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TWO SCENARIOS ON A POTENTIAL SMOOTHNESS BREAKDOWN FOR THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS

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Abstract. In this paper we construct two families of initial data being arbitrarily large under any scaling-invariant norm for which their corresponding weak solution to the three-dimensional Navier-Stokes equations become smooth on either $[0, T_1]$ or $[T_2, \infty)$, respectively, where T_1 and T_2 are two times prescribed previously. In particular, T_1 can be arbitrarily large and T_2 can be arbitrarily small. Therefore, possible formation of singularities would occur after a very long or short evolution time, respectively.

We further prove that if a large part of the kinetic energy is consumed prior to the first (possible) blow-up time, then the global-in-time smoothness of the solutions follows for the two families of initial data.

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1. Introduction

The Cauchy problem of the Navier-Stokes equations for the flow of a viscous, incompressible, Newtonian fluid can be written as

$$\left\{ \begin{array}{rcll} \partial_t \boldsymbol{v} - \Delta \boldsymbol{v} + \nabla p + \boldsymbol{v}\cdot\nabla\boldsymbol{v} &=& \mathbf{0} & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \nabla\cdot\boldsymbol{v} &=& 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty). \end{array} \right. \tag{1}$$

Here $\boldsymbol{v}$ represents the velocity of the fluid and p its pressure. It should be noted that the density and the viscosity have been normalized, as is always possible, by the rescaling argument on the time and space variable $\boldsymbol{u}(\frac{\nu t}{\rho}, \frac{\nu \boldsymbol{x}}{\rho})$ and $\frac{1}{\rho} p(\frac{\nu t}{\rho}, \frac{\nu \boldsymbol{x}}{\rho})$.

To these equations we add an initial condition

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in} \quad \mathbb{R}^3, \tag{2}$$

where $\boldsymbol{v}_0$ is a smooth, divergence-free vector field.

Despite considerable effort invested by scientific community, the mechanisms governing the solutions to the three-dimensional Navier-Stokes equations remain unsolved. At the present time, we do not know yet whether smooth solutions to the three-dimensional Navier-Stokes on $\mathbb{R}^3$ exist for all time. In other words, we do not know whether there are initially smooth solutions with finite energy of the Navier-Stokes equations that develop singularities in finite time.

The mathematical existence theory developed so far supplies only partial answers to the smoothness of the Navier-Stokes equations. It is known that Navier-Stokes solutions are smooth on $[0, \infty)$ provided the initial velocity $\boldsymbol{v}_0$ satisfies a smallness condition for certain norm. Instead, if the initial data $\boldsymbol{v}_0$ are not assumed to be small, it is known that the time interval of existence is reduced to $[0, T)$, where T depends badly on some norm of $\boldsymbol{v}_0$.

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1.1. **Previous results.** In 1934 Leray in his ground-breaking paper [14] established the first result of local- and global-in-time existence of smooth solutions to the three-dimensional Navier-Stokes equations on $\mathbb{R}^3$. More precisely, Leray showed that there was a time interval $[0,T)$ for which $\boldsymbol{L}^\infty(\mathbb{R}^3)$-norm solutions existed and hence they were smooth. He also proved that the Navier-Stokes equations had smooth solutions for all time under a smallness condition for $\|\boldsymbol{v}_0\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\nabla\boldsymbol{v}_0\|_{\boldsymbol{L}^2(\mathbb{R}^3)}$ or $\|\boldsymbol{v}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\boldsymbol{v}_0\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}$. Since that time, there has been quite a vast literature addressing local- and global-in-time existence results in different contexts. We will briefly discuss some works for critical spaces, which are those whose associated norm is invariant under the scaling $\lambda\boldsymbol{u}(\lambda\boldsymbol{x},\lambda t^2)$ for all $\lambda>0$.

Fujita and Kato (1964) [6] established the local- and global-in-time existence of $\dot{\boldsymbol{H}}^{\frac{1}{2}}(\mathbb{R}^3)$-solutions. Twenty years later Kato [10] demonstrated that the three-dimensional Navier-Stokes equations are locally and globally well-posed in the $\boldsymbol{L}^3(\mathbb{R}^3)$ space. The smoothness of $\boldsymbol{L}^3(\mathbb{R}^3)$-solutions being Leray-Hopf weak solutions is due to Escauriaza, Seregin and Sverak (2003) [5].

Afterwards came the work of Cannone (1995) [2] in the Besov spaces $\dot{\boldsymbol{B}}^{-1+3/q}_{q,\infty}(\mathbb{R}^3)$ for $q<\infty$. The next progress was the work of Koch and Tataru (2001) [11] in the $BMO^{-1}(\mathbb{R}^3)$ space. Solving the Navier-Stokes problem in $\dot{\boldsymbol{B}}^{-1+3/q}_{q,\infty}(\mathbb{R}^3)$ or $BMO^{-1}(\mathbb{R}^3)$ allowed to construct highly oscillating initial data $\boldsymbol{v}_0$ with $\|\boldsymbol{v}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$ being large as long as $\|\boldsymbol{v}_0\|_{\dot{\boldsymbol{B}}^{-1+3/q}_{q,\infty}(\mathbb{R}^3)}$ or $\|\boldsymbol{v}_0\|_{BMO^{-1}(\mathbb{R}^3)}$ was small. Moreover, the smallness condition either on $\|\boldsymbol{v}_0\|_{\dot{\boldsymbol{B}}^{-1+3/q}_{q,\infty}(\mathbb{R}^3)}$ or $\|\boldsymbol{v}_0\|_{BMO^{-1}(\mathbb{R}^3)}$ led to global $\boldsymbol{L}^3(\mathbb{R}^3)$-solutions which combined with being Leray-Hopf solutions implied smoothness globally in time. Finally, we mention the work of Lei and Lin (2014) [12] who proved the global-in-time well-posedness of solutions in the scaling invariant space

$$\{\boldsymbol{f}\in\boldsymbol{\mathcal{D}}'(\mathbb{R}^3):\int_{\mathbb{R}^3}|\boldsymbol{\xi}|^{-1}|\mathcal{F}\boldsymbol{f}(\boldsymbol{\xi})|\mathrm{d}\boldsymbol{\xi}\},$$

where $\mathcal{F}$ stands for the Fourier transform.

A turning point appeared with the result of Bourgain and Pavlovic (2008) [1] dealing with the Navier-Stokes problem in $\dot{\boldsymbol{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. They showed that there were initial data in the Schwartz class $\boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ being arbitrarily small in $\dot{\boldsymbol{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ whose $\dot{\boldsymbol{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$-solutions become arbitrarily large after an arbitrarily short time. On the contrary, Chemin and Gallagher (2009) [3] showed that there existed global $\dot{\boldsymbol{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$-solutions if a certain nonlinear smallness condition was satisfied. These two last results broke the pattern followed for scaling invariant spaces in the above indicated references – Global-in-time well-posedness under a linear smallness condition for initial data. Even though Leray [14] already found nonlinear smallness conditions for proving the global-in-time existence of $\boldsymbol{L}^\infty(\mathbb{R}^3)$-solutions. In this sense, Robinson and Sadowski (2014)[16] have recently been published a result of local well-posedness under a smallness condition for $\|\boldsymbol{v}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\int_0^T\int_{\mathbb{R}^3}|\nabla\boldsymbol{u}(s)|^2|\boldsymbol{u}(s)|\mathrm{d}s$ where $\boldsymbol{u}(t)$ is the solution of the heat equation with the initial condition $\boldsymbol{v}_0$.

A change in the philosophy of constructing large initial data $\boldsymbol{v}_0$ was to look for special structures which allowed to prove global-in-time existence. In this sense, Mahalov and Nicolaenko (2003) [15] constructed large initial data $\boldsymbol{v}_0$ which transformed the Navier-Stokes equations into a rotating fluid equation. In such a setting, it is known that Navier-Stokes solutions are globally well-posed. Chemin and Gallagher (2009) [3] proposed initial data which varied slowly in one direction. In these two examples the global well-posedness of two-dimensional Navier-Stokes equations is the crucial point in their proof.

