NUMERICAL TREATMENT OF THICK SHELLS WITH HOLES

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A Boundary Integral Equation Method (B.I.E.M.) formulation is presented. After a general situation of the method among other usual numerical ones, the possibilities of discretizat ion are developed. As this is done only in the boundarythe treatment of tridimensional problems is greatly simplified in comparison with other methods.

Some results on a simple shell with holes are finally presented.

1. SCHEME OF THE PROBLEM.

Use of thick shells arises naturally in some physical situations either in strict sense or as a subregion of a general structure. An exemple of the first situation is provided by some concrete nuclear reactor pressure vessels and of the second by the zone of some shell dams which are thicker in the interface with the soil.

The modelization of general situations is , in addition , generally complicated by the presence of perforations which are necessary for the structure functionalism.

Analytical approaches to actual problems is clearly out of question ,because the problem geometry is never simple . The classical method of reduced physical models is also problematic if some information is needed on points with high stress gradients. (At most the mesure gages can only give a weighted value of the variable of interest).

Numerical methods appear as the only powerful alternative and that's the reason why the most famous of them i.e: the Finite Element method (F. E. M.) has been continuosly used.

Nevertheless , F.E.M. has several drawbacks in the particular problem under study. First of all , the tridimensional character of the model imposes a very high number of elements and nodal variables if one wishes to obtain reasoneable results. On the other hand a realist design of high gradient stress zones imposes the use of a fine mesh near them , and because the special features of the method , a smooth growth of the element sizes in the neighbourhood areas.

Both reasons contribute to increment the number of equations to be solved with the result that , even in simple cases , the numerical effort is unproportionally big.

Item more , as it is well known , in the stiffnes path of approach , the degree of accuracy in stresses is always less than in displacements , just in a problem which this accuracy is foundamental.

It is then logical to ask oneself about the possibilities of an alternative numerical method without those problems.

The Boundary Integral Equation Method (B.I.E.M.) is, in this respect, a perfectly suited tool. As we shall show in the next item the discretization is based on a formulation established on the boundary domain and this why a three dimensional problem can be solved with two dimensional elements. The results are obtained only at the boundary and if some information is needed inside the domain, some auxiliar formulaes have to be used. This, that at the first sight is a drawback, is a very advantageous feature; infact the concentration problems always happen because of sudden changes in boundary geometry or in boundary conditions and the inside information provided for instance by the F.E.M. is generally useless.

If, nevertheless, interior information is sought in some places, there is no aditional error introduced by the formulaes and the results are, in general, even better than in the boundary due to the regularity characteristics of the mathematical operator that represents the physical problem. Puntual (and no wheigthed) information is then easily obtained with great accuracy. Stress gradients are also accurately modelled and the size of the problem is greatly reduced, in comparison with F.E.M. models, due to the change 3-D → 2-D in the elements.

We have tried then, to explain how the method can be applied to the problem expressed by the title and show some simple example which can give and idea of the enormous possibilies of the relative new procedure.

2. MATHEMATICAL MODEL.

Given the field equation of a problem

$$Au = f \qquad \ldots(1)$$

the weak formulation of it can be established with an inner product

$$(Au, \psi) = (f, \psi) \qquad \ldots(2)$$

In the F.E.M. an integration by parts of the left hand side produces a bilinear form plus some boundary conditions

$$a(u, \psi) + b_1(u, \psi) = (f, \psi) \qquad \ldots(3)$$

In this process some derivatives are transferred from the field variable u to the auxiliary one ψ, allowing the use of less rectrictive conditions for the aproximation. At this point a discretization of the kind

$$u \approx u_n = c_i \psi_i \qquad i = 1, 2, \ldots, n \qquad \ldots(4)$$

which approaches the infinite dimensional solution from a finite dimensional subespace produces the typical consistent matrices of the problem.

If the operator is symmetric another series of integrations leads to the mirror image of (2)

$$(u, A\psi) + b_1(u, \psi) + b_2(u, \psi) = (f, \psi) \qquad \ldots(5)$$

here the conditions on the smoothness of u have been transfered to ψ. Moreover, if ψ is chosen in such a way that

$$A\psi = 0 \qquad \ldots(6)$$

the left hand side of equation (5) is reduced to conditions on the boundary and the right hand side is a vector of known values. In the homogeneous case $f = 0$ and equation (5) is written as simple as

$$b_1(u, \psi) + b_2(u, \psi) = 0 \qquad \ldots(7)$$

It is interesting to notice that this result is nothing new. In potential theory equation (3) is the first Green's formula and equation (5) is the second Green's one.

