_\partial(x',\xi') = \Psi_\partial(x')\cdot\xi'$, where $\Psi_\partial(x')$ is an element of $\mathrm{GL}(n-1)$, and the same holds for $\Psi'_\partial(x')$. Moreover, $|\Psi'_\partial(x')\xi| \sim |\xi'|$. Then, notice that it is possible to consider asymptotic expansions of vector-valued symbols as in the proof of Lemma 6.3.3 and, following the scheme of [49], Ch. 25, to obtain the desired left and right quantization. $\square$

Lemma 6.3.3. *Let P be a pseudodifferential operator of order m whose symbol p satisfies the transmission property and is compactly supported with respect to the space variable. Let us consider an admissible symplectomorphism χ with the associated phase function ψ, ψ_∂ being the corresponding phase function at the boundary, and a singular Green symbol of type zero $g \in S^{m'}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$. Then, $r^+Pe^+ \circ \mathrm{Op}^{\psi_\partial}(g)$ is a FIO with phase ψ_∂ and with a singular Green symbol $\tilde{g} \in S^{m+m'}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$.*

Proof. We set $r^+\mathrm{Op}_n(p)e^+ = \mathrm{Op}_n^+(p)$. It is well-known and follows, e.g., from Theorem 6.2.3 that

$$\mathrm{Op}_n^+(p) \in S^m(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+)).$$

Moreover, we can write

$$\begin{aligned}
&\left(\mathrm{Op}_n^+(p) \circ \mathrm{Op}^{\psi_\partial}(g)\right) u(x') = \\
&= \int e^{ix'\cdot\zeta'} \mathrm{Op}_n^+(p)(x', \zeta') \int e^{-iz'\cdot\zeta'} \int e^{i\psi_\partial(z',\xi')} g_L(z', \xi')\hat{u}(\xi') đ\xi' dz' đ\zeta' \\
&= \int e^{i\psi_\partial(x',\xi')} \left[\iint e^{i(x'-z')\cdot\zeta' + i(\psi_\partial(z',\xi') - \psi_\partial(x',\xi'))} \mathrm{Op}_n^+(p)(x', \zeta') g_L(z', \xi') dz' đ\zeta' \right] \hat{u}(\xi') đ\xi' \\
&= \int e^{i\psi_\partial(x',\xi)} \tilde{g}_L(x', \xi')\hat{u}(\xi) đ\xi'.
\end{aligned}$$

By an argument similar to the one valid for pseudodifferential and Fourier operators with scalar symbols, using the properties of oscillatory integrals involving operator-valued symbols, we obtain that

$$\tilde{g}_L(x', \xi') \sim \sum_{\alpha' \in \mathbb{Z}_+^{n-1}} \frac{1}{\alpha'!} (\partial_{\zeta'}^{\alpha'} \mathrm{Op}_n^+(p))(x', d_{x'}\psi_\partial(x', \xi')) D_{z'}^{\alpha'} \left[e^{i\Phi(x',z',\xi')} g_L(z', \xi') \right]_{z'=x'},$$

where

$$\Phi(x', z', \xi') = \psi_\partial(z', \xi') - \psi_\partial(x', \xi') - d_{x'}\psi_\partial(x', \xi)(z' - x').$$

Since the terms appearing in the asymptotic expansion are singular Green symbols of order $m + m' - \left[\frac{|\alpha|}{2}\right]$ and type zero, $\tilde{g}_L \in S^{m+m'}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$, modulo operators with kernel in $C^\infty(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{L}(\mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+)))$, and is a singular Green symbol of order $m + m'$ and type zero, as stated. $\square$