Since our results rely on different ways of perturbing the Navier-Stokes equations for obtaining large solutions, we would like to mention some related works that study the concept of stability of solutions in certain spaces. Gallagher (2001) [7] proved that, for any sequence of initial data, their corresponding solution can be decomposed into a sum of orthogonal profiles bounded in $\dot{\boldsymbol{H}}^{\frac{1}{2}}(\mathbb{R}^3)$ plus a remainder which is small with respect to the $\boldsymbol{L}^3(\Omega)$-norm. As a result, the stability of solutions in $\dot{\boldsymbol{H}}^{\frac{1}{2}}(\mathbb{R}^3)$ is proved for initial data in $\dot{\boldsymbol{H}}^{\frac{1}{2}}(\mathbb{R}^2) \cap \boldsymbol{L}^3(\mathbb{R}^3)$ being bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and providing $\boldsymbol{L}^3(\mathbb{R}^3)$-solutions. The space $\boldsymbol{L}^3(\mathbb{R}^3)$ could be changed by $\dot{\boldsymbol{B}}_{q,\infty}^{-1+3/q}(\mathbb{R}^3)$ or $BMO^{-1}(\mathbb{R}^3)$. This last result was extended, in [8], by Gallagher, Iftimie and Planchon (2003) to the stability of solutions in $\boldsymbol{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ and $\boldsymbol{L}^3(\mathbb{R}^3)$.

1.2. **The contribution of this paper.** Let us highlight the main contributions and how they differ from existing works concerning stability.

In this paper we will construct smooth initial data $\boldsymbol{v}_0$ being arbitrarily large in any critical space that do not develop singularities up to a given time T_1 without appealing to the two-dimensional Navier-Stokes equations. To achieve such a result we make use of Kato's technique. More precisely, the method of proof is based on mild-solution theory for proving the global-in-time existence of $\boldsymbol{L}^3(\mathbb{R}^3)$-solutions for small data $\boldsymbol{v}_0$. The main difference is that we do not directly impose a smallness condition on the $\boldsymbol{L}^3(\mathbb{R}^3)$-norm for $\boldsymbol{v}_0$. In doing so, we decompose the original problem into a Stokes problem with an initial datum $\boldsymbol{u}_0$ and a perturbed Navier-Stokes-like problem with an initial datum $\boldsymbol{w}_0$. From these two subproblems, we will prove that the three-dimensional Navier-Stokes problem possesses $\boldsymbol{L}^3(\mathbb{R}^3)$-solutions with initial data $\boldsymbol{v}_0 = \boldsymbol{u}_0 + \boldsymbol{w}_0$, where $\boldsymbol{u}_0$ has to be small concerning the $\boldsymbol{L}^3(\mathbb{R}^3)$-norm and $\boldsymbol{w}_0$ has to be small concerning the $\boldsymbol{L}^q(\mathbb{R}^3)$-norm. As a consequence, $\boldsymbol{v}_0$ is no longer small in any critical space such as $\dot{\boldsymbol{H}}^{\frac{1}{2}}(\mathbb{R}^2)$, $\boldsymbol{L}^3(\mathbb{R}^3)$, $\dot{\boldsymbol{B}}_{q,\infty}^{-1+\frac{3}{q}}(\mathbb{R}^3)$ or $BMO^{-1}(\mathbb{R}^3)$. This way we will rule out the smallness conditions for $\boldsymbol{v}_0$. The result of Escauriaza, Seregin, and Šverák is the final ingredient to conclude with the construction of large initial data $\boldsymbol{v}_0$ for the Navier-Stokes equations which provides smooth solutions on $[0, T_1]$, for T_1 being arbitrarily large. Consequently, the formation of potential singularities would have to be after T_1. This means that the system would preserve an enough amount of kinetic energy so that the solutions could blow up. On the other hand, if the $\boldsymbol{L}^2(\mathbb{R}^2)$-value of the vorticity would keep large without blowing up so that the kinetic energy would decay under a certain threshold on $[0, T_1]$, the solutions starting from our initial data remained smooth for all time.

Moreover, if a different decomposition of (1) into a Navier-Stokes problem and a perturbed Navier-Stokes-like problem is used, we will be able to prove that there exist Leray-Hopf weak solutions becoming smooth on $[T_2, \infty)$ for any given time T_2. Then we infer that potential singularities would have to occur on $(0, T_2)$, for T_2 being arbitrarily small. The most kinetic energy would be consumed on $(0, T_2)$ so that the solutions can not experience new singularities on $[T_2, \infty)$.

In this paper we do not use the perturbation theory as a way of studying stability of solutions but a way of constructing large solutions to the Navier-Stokes equations. Particularly, if we used the stability theory developed for some space X, with X being $\dot{\boldsymbol{H}}^{\frac{1}{2}}(\mathbb{R}^2)$, $\boldsymbol{L}^3(\mathbb{R}^3)$, $\dot{\boldsymbol{B}}_{q,\infty}^{-1+\frac{3}{q}}(\mathbb{R}^3)$ or $BMO^{-1}(\mathbb{R}^3)$, we would obtain that there exists a number ε (small enough) such that if $\|\boldsymbol{v}_0 - \boldsymbol{u}_0\| \le \varepsilon$ we have

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_X \le E\|\boldsymbol{v}_0 - \boldsymbol{u}_0\|_X \quad \text{for all} \quad t \in [0, T],$$

where $\varepsilon > 0$ and $E > 0$ depend on some energy norms of the solution $\boldsymbol{u}(t)$. This would provide that the perturbed solution $\boldsymbol{v}(t)$ would have an initial datum satisfying $\|\boldsymbol{v}_0\|_X \le \varepsilon + \|\boldsymbol{u}_0\|_X$. But in order for the solution $\boldsymbol{u}(t)$ to exist on $[0, T]$ one requires some smallness condition for $\boldsymbol{u}_0$. Then, the solution $\boldsymbol{v}(t)$ would inherit a smallness condition for $\boldsymbol{v}_0$ and hence would not be large.

2. Statement of Problem

2.1. **Notation.** As usual, $\boldsymbol{L}^p(\mathbb{R}^3)$, $1 \le p \le +\infty$, denotes the space of p-integrable, Lebesgue-measurable, $\mathbb{R}^3$-valued functions defined on $\mathbb{R}^3$, and $\boldsymbol{H}^1(\mathbb{R}^3)$ denotes the space of functions $\boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^3)$ such that $\nabla \boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^3)$, where ∇ is the gradient operator in the distributional sense. Moreover, $\boldsymbol{C}_c(\mathbb{R}^3 \times (0, T))$ is the space of infinitely continuously differentiable functions with compact supports in $\mathbb{R}^3 \times (0, T)$. The Schwartz space is denoted as $\boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ representing the space of rapidly decreasing infinitely continuously differentiable functions on $\mathbb{R}^3$.

For X a Banach space, $L^p(0, T; X)$ denotes the space of p-integrable, Bochner-measurable, X-valued functions on $(0, T)$.

We let $\mathcal{P}$ be the Helmholtz-Leray operator onto the space of divergence-free functions in $\boldsymbol{L}^p(\mathbb{R}^3)$ with $1 < p < \infty$.

2.2. **The Navier-Stokes equations.** In this paper the concept of *weak solutions* for the Navier-Stokes problem (1)–(2) will be understood in the sense of Leray and Hopf (see [14, 9]).

Definition 2.1. *A function $\boldsymbol{v}(t)$ is said to be a Leray-Hopf weak solution of problem (1)–(2) if:*

$$\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \quad \text{with} \quad \nabla\cdot\boldsymbol{v} = 0, \tag{3}$$

and

$$\begin{aligned} -\int_0^T\int_{\mathbb{R}^3} \boldsymbol{v}(s,\boldsymbol{x})\cdot\partial_t\boldsymbol{\varphi}(s,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\,\mathrm{d}s &+ \int_0^T\int_{\mathbb{R}^3} \nabla\boldsymbol{v}(s,\boldsymbol{x}):\nabla\boldsymbol{\varphi}(s,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ &+ \int_0^T\int_{\mathbb{R}^3} \boldsymbol{v}(s,\boldsymbol{x})\cdot\nabla\boldsymbol{v}(s,\boldsymbol{x})\cdot\boldsymbol{\varphi}(s,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = (\boldsymbol{u}_0,\boldsymbol{\varphi}(0)), \end{aligned} \tag{4}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{C}_c^\infty(\mathbb{R}^3\times[0,T))$ with $\nabla\cdot\boldsymbol{\varphi} = 0$. Moreover, the energy inequality

$$\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} + \int_0^t \|\nabla\boldsymbol{v}(s)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\,\mathrm{d}s \le \frac{1}{2}\|\boldsymbol{v}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} \tag{5}$$

holds a. e. in $[0,T]$.

Leray proved the global-in-time existence of weak solutions [14].

Theorem 2.1. *Let $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$ be a divergence-free vector field. Then there exists at least a Leray-Hopf weak solution to (1)–(2) on $[0,T]$.*

Next we introduce the concept of strong (or regular) solutions to (1)-(2).

Definition 2.2. *A weak solution $\boldsymbol{v}(t)$ to problem (1)–(2) is said to be a strong solution if there exists a number $M_{\boldsymbol{v}} > 0$ such that*

$$\sup_{t\in[0,T]} \|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le M_{\boldsymbol{v}}.$$

The key point for proving that solutions to the Navier-Stokes equations are smooth is to obtain that Leray-Hopf weak solutions are strong indeed, of course, for smooth initial data.

