A NEW BOUND ON THE MORSE INDEX OF CONSTANT MEAN CURVATURE TORI OF REVOLUTION IN $\mathbb{S}^3$

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ABSTRACT. In this work we give a new lower bound on the Morse index for constant mean curvature tori of revolution immersed in the three-sphere $\mathbb{S}^3$, by computing some explicit negative eigenvalues for the corresponding Jacobi operator.

INTRODUCTION

Given a closed surface M with *constant mean curvature* (CMC), immersed in a three-dimensional manifold N, it is well known that M is a critical point of the area functional, when we consider variations preserving the enclosed volume [3], [4]. For this kind of surfaces, we can discuss the *stability* by studying the second variation of the area, which can be expressed by means of an useful and classical functional operator L; more precisely, denoting by $f \in \mathcal{C}^\infty(M, \mathbb{R})$ to the normal component of the vector field associated to a variation of M, the second variation formula will be given by

$$(1) \qquad -\int_M f\,Lf\,da,$$

where $L = \Delta + \overline{R} + |\sigma|^2$, Δ is the Laplacian operator of M, $\overline{R}$ is the Ricci curvature of N, and σ is the second fundamental form of M. Such operator $L : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathcal{C}^\infty(M, \mathbb{R})$ is usually called the *Jacobi* (or stability) *operator*.

For CMC surfaces, M is said to be *stable* if and only if the above expression (1) is greater than or equal to zero, for any volume-preserving variation [3, § 2]. In this setting, a usual way of measuring the *instability* of a surface M is given by the *Morse index* of M (see for instance [2], [20]), which is defined as the number of negative eigenvalues, counted with their multiplicities, of the Jacobi operator L (throughout this paper, a value $\lambda \in \mathbb{R}$ will be an eigenvalue of L if there exists a function $f_\lambda \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $Lf_\lambda + \lambda f_\lambda = 0$).

This intrinsic relation with stability (see [13] for further details) has stimulated the study of the Morse index of CMC surfaces in several works. The main approaches have focused on the minimal case, that is, surfaces with zero mean curvature (see [14], [8], [15], [23], [6]). In the euclidean space $\mathbb{R}^3$, for instance, planes have zero index [7], meanwhile catenoids and Enneper's surfaces have index one [10], [13]. In the sphere $\mathbb{S}^3$, an interesting result from J. Simons [22] states that the index of any compact minimal surface M is always greater than or equal to one, with equality if and only if M is a totally geodesic 2-sphere (in fact, such a result was stated in general dimension). Later on, F. Urbano [24] proved that any compact minimal surface (not totally geodesic) in $\mathbb{S}^3$ has index greater than or equal to five, with equality

uniquely for the Clifford torus. The analogous result in the n-dimensional case has been partially demonstrated in [9], [11], [16]. A nice review of these results can be found in [1].

However, in the family of nonminimal CMC surfaces, the Morse index has been less discussed in literature, and the main results refer to bounds for the index. In $\mathbb{R}^3$, apart from the spheres (which have index one), only lower bounds for tori of revolution [19], and upper and lower bounds for the Wente tori [18] are known. Moreover, for CMC immersions of revolution in $\mathbb{S}^3$, W. Rossman and N. Sultana [20] have recently computed the index of flat tori of revolution (in terms of the mean curvature), and have also found a lower bound for non-flat tori, giving as well numerical methods for explicit calculations [21]. It follows from their work that the index of these tori is at least five, in any case.

In this work, we focus on the index of CMC tori of revolution immersed in $\mathbb{S}^3$. By using an approach different from the one used in [20], we shall explicitly find some negative eigenvalues for the Jacobi operator of these surfaces, obtaining new bounds for the Morse index. Our technique implies a suitable arrangement of the metric of the surface. This will lead us to have a nice expression of the Jacobi operator, which allows to determine some specific eigenfunctions and eigenvalues. We remark that these eigenvalues will depend on some geometric quantities associated to the surface (as the energy and the mean curvature), and we will discuss analytically their sign, improving in most of the cases the previous known bounds for the Morse index (see Theorem 2.14). For instance, we obtain that when the mean curvature of the torus is less than -1 or greater than $3/2$, the Morse index is, at least, eight.

We point out that an interesting question, in this CMC setting, is finding a similar result to that of F. Urbano, who characterized Clifford tori in $\mathbb{S}^3$ by means of a minimum value of the Morse index [24]. Some partial progresses have been made in this direction [2].

