A Direct Approach for Hypersingular Integral Evaluation in 3-D B.E.M.

Article · March 1999

Source: CiteSeer

3 authors, including:

Pilar Ariza
Universidad de Sevilla

Jose Dominguez
Universidad de Sevilla

A Direct Approach for Hypersingular Integral Evaluation in 3-D B.E.M.

M.P. Ariza and J. Dominguez, Member ASCE
Escuela Superior de Ingenieros, Universidad de Sevilla,
e-mail: mpariza@cica.es and pepon@cica.es

1 Introduction

There are two main difficulties behind the use of Hypersingular Boundary Elements (HBE): One is the continuity conditions that must be satisfied by the density functions; and the other, the integration of the strongly singular and hypersingular kernels obtained by differentiation of the classical ones.

To satisfy the requirement that surface derivatives of displacement should be continuous at the boundary collocation point, several strategies have been proposed (see for instance, Martha *et al.*,1992). In the current paper a simple and efficient approach is presented. It is an extension to three dimensions of the idea introduced by Gallego and Dominguez (1996). The elements are continuous with the classical representation of the variables. However, collocation is done at points inside the elements where the $C^{1,\alpha}$ continuity requirement is satisfied. Using this strategy the difficulties produced by the continuity requirements at collocation points are avoided and the number of nodal unknowns remains the same as in the classical formulation.

The second important difficulty of the HBE formulation is the integration of hypersingular and strongly singular kernels. This problems has been dealt with from different points of view and several regularization techniques have been proposed (see the review paper by Tanaka *et al.*,1994).

In this paper, Laplace's potential problem is treated first. The classical integral representation is differentiated with respect to a general direction and taken to the boundary to obtain a hypersingular BIE in terms of potential, flux and surface derivatives at the boundary collocation point. The hypersingular and strongly singular integrals are regularized by subtracting two terms of Taylor's series expansion of the potential at collocation point. The remaining hypersingular and strongly singular integrals are analytically transformed into regular or weakly singular integrals which can be numerically computed without special difficulties. In a similar way, the hypersingular BIE is obtained for elasticity taking to the boundary the traction integral representation. The regularization is done by using two terms of the Taylor's series expansion of the displacement and one term of the traction expansion, at collocation point. All the remaining integrals containing hypersingular and strongly singular kernels are transformed also in this case into regular or weakly singular integrals by analytical procedures.

The method presented is general for boundary surfaces and boundary elements of any shape. The regularization is done prior to any boundary discretization. All the subsequent integrations are performed analytically in a direct way without any change of coordinates.

2 Flux BIE in potential problems

Consider the integral representation of the potential u of a Laplace's problem at any point $\mathbf{y}$ in the interior of a 3-D domain Ω bounded by a surface Γ with unit outward normal $\mathbf{n}(\mathbf{x})$ at boundary point $\mathbf{x}$.Taking derivatives at point $\mathbf{y}$ along a direction defined by the unit vector $\mathbf{N}(\mathbf{y})$ and calling $q(\mathbf{y}) = (\partial u/\partial y_i)N_i(\mathbf{y})$ one obtains a flux integral representation which contains hypersingular and strongly singular integrals. This representation can be transformed into a new boundary integral equation where all the hypersingular and strongly singular integrals have been transformed into regular and/or weakly singular integrals.

$$\begin{aligned}
&\frac{1}{2}q(\mathbf{y}) + \int_\Gamma \left\{V_i N_i \left[u\left(\mathbf{x}\right) - u(\mathbf{y}) - u_{,k}(\mathbf{y})(x_k - y_k)\right] - W_i N_i\, q\left(\mathbf{x}\right)\right\} d\Gamma + \\
&u_{,k}(\mathbf{y}) \int_\Gamma -\frac{3}{4\pi r^3} r_{,i} N_i \frac{\partial r}{\partial n}(x_k - y_k) d\Gamma + u_{,k}(\mathbf{y}) \int_\Gamma \frac{N_i n_k r_{,i}}{4\pi r^2} d\Gamma + \\
&u(\mathbf{y}) \oint_{\partial\Gamma} \frac{\mathbf{r}\times\mathbf{N}}{4\pi r^3} d\mathbf{l} + u_{,k}(\mathbf{y}) \oint_{\partial\Gamma} \frac{1}{4\pi r}\left(\mathbf{e}_k \times \mathbf{N}\right) d\mathbf{l} = 0
\end{aligned} \tag{1}$$

The two line integrals extend over the contour of the boundary Γ when this boundary is open and are zero when it is closed.Explicit expressions for V_i and W_i can be found in Dominguez et *al*. (1999). Integral equation (1) can be discretized to obtain a BE formulation whose integrals over the elements can be handled without special numerical difficulties.

