Tampereen yliopisto

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QUANTUM SCARRING IN A TWO-DIMENSIONAL ELLIPTICAL HARMONIC OSCILLATOR

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ABSTRACT

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Perturbation-induced quantum scarring is a recently discovered phenomenon in a quantum well perturbed by local potential bumps. In this phenomenon, the probability density of an eigenstate in a perturbed system is enhanced along a periodic orbit of the unperturbed classical counterpart. In this Thesis, we examine perturbation-induced scarring in a two-dimensional anisotropic (elliptic) oscillator characterized by the anisotropy parameter, i.e., the frequency ratio of the confining potential. In particular, we show that some eigenstates of the perturbed oscillator are scarred by the so-called Lissajous orbits occurring at specific anisotropy parameters. Furthermore, the stability of these Lissajous scars is preliminarily analyzed.

TIIVISTELMÄ

Aleksi Ruhanen: Kvanttiarpeutuminen kaksiulotteisessa elliptisessä harmonisessa oskillaattorissa
Tampereen yliopisto
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Häiriöillä tuotettu kvanttiarpeutuminen on ilmiö, jossa klassisen mekaniikan mukaiset jaksolliset liikeradat jättävät jäljen – kvanttiarven – vastaavan kvanttimekaanisen, häirityn järjestelmän ominaistiloihin. Tässä työssä tutkitaan häiriöillä tuotettua arpeutumista kaksiulotteisessa anisotrooppisessa (elliptisessä) harmonisessa oskillaattorissa. Erityisesti osoitetaan, että osa paikallisesti häirityn elliptisen oskillaattorin ominaistiloista on arpeutunut muistuttamaan häiriöttömän systeemin mukaisia Lissajous'n ratoja. Systeemiä häiritään ripottamalla satunnaisesti harmoniseen potentiaaliin paikallisia ”töyssyjä”, jotka mallintavat vastaavan nanorakenteen epäpuhtauksia. Nämä Lissajous'n arvet kytkeytyvät vastaaviin klassisiin ratoihin, joiden geometria ja olemassaolo taas kytkeytyvät oskillaattorin elliptisyyteen. Lisäksi alustava tutkimus osoittaa Lissajous'n arpien olevan stabiilimpia kuin vastaavat klassiset radat systeemin elliptisyyden muutoksille.

TABLE OF CONTENTS

1. INTRODUCTION

In classical mechanics, chaotic systems are characterized by their sensitivity to initial conditions. Two initially nearby trajectories will separate at an exponential rate described by a positive Lyapunov exponent, and the chaotic trajectories exhibit aperiodic behavior. These systems are still deterministic, and their complex motion arises from nonlinearities in the system than from truly random behavior [1].

Instead of tracing the precise trajectory of a particle, the state of a quantum system is determined by a wave function. In the physical interpretation, its absolute value squared describes the probability density of finding the particle at a given position [2]. The classical definition of chaos becomes ambiguous due to the probabilistic nature of quantum mechanics coupled with the uncertainty principle and the unitary time evolution of the quantum system. Hence the term “quantum chaos” refers to the study of quantum systems which are classically chaotic [3]. Nevertheless, classical-like behavior should start to emerge in the semiclassical limit according to the correspondence principle.

Perturbation-induced (PI) quantum scars are enhancements of the probability density found in some high-energy eigenstates of a quantum system perturbed by local potential bumps [4]. These enhancements resemble the periodic orbits (PO) of the corresponding unperturbed classical system. In this Thesis, PI scarring in an anisotropic (elliptic) harmonic oscillator (HO) is studied. This is an interesting subject of study because real potential wells are seldom perfectly circularly symmetric. We demonstrate that some eigenstates of the perturbed oscillator show strong scarring resembling the POs of the unperturbed classical system, which are known as Lissajous curves (see, e.g., Refs. [5,6]). We demonstrate that the strength, shape and abundance of the scars depend on the ratio of anisotropy of the oscillator. We identify that the optimal scarring condition is related to the existence of classical POs.

2. THEORETICAL BACKGROUND

The eigenstates of a time-independent quantum system can be solved from the Schrödinger equation

$$H\psi = E\psi, \tag{1}$$

where ψ is a wave function, E is the corresponding energy, and H is the Hamiltonian operator. The eigenstates of a classically chaotic quantum system were assumed to be formless and random, until certain tracks of enhanced probability density were discovered [7] in the quantum version of the Bunimovich stadium, a well-known example of a chaotic billiard [7]. These tracks, also known as scars, trace clear geometric shapes related to certain unstable periodic orbits of the corresponding chaotic classical system. The instability of the PO separates quantum scarring from being a direct consequence of the correspondence principle, which predicts that probability density is enhanced in the vicinity of a stable PO. While being theoretically peculiar, scars have also been experimentally observed, see, e.g., Ref. [3].