Theorem 6.3.4. *Let $\mathcal{P} \in \mathscr{B}^{m_\mathcal{P}, d_\mathcal{P}}(\overline{\mathbb{R}^n_+})$ and $\mathcal{A} \in \mathscr{B}_\chi^{m_\mathcal{A}, d_\mathcal{A}}(\overline{\mathbb{R}^n_+})$ be a Boutet de Monvel pseudodifferential operator and a FIO of Boutet de Monvel type, respectively. Then $\mathcal{P} \circ \mathcal{A}$ is a FIO of Boutet de Monvel type of order $m_\mathcal{P} + m_\mathcal{A}$ and type $(m_\mathcal{A} + d_\mathcal{P}, d_\mathcal{A})_+ = \max\{(m_\mathcal{A} + d_\mathcal{P}), d_\mathcal{A}\}$ defined again by the symplectomorphism χ, that is $\mathcal{P} \circ \mathcal{A} \in \mathscr{B}_\chi^{m_\mathcal{P}+m_\mathcal{A}, (m_\mathcal{A}+d_\mathcal{P}, d_\mathcal{A})_+}(\overline{\mathbb{R}^n_+})$.*

Proof. Let us consider a phase function ψ which represents χ close to the boundary. We have to analyze the composition

$$\begin{pmatrix} r^+\mathrm{Op}(p)e^+ + G_{\mathcal{P}} & K_{\mathcal{P}} \\ T_{\mathcal{P}} & S_{\mathcal{P}} \end{pmatrix} \circ \begin{pmatrix} r^+\mathrm{Op}^{\psi}(a)e^+ + G_{\mathcal{A}}^{\psi_\partial} & K_{\mathcal{A}}^{\psi_\partial} \\ T_{\mathcal{A}}^{\psi_\partial} & S_{\mathcal{A}}^{\psi_\partial} \end{pmatrix}.$$

We start with the composition of elements in the upper-left corner. We can write

$$\begin{aligned} &r^+\mathrm{Op}(p)e^+ \circ r^+\mathrm{Op}^{\psi}(a)e^+ = \\ &r^+\mathrm{Op}(p) \circ \mathrm{Op}^{\psi}(a)e^+ + r^+\mathrm{Op}(p)(e^+r^+ - 1)\mathrm{Op}^{\psi}(a)e^+ \end{aligned}$$

$\mathrm{Op}(p) \circ \mathrm{Op}(a)^{\psi}$, by the general theory of FIOs, is a FIO of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ with canonical transformation χ. We prove next that the operator $r^+\mathrm{Op}(p)(e^+r^+ - 1)\mathrm{Op}^{\psi}(a)e^+$ is a FIO on the boundary with vector-valued symbol, associated with the canonical transformation χ_∂ and with a Green symbol of order $m_{\mathcal{P}}+m_{\mathcal{A}}$ and type $(m_{\mathcal{A}})_+ = \max\{m_{\mathcal{A}}, 0\}$. Thus, we have to study the composition in the normal direction. We decompose the symbol $a = a_d + a_0$ as in Proposition 6.2.5. First, we analyze the differential part

$$r^+\mathrm{Op}_n(p)(e^+r^+ - 1)\mathrm{Op}_n^{\psi}\left(\sum_{j=1}^{m_{\mathcal{A}}} a_j(x', x_n, \xi')\xi_n^j\right)e^+u, \tag{6.3.36}$$

where $a_j(x', x_n, \xi') \in S^{m_{\mathcal{A}}-j}(\mathbb{R}^n, \mathbb{R}^{n-1})$. Since, on $\mathscr{S}(\mathbb{R}_+)$

$$\xi_n \widehat{e^+u(y_n)}(\xi_n) = -i\widehat{e^+\partial_{y_n}u}(\xi_n) - iu(0)\widehat{\delta_0}, \tag{6.3.37}$$

by induction, we have that

$$\xi_n^k \widehat{e^+u(y_n)} = (-i)^k\left(\widehat{e^+\partial_{x_n}^k u} + \sum_{l=0}^{k-1} u^{(l)}(0)\widehat{\delta_0^{(k-l-1)}}\right). \tag{6.3.38}$$

So, (6.3.36) turns into

$$\begin{aligned} &r^+\mathrm{Op}_n(p)(e^+r^+ - 1) \\ &\int e^{i\psi - i\psi_\partial} \sum_{j=1}^{m_{\mathcal{A}}} (-i)^j a_j(x', x_n, \xi')\left(\widehat{e^+\partial_{y_n}^j u}(\xi_n) - \sum_{l=0}^{j-1} u^{(l)}(0)\widehat{\delta_0^{(j-l-1)}}\right) đ\xi_n. \end{aligned}$$