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1. PRELIMINARIES

Consider a torus of revolution M with constant mean curvature H, immersed in the 3-dimensional unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. Let us fix a geodesic curve ℓ in $\mathbb{S}^3$, given by

$$\ell = \{(\cos(t), \sin(t), 0, 0), \ t \in \mathbb{R}\}.$$

Then, the torus can be seen as generated by the rotation about ℓ of a given curve parameterized by arc-length

$$\gamma : [0, t_0] \longrightarrow \mathbb{S}^3$$

defined by

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), 0), \quad t \in [0, t_0].$$

We can further assume that $\gamma_3(t) > 0$ for all t.

Remark 1.1. The description of the generating curve of a CMC torus of revolution can be found in [12, Th. 3] (see also [20, § 1]), being unduloidal or nodoidal.

Hence, an immersion $\psi : M \to \mathbb{S}^3$ of the torus into $\mathbb{S}^3$ will be given by

$$\psi(t, \theta) = (\gamma_1(t), \gamma_2(t), \gamma_3(t) \cos\theta, \gamma_3(t) \sin\theta), \quad t \in [0, t_0], \ \theta \in [0, 2\pi].$$

Then we have that the tangent space to $\psi(t, \theta)$ is generated by the vectors

$$(2) \qquad \partial_t = (\gamma_1'(t), \gamma_2'(t), \gamma_3'(t) \cos\theta, \gamma_3'(t) \sin\theta),$$
$$\partial_\theta = (0, 0, -\gamma_3(t) \sin\theta, \gamma_3(t) \cos\theta),$$

with $|\partial_t| = 1, |\partial_\theta| = \gamma_3(t)$. Moreover, it is straightforward checking that the unit normal vector to $\psi(t, \theta)$ will be given by

$$(3) \qquad N(t, \theta) = \begin{pmatrix} \gamma_2(t)\gamma_3'(t) - \gamma_2'(t)\gamma_3(t) \\ \gamma_3(t)\gamma_1'(t) - \gamma_3'(t)\gamma_1(t) \\ (\gamma_1(t)\gamma_2'(t) - \gamma_1'(t)\gamma_2(t))\cos\theta \\ (\gamma_1(t)\gamma_2'(t) - \gamma_1'(t)\gamma_2(t))\sin\theta \end{pmatrix} = \begin{pmatrix} a(t) \\ b(t) \\ c(t)\cos\theta \\ c(t)\sin\theta \end{pmatrix}.$$

Since M is rotationally symmetric, we can consider in M a metric of type

$$ds^2 = dt^2 + f(t)^2 \, d\theta^2,$$

for certain positive function $f : M \to \mathbb{R}$. We are interested in computing the Hopf differential of our immersion ψ, and so we shall need conformal (isothermal) coordinates in M. In order to have this, we use the following change of coordinates: consider a new coordinate t' defined by $t = G(t')$, where G is a diffeomorphism of $\mathbb{R}$ determined by $G'(t') = f(G(t'))$ and with initial condition $G(0) = 0$. Defining now

$$g(t', \theta) = (G(t'), \theta),$$

it is not difficult to check that the new immersion of M given by

$$(\psi \circ g)(t', \theta) = \psi((G(t'), \theta))$$

is conformal, with

$$ds^2 = f(G(t'))^2 \left((dt')^2 + d\theta^2\right).$$

Observe that we have properly replaced coordinates (t, θ) with (t', θ). It also follows from above that

$$(4) \qquad |\partial_\theta| = f(t) = \gamma_3(t), \text{ for all } t.$$

Now, by considering the flat metric ds_0^2 associated to the Hopf differential of the immersion $\psi \circ g$, it follows that

$$(5) \qquad ds^2 = \exp(2w)b^{-2} \, ds_0^2,$$

where $b^2 = 4(1 + H^2)$, and $w : M \to \mathbb{R}$ is a smooth function defined on the torus (see [17, § 1]).

Remark 1.2. We point out that

$$(6) \qquad ds_0^2 = \beta \left((dt')^2 + d\theta^2\right),$$

for certain $\beta \in \mathbb{R}$, due to the flatness of the metric ds_0^2. From the above expressions (5), (6) we get

$$(7) \qquad \beta = \exp(-2w(t')) \, b^2 f(t)^2 > 0,$$

or equivalently

$$f(t) = \exp(w(t')) \, b^{-1} \beta^{1/2}.$$

1.1. **On the value of the constant β.** The value of the constant β can be computed in terms of the principal curvatures k_t, k_θ of the immersion ψ, taking into account the construction of the Hopf differential. More precisely, denoting by σ' the second fundamental form of the conformal immersion $\psi \circ g$, we have (see [17, § 1])