A new type of quantum scarring was recently reported in two-dimensional quantum wells perturbed by randomly scattered localized potential “bumps” [4]. In this PI scarring, some of the high-energy eigenstates of the perturbed system resemble the POs of the corresponding unperturbed classical system. Thus, it is a phenomenon related to conventional scarring only by appearance. The existence of PI scarring can be understood within the perturbation theory, arising as a combined effect of the special near-degeneracies in the unperturbed system and the local nature of the perturbation [4]. Interestingly, it has been shown [4] that PI scarring can be utilized to propagate a quantum wave packet with high occurrence. A later research in Ref. [8] demonstrates a high degree of controllability over the shape and orientation of the scars by employing a focused perturbation and an external magnetic field.

On the classical side, the POs, which scars resemble, are associated with resonances in the oscillation frequencies of radial and angular motion [4] in an isotropic potential well. In an elliptic oscillator however, the POs are connected to the ratio of ellipticity of the potential. These POs are known as Lissajous orbits.

3. MODEL SYSTEM

Harmonic oscillators (HOs) are systems where a displacement from the equilibrium results in a restoring force proportional to the displacement. Despite the simplicity of the system, HOs are found in many fields of physics. Furthermore, they can be used to approximate a potential function near the equilibrium [9]. Examples of HOs include Hooke's law which applies to springs and the Lennard-Jones potential near its equilibrium, which describes interatomic interaction. Likewise, the harmonic oscillator acts as an important model in quantum mechanics. It also bears major historical significance, being studied by Born [10] and Schrödinger [11] in the early formulation of quantum mechanics.

In this Thesis, we study a two-dimensional elliptical HO, meaning that the restoring force is dependent on both the direction and magnitude of the displacement. Atomic units with $m = \hbar = e = 1$ are used for simplicity.

3.1 Classical Elliptical Oscillator

A classical elliptical harmonic oscillator is defined by the Hamilton function

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \omega_x^2 x^2 + \omega_y^2 y^2, \tag{2}$$

where ω_x and ω_y are parameters defining the strength of the confining potential. The corresponding classical equations of motion are

$$\ddot{x} + \omega_x^2 x = 0 \;\text{ and }\; \ddot{y} + \omega_y^2 y = 0.$$

The equations above describe two uncoupled, one-dimensional simple harmonic oscillators, which have the general solutions

$$x(t) = \mathrm{A}\sin(\omega_x t + \delta) \;\text{ and }\; y(t) = \mathrm{B}\sin\left(\omega_y t\right), \tag{3}$$

where δ presents a phase factor, and A and B are constants determined by the initial conditions. The period of the individual sine functions is $\frac{2\pi}{\omega_{x/y}}$, where $\omega_{x/y}$ is either ω_x or ω_y depending on the axis. For an orbit to close, the two periods of the sine functions have to be commensurable. The period T of a closed orbit can thus be written as

$$T = a\frac{2\pi}{\omega_x} = b\frac{2\pi}{\omega_y}, \tag{4}$$

where a and b are relatively prime positive integers, i.e., their greatest common divisor is one. Consequently, if the greatest common divisor of ω_x and ω_y is one, the period of a closed orbit is 2π, and a and b are equal to ω_x and ω_y, respectively. Furthermore, each a:b pair uniquely labels a set of Lissajous curves with a horizontal oscillations and b vertical oscillations in a single period. The phase δ describes how the curve would seem to be rotated as interpreted as a three-dimensional object.

For the periods of the sine functions to be commensurable, the integers a and b are required to be commensurable, i.e., their ratio is a rational number. This is true when neither ω_x or ω_y are irrational. If the frequencies are incommensurate, a trajectory does not close in coordinate space. In phase space the trajectory is dense on a toroid, passing arbitrarily close to each point on the allowed surface without ever passing over the same point twice [12].