Following the scheme of the proof of Theorem 6.2.1, one gets that

$$\begin{aligned} &(e^+r^+ - 1)\int e^{i\psi - i\psi_\partial} \sum_{j=1}^{m_{\mathcal{A}}} (-i)^j a_j(x', x_n, \xi') \sum_{l=0}^{j-1} u^{(l)}(0)\widehat{\delta_0^{(j-l-1)}} đ\xi_n \\ &= \sum_{j=1}^{m_{\mathcal{A}}} \sum_{l=0}^{j-1} \tilde{b}_{j,l}(x', \xi')u^{(l)}(0)\delta_0^{(j-l-1)} + \sum_{l=0}^{m_{\mathcal{A}}} c_l(x', x_n, \xi')u^{(l)}(0), \end{aligned}$$

where $\tilde{b}_{j,l}(x', \xi') \in S^{m_{\mathcal{A}}-j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ and $c_l \in S^{m_{\mathcal{A}}-l-\frac{1}{2}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}(\mathbb{R}_-))$. Observing that γ_l is a trace symbol of order $l + \frac{1}{2}$ and type l, we get that

$$r^+\mathrm{Op}_n(p)(e^+r^+ - 1)\int e^{i\psi - i\psi_\partial} \sum_{j=1}^{m_{\mathcal{A}}} (-i)^j a_j(x', x_n, \xi') \sum_{l=0}^{j-1} u^{(l)}(0)\widehat{\delta_0^{(j-l-1)}} đ\xi_n$$

is a Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and of type $(m_{\mathcal{A}})_+$. We have now to analyze

$$\begin{aligned} &\mathrm{Op}_n(p)(e^+r^+ - 1)\sum_{j=1}^{m_{\mathcal{A}}} \int e^{i\psi - i\psi_\partial} a_j(x', x_n, \xi') \widehat{e^+\partial^j_{y_n} u}(\xi_n) đ\xi_n \\ &= \mathrm{Op}_n(p)(e^+r^+ - 1)\sum_{j=1}^{m_{\mathcal{A}}} a_j \mathrm{Op}^{\psi}_n(1) e^+ \circ \partial^j_+ u \end{aligned} \tag{6.3.39}$$

Recall that $e^+r^+ - 1 = -e^-r^-$ on regular distributions. Then, by Theorem 6.2.7 we get that $r^- a_j \mathrm{Op}^{\psi}_n(1)e^+$ is a symbol in $S^{m_{\mathcal{A}}-j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_-))$; since $r^+\mathrm{Op}_n(p)e^- \in S^{m_{\mathcal{P}}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_-), \mathscr{S}(\mathbb{R}_+))$, the symbol in (6.3.39) is a Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type $(m_{\mathcal{A}})_+$.
We have now to consider

$$r^+\mathrm{Op}_n(p)(e^+r^+ - 1)\mathrm{Op}^{\psi}_n(a_0)e^+. \tag{6.3.40}$$

Theorem 6.2.7 implies $r^-\mathrm{Op}^{\psi}_n(a_0)e^+ \in S^{m_{\mathcal{A}}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_-))$. Observing that $r^+\mathrm{Op}_n(p)e^-$ is an element of $S^{m_{\mathcal{P}}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}(\mathbb{R}_-), \mathscr{S}(\mathbb{R}_+))$, we get that the symbol in (6.3.40) is a Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type zero.

The other compositions can be analyzed in a similar way, we omit most of the details.

1. $r^+\mathrm{Op}(p)e^+ \circ G^{\psi_\partial}_{\mathcal{A}}$, is a FIO a with phase function that represents χ_∂ and a singular Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type $d_{\mathcal{A}}$.

2. $G_{\mathcal{P}} \circ e^+\mathrm{Op}^{\psi}(a)e^+$ is a FIO with a phase function that represents χ_∂ and a singular Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and of type $(m_{\mathcal{A}} + d_{\mathcal{P}})_+ = \max\{m_{\mathcal{A}} + d_{\mathcal{P}}, 0\}$.