3 Traction BIE in elasticity

A similar approach can be followed for the hypersingular traction boundary integral equation in elasticity to obtain a general traction BIE in terms only of regular and weakly singular integrals:

$$\begin{aligned}
&\frac{1}{2}p_l(\mathbf{y}) + \int_\Gamma \{s^*_{lmk} N_m \left[u_k\left(\mathbf{x}\right) - u_k(\mathbf{y}) - u_{k,h}(\mathbf{y})(x_h - y_h)\right] - \\
&d^*_{lmk} N_m \left[p_k\left(\mathbf{x}\right) - p_k(\mathbf{y})\right]\} d\Gamma + \\
&\frac{\mu}{4\pi\left(1-\upsilon\right)}\left[u_k(\mathbf{y}) I_{lk} + u_{k,h}(\mathbf{y}) J_{lhk} + p_k(\mathbf{y}) K_{lk}\right] = 0
\end{aligned} \tag{2}$$

where u_k and p_k stand for the k component of displacement and traction vectors, respectively, d^*_{lmk} and s^*_{lmk} are linear combinations of derivatives of the fundamental solution displacements and tractions, respectively. Explicit expressions for d^*_{lmk} and s^*_{lmk}, and for the integrals I_{lk}, J_{lhk} and K_{lk} can be found in Dominguez et al.(1999). This BIE is valid for open or closed, plane or curved boundaries and can be discretized in a straightforward manner using a boundary element formulation.

4 Boundary element formulation

The two facts which make the discretization of the tractions BIE different to the discretization of the classical BIE are: (1), the boundary displacement u_k must satisfy the Holder continuity condition $u_k \in C^{1,\alpha}$, at $\mathbf{y}$; and (2), derivatives, at $\mathbf{y}$, of the displacement components exist in the boundary integral equation. As a consequence of the first of this two characteristics, one should not use standard linear or quadratic elements with collocation at nodal points located at the contour of the element. Due to the second, one can not use constant elements.

Following the idea of Gallego and Domínguez(1996) for two dimensions, the boundary integral equation is discretized into quadratic surface elements with six or nine nodes located at their usual position to represent the geometry and the boundary variables. However, the collocation is not done at the contour nodes; i.e. at ξ_1, $\xi_2 = \pm 1$, but at certain points close to the nodes, inside the element.

It should be noticed that elements of this type are continuous C^o since the boundary variables are written as usual in terms of their values at nodes located at the contour of the element. Only the collocation points are shifted to the interior of the element. Tractions and displacements at each collocation point are expressed in terms of nodal values and shape function values at collocation point. The displacement derivatives are expressed in terms of nodal displacements and shape function derivatives at collocation point.

As a consequence of the collocation strategy, one may have two or more equations for each nodal component, obtained by collocation at as many points as elements contain the node. These equations are added up to yield only one per nodal component. This Multiple Collocation Approach (MCA) is only used for the nodes on one crack surface except for those inside an element and nodes at the crack front where no collocation at all is required ($\Delta u_k = 0$). The integration process can be implemented in a way such that the integrals corresponding to multiple collocation points are computed simultaneously.

It is also important to mention that since the regularization process leads to expressions of the boundary integrals with more terms than the original hypersingular and strongly singular integrals, the regular expressions of the integral will only be used over a part of the surface Γ close to the collocation point whereas the original expressions of the integrals are used in the rest of the boundary where they are non-singular.

5 Numerical example

To show the robustness and simplicity of the present approach, a 3-D fracture mechanics problem is analyzed.. A penny shape crack subject to internal pressure and included in an infinite elastic domain is studied (shear modulus, $\mu = 10^6 Pa$, Poisson Ratio, $\upsilon = 0.25$). Only one surface of the crack is discretized into quadratic nine node quadrilateral elements (Figure 1). The elements at the crack front are quarter-point.