$$(8) \qquad \beta = \sigma'(\partial_{t'}, \partial_{t'}) - \sigma'(\partial_\theta, \partial_\theta) = f(t)^2 \, (k_t - k_\theta),$$

where, taking into account (2), (3), the principal curvatures are given by

$$(9) \qquad k_\theta = \sigma(\partial_\theta, \partial_\theta) = \frac{\gamma_1'(t)\gamma_2(t) - \gamma_1(t)\gamma_2'(t)}{\gamma_3(t)} = \frac{-c(t)}{\gamma_3(t)},$$

$$k_t = \sigma(\partial_t, \partial_t) = - \begin{vmatrix} \gamma_1(t) & \gamma_2(t) & \gamma_3(t) \\ \gamma_1'(t) & \gamma_2'(t) & \gamma_3'(t) \\ \gamma_1''(t) & \gamma_2''(t) & \gamma_3''(t) \end{vmatrix}.$$

In order to calculate explicitly the value of β (which we shall need later), we will take into account some particular parametrization of the points in $\mathbb{S}^3$ appearing in [12, § 1]. This will provide, in particular, new useful expressions for the principal curvatures k_t, k_θ.

For any point $p \in M \subset \mathbb{S}^3$, there exists a point $q \in \ell$ such that $\overline{pq}$ is a geodesic arc whose length is equal to the distance between p and ℓ. Fixing a base point in ℓ and considering the arc length on ℓ with respect to the base point, we can assign to p new coordinates (x, y), where x is the coordinate of q in ℓ, and y is the length of the geodesic $\overline{pq}$. This procedure allows to express the generating curve $\gamma \subset \mathbb{S}^3$ in terms of coordinates x, y. Straightforward computations yields

$$(10) \qquad \begin{aligned} \gamma_1(t) &= \cos(x(t)) \cos(y(t)), \\ \gamma_2(t) &= \sin(x(t)) \cos(y(t)), \\ \gamma_3(t) &= \sin(y(t)). \end{aligned}$$

Moreover, since the mean curvature H is constant, the following relations hold ([12, eq. (4)]):

$$(11) \qquad x'(t) = \frac{\sin(\alpha(t))}{\cos(y(t))}, \quad y'(t) = \cos(\alpha(t)),$$

where $\alpha(t)$ is the angle between the tangent vector to $\gamma(t)$ and the vertical direction.

Then, from (9), (10) we get that

$$(12) \qquad \begin{aligned} k_\theta &= -\sin(\alpha(t)) \, \cot(y(t)), \\ k_t &= \sin(\alpha(t)) \, \tan(y(t)) - \alpha'(t). \end{aligned}$$

We recall that

$$(13) \qquad 2H = k_t + k_\theta = -\sin(\alpha(t)) \, \cot(y(t)) + \sin(\alpha(t)) \, \tan(y(t)) - \alpha'(t).$$

On the other hand, it is easy to check that

$$(14) \qquad E = \sin(y(t)) \left(\cos(y(t) \sin\alpha(t) + H \sin(y(t)) \right)$$

is constant (just compute the derivative with respect to t, using the above relations (11)).

Finally, taking into account previous expressions (8), (12), (13) and (14), it follows that

$$(15) \qquad \beta = f(t)^2 \left(k_t - k_\theta\right) = \sin^2(y(t)) \left(2 \sin \alpha(t) \cot(y(t)) + 2H\right) = 2E.$$

Remark 1.3. In fact, the above constant E is a first integral associated to the system of equations (11) of the generating curve γ (see [12, Th. 1]).

Remark 1.4. In the previous computations of k_t, k_θ, we have considered the normal vector $N(t,\theta)$ defined by (3). If we consider that normal vector with opposite sign, the values of k_t, k_θ will change the sign too. The appropriate choice of $N(t,\theta)$ will be determined by the positivity of β, in view of (8).

Remark 1.5. It follows from (4), (10) that

$$(16) \qquad f(t) = \gamma_3(t) = \sin(y(t)), \ t \in [0, t_0].$$

1.2. **On the function w.** The function w arising in (5) establishes the relation between the metric ds^2 in M and the flat metric ds_0^2 associated to the Hopf differential of the immersion. It only depends on the variable t since M is rotationally symmetric, and it satisfies the sinh-Gordon equation for the laplacian associated to ds_0^2 ([17, eq. (2)]); equivalently, by using (6) we have

$$(17) \qquad \begin{cases} w'' + \beta \sinh(w)\cosh(w) = 0, \\ w(0) = c, w'(0) = 0, \end{cases}$$

where $c \in \mathbb{R}$, and the derivatives are taken with respect to the standard flat metric $(dt')^2 + d\theta^2$.