In elasticity they are respectively the principle of virtual work and the Maxwell-Betti reciprocity theorem.

The process of discretization is something different of (4). There ψ_i were compact support functions in order to produce banded matrices after the scalar product. Here the ψ are solutions of (6) and then extended to the whole domain. It could seem reasonable to use globally defined trial functions as a basis of the finite dimensional subspace. This produces a set of equations whose variables have no physical meaning. So we prefer to use locally based functions

$$\varphi_i \neq \psi_i \qquad \ldots(8)$$

and to establish

$$u \simeq u_n = c_i \varphi_i \qquad \ldots(9)$$

It is important to notice that, because (5), any condition of smoothness is required in the u and this is because one may use constant φ_i elements to interpolate the field variable.

The preceeding approach is known as TREFFTZ method. The B.I.E.M. uses as auxiliar function the solution to

$$A\psi = \delta \qquad \ldots(10)$$

in place of (6).

That is ψ is taken as the solution to the differential operator. In potential theory is the Coulomb solution of an electrostatic elemental change and in elasticity theory ψ is the point load solution to KELVIN problem.

The equation (7) is transformed then in

$$u_p + b_1(u,\psi) + b_2(u,\psi) = 0 \qquad \ldots(11)$$

where u_p is the field value at the point in which the load is applied. (This point is inside the doamin).

In elastostatics equation (11) is the SOMIGLIANAS' identity.

$$u_j(\underline{x})+\int_{\partial\Omega} T_{ij}(\underline{x},\underline{y})\cdot u_j(\underline{y})\,da_y=\int_{\partial\Omega} U_{ij}(\underline{x},\underline{y})\cdot t_j(\underline{y})\,da_y \quad \ldots(12)$$

where

$$U_{ij}(\underline{x},\underline{y})=\frac{1+\nu}{8\pi E(1-\nu)r}\left\{(3-4\nu)\delta_{ij}+\frac{(x_i-y_i)(x_j-y_j)}{r^2}\right\} \quad \ldots(13)$$

$$T_{ij}(\underline{x},\underline{y})=\frac{1}{8\pi(1-\nu)r^2}\left\{(1-2\nu)\left[n_i(\underline{y})\frac{x_j-y_j}{r}-n_j(\underline{y})\frac{x_i-y_i}{r}\right]+\right.$$
$$\left.+\left[(1-2\nu)\delta_{ij}+3\frac{(x_i-y_i)(x_j-y_j)}{r^2}\right]n_s(\underline{y})\frac{x_s-y_s}{r}\right\} \quad \ldots(14)$$

In equation (13) $u_j(\underline{x})$ is measured at an interior point $\underline{x}$ while point $\underline{y}$ is at the boundary. In order to produce equations in terms only of values at the boundary, it is necessary to establish a limiting process by which equation (12) moves to

$$c_{ij}(\underline{x})\,u_j(\underline{x})+\int_{\partial\Omega} T_{ij}(\underline{x},\underline{y})\,u_j(\underline{y})\,da_y=\int_{\partial\Omega} U_{ij}(\underline{x},\underline{y})\cdot t_j(\underline{y})\,da_y \quad \ldots(15)$$

where both $\underline{x},\underline{y}\in\partial\Omega$ and for smooth boundaries

$$c_{ij}=\frac{1}{2}\delta_{ij} \quad \ldots(16)$$

Equation (15) is now ready for discretization.

3. DISCRETIZATION.

As was previously said we are using locally based functions for the interpolation of field values in order to maintain a physical significance of the parameters.

In a linear triangular element approach, for instance:

$$\underline{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \left[\begin{array}{ccc|ccc|ccc} N_1 & N_2 & N_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & N_1 & N_2 & N_3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \hline v_1 \\ v_2 \\ v_3 \\ \hline w_1 \\ w_2 \\ w_3 \end{Bmatrix}^e \qquad \ldots(17)$$

i.e.

$$\underline{u} = \underline{N}\,\underline{u}^e \qquad \ldots(18)$$

Similarly

$$\underline{t} = \begin{Bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{Bmatrix} = \underline{M}\,\underline{t}^e \qquad \ldots(19)$$

where the N_i, M_i are the well known interpolation functions used in F.E.M..