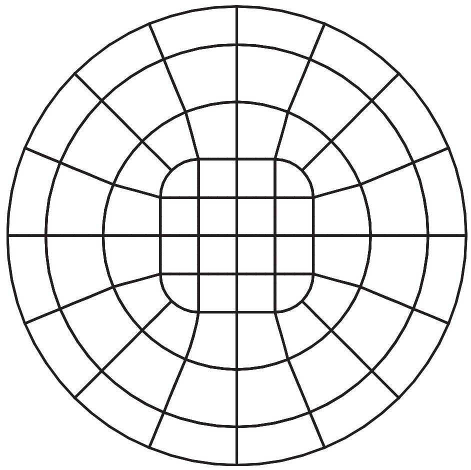

Figure 1: Boundary Element discretization for penny shape crack.

The computed Crack Opening Displacements (COD) at nodes along a radius are shown in Figure 2. The Stress Intensity Factor evaluated from the COD at the first row of nodes from the crack front (quarter point nodes) $K_I = 1.989\sigma\sqrt{\frac{a}{\pi}}$ is within a 1% range from the known analytical solution for this parameter.

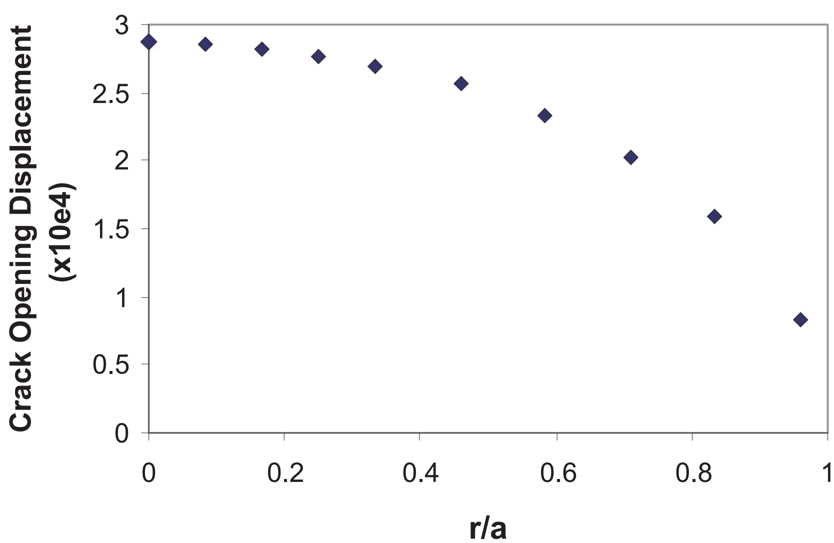

Figure 2: Crack Opening Displacement along a radius line.

6 Conclusions

A formulation which permits the numerical treatment of potential derivative and traction BIE without special difficulties has been presented in this paper.One of the main issues is to contribute to transform hypersingular formulations of the BEM in something which is as clear, general and easy to handle as the classical formulation. Ideas such as Hadamart Finite Part or Cauchy Principal Values are not needed. All the integrals can be numerically evaluated in an easy way. The boundary of the problem can be curved or plane, closed or open.

A simple BE discretization strategy is adopted to fulfil the regularity condition without increasing the number of equations or introducing more complicated continuous elements. The boundary approximation is done in the usual way with nodes located in most cases on the contour of the elements. Thus, the displacement and traction (potential and potential derivative) continuity over the boundary is retained, contrary to non-conforming elements. The collocation points are located inside the element with multiple collocation for nodes between two or more elements. The generality of the current integration procedure allows for the use of special crack elements such as quarter point.

7 Acknowledgments

This work was supported by the Comisión Interministerial de Ciencia y Tecnología of Spain. (PB96-1380 and PB96-1322-C03-01). The financial support is gratefully acknowledged.

8 References

1. Dominguez,J., Ariza, M.P. and Gallego,R. (1999), "Flux and Traction Boundary Elements without Hypersingular or Strongly-singular Integrals", Int. J. Numer. Meth. Eng., (in press).

2. Gallego, R. and Domínguez, J. (1996) "Hypersingular BEM for Transient Elastodynamics", Int. J. Numer. Meth. Eng., **39**, 1681-1705.

3. Martha,L. F., Gray, L. J.and Ingraffea,A. R. (1992) "Three-Dimensional Fracture Simulation with a Single-Domain Direct Boundary Element Formulation", Int. J. Numer. Meth. Eng., **35**, 1907-1921.

4. Tanaka,M., Sladek V. and Sladek, J. (1994) "Regularization Techniques Applied to Boundary Element Methods", Appl. Mech. Reviews, **47**,457-499.