3. $G_{\mathcal{P}} \circ G^{\psi}_{\mathcal{A}}$ is a FIO on the boundary, with a phase function that represents χ_∂ and a Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type $d_{\mathcal{A}}$.

4. $r^+\mathrm{Op}(a)e^+ \circ K^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ with a potential symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$.

5. $G_{\mathcal{P}} \circ K^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ and a potential symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$.

6. $K_{\mathcal{P}} \circ T^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ and a Green symbol of type $d_{\mathcal{A}}$ and order $m_{\mathcal{P}} + m_{\mathcal{A}}$.

7. $K_{\mathcal{P}} \circ S^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ and a potential symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$.

8. $T_{\mathcal{P}} \circ r^+\mathrm{Op}^{\psi}(a)e^+$ is a FIO on the boundary with phase function that represents χ_∂ and a trace operator of type $(m_{\mathcal{A}} + d_{\mathcal{P}})_+$ and order $m_{\mathcal{P}} + m_{\mathcal{A}}$.

9. $T_{\mathcal{P}} \circ G^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ and a trace symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type $d_{\mathcal{A}}$.

10. $S_{\mathcal{P}} \circ T^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ and a trace symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type $d_{\mathcal{A}}$.

11. $T_{\mathcal{P}} \circ K^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with a phase function that represents χ_∂ and a symbol in $S^{m_{\mathcal{P}}+m_{\mathcal{A}}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$.

12. $S_{\mathcal{P}} \circ S^{\psi_\partial}_{\mathcal{A}}$ is a FIO on the boundary with phase function that represents χ_∂ and a symbol in $S^{m_{\mathcal{P}}+m_{\mathcal{A}}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$.

The composition in 1) follows by Lemma 6.3.3. The other compositions in 3), 4), 5), 6), 7), 8), 10), 11), 12), can be treated similarly exploiting the properties of vector-valued symbols, in particular the possibility to write asymptotic expansions. The compositions in 2) and 9) are slightly more delicate. Let us analyze the composition in 2). We suppose $d_{\mathcal{P}} = 0$. The operator $r^+\mathrm{Op}^{\psi}(a)e^+$, by Proposition 6.2.5, can be split into $r^+\mathrm{Op}^{\psi}(a_d + a_0)e^+$. We first analyze the differential part, obtaining

$$\mathrm{Op}(g_A) \circ \mathrm{Op}^{\psi_\partial}(r^+\mathrm{Op}^{\psi}_n(a_d)e^+).$$

We analyze the composition in the normal direction of the involved vector-valued symbols. We have that

$$(r^+\mathrm{Op}^{\psi}_n(a_d)e^+)u = r^+ \int e^{i\psi-\psi_\partial} \sum_{j=1}^{m_{\mathcal{A}}} a_j(x', x_n, \xi')\xi_n^j \widehat{e^+u}(\xi_n) đ\xi_n. \tag{6.3.41}$$

Using (6.3.37) and (6.3.38), we can write

$$\begin{aligned}&(r^+\mathrm{Op}^{\psi}_n(a_d)e^+)u = \\ &\sum_{j=1}^{m_{\mathcal{A}}}(-i)^j \sum_{l=1}^{j-1} r^+a_j(x', x_n, \xi')\left(\mathrm{Op}^{\psi}_n(1)(\delta_0^{(j-l-1)})\gamma_l(u) + \mathrm{Op}^{\psi}_n(1)e^+\partial^l_{y_n}u\right).\end{aligned} \tag{6.3.42}$$

By Theorem 6.2.1, Remark 6.3.1 and the properties of trace operators, we get that the sum in j, l can be written as

$$(r^+\mathrm{Op}^{\psi}_n(b_d)e^+) = \sum_{j=1}^{m_{\mathcal{A}}} \tilde{a}_j(x', \xi')\partial^j_+, \quad \tilde{a}_j \in S^{m_{\mathcal{A}}-j}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}'(\mathbb{R}_+), \mathscr{S}'(\mathbb{R}_+)).$$