Here we announce our two main results.

Theorem 2.2. *Let $T > 1$ be given. Then there exist smooth, divergence-free initial data $\boldsymbol{v}_0$ arbitrarily large under any critical norm such that their corresponding Leray-Hopf solution $\boldsymbol{v}(t)$ to (1)–(2) is smooth on $[0,T]$.*

Theorem 2.3. *Let $0 < T < 1$ be given. Then there exist initial data $\boldsymbol{v}_0$ arbitrarily large under any critical norm such that there exists at least a Leray-Hopf solution $\boldsymbol{v}(t)$ to (1)–(2) which is smooth on $[T,\infty)$.*

Throughout this paper, different positive constants will appear due to interpolations and embeddings among spaces. Thus, C will always be the maximum of all of these constants in the previous steps, and K and K' will stand for constants depending on the initial data.

3. Proof of Theorem 2.2

In proving Theorem 2.2 we need to introduce a suitable approximation procedure so that all the estimates that follow are rigorously set up. To do this, we use a regularization *à la Leray.* That is, we replace the nonlinearity $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ by $(\rho_\varepsilon*\boldsymbol{v})\cdot\nabla\boldsymbol{v}$, where $\rho\in C_c^\infty(\mathbb{R}^3)$ such that $\rho\ge 0$ and $\int_{\mathbb{R}^3}\rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$ and $\rho_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^2}\rho(\frac{\boldsymbol{x}}{\varepsilon})$ for all $\varepsilon > 0$, to get

$$\left\{\begin{array}{rcll} \partial_t\boldsymbol{v}_\varepsilon - \Delta\boldsymbol{v}_\varepsilon + \nabla p_\varepsilon + (\rho_\varepsilon*\boldsymbol{v}_\varepsilon)\cdot\nabla\boldsymbol{v}_\varepsilon &=& \boldsymbol{0} & \text{in} \quad \mathbb{R}^3\times(0,\infty), \\ \nabla\cdot\boldsymbol{v}_\varepsilon &=& 0 & \text{in} \quad \mathbb{R}^3\times(0,\infty), \end{array}\right. \tag{6}$$

associated with the regularized initial condition $\boldsymbol{v}_\epsilon(0) = \boldsymbol{v}_0$. This procedure gives rise to a solution pair $(\boldsymbol{v}_\varepsilon, p_\varepsilon) \in \boldsymbol{C}^\infty(\mathbb{R}^3 \times [0,\infty)) \times C^\infty(\mathbb{R}^3 \times [0,\infty))$. On dealing with above equations, it is preferably better to avoid the pressure. For this, we apply the Helmholtz-Leray operator $\mathcal{P}$ to (6) to get

$$\begin{cases} \partial_t \boldsymbol{v}_\varepsilon - \Delta \boldsymbol{v}_\varepsilon + \mathcal{P}((\rho_\varepsilon * \boldsymbol{v}_\varepsilon) \cdot \nabla \boldsymbol{v}_\varepsilon) = \boldsymbol{0}, \\ \boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_0, \end{cases} \tag{7}$$

where we have utilized the fact that $-\mathcal{P}\Delta\boldsymbol{v} = -\Delta\mathcal{P}\boldsymbol{v} = -\Delta\boldsymbol{v}$ since $\mathcal{P}$ commutes with derivatives of any order.

From now on, for simplicity in exposition, we handle (7) without regularizing, although it must be taken into account in order to justify all the computations in this work.

Our first step is to modify equation (7) in order to easily produce a family of global smooth solutions. We first decompose (7) into two subproblems: a Stokes problem and a Navier-Stokes-like perturbation as follows. Let $\boldsymbol{u}$ be the solution to the Stokes problem

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} &= \boldsymbol{0}, \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, \end{cases} \tag{8}$$

and let $\boldsymbol{w}$ be the solution to the perturbation problem

$$\begin{cases} \partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}) + \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u}) + \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}) + \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) &= \boldsymbol{0}, \\ \boldsymbol{w}(0) &= \boldsymbol{w}_0. \end{cases} \tag{9}$$

Observe that defining $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ and adding (8) and (9), we obtain (7) for $\boldsymbol{v}_0 = \boldsymbol{u}_0 + \boldsymbol{w}_0$. In order to prove our main result, we need to write (8) and (9), by using the Fourier transform, as

$$\boldsymbol{u}(t) = K_t * \boldsymbol{u}_0 \tag{10}$$

and

$$\boldsymbol{w}(t) = K_t * \boldsymbol{w}_0 + \int_0^t K_{t-s} * (\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}) + \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}) + \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}) + \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}))\mathrm{d}s, \tag{11}$$

where $K_t = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4t}}$, for all $t > 0$, is the heat kernel.

At this point we emphasize that, from (10) and (11), we obtain the Duhamel integral form of (7):

$$\boldsymbol{v}(t) = K_t * \boldsymbol{v}_0 + \int_0^t K_{t-s} * (\mathcal{P}(\boldsymbol{v}\cdot\nabla\boldsymbol{v}))\mathrm{d}s, \tag{12}$$

with $\boldsymbol{v}_0 = \boldsymbol{u}_0 + \boldsymbol{w}_0$. The equivalence between equations (7) and (12) and equations (10) and (11) are ensured due to the regularity of $\boldsymbol{v}$ or, more precisely, $\boldsymbol{v}_\varepsilon$.

The following proposition is concerned with some properties of K_t. The proof is straightforward by using the properties of the convolution operator and the particular structure of K_t.

Proposition 3.1. *It follows that, for all $1 < p \le q < \infty$,*

$$\|K_t * \boldsymbol{f}\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \le C t^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}, \tag{13}$$

$$\|\nabla K_t * \boldsymbol{f}\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \le C t^{-(1+\frac{3}{p}-\frac{3}{q})\frac{1}{2}} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}, \tag{14}$$

where $C > 0$ is a constant that does not depend on $\boldsymbol{f}$.

Proof. We will use the following property for the convolution operator:

$$\|\boldsymbol{f} * \boldsymbol{g}\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \le \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\mathbb{R}^3)} \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \quad \text{for} \quad \frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{q}.$$

Then we have

$$\|K_t * \boldsymbol{f}\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \le \|K_t\|_{\boldsymbol{L}^r(\mathbb{R}^3)} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}.$$

The proof of (13) follows by observing that $\|K_t\|_{L^r(\mathbb{R}^3)} \le C t^{-(\frac{1}{p}-\frac{1}{q})}$. In the same way, we obtain that (14) holds since $\|\nabla K_t\|_{\boldsymbol{L}^r(\mathbb{R}^3)} \le C t^{-(1+\frac{3}{p}-\frac{3}{q})\frac{1}{2}}$.

□

By Hölder's inequality and Hodge's decomposition, we have

$$\|\mathcal{P}(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\boldsymbol{v}\|_{\boldsymbol{L}^r(\mathbb{R}^3)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^s(\mathbb{R}^3)}. \tag{15}$$

for $\frac{1}{p}=\frac{1}{r}+\frac{1}{s}$.

From now on, we will assume $3<q$ and $\frac{1}{p}=\frac{1}{q}+\frac{1}{3}$ which implies that $3>p>\frac{3}{2}$.

We will denote

$$\beta(a,b)=\int_0^1 \gamma^{a-1}(1-\gamma)^{b-1}$$

for all $a,b>0$.

Next we provide some estimates for the solution to problem (9) under a certain smallness condition for $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$, respectively.

Lemma 3.1. *Let $T>1$ be given, and let $\boldsymbol{u}_0\in\boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\boldsymbol{w}_0\in\boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be two divergence-free vector fields. Then there exists $K>0$ such that if*

$$T^{\frac{1}{2}}\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}<\frac{K}{2^{\frac{1}{2}}4} \tag{16}$$

and

$$\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}<\frac{K}{2^{\frac{1}{2}}4}, \tag{17}$$

we have

$$t^{\frac{1}{2}(1-\frac{3}{q})}\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\le K \tag{18}$$

and

$$t^{\frac{1}{2}}\|\nabla\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\le K \tag{19}$$

for all $t\in[0,T]$.