Remark 1.6. The above value $c \in \mathbb{R}$ in (17) is related with the length of the parallel $\mathbb{S}^1 \times \{0\} \subset M$, since that length is equal to $2\pi f(0) = 2\pi \exp(c) b^{-1} \beta^{1/2}$. On the other hand, by using [12, eq. (1)] and (16), we have that $L(\mathbb{S}^3 \times \{0\}) = 2\pi \cos(y(0)) = 2\pi\sqrt{1 - f(0)^2}$. Thus, the value c equals

$$\log\left(\frac{b\sqrt{1 - \gamma_3^2(0)}}{\beta^{1/2}}\right) = \log\left(\frac{b\sqrt{1 - f(0)^2}}{\beta^{1/2}}\right).$$

In this setting, we recall that the Gauss curvature K of M, with respect to the metric ds^2, only depends on the t-coordinate and is given by

$$(18) \qquad K = (b^2/4)(1 - \exp(-4w)).$$

Remark 1.7. By integrating equality (17), multiplied by $2w'$, we obtain

$$(19) \qquad (w')^2 + \beta \cosh^2(w) = \beta \cosh^2(c),$$

which is a first integral of equation (17), where the derivative is with respect to the flat metric $(dt')^2 + d\theta^2$.

1.3. **Index form and Jacobi operator.** Recall that N denotes the normal vector field of M, and K is the Gauss curvature. Then, it is well known that the second variation formula of the area, for variations preserving the volume enclosed by M, is given in general by [3, Prop. 2.5]

$$(20) \qquad \begin{aligned} I(f,f) &= -\int_M \left(f\Delta f + (\overline{R} + |\sigma|^2)f^2\right) da \\ &= -\int_M \left(f\Delta f + (4 + 4H^2 - 2K)f^2\right) da, \end{aligned}$$

where Δ is the laplacian operator associated to the metric ds^2, f is the normal component of the vector field associated to the variation, $\overline{R} = 2\,Ric(N)$ is the Ricci curvature of the ambient space S^3, and $|\sigma|^2$ is the square of the norm of the second fundamental form σ. We shall refer to the quadratic form defined by (20) as the *index form* of M. The associated *Jacobi operator* is thus given by

$$Lf = \Delta f + (4 + 4H^2 - 2K)f,$$

for any $C^\infty(M)$ function f, and so

$$I(f,f) = -\int_M f\,Lf\,da.$$

Taking into account expression (18) we have that

$$
\begin{aligned}
4 + 4H^2 - 2K = b^2 - 2K &= (b^2/2)\,(1 + \exp(-4w)) \\
&= b^2\,\exp(-2w)\,\cosh(2w) \\
&= b^2\,\exp(-2w)\,(\cosh^2(w) + \sinh^2(w)).
\end{aligned}
$$
(21)

In view of expression (21), we now define a new operator L_0 by

$$L_0 f = \exp(2w)b^{-2}Lf, \quad f \in C^\infty(M).$$

Then, it is clear, in view of (5) and (6), that

(22)
$$
\begin{aligned}
L_0 f &= \Delta_0 f + (\cosh^2(w) + \sinh^2(w))f \\
&= \frac{1}{\beta}\left(\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial \theta^2}\right) + (\cosh^2(w) + \sinh^2(w))f,
\end{aligned}
$$

where Δ_0 represents the laplacian of ds_0^2. An important fact that we will use later is that both operators L, L_0 only differ in a positive scalar factor.

1.4. **Morse index of CMC surfaces.**

The *Morse index* of a closed constant mean curvature (CMC) surface $\mathcal{S}$ is defined by means of the Jacobi operator and, as indicated in [20], provides a degree of the instability of $\mathcal{S}$ with respect to the area. We first recall the following definition.

Definition 1. *Given a function $f : M \to \mathbb{R}$, we shall say that f is an eigenfunction of the Jacobi operator L, with associated eigenvalue $\lambda \in \mathbb{R}$, if*

$$Lf + \lambda f = 0.$$

It is known that the set of eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ of the Jacobi operator L consists of an increasing sequence, diverging to $+\infty$, and that the first eigenvalue λ_1 has multiplicity one (see [5] for further details). We can now define the Morse index of a CMC surface.