Then, by the definition of Green symbols of type zero, we obtain that $G_{\mathcal{P}} \circ (r^+\mathrm{Op}^{\psi}_n(a_d)e^+)$ is a Green symbol of order $m_{\mathcal{P}} + m_{\mathcal{A}}$ and type $(m_{\mathcal{A}})_+$. To prove the same result for a_0, we only have to notice that $r^+\mathrm{Op}^{\psi}(a_0)e^+$ extends to a symbol in $S^{m_{\mathcal{A}}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}'(\mathbb{R}_+))$, and the result follows from the definition of Green symbol. If $d_{\mathcal{P}} \neq 0$, we see that

$$\begin{aligned}&\partial_+ \int e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}a(x', x_n, \xi', \xi_n)\hat{u}(\xi_n)đ\xi_n \\ &= r^+ \int e^{i\psi(x',x_n,\xi',\xi_n)-i\psi_\partial(x',\xi')}\tilde{a}(x', x_n, \xi', \xi_n)\hat{u}(\xi_n)đ\xi_n,\end{aligned}$$

where $\tilde{a} = \partial_{x_n}a + (i\partial_{x_n}\psi)a$, which implies $\tilde{a} \in S^{m_{\mathcal{A}}+1}(\mathbb{R}^n, \mathbb{R}^n)$. Using an iterative scheme we can reduce to the case $d_{\mathcal{P}} = 0$, raising the order from $m_{\mathcal{A}}$ to $m_{\mathcal{A}} + d_{\mathcal{P}}$. To handle the composition 9), we can repeat the same scheme. □

Remark 6.3.2. *Using essentially the same scheme of Theorem 6.3.4, it is possible to prove that if* $\mathcal{A} \in \mathscr{B}_\chi^{m_\mathcal{A}, d_\mathcal{A}}(\overline{\mathbb{R}^n_+})$ *and* $\mathcal{P} \in \mathscr{B}^{m_\mathcal{P}, d_\mathcal{P}}(\overline{\mathbb{R}^n_+})$ *then* $\mathcal{A} \circ \mathcal{P} \in \mathscr{B}_\chi^{m_\mathcal{A}+m_\mathcal{P}, d}(\overline{\mathbb{R}^n_+})$*,* $(d = \max\{m_\mathcal{P} + d_\mathcal{A}, d_\mathcal{P}\})$*,*

As in the case of FIOs on closed manifolds, one could look for an Egorov type Theorem. To this aim we have to analyze the adjoint of operators in $\mathscr{B}_\chi^{m,d}(\overline{\mathbb{R}^n_+})$.

Theorem 6.3.5. *Let us consider* $\mathcal{A} \in \mathscr{B}_\chi^{m,0}(\overline{\mathbb{R}^n_+})$*,* $m \leq 0$*. Then,* $\mathcal{A}^*$*, the formal adjoint of Q, is a FIO of Boutet de Monvel type, namely* $\mathcal{A}^* \in \mathscr{B}_{\chi^{-1}}^{m,0}(\overline{\mathbb{R}^n_+})$*. Moreover, locally close to the boundary*

$$\mathcal{A}^* = \begin{pmatrix} r^+(\mathrm{Op}^\psi(a))^* e^+ + (G^{\psi_\partial})^* & (T^{\psi_\partial})^* \\ (K^{\psi_\partial})^* & (S^{\psi_\partial})^* \end{pmatrix}, \tag{6.3.43}$$

where $(\mathrm{Op}^\psi(a))^*$ *is the formal adjoint of* $\mathrm{Op}^\psi(a)$*, so its phase function is* ψ^{-1}*. The operators* $(G^{\psi_\partial})^*$*,* $(K^{\psi_\partial})^*$*,* $(T^{\psi_\partial})^*$*,* $(S^{\psi_\partial})^*$ *appearing in* (6.3.43) *are the adjoints of* G^{ψ_∂}*,* K^{ψ_∂}*,* T^{ψ_∂}*,* S^{ψ_∂}*, respectively, that is, they are FIOs with vector-valued symbols, with phase function* ψ_∂^{-1} *that represents* χ_∂^{-1}*, the inverse of the boundary symplectomorphism.*