3.2 Quantum Oscillator

In quantum mechanics, the Hamiltonian of an elliptical HO is

$$H = -\frac{1}{2}\nabla^2 + V, \tag{5}$$

where the potential V has the form

$$V = \omega_x^2 x^2 + \omega_y^2 y^2.$$

The stationary states of the HO can be solved analytically [13] from the Schrödinger equation (1), and the solution is given by

$$\psi_{nm}(x,y) = \frac{\left(\omega_x\omega_y\right)^{\frac{1}{4}}}{\sqrt{2^{(n+m)} n!\, m!\, \pi}} H_n\left(x\sqrt{\omega_x}\right) H_m\left(y\sqrt{\omega_y}\right) e^{-\frac{1}{2}\left(\omega_x x^2 + \omega_y y^2\right)}, \tag{6}$$

where H_n is the Hermite polynomial of the nth order. The corresponding energies are given by

$$E_{nm} = \omega_x\left(n + \frac{1}{2}\right) + \omega_y\left(m + \frac{1}{2}\right),$$

which shows degeneracies at commensurable frequency ratios.

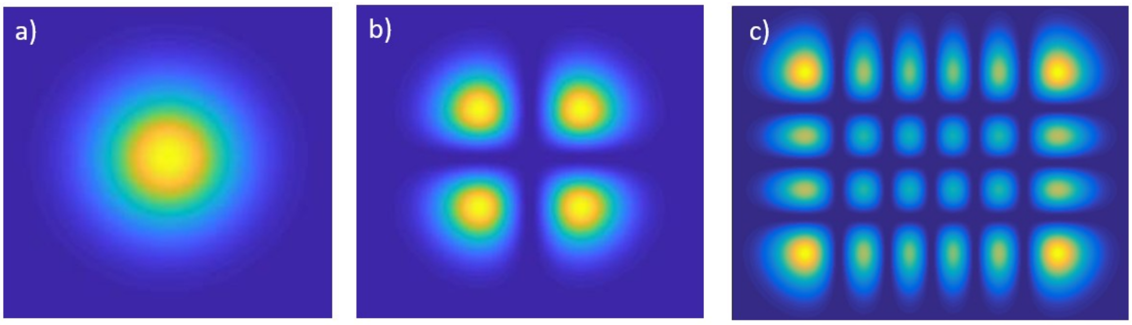

Figure 1: *Examples of eigenstates in an elliptical harmonic oscillator at commensurable frequency ratio* $2:3$. *Subplots (a)-(c) show the probability densities of the quantum states* ψ_{00}, ψ_{11} *and* ψ_{34} *respectively.*

Figure 1 shows the probability densities of some eigenstates at the commensurable frequency ratio 2:3. The solution (6) exhibits similar rectangular symmetry up to the semi-classical limit where classical behavior should start to appear.

3.3 Perturbed Classical and Quantum Systems

In this Thesis, the perturbed elliptical harmonic oscillator is modeled by distributing randomly located, Gaussian-like "bumps" into the system introduced in Eq. (2) and Eq. (5). The perturbation is described as a sum of individual bumps:

$$V_P(\boldsymbol{r}) = \sum_i \mathrm{A} e^{\frac{-|\boldsymbol{r}-\boldsymbol{r}_i|^2}{2\sigma^2}}, \tag{7}$$

where A is the bump amplitude and σ the bump width, which corresponds to the full width at half maximum (FWHM) by

$$\mathrm{FWHM} = 2\sqrt{2\ln 2}\,\sigma.$$

The quantum Hamiltonian of the perturbed system is

$$H = -\frac{1}{2}\nabla^2 + V + V_p. \tag{8}$$

In general, numerical methods are required to solve the eigenstates of the perturbed Hamiltonian (8), as well as to study the classical trajectories in the corresponding classical system.

4. RESULTS

4.1 Classical System

The classical solution presented in Eq. (3) for the unperturbed system produces the Lissajous curves, as illustrated in Fig. 2.