The sustitution of (18) and (19) into (15) enables one to write a system of 3-n equations if n is the number of nodes

$$\underline{A}\,\underline{u} = \underline{B}\,\underline{t} \qquad \ldots(20)$$

In a well posed mixed problem and after a reordering of data and unknowns (20) can be writen in the classical form

$$\underline{K}\,\underline{X} = \underline{F} \qquad \ldots(21)$$

where $\underline{X}$ is the vector of unknowns (in general with both stresses and displacements) and $\underline{F}$ collects the data. A routine solution of (21) completes the program, and the discretization of (12) produces the results at the soght interior points.

Unfortunatly $\underline{K}$ is neither banded (unless some precautions are taken) nor symmetric and this probably is the worse drawback of the method.

4. SAMPLE PROBLEM.

A simple problem has been run in order to show some of the method capabilities. It is a rectangular vessel with to openings, under axial loading.

In figure 1 the discretization used is shown. Rectangular elements of equal size have been used. The hypothesis of constant values for both stresses and displacements within each element has been done. The results obtained are assumed at the mid-point of the elements. The elastic characteristics of the material are $\nu = \frac{1}{3}$, $G = 1$ and the load is assumed to act regularly with unit intensity in order to show relative values.

In figure 1 the results of stresses along the horizontal plain of symmetry are shown. In order to compare the influence of the tridimensional effect we have also plotted the results got after a plain stress analysis of a similar problem. As can be seen due to the relative sizes of hole and shell there is practically no difference betwen both cases.

In figure 3 we have plotted the results of stresses along different horizontal and vertical lines. The influence of the isostatics reagroupment is clear near the corner of the hole.

In figure 4 the displacements at the boundary are shown. Again a comparison is established with respect to the plane stress case.

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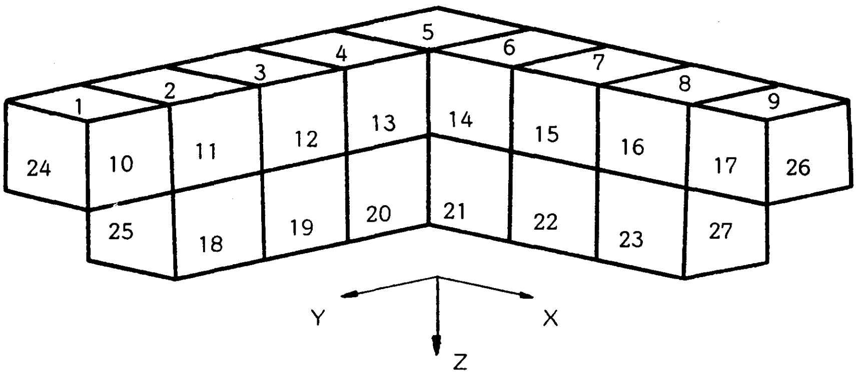
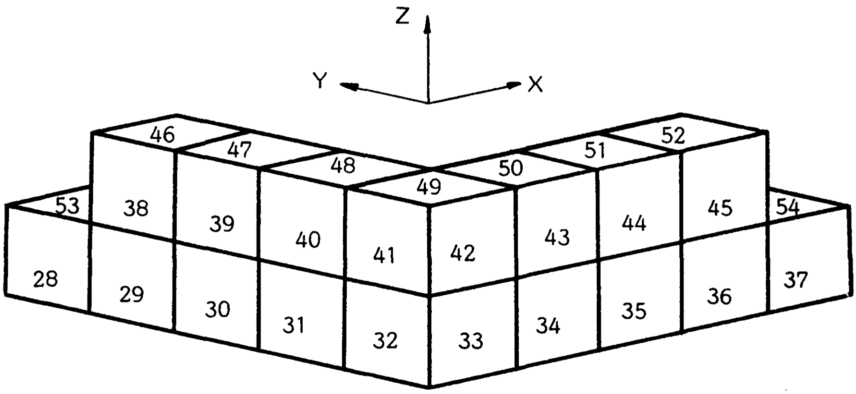
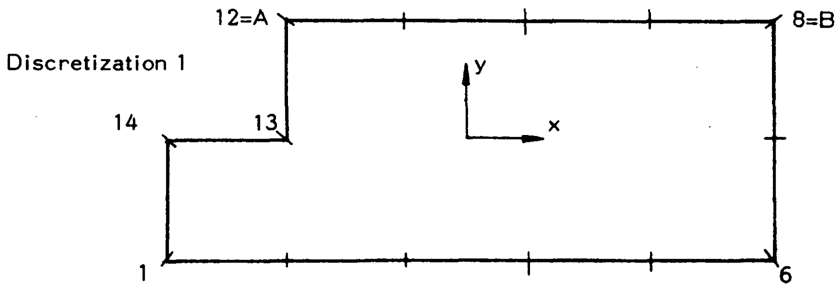
BOUNDARY DISCRETIZATION FOR