Definition 2. *Given a closed CMC surface $\mathcal{S}$, the Morse index $Ind(\mathcal{S})$ is the number of negative eigenvalues of the Jacobi operator L, each one counted with its multiplicity.*

Our purpose is giving a lower bound for the Morse index for CMC tori of revolution immersed in S^3. To do that, we will focus on the operator L_0, since due to its definition, it will have the same number of negative eigenvalues that the Jacobi operator L.

2. EXPLICIT COMPUTATION OF SOME EIGENVALUES OF L_0

In this Section we shall compute directly some (negative) eigenvalues of the operator L_0, taking into account its expression (22), by using certain functions on M with independent variables. We first define the operator L_t, on the set of functions of real variable, as

$$L_t f = \frac{1}{\beta} f''(t) + (\cosh^2(w) + \sinh^2(w)) f(t), \quad f : \mathbb{R} \to \mathbb{R}.$$

It is clear that, for functions defined on M, we have that $L_0 = \frac{1}{\beta} \frac{\partial^2}{\partial \theta^2} + L_t$. We begin with the following key result.

Lemma 2.1. *Let $u = u(t)$ be an eigenfunction of L_t, with associated eigenvalue $\lambda \in \mathbb{R}$, and let $v = v(\theta)$ be an eigenfunction of the laplacian $\frac{\partial^2}{\partial \theta^2}$, with associated eigenvalue $\mu \in \mathbb{R}$. Then, the function $f : M \to \mathbb{R}$ given by $f(t, \theta) = u(t)v(\theta)$ is an eigenfunction of L_0, with associated eigenvalue $\lambda + \mu/\beta$.*

Proof. Applying the operator L_0 to f, we have that

$$L_0 f = \frac{1}{\beta} \frac{\partial^2 v}{\partial \theta^2} u + v L_t u = -(\lambda + \frac{\mu}{\beta}) u v = -(\lambda + \frac{\mu}{\beta}) f,$$

and so the result follows. $\qquad \square$

We now proceed to find convenient functions for applying Lemma 2.1. It is clear that $v(\theta)$ can be taken equal to a constant, or equal to $\cos(\theta)$, $\sin(\theta)$, which are eigenfunctions of the laplacian (with $\mu = 0$ and $\mu = 1$ as eigenvalues, respectively). The next result gives some eigenfunctions for the operator L_t.

Proposition 2.2. *The functions $u_1, u_2 : [0, t_0] \to \mathbb{R}$ defined by*

$$u_1(t) = \cosh(w(t)), \quad u_2(t) = \sinh(w(t)),$$

are independent eigenfunctions of L_t, with associated eigenvalues

$$\lambda_1 = -\cosh^2(c), \quad \lambda_2 = 1 - \cosh^2(c),$$

respectively.

Proof. The proof is straightforward, taking into account (17) and (19). $\qquad \square$

Remark 2.3. We point out that when w is identically zero (which corresponds to flat metric ds^2), previous Proposition 2.2 only shows that constant functions will be eigenfunctions of L_t, with eigenvalue $\lambda = -1$.

Our idea consists of using functions with independent variables. By combining the functions $u_1(t)$, $u_2(t)$ from Proposition 2.2 with a constant function, with $\sin(\theta)$, or with $\cos(\theta)$, we can apply Lemma 2.1 to obtain some eigenvalues for the operator L_0.

Theorem 2.4. *Let M be a CMC torus of revolution immersed in $\mathbb{S}^3$ and L_0 the operator defined previously. Then,*

   i) *the first eigenfunction of L_0 is $f_1(t, \theta) = \cosh(w(t))$, with associated eigenvalue $\lambda_1 = -\cosh^2(c) < 0$,*
   ii) *$f_2(t, \theta) = \sinh(w(t))$ is an eigenfunction of L_0, with associated eigenvalue $1 - \cosh^2(c) < 0$,*

iii) $f_3(t, \theta) = \cosh(w(t)) \sin(\theta)$, $\overline{f_3}(t, \theta) = \cosh(w(t)) \cos(\theta)$ *are two eigenfunctions of L_0, with associated eigenvalue* $-\cosh^2(c) + 1/\beta$,

iv) $f_4(t, \theta) = \sinh(w(t)) \sin(\theta)$, $\overline{f_4}(t, \theta) = \sinh(w(t)) \cos(\theta)$ *are two eigenfunctions of L_0, with associated eigenvalue* $1 - \cosh^2(c) + 1/\beta$.

Furthermore, these six eigenfunctions are independent, and Ind(M)$\geqslant$ 2.