Proof. First of all, observe, from (13) and (14), that

$$\|K_t*\boldsymbol{w}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\le Ct^{-(1-\frac{3}{q})\frac{1}{2}}\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$$

and

$$\|\nabla K_t*\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\le Ct^{-\frac{1}{2}}\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}.$$

Next, assume that (18) and (19) hold. Then we will see that it requires that (16) and (17) are to be satisfied. Let us bound the right-hand side of (9). We have, by (13) and (15), that

$$\begin{aligned}
\|\int_0^t K_{t-s}*\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\mathrm{d}s\|_{\boldsymbol{L}^q(\mathbb{R}^3)} &\le \int_0^t\|K_{t-s}*\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\mathrm{d}s\\
&\le C\int_0^t(t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}\|\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s\\
&\le C\int_0^t(t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}\|\boldsymbol{w}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s\\
&\le CK^2\int_0^t(t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}s^{-\frac{1}{2}(1-\frac{3}{q})}s^{-\frac{1}{2}}\mathrm{d}s\\
&\le CK^2t^{-\frac{1}{2}(1-\frac{3}{q})}\beta(\tfrac{3}{2q},\tfrac{1}{2})\le CK^2t^{-\frac{1}{2}(1-\frac{3}{q})},
\end{aligned}$$

where we have utilized the change of variable $s = t\gamma$ to obtain $\beta(\frac{3}{2q}, \frac{1}{2})$. Analogously, we obtain, from $\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \leq \|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}$ and $\|\nabla\boldsymbol{u}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq \|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$:

$$\begin{aligned}
\int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\mathrm{d}s &\leq \int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}\|\mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\int_0^t (t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}s^{-\frac{1}{2}}\mathrm{d}s \\
&\leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\beta(\tfrac{1}{2},\tfrac{1}{2}) \\
&\leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}T^{\frac{1}{2}(1-\frac{3}{q})}t^{-\frac{1}{2}(1-\frac{3}{q})},
\end{aligned}$$

$$\begin{aligned}
\|\int_0^t K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\mathrm{d}s\|_{\boldsymbol{L}^q(\mathbb{R}^3)} &\leq \int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}\|\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}\|\boldsymbol{w}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\int_0^t (t-s)^{-(\frac{1}{p}-\frac{1}{q})\frac{3}{2}}s^{-\frac{1}{2}(1-\frac{3}{q})}\mathrm{d}s \\
&\leq CK\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}t^{\frac{1}{2}}t^{-\frac{1}{2}(1-\frac{3}{q})}\beta(\tfrac{1}{2}(1+\tfrac{3}{q}),\tfrac{1}{2}) \\
&\leq CK\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}T^{\frac{1}{2}}t^{-\frac{1}{2}(1-\frac{3}{q})}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\mathrm{d}s &\leq C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}t^{\frac{1}{2}}\beta(1,\frac{1}{2}) \\
&\leq C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}T^{1-\frac{3}{2q}}t^{-\frac{1}{2}(1-\frac{3}{q})}.
\end{aligned}$$

Applying the above estimates to (11), we obtain

$$\begin{aligned}
t^{\frac{1}{2}(1-\frac{3}{q})}\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^3)} &\leq C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + CT^{\frac{1}{2}(1-\frac{3}{q})}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}K \\
&\quad + CT^{\frac{1}{2}}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}K + CT^{1-\frac{3}{2q}}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}.
\end{aligned}$$

Since $T > 1$ and $q > 3$, we also have:

$$\begin{aligned}
t^{\frac{1}{2}(1-\frac{3}{q})}\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^3)} &\leq C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + CT^{\frac{1}{2}}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}K \\
&\quad + CT^{\frac{1}{2}}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}K + CT\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \\
&\leq C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + 2CT^{\frac{1}{2}}\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}, \|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}K \\
&\quad + CT\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}, \|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}^2.
\end{aligned}$$

Moreover, we have, by (14) and (15), that

$$\begin{aligned}
\int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\leq \int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2p}}\|\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2p}}\|\boldsymbol{w}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK^2\int_0^t (t-s)^{-\frac{3}{2p}}s^{-\frac{1}{2}(1-\frac{3}{q})}s^{-\frac{1}{2}}\mathrm{d}s \\
&\leq CK^2t^{-\frac{1}{2}}\beta(\tfrac{1}{2}(1-\tfrac{3}{q}),\tfrac{3}{2q}) \leq CK^2t^{-\frac{1}{2}}.
\end{aligned}$$

Analogously,

$$\begin{aligned}\int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\le \int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{3}{2p}}\|\mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{3}{2p}}\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s\\
&\le CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\int_0^t (t-s)^{-\frac{3}{2p}}s^{-\frac12}\mathrm{d}s\\
&\le CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}t^{\frac12(1-\frac3p)}\beta(\tfrac12,1-\tfrac{3}{2p})\\
&\le CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}T^{1-\frac{3}{2p}}t^{-\frac12},\end{aligned}$$

$$\begin{aligned}\int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\le \int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{3}{2p}}\|\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-\frac{3}{2p}}\|\boldsymbol{w}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s\\
&\le CK\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\int_0^t (t-s)^{-\frac{3}{2p}}s^{-\frac12(1-\frac3q)}\mathrm{d}s\\
&\le CK\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\beta(\tfrac12+\tfrac{3}{2p},1-\tfrac{3}{2p})\\
&\le CK\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}T^{\frac12}t^{-\frac12}\end{aligned}$$

and

$$\int_0^t \|\nabla K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \le C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}t^{-\frac12}T^{\frac32(1-\frac1p)}.$$

Applying the above estimates to (11), we obtain

$$\begin{aligned}t^{\frac12}\|\nabla\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} &\le C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + CT^{1-\frac{3}{2p}}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}K\\
&\quad + CT^{\frac12}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}K + CT^{\frac32(1-\frac1p)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}.\end{aligned}$$

From the relation $\frac1p = \frac1q + \frac13$, we write

$$\begin{aligned}t^{\frac12}\|\nabla\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} &\le C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + CT^{\frac12(1-\frac3q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}K\\
&\quad + CT^{\frac12}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}K + CT^{1-\frac{3}{2q}}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\end{aligned}$$

and hence

$$\begin{aligned}t^{\frac12}\|\nabla\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} &\le C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + CT^{\frac12}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}K\\
&\quad + CT^{\frac12}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}K + CT\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}.\\
&\le C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + 2CT^{\frac12}\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}K\\
&\quad + CT\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}^2.\end{aligned}$$

To close the bootstrap argument, we need to find $K>0$ such that

$$\begin{aligned}K &= C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + 2CT^{\frac12}\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}K\\
&\quad + CT\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}^2.\end{aligned}$$

or, equivalently,

$$\begin{aligned}0 &= C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + (2CT^{\frac12}\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}-1)K\\
&\quad + CT\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}^2.\end{aligned}$$

Let us choose $2CT^{\frac12}\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}-1 < -\frac12$. As a result, we obtain that K satisfies

$$0 < K \le \frac{\frac12 - \sqrt{\frac14 - 4C^2(\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + T\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)},\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}^2)}}{2C}.$$

Let us choose $\frac{1}{4} - 4C^2(\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + T\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}, \|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}^2) > 0$. From conditions (16) and (17) for $\max\{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\mathbb{R}^3)}, \|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\}$ and $\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$, the two above conditions hold. Thus, (18) and (19) are also satisfied. It completes the proof. □

As a consequence of Lemma 3.1, we will infer an estimate uniform in time for the $\|\cdot\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$-norm of $\boldsymbol{w}$ on [0,T].

Lemma 3.2. *Let $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\boldsymbol{w}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be two divergence-free vector fields satisfying (16) and (17). Then there exists a number $M_{\boldsymbol{w}} > 0$ such that the solution $\boldsymbol{w}(t)$ to (9) satisfies*

$$\sup_{t\in[0,T]} \|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq M_{\boldsymbol{w}}.$$

Proof. From (11), we have

$$\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq \|K_t*\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + \int_0^t \|K_{t-s}*(\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})+\mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})+\mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})+\mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}))\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s.$$

Let us now bound each term on the right-hand side. We have, by (13), (16), and (17), that

$$\|K_t * \boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$$

$$\begin{aligned}
\int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\leq C\int_0^t (t-s)^{-(\frac{3}{p}-1)\frac{1}{2}}\|\boldsymbol{w}\cdot\nabla\boldsymbol{w}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2q}}\|\boldsymbol{w}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK^2\int_0^t (t-s)^{-\frac{3}{2q}}s^{-(1-\frac{3}{q})\frac{1}{2}}s^{-\frac{1}{2}}\mathrm{d}s \\
&\leq CK^2\beta(\tfrac{1}{2}(2-\tfrac{3}{q}), \tfrac{3}{q}) \leq CK^2,
\end{aligned}$$

$$\begin{aligned}
\int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\leq C\int_0^t (t-s)^{-(\frac{3}{p}-1)\frac{1}{2}}\|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2q}}\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK\int_0^t (t-s)^{-\frac{3}{2q}}s^{-(1-\frac{3}{q})\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}s^{-\frac{1}{2}}\mathrm{d}s \\
&\leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\beta(\tfrac{1}{2}(2-\tfrac{3}{q}), \tfrac{3}{q}) \leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)},
\end{aligned}$$

$$\begin{aligned}
\int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\leq C\int_0^t (t-s)^{-(\frac{3}{p}-1)\frac{1}{2}}\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2q}}\|\boldsymbol{w}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK\int_0^t (t-s)^{-\frac{3}{2q}}s^{-\frac{1}{2}(1-\frac{3}{q})}s^{-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\beta(\tfrac{1}{2}(2-\tfrac{3}{q}), \tfrac{3}{q}) \leq CK\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|K_{t-s} * \mathcal{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s &\leq C\int_0^t (t-s)^{-(\frac{3}{p}-1)\frac{1}{2}}\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2q}}\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2q}}s^{-(\frac{1}{3}-\frac{1}{q})\frac{3}{2}}s^{-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}s \\
&\leq C\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^3(\mathbb{R}^3)}\beta(\tfrac{1}{2}(2-\tfrac{3}{q}), \tfrac{3}{q}) \leq C\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^3(\mathbb{R}^3)}.
\end{aligned}$$