Proof. Since $m \leq 0$, $\mathrm{Op}^\psi(a)$ is continuous from $L^2(\mathbb{R}^n)$ to itself. Moreover, $e^+ : L^2(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n)$ is continuous and its adjoint is r^+. So we can write

$$\begin{aligned}(r^+\mathrm{Op}^\psi(a)e^+u, v)_{L^2(\mathbb{R}^n_+)} &= (\mathrm{Op}^\psi(a)e^+u, e^+v)_{L^2(\mathbb{R}^n)} \\ &= (e^+u, (\mathrm{Op}^\psi(a))^* e^+ v)_{L^2(\mathbb{R}^n)} = (u, r^+(\mathrm{Op}^\psi(a))^* e^+ v)_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

For the other components of $\mathcal{A}^*$, one can use the properties of adjoints of *FIOs*, noticing that the adjoint of a Green operator of order m and type 0 is still a Green operator of the same order and type, the adjoint of a potential operator of order m is a trace operator of order m and type 0 and the adjoint of a trace operator of order m and type 0 is a potential operator of order m. This a consequence of the adjoint property of Green, potential and trace symbols, see [94]. □

Definition 6.3.2. For every $m \in \mathbb{Z}$ we can define the operator

$$[\Lambda_+^m] := \begin{pmatrix} r^+\Lambda^m e^+ & 0 \\ 0 & \mathrm{Op}(\langle\xi'\rangle^m) \end{pmatrix}$$

where $r^+\Lambda^m e^+ : H^m(\mathbb{R}^+_n) \to L^2(\mathbb{R}^+_n)$ is an isomorphism. The operator $[\Lambda_+^m]$ is an element of $\mathscr{B}^{m,0}(\overline{\mathbb{R}^n_+})$ and it is invertible.

Now we can state in this case a version of Egorov Theorem for FIOs of Boutet de Monvel type.

Theorem 6.3.6. *Let* $\mathcal{A} \in \mathscr{B}_\chi^{m,d}(\overline{\mathbb{R}^n_+})$ *be a FIO of Boutet de Monvel type. Then, provided* $d \leq m_+$*,* $m \in \mathbb{Z}$

i) If $m \leq 0$ *and* $d = 0$ *then* $\mathcal{A} \circ \mathcal{A}^*$ *is a an element of* $\mathscr{B}^{2m,0}(\overline{\mathbb{R}^n_+})$*.*

ii) If $m > 0$ *then we have that* $(\mathcal{A} \circ [\Lambda_+^{-m}])(\mathcal{A} \circ [\Lambda_+^{-m}])^*$ *is an element of* $\mathscr{B}^{0,0}(\overline{\mathbb{R}^n_+})$*.*

Proof. The proof of part *i)* essentially follows from Theorem 6.3.5 and from Egorov Theorem for standard FIOs. The second part follows from the first, noticing that, from Theorem 6.3.4, $Q \circ [\Lambda_+^{-m}]$ belongs to $\mathscr{B}_\chi^{0,0}(\overline{\mathbb{R}^n_+})$. □

In general, one cannot prove an Egorov type Theorem for FIOs of Boutet de Monvel type of all orders and types, in fact such a statement would not even be true for an operator of $\mathscr{B}^{m,d}(\overline{\mathbb{R}^n_+})$, provided $m > 0$ or $d > 0$ or $d > m$. Namely, if we consider a Boutet de Monvel operator $\mathcal{P}$ of positive order, then its formal adjoint $\mathcal{P}^*$, in general, is not even a Boutet de Monvel operator.

Remark 6.3.3. *By means of Theorems 6.3.4 and 6.3.6, it is possible to prove that, if $\mathcal{P} \in \mathscr{B}^{m',d'}(\overline{\mathbb{R}^n_+})$ and $\mathcal{A} \in \mathscr{B}_\chi^{m,d}(\overline{\mathbb{R}^n_+})$, $m \leq 0$, $d = 0$, then $\mathcal{A} \circ \mathcal{P} \circ \mathcal{A}^*$ belongs to $\mathscr{B}^{m',d'}(\overline{\mathbb{R}^n_+})$.*

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