Proof. Just apply Lemma 2.1 with the corresponding functions in order to obtain the six independent eigenfunctions. Moreover, f_1 is the first eigenfunction of L_0 since it does not vanish. Finally, $\lambda_1 = -\cosh^2(c)$ is always negative, and since $c \neq 0$ (otherwise, (17) yields $w = 0$ and so f is constant), we have that $1 - \cosh^2(c)$ is also a negative eigenvalue for L_0. $\square$

Another negative eigenvalue of L_0 can be obtained by following some ideas from [20]. For a given geodesic curve ℓ' in S^3, orthogonal to ℓ, we can consider the Killing vector field K associated to the rotations about ℓ' in S^3. Then, the normal component $f = \langle K, N \rangle$ of K satisfies $L_0(f) = 0$ (that is, f is a Jacobi function for L_0), and can be expressed as $f(t, \theta) = u(t) \cos \theta$ (see [20, Lemma 4.1]). In this situation, it is easy to check that u is an eigenfunction of L_0, with associated eigenvalue $-1/\beta$ (which is negative since $\beta > 0$). Therefore, as in [20, Lemma 4.2], by taking the two geodesic curves orthogonal to ℓ, we get two independent eigenfunctions of L_0 (depending only on variable t) with eigenvalue $-1/\beta$. Straightforward computations show that these two eigenfunctions are given by

(23)
$$u(t, \theta) = -\gamma_3(t) b(t) + \gamma_2(t) c(t) = -\gamma_1'(t),$$
$$\overline{u}(t, \theta) = -\gamma_3(t) a(t) + \gamma_1(t) c(t) = \gamma_2'(t),$$

where $a(t), b(t), c(t)$ are the real functions provided by (3). Lemma 2.5 summarizes these properties.

Lemma 2.5. ([20, Lemmata 4.1 and 4.2]) *The functions $u, \overline{u}$ defined by (23) are two independent eigenfunctions of L_0 with $-1/\beta$ as associated (negative) eigenvalue.*

From above lemma we have the following interesting consequence related with Theorem 2.4.

Lemma 2.6. *The eigenvalue $-\cosh^2(c) + 1/\beta$ is negative.*

Proof. From Theorem 2.4 we have that $\lambda_1 = -\cosh^2(c)$ is the first eigenvalue of L_0. Then, since $-1/\beta$ is another eigenvalue of L_0, we necessarily have $-\cosh^2(c) < -1/\beta$, and so $-\cosh^2(c) + 1/\beta < 0$. $\square$

The Morse index is defined taking into account the multiplicities of the negative eigenvalues. In this sense, we have to study carefully whether u or $\overline{u}$ belong to the eigenfunctions space associated to one of the eigenvalues described in Theorem 2.4 (if this is the case, they will not contribute to the Morse index). For having that, a necessary condition is that $-1/\beta$ coincides with one of the eigenvalues previously obtained. It is clear that it cannot be equal to $\lambda_1 = -\cosh^2(c)$, otherwise the first eigenvalue of L_0 would have multiplicity greater than one, which is a contradiction (recall that $u, \overline{u}$ are independent). If $-1/\beta$ coincides with $\lambda = -\cosh^2(c) + 1/\beta$, as $u, \overline{u}$ only depend on variable t, they will be independent from $f_3, \overline{f_3}$, and so the multiplicity of λ will be greater than (or equal to) four (hence contributing to the Morse index, since $\lambda < 0$ from Lemma 2.6). The same reasoning is valid if $-1/\beta$

coincides with $\lambda' = 1 - \cosh^2(c) + 1/\beta$ (observe that the sign of λ' has not been discussed yet). Finally, we have to study if $-1/\beta$ coincides with $1 - \cosh^2(c)$, equivalently $1 - \cosh^2(c) + 1/\beta = 0$. This case of equality is treated in Subsection 2.1. However, if this equality does not occur, we immediately have the following result, which establishes lower bounds for the Morse index.

Theorem 2.7. *Let M be a CMC torus of revolution immersed in $\mathbb{S}^3$. With the previous notation, assume that $1 - \cosh^2(c) + 1/\beta \neq 0$. Then,*

  i) *Ind(M)$\geqslant 8$, if $1 - \cosh^2(c) + 1/\beta < 0$.*
  ii) *Ind(M)$\geqslant 6$, if $1 - \cosh^2(c) + 1/\beta > 0$.*

Proof. From the hypothesis we have that the eigenfunctions shown in Theorem 2.4 and Lemma 2.5 are independent. An analysis of the eigenvalue $1 - \cosh^2(c) + 1/\beta$ yields the statement. $\square$

Remark 2.8. Note that Theorem 2.7 establishes bounds on the Morse index of CMC tori of revolution in $\mathbb{S}^3$ which improve the ones given in [20, Th. 1.1].