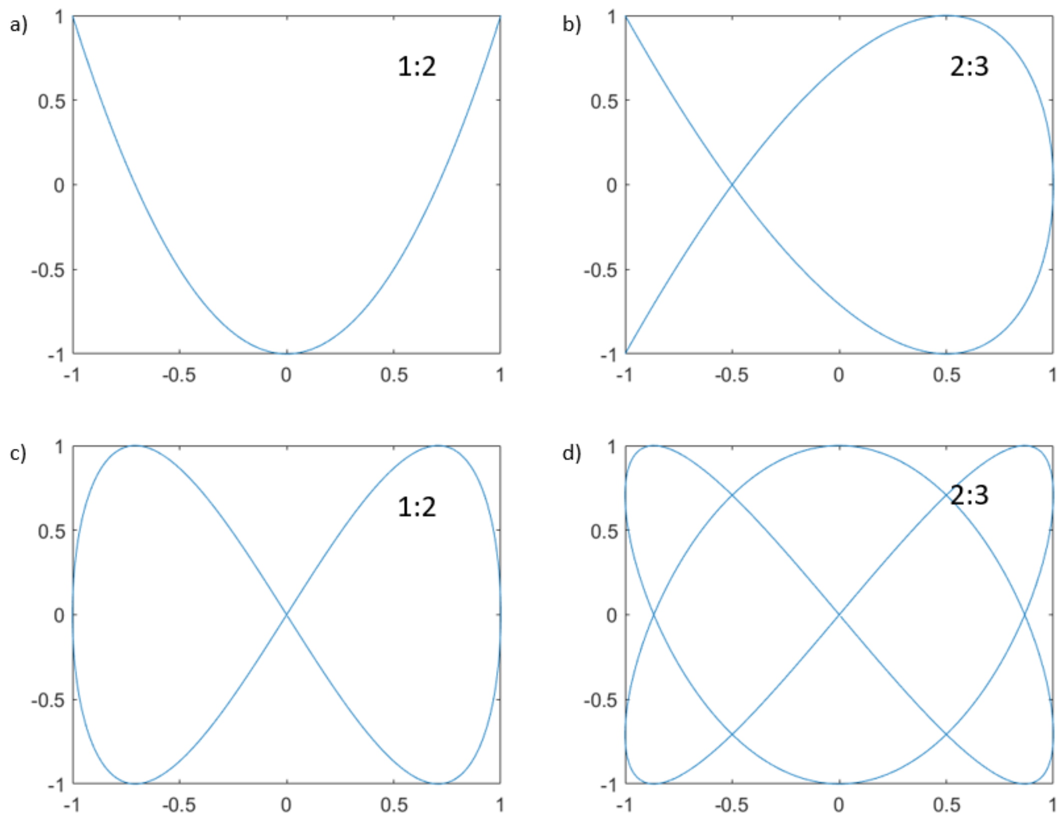

Figure 2: *Examples of Lissajous curves at commensurable frequency ratio* $1:2$ *and* $2:3$ *as labelled in the subplots. The amplitudes A and B are set to unity. A string-like Lissajous curve is depicted in subplots (a) and (b), and subplots (c) and (d) show a loop-like curve. The different appearance of the curve with the same frequency ratio stems from the selection of the phase factor* δ *in Eq. (3).*

4.2 Quantum System

We utilize the `itp2d` program [14] to numerically solve the 4000 lowest eigenstates of a perturbed anisotropic HO described by the Hamiltonian (8). The bumps [see Eq. (7)] are randomly scattered across the system with a uniform mean density of two bumps per unit square. In the studied energy range $E = 100 - 230$, the classical allowed region contains hundreds of bumps. The bump amplitude was set to A = 4, which causes strong scarring in the studied energy range. The bump width is fixed to σ = 0.1, which corresponds to FWHM of 0.235, comparable to the local wavelength of the considered eigenstates.