Therefore, we obtain

$$\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 + 2C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}K + C\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^3(\mathbb{R}^3)} := M_{\boldsymbol{w}}. \tag{20}$$

□

In view of Lemmas 3.1 and 3.2, we have proved the existence of an $\boldsymbol{L}^3(\mathbb{R}^3)$-solution to (7) on $[0,T]$ under certain smallness conditions for $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$.

Lemma 3.3. *Let $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\boldsymbol{w}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be two divergence-free vector fields satisfying (16) and (17), respectively. Then there exists $M_{\boldsymbol{v}} > 0$ such that the solution $\boldsymbol{v}(t)$ to (7) with $\boldsymbol{v}_0 = \boldsymbol{u}_0 + \boldsymbol{w}_0$ satisfies*

$$\sup_{t\in[0,T]} \|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le M_{\boldsymbol{v}},$$

Proof. First notice that, from (10), we have $\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le \|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} := M_{\boldsymbol{u}}$ for $t \in [0,T]$. By Lemma 3.2, we have $\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le M_{\boldsymbol{w}}$ for all $t \in [0,T]$. Therefore, if we define $\boldsymbol{v}(t) = \boldsymbol{u}(t) + \boldsymbol{w}(t)$, we obtain $\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le M_{\boldsymbol{u}} + M_{\boldsymbol{w}} := M_{\boldsymbol{v}}$ for all $t \in [0,T]$, where $\boldsymbol{v}$ satisfies (7) on $[0,T]$ with $\boldsymbol{v}_0 = \boldsymbol{u}_0 + \boldsymbol{w}_0$. □

Remark 3.1. *It is not hard to see that the above estimate obtained for the regularized solutions are independent of ε and hence are also true for the solutions of the unregularized Navier-Stokes equations. From now on, we are allowed to work without any regularization procedure.*

Our next goal is to provide a family of smooth initial data $\boldsymbol{v}_0$ which can be split into $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ satisfying (16) and (17), respectively. In doing so, we take advantage of the "scissors effect " of the scaling property of the Navier-Stokes solutions. That is, we will use different scalings for $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ so that supercritical and subcritical norms increase and decrease oppositely with the $\boldsymbol{L}^3(\mathbb{R}^3)$-norm being invariant. This way we avoid that the size of any norm of $\boldsymbol{v}_0$ is no longer small but large.

Let $\boldsymbol{\vartheta}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be a divergence-free vector field. We are allowed to take $\varepsilon > 0$ such that $\boldsymbol{\vartheta}_0 = (1-\varepsilon)\boldsymbol{\vartheta}_0 + \varepsilon\boldsymbol{\vartheta}_0 := \boldsymbol{u}_{0,\varepsilon} + \boldsymbol{w}_{0,\varepsilon}$ so that $\boldsymbol{w}_{0,\varepsilon}$ satisfies condition (17). Next we define $\boldsymbol{u}_{0,\varepsilon}^{\tilde\lambda} = \tilde\lambda \boldsymbol{u}_{0,\varepsilon}(\tilde\lambda \boldsymbol{x})$ and $\boldsymbol{w}_{0,\varepsilon}^{\hat\lambda} = \hat\lambda \boldsymbol{w}_{0,\varepsilon}(\hat\lambda \boldsymbol{x})$ for $\tilde\lambda, \hat\lambda > 0$. Letting $\tilde\lambda$ go to 0, we find that there exists $\tilde\lambda_0$ such that, for all $\tilde\lambda \le \tilde\lambda_0$, it follows that condition (16) holds for $\boldsymbol{u}_{0,\varepsilon}^{\tilde\lambda}$. Moreover, for any $\hat\lambda$, we find that $\boldsymbol{w}_{0,\varepsilon}^{\tilde\lambda}$ fulfills condition (17) since the $\boldsymbol{L}^3(\mathbb{R}^3)$-norm is scaling invariant. This last rescaling is not really necessary, but it allows us to construct initial data arbitrarily large under any supercritical norm. Thus we define $\boldsymbol{v}_0 = \boldsymbol{u}_{0,\varepsilon}^{\tilde\lambda} + \boldsymbol{w}_{0,\varepsilon}^{\hat\lambda}$ whose $\boldsymbol{L}^3(\mathbb{R}^3)$-norm remains almost invariant due to our special choice, i.e. $\|\boldsymbol{v}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le \|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + \|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le (1-\varepsilon)\|\boldsymbol{\vartheta}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + \varepsilon\|\boldsymbol{\vartheta}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} = \|\boldsymbol{\vartheta}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}$. Instead, supercritical norms can be arbitrarily large by doing $\hat\lambda$ to tend to ∞ and subcritical norms by doing $\tilde\lambda$ to tend to 0.

Another possibility to construct smooth initial data $\boldsymbol{v}_0$ is as follows. Consider $\tilde{\boldsymbol{u}}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\tilde{\boldsymbol{w}}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ to be two divergence-free vector fields and define $\boldsymbol{v}_0 = \tilde{\boldsymbol{u}}_0^{\tilde\lambda} + \varepsilon\tilde{\boldsymbol{w}}_0^{\lambda} := \boldsymbol{u}_0^{\tilde\lambda} + \boldsymbol{w}_{0,\varepsilon}$. Pick $\tilde\lambda$ to be such that $\boldsymbol{u}_0^{\tilde\lambda}$ satisfies condition (16) and ε to be such that $\boldsymbol{w}_{0,\varepsilon}$ satisfies condition (17).

The following theorem was proved in [5] by Escauriaza, Seregin, and Šverák.

Theorem 3.1. *Let $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be a divergence-free vector field. Assume that $\boldsymbol{v}(t)$ is a weak Leray-Hopf solution to (1)–(2) and satisfies the additional condition*

$$\sup_{t\in[0,T]} \|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < \infty.$$

Then $\boldsymbol{v}(t)$ is a strong solution to (1)–(2) on $[0,T]$.

Therefore, Lemma 3.3 and Theorem 3.1 combined with Theorem 2.1 give that the solutions $\boldsymbol{v}(t)$ whose initial data $\boldsymbol{v}_0$ can be decomposed as, for instance, $\boldsymbol{v}_0 = \boldsymbol{u}_{0,\varepsilon}^{\tilde\lambda} + \boldsymbol{w}_{0,\varepsilon}^{\hat\lambda}$ with $\boldsymbol{u}_{0,\varepsilon}^{\tilde\lambda} \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\boldsymbol{w}_{0,\varepsilon}^{\hat\lambda} \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ being divergence-free vector fields fulfilling (16) (for certain $\tilde\lambda$) and (17) (for certain ε) are strong, and hence they are smooth on $[0,T]$. It proves Theorem 2.2.

Remark 3.2. *It is easy to see that the solutions given in Theorem 2.2 satisfy the estimate:*

$$\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \le (K + C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)})t^{-\frac{1}{2}(1-\frac{3}{q})} \quad \text{for all } t \in [0,T].$$

This implies that $\|\boldsymbol{v}(T)\|_{\boldsymbol{L}^q(\mathbb{R}^3)}$ can be as small as required provided that T is large. As a result, we can extend our solution to $[0,T^*)$ for T^* being possible large. See [13, Thm 15.3].