Observe that above Theorem 2.7 yields a numerical criterion (based only on the sign of $1 - \cosh^2(c) + 1/\beta$, which depends on the constant values of c and β) that provides lower bounds for the Morse index. Now, we shall express the eigenvalue $1 - \cosh^2(c) + 1/\beta$ in terms of the mean curvature H and the value $f(0)$, in order to study its sign.

From Remark 1.6 we have that

$$\cosh(c) = \frac{e^c + e^{-c}}{2} = \frac{b^2 \left(1 - f(0)^2\right) + \beta}{2\,\beta^{1/2}\,b\,\sqrt{1 - f(0)^2}},$$

and so

$$(24) \qquad \cosh^2(c) = \frac{\left(4(1 + H^2)(1 - f(0)^2) + \beta\right)^2}{16\,\beta\,(1 + H^2)(1 - f(0)^2)}.$$

On the other hand, the constant β can be expressed only in terms of H and $f(0)$: in fact, its value can be obtained by taking $t = 0$ in (14), and so

$$\beta = 2\,\sin(y(0))\Big(\cos(y(0))\,\sin(\alpha(0)) + H\,\sin(y(0))\Big).$$

Since $f(t) = \sin(y(t))$, and taking $t = 0$ such that $\alpha(0) = \pi/2$, it follows that

$$(25) \qquad \beta = 2\,f(0)\left(\sqrt{1 - f(0)^2} + H\,f(0)\right).$$

Using equalities (24) and (25), the eigenvalue $1 - \cosh^2(c) + 1/\beta$ can be expressed only in terms of H and $f(0)$. Moreover, since $\beta > 0$, it follows that $\sqrt{1 - f(0)^2} + H\,f(0)$ must be positive, and so necessarily

$$f(0) < \sqrt{\frac{1}{1 + H^2}},$$

if $H < 0$ (in the case $H \geqslant 0$, then $f(0) \in (0, 1)$).

Above equalities allow to calculate explicitly that eigenvalue, for each $H \in \mathbb{R}$ and each considered value of $f(0)$. Numerical computations show that both possibilities

from Theorem 2.7 may occur for different tori. For instance, when $H = 1$ and $f(0) = 0.3$, we have that $1 - \cosh^2(c) + 1/\beta$ is negative, and then we can claim that the Morse index of the corresponding torus is greater than or equal to 8. On the other hand, when $H = 0.5$ and $f(0) = 0.5$, that eigenvalue is positive, and so the Morse index will be greater than or equal to 6. More examples of these two situations arise for different values of H and $f(0)$.

Moreover, we have checked numerically that the following statement holds.

Corollary 2.9. *Let M be a torus of revolution immersed in* $\mathbb{S}^3$ *with constant mean curvature H. If* $H \geqslant 3/2$ *or* $H \leqslant -1$, *then Ind(M)$\geqslant 8$.*

Remark 2.10. By using (24) and (25), it is also possible to express the eigenvalue $1 - \cosh^2(c) + 1/\beta$ in terms of the mean curvature H and the constant β (which is equivalent to the first integral or energy of the generating curve γ, see (15)).

2.1. **A degenerate case.** We will now focus on the equality case $-1/\beta = 1 - \cosh^2(c)$. We have verified that this equality may occur in our surfaces (for instance, when $H = 1$ and $f(0) = 0.4658$), but in very few situations. In fact, it only holds when H lies approximately in the interval $(-1, 1.4)$, and just for a unique value of $f(0)$ in each case. Consequently, it can be considered as a degenerate possibility. In these situations, we just have the following result:

Theorem 2.11. *Let M be a CMC torus of revolution immersed in* $\mathbb{S}^3$. *If* $1 - \cosh^2(c) = -1/\beta$, *then Ind(M)$\geqslant 5$.*

Proof. Observe that, in this case, u, $\overline{u}$, f_2 are three eigenfunctions depending only on t, with the same eigenvalue $-1/\beta$ for the second order differential equation given by L_t. Then necessarily they cannot be independent, and so, taking into account Lemma 2.5, we can only assure that $-1/\beta$ is an eigenvalue of multiplicity two, at least. Since $1 - \cosh^2(c) + 1/\beta = 0$, from Theorem 2.4 and Lemma 2.6 we conclude that Ind(M)$\geqslant 5$. $\qquad\square$

Remark 2.12. Note that the previous Theorem 2.11 gives the same lower bound for the Morse index as in [20].