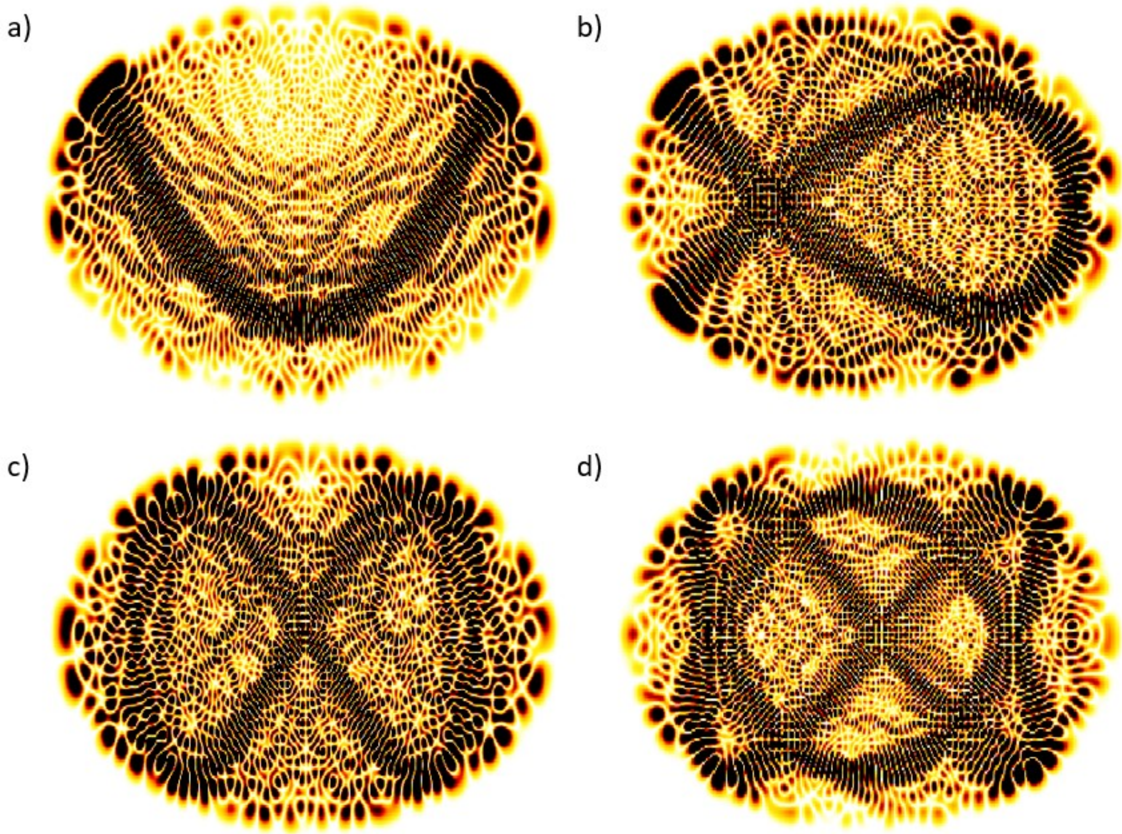

Figure 3: *Examples of eigenstates in a perturbed anisotropic oscillator showing strong perturbation-induced scarring related to the Lissajous curves of the unperturbed classical oscillator. Subplots (a) and (c) are calculated with frequency ratio* $1:2$ *and subplots (b) and (d) with ratio* $2:3$.

As expected based on the generalized PI scar theory [15], some of the high-energy eigenstates of the perturbed oscillator are strongly scarred by the Lissajous curves of the classical, unperturbed counterpart [see Fig. 2]. Examples of these Lissajous scars are shown in Fig. 3. The scarring appears in the geometries reflecting the classical solution [see Eq. (3)] with different phases δ exhibiting two distinct geometries, named here as strings [see Figs. 3 (a) and (b)] and loops [see Figs. 3 (c) and (d)].

4.3 Scar Stability

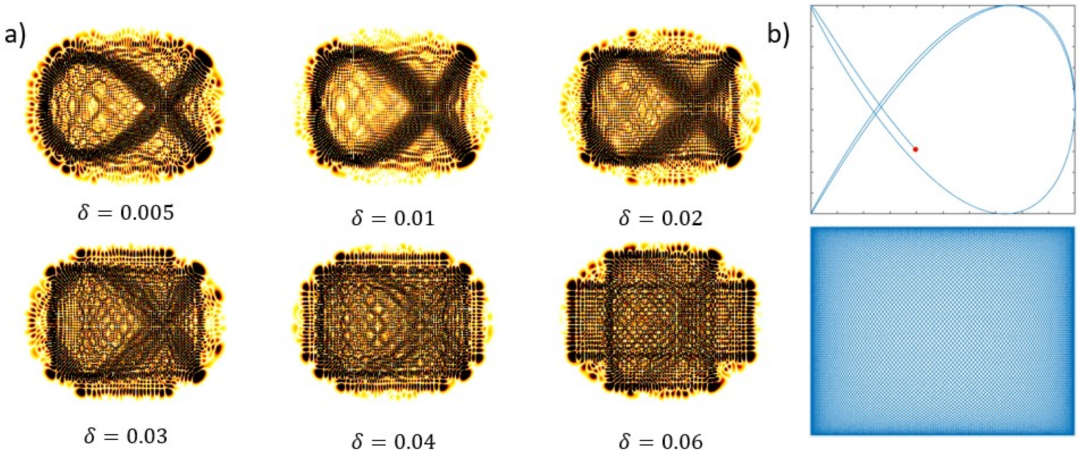

Figure 4: *(a) Examples of PI scarred eigenstates with increasing deviation δ at frequency ratio $\omega_x : \omega_y = (2 + \delta) : 3$. (b) Classical, unperturbed (no bumps) trajectory at frequency ratio $\omega_x : \omega_y = 2.01 : 3$. The top figure features an incomplete orbit, exhibiting similarity to the Lissajous curve seen in Fig. 2 (b). The bottom figure is a full orbit with the same frequency ratio.*