4. Proof of Theorem 2.3

We first decompose (7) as follows. Let $\boldsymbol{w}_\varepsilon$ be the solution to the Navier-Stokes problem

$$\left\{ \begin{array}{rcl} \partial_t \boldsymbol{w}_\varepsilon - \Delta \boldsymbol{w}_\varepsilon + \mathcal{P}((\rho_\varepsilon * \boldsymbol{w}_\varepsilon)\cdot\nabla \boldsymbol{w}_\varepsilon) & = & \mathbf{0}, \\ \boldsymbol{w}_\varepsilon(0) & = & \boldsymbol{w}_0, \end{array} \right. \tag{21}$$

and let $\boldsymbol{u}_\varepsilon$ be the solution to the perturbation problem

$$\left\{ \begin{array}{rcl} \partial_t \boldsymbol{u}_\varepsilon - \Delta \boldsymbol{u}_\varepsilon + \mathcal{P}((\rho_\varepsilon * \boldsymbol{w}_\varepsilon)\cdot\nabla \boldsymbol{u}_\varepsilon) + \mathcal{P}((\rho_\varepsilon * \boldsymbol{u}_\varepsilon)\cdot\nabla \boldsymbol{w}_\varepsilon) + \mathcal{P}((\rho_\varepsilon * \boldsymbol{u}_\varepsilon)\cdot\nabla \boldsymbol{u}_\varepsilon) & = & \mathbf{0}, \\ \boldsymbol{u}_\varepsilon(0) & = & \boldsymbol{u}_0. \end{array} \right. \tag{22}$$

As before, we drop the subscript ε and the convolution operator from (21) and (22).

The following result is a consequence of Lemmas 3.1 and 3.2. In particular, we assume that we have $C > 1$ in Lemma 3.1.

Lemma 4.1. *Let $\boldsymbol{w}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be a divergence-free vector field such that*

$$\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq \frac{1}{4C}. \tag{23}$$

Then there exists a smooth solution $\boldsymbol{w}(t)$ to (21) on $[0,\infty)$ such that

$$\sup_{t\in[0,\infty)} \|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < K := \frac{1-\sqrt{1-4C^2\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}}}{2C}. \tag{24}$$

Proof. From (8) for $\boldsymbol{u}_0 = \mathbf{0}$ and (9) , we recover (21). By following the proof of Lemma 3.1, we obtain that (18) and (19) hold for K such that

$$0 = C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + CK^2 - K.$$

Therefore,

$$K = \frac{1-\sqrt{1-4C^2\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}}}{2C}.$$

In virtue of (20), we obtain (24). □

Lemma 4.2. *Let $0 < T < 1$ be given. Let $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\boldsymbol{w}_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ be two divergence-free vector fields such that*

$$\max_{t\in[0,T]} \|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < \frac{1}{8C} \tag{25}$$

and

$$T^{-\frac{1}{4}}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\mathbb{R}^3)} < \frac{1}{8C}. \tag{26}$$

Then there exists $t^* \in (0,T]$ such that

$$\|\boldsymbol{u}(t^*)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < \frac{1}{8C}. \tag{27}$$

Proof. Multiplying (22) by $\boldsymbol{u}$ and integrating over $\mathbb{R}^3$ gives, after integration by parts,

$$\begin{array}{rcl} \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} + \|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} & = & \displaystyle\int_{\mathbb{R}^3} \boldsymbol{u}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = -\int_{\mathbb{R}^3} \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \\ & \leq & \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \\ & \leq & C\|\boldsymbol{w}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}. \end{array}$$

From (25), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} + \|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} \leq 0.$$

Integrating with respect to time, we get

$$\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} + \int_0^T \|\nabla\boldsymbol{u}(s)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\mathrm{d}s \leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)},$$

whence

$$\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+2\int_0^T\|\nabla\boldsymbol{u}(s)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\mathrm{d}s\le C\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

By interpolation, we write

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\le C\|\boldsymbol{u}(t)\|^{\frac12}_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\nabla\boldsymbol{u}(t)\|^{\frac12}_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Therefore,

$$\int_0^T\|\boldsymbol{u}(t)\|^4_{\boldsymbol{L}^3(\mathbb{R}^3)}\mathrm{d}t\le C\|\boldsymbol{u}_0\|^4_{\boldsymbol{L}^2(\mathbb{R}^3)}$$

and hence

$$T\inf_{s\in[0,T]}\|\boldsymbol{u}\|^4_{\boldsymbol{L}^3(\mathbb{R}^3)}\le C\|\boldsymbol{u}_0\|^4_{\boldsymbol{L}^2(\mathbb{R}^3)}$$

and

$$\inf_{s\in[0,T]}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(\mathbb{R}^3)}\le CT^{-\frac12}\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

If conditions (25) and (26) hold, there exists $t^*\in(0,T]$ such that condition (27) is satisfied. □

In order for condition (25) to hold, we need

$$1-\sqrt{1-4C^2\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}}<\frac14, \tag{28}$$

which holds from (24). Let us choose $\tilde\lambda$ and ε such that $\boldsymbol{v}_0=\boldsymbol{u}^{\tilde\lambda}_{0,\varepsilon}+\boldsymbol{w}^{\hat\lambda}_{0,\varepsilon}$ with $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ satisfying (26) and (28), respectively. Thus, we arrive at

$$\|\boldsymbol{v}(t^*)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\le\|\boldsymbol{w}(t^*)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}+\|\boldsymbol{u}(t^*)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}<\frac{1}{4C}.$$

Then, by Lemma 4.1, we obtain

$$\sup_{t\in[T,\infty)}\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\le\frac{1}{2C}.$$

since $\boldsymbol{v}(t)$ is a solution of the regularized Navier-Stokes equations as $\boldsymbol{w}$.

As a result of Theorem 3.1, we have accomplished to prove that the unregularized solutions $\boldsymbol{v}(t)$ whose initial data $\boldsymbol{v}_0$ can be decomposed as $\boldsymbol{v}_0=\boldsymbol{u}^{\tilde\lambda}_{0,\varepsilon}+\boldsymbol{w}^{\hat\lambda}_{0,\varepsilon}$ with $\boldsymbol{u}^{\tilde\lambda}_{0,\varepsilon}\in\boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ and $\boldsymbol{w}^{\hat\lambda}_{0,\varepsilon}\in\boldsymbol{\mathcal{S}}(\mathbb{R}^3)$ being divergence-free vector fields fulfilling (28) and (26), respectively, are strong, and hence they are smooth on $[T,\infty)$. We have used the same decomposition for $\boldsymbol{v}_0$ as in the proof of Theorem 2.2. This way our initial conditions are arbitrarily large under any critical norm. It proves Theorem 2.3.

5. Additional results

To complete the proof of Theorem 2.2 we show there exist initial data $\boldsymbol{v}_0$ which can not be *a priori* decomposed as above. To do this, we just need to use, for instance, an $\boldsymbol{L}^3(\mathbb{R}^3)$-stability result. The proof combines ideas from [5] for establishing local-in-time existence of $\boldsymbol{L}^3(\mathbb{R}^3)$-solutions and from [8] for proving stability in Bevov spaces.

Theorem 5.1. *Let $\boldsymbol{v}(t)$ be a smooth solution to (1)–(2) with an initial datum $\boldsymbol{v}_0=\boldsymbol{u}_0+\boldsymbol{w}_0$, where $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ are two smooth, divergence-free vector fields fulfilling (16) and (17), respectively. Then there exists $\varepsilon=\varepsilon(\boldsymbol{v})$ such that, for all initial data $\tilde{\boldsymbol{v}}_0$ with $\|\boldsymbol{v}_0-\tilde{\boldsymbol{v}}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}<\varepsilon$, the corresponding solution $\tilde{\boldsymbol{v}}(t)$ with $\tilde{\boldsymbol{v}}(0)=\tilde{\boldsymbol{v}}_0$ satisfies*

$$\|\boldsymbol{v}(t)-\tilde{\boldsymbol{v}}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\le C(\boldsymbol{v})\|\boldsymbol{v}_0-\tilde{\boldsymbol{v}}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\text{ for all }t\in[0,T]. \tag{29}$$

Proof. The proof is divided into two parts:

Part I: A priori estimates

To start with, define $\boldsymbol{w}(t):=\boldsymbol{v}(t)-\tilde{\boldsymbol{v}}(t)$ to be the solution to

$$\left\{\begin{array}{rcl}\partial_t\boldsymbol{w}-\Delta\boldsymbol{w}+\nabla q+\boldsymbol{v}\cdot\nabla\boldsymbol{w}+\boldsymbol{w}\cdot\nabla\boldsymbol{v}+\boldsymbol{w}\cdot\nabla\boldsymbol{w}&=&\boldsymbol{0},\\ \boldsymbol{w}(0)&=&\boldsymbol{w}_0:=\boldsymbol{v}_0-\tilde{\boldsymbol{v}}_0.\end{array}\right. \tag{30}$$

Testing by $|\boldsymbol{w}|\boldsymbol{w}$, we obtain

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\|\boldsymbol{w}\|^3_{\boldsymbol{L}^3(\mathbb{R}^3)}+\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\frac{4}{9}\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}\nabla q\cdot|\boldsymbol{w}|\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\\ &-\int_{\mathbb{R}^3}\nabla\cdot(\boldsymbol{v}\boldsymbol{w}+\boldsymbol{w}\boldsymbol{v}+\boldsymbol{w}\boldsymbol{w})\cdot|\boldsymbol{w}|\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\end{aligned}$$