Remark 2.13. In [24], it is proven that the Clifford torus has Morse index equal to five. For this torus we have the degenerate situation described in Subsection 2.1.

Finally, we summarize the main results we have obtained (Theorems 2.7 and 2.11) in the following Theorem.

Theorem 2.14. *Let M be a CMC torus of revolution immersed in* $\mathbb{S}^3$. *With the previous notation, assume that* $-1/\beta \neq 1 - \cosh^2(c)$. *Then,*

- *if* $1 - \cosh^2(c) + 1/\beta < 0$, *then Ind(M)$\geqslant 8$.*
- *If* $1 - \cosh^2(c) + 1/\beta > 0$, *then Ind(M)$\geqslant 6$.*

In particular, if $H \geqslant 3/2$ *or* $H \leqslant -1$, *then Ind(M)$\geqslant 8$.*

On the other hand, if $-1/\beta = 1 - \cosh^2(c)$, *we have that Ind(M)$\geqslant 5$.*

With this result, taking into account that for each torus of revolution, the constant values c and β can be computed by means of the expressions (24) and (25), we obtain a lower bound for the Morse index in our surfaces.

3. Some final comments

3.1. **Study of equality of eigenvalues.** It is difficult to describe geometrically the tori of revolution in $\mathbb{S}^3$ satisfying $-1/\beta = 1 - \cosh^2(c)$, which is the situation corresponding to the degenerate case from Subsection 2.1. Anyway, this condition will impose some restrictions to the surface.

For the function $w(t)$, we have from (7) that it depends on b, or equivalently on the mean curvature H, thus involving the generating curve γ and its derivatives γ', γ'' (observe the equations of the principal curvatures in (9)). In the degenerate case, $w(t)$ has a simpler expression: since f_2 must be a linear combination of u, $\overline{u}$ (see the proof of Th. 2.11), it follows that

$$w(t) = \text{arsinh}(-\rho_1 \gamma_1'(t) + \rho_2 \gamma_2'(t)),\ \rho_1, \rho_2 \in \mathbb{R},$$

that is, w just depends essentially on γ'.

Additionally, some restrictions appear between the derivatives of γ. Using again (7) and the linear dependence of $f_2(t) = \sinh(w(t))$, $u(t)$ and $\overline{u}(t)$, it is not difficult to find others analytic relations between γ_3, γ_1' and γ_2'. Unfortunately, the geometrical meaning of them is not clear at all.

3.2. **An application of the Courant's Nodal Domain Theorem.** A further estimate of the Morse index of a surface can be obtained by means of Courant's Nodal Domain Theorem. This result states that for the ordered sequence $\{\lambda_n\}_n$ of eigenvalues, the eigenfunction associated to an eigenvalue λ_k has at most k nodal domains [5, page 19].

In our case, fix for instance the eigenfunction $u(t) = -\gamma_1'(t)$, with associated eigenvalue $-1/\beta$. The nodal domains for $u(t)$ will be determined by the number of critical points (local minima or maxima) of the first coordinate of the generating curve $\gamma(t)$. Hence, if $\gamma(t)$ have k_0 critical points for the first coordinate, then $u(t)$ will have k_0 nodal domains, and so its associated eigenvalue $-1/\beta = \lambda_k$ will satisfy $k \geqslant k_0$. Consequently, since that eigenvalue is negative, we shall conclude that the Morse index is, at least, k_0. Observe that an analogous reasoning can be done by taking the eigenfunction $\overline{u}(t) = \gamma_2'(t)$. Following this idea, a deeper analysis of the generating curves of particular tori of revolution in $\mathbb{S}^3$ should improve the bounds for the Morse index in each case.

3.3. **The Morse index and instability.** As commented in the Introduction, the stability notion for CMC surfaces is usually studied by considering volume-preserving variations. This fact makes that the index form (20) must be defined on the set of zero-mean functions (see [3]). As the eigenfunctions for computing the Morse index do not satisfy, in general, this restriction, we recall the precise relation with stability (see [4, Prop. 3.3] or [20]): if a surface M is stable, then $Ind(M) \leqslant 1$. An obvious consequence from our results is that any torus of revolution with constant mean curvature immersed in $\mathbb{S}^3$ is not stable (the only stable CMC surfaces in $\mathbb{S}^3$ are, in fact, the round spheres [3]).

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