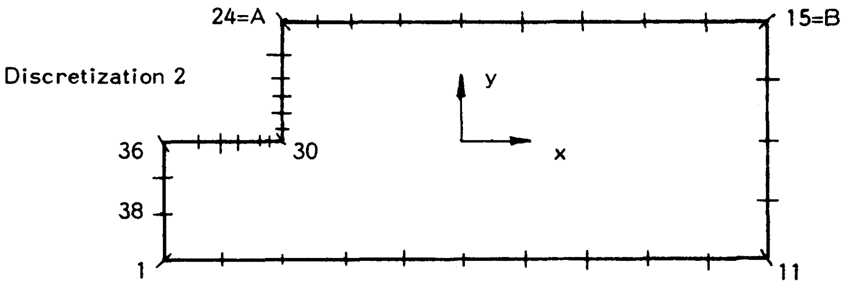
SHELL WITH OPENINGS

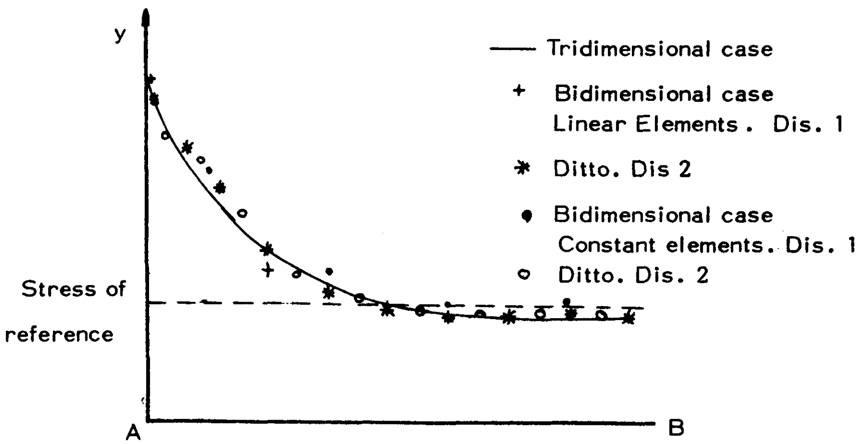


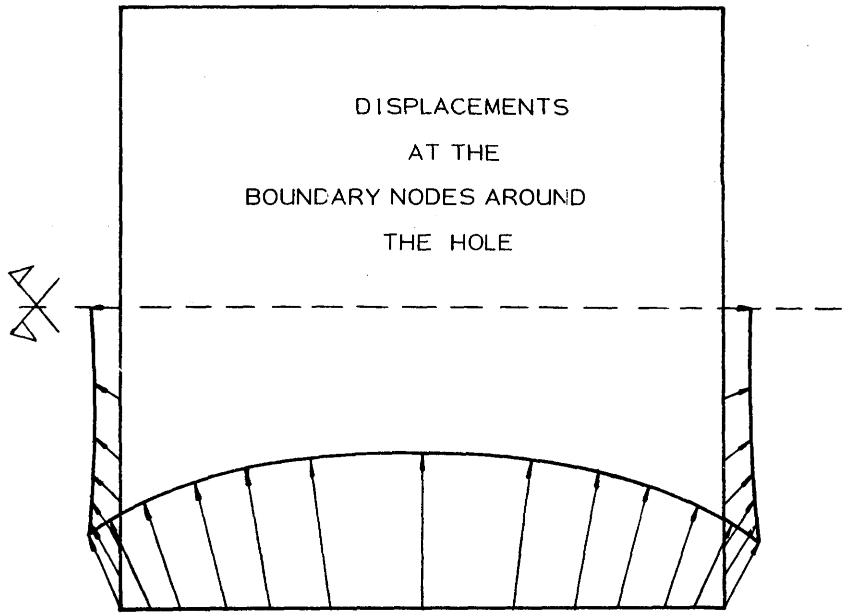


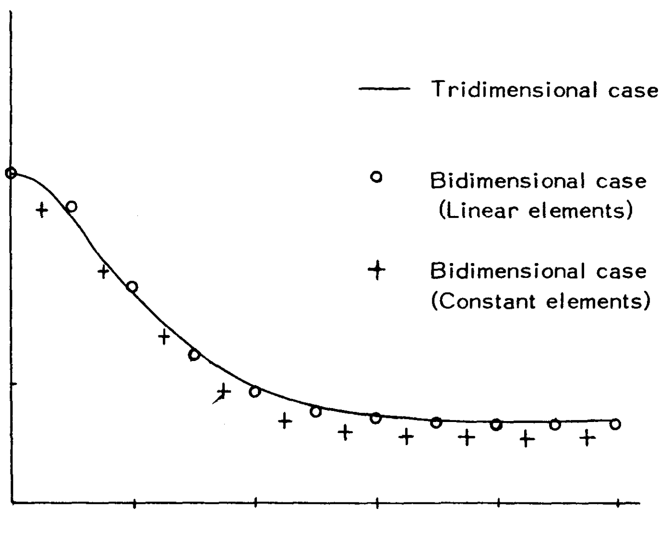
VERTICAL STRESSES ALONG HORIZONTAL MIDDLE PLANE