Integrating by parts, we estimate each term on the right hand side as follows. For the pressure term, applying the divergence operator to (30), we first observe that

$$-\Delta q=\nabla\cdot\nabla\cdot(\boldsymbol{u}\boldsymbol{w}+\boldsymbol{u}\boldsymbol{w}+\boldsymbol{w}\boldsymbol{w})\quad\text{in}\quad\mathbb{R}^3. \tag{31}$$

The Calderon-Zygmund inequality applied to (31) (see [16, Lm 5.1] for a proof) implies that

$$\|q\|_{\boldsymbol{L}^{\frac{5}{2}}(\mathbb{R}^3)}\le C\|\boldsymbol{w}\|_{\boldsymbol{L}^5(\mathbb{R}^3)}(\|\boldsymbol{w}\|_{\boldsymbol{L}^5(\mathbb{R}^3)}+\|\boldsymbol{v}\|_{\boldsymbol{L}^5(\mathbb{R}^3)}).$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla q\cdot|\boldsymbol{w}|\boldsymbol{w}\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^3}q\nabla\cdot(\boldsymbol{w}|\boldsymbol{w}|)\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^3}q\nabla\cdot\boldsymbol{w}|\boldsymbol{w}|\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3}q\boldsymbol{w}\nabla\boldsymbol{w}\frac{\boldsymbol{w}}{|\boldsymbol{w}|}\mathrm{d}\boldsymbol{x}\\ &\le \|q\|^{\frac{1}{2}}_{\boldsymbol{L}^{\frac{5}{2}}(\mathbb{R}^3)}\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}.\end{aligned}$$

Next the interpolation inequality $\|\cdot\|_{\boldsymbol{L}^{\frac{10}{3}}(\mathbb{R}^3)}\le C\|\cdot\|^{\frac{2}{5}}_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\nabla\cdot\|^{\frac{3}{5}}_{\boldsymbol{L}^2(\mathbb{R}^3)}$ leads to

$$\|\boldsymbol{w}\|_{\boldsymbol{L}^5(\mathbb{R}^3)}=\||\boldsymbol{w}|^{\frac{3}{2}}\|^{\frac{2}{3}}_{\boldsymbol{L}^{\frac{10}{3}}(\mathbb{R}^3)}\le C\|\boldsymbol{w}\|^{\frac{2}{5}}_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^{\frac{2}{5}}_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{32}$$

From (32) and Young's inequality, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla q\cdot|\boldsymbol{w}|\boldsymbol{w}\mathrm{d}\boldsymbol{x} &\le C\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}(\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}+\|\boldsymbol{v}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^5)})\\ &+\gamma\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\delta\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.\end{aligned}$$

The other term for the pressure term is also bounded as:

$$\begin{aligned}\int_{\mathbb{R}^3}q\boldsymbol{w}\nabla\boldsymbol{w}\frac{\boldsymbol{w}}{|\boldsymbol{w}|}\mathrm{d}\boldsymbol{x} &\le C\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}(\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}+\|\boldsymbol{v}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^5)})\\ &+\gamma\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\delta\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.\end{aligned}$$

In the same way, we bound the remainder terms:

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla\cdot(\boldsymbol{v}\boldsymbol{w})\cdot|\boldsymbol{w}|\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^3}\boldsymbol{v}\boldsymbol{w}\nabla\boldsymbol{w}|\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x}-\int_{\mathbb{R}^3}\boldsymbol{v}\boldsymbol{w}\nabla\boldsymbol{w}\frac{\boldsymbol{w}}{|\boldsymbol{w}|}\,d\boldsymbol{x}\\ &\le C\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}\|\boldsymbol{v}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}\\ &\quad+\gamma\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\delta\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)},\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla\cdot(\boldsymbol{w}\boldsymbol{v})\cdot|\boldsymbol{w}|\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} &\le C\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}\|\boldsymbol{v}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}\\ &\quad+\gamma\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\delta\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla\cdot(\boldsymbol{w}\boldsymbol{w})\cdot|\boldsymbol{w}|\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} &\le C\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|^5_{\boldsymbol{L}^5(\mathbb{R}^3)}\\ &\quad+\gamma\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\delta\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.\end{aligned}$$

Adjusting γ and δ adequately, we find that

$$\frac{1}{3}\frac{d}{dt}\|\boldsymbol{w}\|^3_{\boldsymbol{L}^3(\mathbb{R}^3)}+\frac{1}{3}\||\boldsymbol{w}|^{\frac{1}{2}}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\frac{2}{9}\|\nabla|\boldsymbol{w}|^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\le C\|\boldsymbol{w}\|^{\frac{1}{2}}_{\boldsymbol{L}^3(\mathbb{R}^3)}(\|\boldsymbol{w}\|^5_{\boldsymbol{L}^5(\mathbb{R}^3)}+\|\boldsymbol{w}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}\|\boldsymbol{v}\|^{\frac{5}{2}}_{\boldsymbol{L}^5(\mathbb{R}^3)}).$$

Integrating over (T_i, T_{i+1}), where $\{T_i\}_{i=1}^M$ are to be determined later on, yields

$$\begin{array}{rcl}\|\boldsymbol{w}(t)\|^3_{\boldsymbol{L}^3(\mathbb{R}^3)} & \le & \|\boldsymbol{w}(T_i)\|^3_{\boldsymbol{L}^3(\mathbb{R}^3)} + C\displaystyle\int_{T_i}^{T_{i+1}} \|\boldsymbol{w}\|^{\frac12}_{\boldsymbol{L}^3(\mathbb{R}^3)}(\|\boldsymbol{w}\|^5_{\boldsymbol{L}^5(\mathbb{R}^3)} + \|\boldsymbol{w}\|^{\frac52}_{\boldsymbol{L}^5(\mathbb{R}^3)}\|\boldsymbol{v}\|^{\frac52}_{\boldsymbol{L}^5(\mathbb{R}^3)})\mathrm{d}s \\ & \le & \|\boldsymbol{w}(T_i)\|^3_{\boldsymbol{L}^3(\mathbb{R}^3)} + C\|\boldsymbol{w}(t)\|^{\frac12}_{L^\infty(T_i,T_{i+1};\boldsymbol{L}^3(\mathbb{R}^3)}\displaystyle\int_{T_i}^{T_{i+1}}(\|\boldsymbol{w}\|^5_{\boldsymbol{L}^5(\mathbb{R}^3)} + \|\boldsymbol{w}\|^{\frac52}_{\boldsymbol{L}^5(\mathbb{R}^3)}\|\boldsymbol{v}\|^{\frac52}_{\boldsymbol{L}^5(\mathbb{R}^3)})\mathrm{d}s \\ & \le & \|\boldsymbol{w}(T_i)\|^3_{\boldsymbol{L}^3(\mathbb{R}^3)} + \dfrac12\|\boldsymbol{w}(t)\|^3_{L^\infty(T_i,T_{i+1};\boldsymbol{L}^3(\mathbb{R}^3))} + C\|\boldsymbol{w}\|^6_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))} \\ & & +C\|\boldsymbol{w}\|^3_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}\|\boldsymbol{v}\|^3_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}.\end{array}$$

In particular, this shows

$$\begin{array}{rcl}\|\boldsymbol{w}\|_{L^\infty(T_i,T_{i+1};\boldsymbol{L}^3(\mathbb{R}^3))} & \le & C\|\boldsymbol{w}(T_i)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + C\|\boldsymbol{w}\|^2_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))} \\ & & +C\|\boldsymbol{w}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}\|\boldsymbol{v}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))},\end{array} \tag{33}$$

which implies that

$$\begin{array}{rcl}\displaystyle\int_{T_i}^{T_{i+1}}(\frac12\||\boldsymbol{w}|^{\frac12}\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} + \frac29\|\nabla|\boldsymbol{w}|^{\frac32}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)})\mathrm{d}s & \le & C\|\boldsymbol{w}(T_i)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + C\|\boldsymbol{w}\|^2_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))} \\ & & +C\|\boldsymbol{w}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}\|\boldsymbol{v}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}.\end{array} \tag{34}$$

We now use (32) together with (34) to get

$$\begin{array}{rcl}\|\boldsymbol{w}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^5))} & \le & C\|\boldsymbol{w}\|^{\frac35}_{L^\infty(T_i,T_{i+1};\boldsymbol{L}^3(\mathbb{R}^3))}\|\nabla|\boldsymbol{w}|^{\frac32}\|^{\frac25}_{L^2(T_i,T_{i+1;\boldsymbol{L}^2(\mathbb{R}^3)})} \\ & \le & C\|\boldsymbol{w}(T_i)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + C\|\boldsymbol{w}\|^2_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))} \\ & & +C\|\boldsymbol{w}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}\|\boldsymbol{v}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}.\end{array} \tag{35}$$

Part II: Induction argument.