We carried out a preliminary, qualitative analysis on the stability of scars against a slight modulation out of the optimal confinement anisotropy. While a trajectory initially follows the Lissajous orbit seen in Fig. 2 (b) in a slightly deformed HO, as illustrated in the upper panel of Fig. 4 (b), it eventually loses its distinct geometry, and forms instead a rectangular, mesh-like structure such as in Fig. 4 (b). This is explained by Eq. (4), as the considered modulation to ω_x or ω_y changes the integers a and b, leading to more vertical and horizontal oscillations in a single period.

Despite the classical unperturbed trajectory being sensitive to the anisotropy of the confinement, the PI scars seem to exhibit resistance against a slight modulation. With increasing modulation however, the scars also appear to fade out and their abundance also diminishes. Furthermore, a rectangular structure appears, which is a reminiscent of the structure arising from the Hermite polynomials seen in Fig. 1. A more in-depth study into scarring under slight modulation can be found in Ref. [15].

5. SUMMARY

Anisotropic, or elliptic harmonic oscillators have most of their properties described by their ratio of anisotropy. The classical solutions to these systems are characterized by so-called Lissajous curves, which bear very distinct geometries. As expressed in terms of the Hermite polynomials, the corresponding quantum systems, on the other hand, display a rectangularly symmetric structure which has no resemblance to the classical solution.

In this Thesis, local, Gaussian-like perturbations were added to an elliptic HO in an effort to investigate perturbation-induced scarring in the system. We have demonstrated that some eigenstates are scarred by the Lissajous orbits of the classical unperturbed oscillator. These scars appear in two distinct geometries called “strings” and “loops”. While the primary focus was on the optimal commensurable ratios where scarring was most expected to appear, we also carried out a preliminary analysis on the scar stability against slight modulation out of the anisotropy ratio. The results indicate that Lissajous scars have resistance against minor deviations. Lissajous scarring extends the concept of perturbation-induced scarring to anisotropic oscillators, in addition to the previously studied circularly symmetric potential wells in Refs. [4,8].

The future interest lies in the experimental detection of the phenomenon and in the analysis of quantum transport in the system. The use of an optical system to detect the Lissajous scarring is suggested in Ref. [15]. In this approach, an optical fiber with certain refractive properties is employed, so that the electromagnetic wave equation inside the fiber is analogous to the Schrödinger equation relevant to the system explored in this Thesis.

BIBLIOGRAPHY

[1] S. H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering* (Perseus Books, Massachusetts, 1994).

[2] D. J. Griffiths, *Introduction to quantum mechanics* (Prentice Hall, Englewood Cliffs, N.J., 1995).

[3] For a review of the field, see, e.g., H. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, 1999).

[4] P. J. J. Luukko, B. Drury, A. Klales, L. Kaplan, E. J. Heller, and E. Räsänen, Sci. Rep. **6**, 37656 (2016).

[5] J. Lissajous, Ann. Chim. Phys. **51**, 147 (1857).

[6] N. Bowditch, Mem. Am. Acad. Arts. Sci. **3**, 413 (1815).

[7] E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).

[8] J. Keski-Rahkonen, P. J. J. Luukko, L. Kaplan, E. J. Heller, and E. Räsänen, Phys. Rev. B **96**, 094204 (2017).

[9] S. C. Bloch, *Introduction to Classical and Quantum Harmonic Oscillators* (John Wiley & Sons, Incorporated, Hoboken, 1997).

[10] M. Born and P. Jordan, Z. Phys. **34**, 858 (1925).

[11] E. Schrödinger, Ann. Phys. **384**, 361 (1926).

[12] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics (3rd Edition)* (Addison-Wesley, 2001).

[13] D. Bonatsos, P. Kolokotronis, D. Lenis, and C. Daskaloyannis, Int. J. Mod. Phys. A **12**, 3335 (1997).

[14] P. J. J. Luukko and E. Räsänen, Comput. Phys. Commun. **184**, 769 (2013).

[15] J. Keski-Rahkonen, A. Ruhanen, E. J. Heller, and E. Räsänen, *Quantum Lissajous Scars* (Submitted).