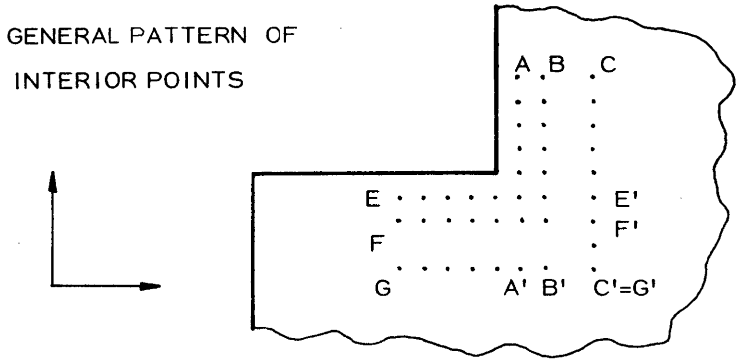


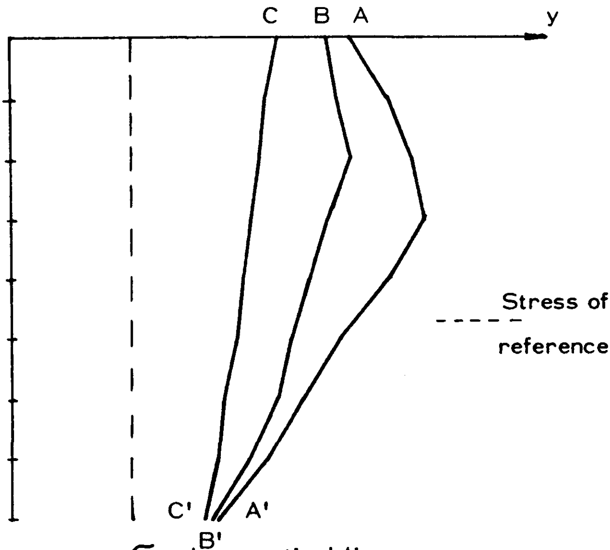




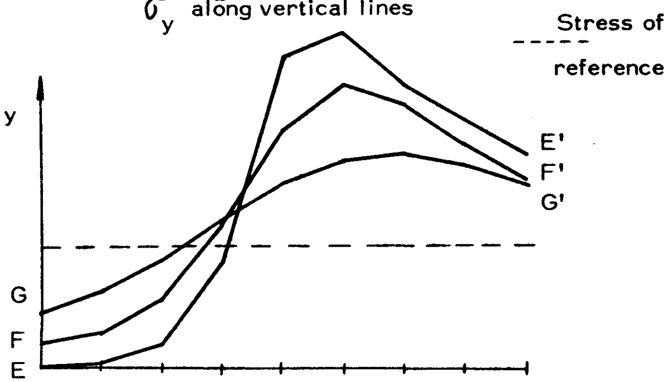


VERTICAL DISPLACEMENTS AT THE LOWER FREE BOUNDARY





σ_y along vertical lines



σ_y along horizontal lines