Since $\boldsymbol{v} \in L^5(0,T;\boldsymbol{L}^5(\mathbb{R}^3))$, there exists a finite sequence $\{T_i\}_{i=0}^M$ such that $[0,T] = \cup_{i=0}^{M-1}[T_i,T_{i+1}]$ satisfying

$$\|\boldsymbol{v}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3)} < \frac{1}{4C}. \tag{36}$$

where $C > 0$ is the constant appearing in (35).

Let us consider

$$\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \le \frac{1}{8C(2C)^M}. \tag{37}$$

Then we claim that

$$\|\boldsymbol{w}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^5))} \le (2C)^{i+1}\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \tag{38}$$

and

$$\|\boldsymbol{w}\|_{L^\infty(T_i,T_{i+1};\boldsymbol{L}^3(\mathbb{R}^3))} \le (2C)^{i+1}\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}, \tag{39}$$

for all $i \in \{0,\cdots,M-1\}$.

For $i = 0$, let us suppose that there exists $K' > 0$ such that

$$\|\boldsymbol{w}\|_{L^5(0,T_1;\boldsymbol{L}^5(\mathbb{R}^3))} \le K'.$$

Then, from (35), (36) and (37), we find that

$$\|\boldsymbol{w}\|_{L^5(0,T_1;\boldsymbol{L}^5(\mathbb{R}^3))} \le C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + C(K')^2 + \frac12\|\boldsymbol{w}\|_{L^5(0,T_1;\boldsymbol{L}^5(\mathbb{R}^3))}.$$

We now impose that K' satisfies

$$K' = C\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)} + C(K')^2 + \frac12\|\boldsymbol{w}\|_{L^5(0,T_1;\boldsymbol{L}^5(\mathbb{R}^3))},$$

which gives

$$0 < K' = \frac{\frac12 - \sqrt{\frac14 - 4C^2\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}}}{2C}.$$

This implies the existence of $\boldsymbol{w}$ on $[0,T_1]$ provided that $\frac{1}{4}-4C^2\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3}>0$ holds, which is true due to (37). In particular, we have

$$\|\boldsymbol{w}\|_{L^5(0,T_1;\boldsymbol{L}^5(\mathbb{R}^3))}\leq\frac{1}{4C}. \tag{40}$$

In view of (33) and (35), estimates (38) and (39) are satisfied for $i=0$ by using (40).

In general, for $i\geq 1$, assume that (38) and (39) hold for $i-1$. Then if we argue as before, we see that

$$0<K'=\frac{\frac{1}{2}-\sqrt{\frac{1}{4}-4C^2\|\boldsymbol{w}(T_i)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}}}{2C}.$$

The induction hypothesis gives

$$\|\boldsymbol{w}(T_i)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\leq(2C)^i\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}<\frac{1}{8C(2C)^{M-i}}\leq\frac{1}{16C^2}.$$

Thus,

$$\|\boldsymbol{w}\|_{L^5(T_i,T_{i+1};\boldsymbol{L}^5(\mathbb{R}^3))}\leq\frac{1}{4C}.$$

Applying this to (33) and (35), we obtain that estimates (38) and (39) hold for i.

To complete the proof, note, by (39), that

$$\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^5))}\leq(2C)^M\|\boldsymbol{w}_0\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\quad\text{for all}\quad t\in[0,T],$$

whence (29) holds. □

The question that remains open is whether our particular solutions provided by Theorem 2.2 can develop singularities on (T,∞). Unfortunately, we are only able to give a partial answer to this question based on the following assumption. Let $\boldsymbol{v}_0=\boldsymbol{u}_{0,\varepsilon}^{\tilde{\lambda}}+\boldsymbol{w}_{0,\varepsilon}^{\hat{\lambda}}$ be as in the proof of Theorem 2.2 and satisfy conditions (16) and (17). Then we suppose that, for each $\varepsilon>0$, there exists $\hat{\lambda}_0$ such that, for all $\hat{\lambda}\geq\hat{\lambda}_0$, it follows that

$$\left|\int_0^{\frac{T}{2}}\|\nabla\boldsymbol{v}(s)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\mathrm{d}s-\frac{1}{2}\|\boldsymbol{v}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\right|<\varepsilon. \tag{A}$$

In other words, we shall look for initial data $\boldsymbol{v}_0$ whose corresponding solution to (1)–(2) has $\boldsymbol{L}^2(\mathbb{R}^3)$-values of the vorticity, i.e. $\|\nabla\times\boldsymbol{v}(t)\|_{\boldsymbol{L}^2(\mathbb{R}^3)}=\|\nabla\boldsymbol{v}(t)\|_{\boldsymbol{L}^2(\mathbb{R}^3)}$, sufficiently high on $(0,T)$ such that assumption (A) holds.

From our special choice of initial data $\boldsymbol{v}_0=\boldsymbol{u}_{0,\varepsilon}^{\tilde{\lambda}}+\boldsymbol{w}_{0,\varepsilon}^{\hat{\lambda}}$, we can take $\tilde{\lambda}$ to tend to ∞ without increasing $\|\boldsymbol{v}_0\|_{\boldsymbol{L}^2(\Omega)}$, but $\|\nabla\boldsymbol{v}_0\|_{\boldsymbol{L}^2(\mathbb{R}^3)}$ does. Then we would expect that the vorticity does keep high via the vortex stretching mechanism for a certain period of time; and hence the kinetic energy would decay up to a certain threshold on $[0,T]$.

Theorem 5.2. *Let $T>1$. Assume that assumption (A) holds. Then the solution $\boldsymbol{v}(t)$ to (1)–(2) provided by Theorem 2.2 with $\boldsymbol{v}_0=\boldsymbol{u}_{0,\varepsilon}^{\tilde{\lambda}}+\boldsymbol{w}_{0,\varepsilon}^{\hat{\lambda}}$ are smooth on $[0,\infty)$.*

Proof. From (5), we find

$$\frac{1}{2}\|\boldsymbol{v}(\frac{T}{2})\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\int_0^{\frac{T}{2}}\|\nabla\boldsymbol{v}(s)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\,\mathrm{d}s=\frac{1}{2}\|\boldsymbol{v}_0\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

In virtue of assumption (A), we infer that $\|\boldsymbol{v}(\frac{T}{2})\|_{\boldsymbol{L}^2(\mathbb{R}^3)}<2\varepsilon$. Moreover, we have

$$\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}+\int_{\frac{T}{2}}^t\|\nabla\boldsymbol{v}(s)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\,\mathrm{d}s=\frac{1}{2}\|\boldsymbol{v}(\frac{T}{2})\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

for all $t\in[\frac{T}{2},T]$.

As in the proof of Lemma 4.2, we write

$$\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\leq C\|\boldsymbol{v}(t)\|^{\frac{1}{2}}_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\nabla\boldsymbol{v}(t)\|^{\frac{1}{2}}_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Taking the fourth power of both sides and integrating over $(\frac{T}{2}, T)$ yields

$$\int_{\frac{T}{2}}^{T} \|\boldsymbol{v}(t)\|^4_{\boldsymbol{L}^3(\mathbb{R}^3)} \mathrm{d}t \leq \frac{C}{4} \|\boldsymbol{v}(\frac{T}{2})\|^4_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Therefore,

$$\frac{T}{2} \inf_{s\in[\frac{T}{2},T]} \|\boldsymbol{v}(s)\|^4_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq \frac{C}{4} \|\boldsymbol{v}(\frac{T}{2})\|^4_{\boldsymbol{L}^2(\mathbb{R}^3)}$$

and hence

$$\inf_{s\in[\frac{T}{2},T]} \|\boldsymbol{v}(s)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq C 2^{-\frac{1}{4}} T^{-\frac{1}{4}} \|\boldsymbol{v}(\frac{T}{2})\|_{\boldsymbol{L}^2(\mathbb{R}^3)} < C \|\boldsymbol{v}(\frac{T}{2})\|_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Let us choose $\varepsilon < \frac{1}{8C^2}$. Then we find that there exists $t^* \in (0, T]$ such that it follows that

$$\|\boldsymbol{v}(t^*)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < \frac{1}{4C}.$$

We are now allowed to apply Lemma 4.1 to obtain that $\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq \frac{1}{2C}$ for all $t \in [T, \infty)$. Moreover, we know that $\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \leq M_{\boldsymbol{v}}$ for all $t \in [0, T]$ by Lemma 3.3. As a result of Theorem 3.1, the solution $\boldsymbol{v}(t)$ is smooth globally in time.

□

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