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New Perspectives in Hyperbolic Function Theory

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Abstract

In this thesis we are working with a function theory on the hyperbolic upper-half space. The function theory is called the hyperbolic function theory and it is studied since 1990's by Heinz Leutwiler and Sirkka-Liisa Eriksson. The advantage of the hyperbolic function theory is that positive and negative powers of hypercomplex variables are included to the theory. Thus the hyperbolic function theory offers a natural generalization of classical complex analysis.

The hyperbolic space is defined as a Riemannian manifold $(\mathbb{R}^{n+1}_{+}, h_k)$, where the manifold is

$$\mathbb{R}^{n+1}_{+} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$$

and the metric is

$$h_k = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^{\frac{2k}{n-1}}}.$$

Using harmonic differential forms and Clifford algebras with a negative signature we obtain the modified Dirac operator defined by

$$M_k f = D_x f + \frac{k}{x_n} Q' f,$$

where D_x is the Euclidean Dirac operator and Q' is a projection type mapping. Null-solutions of M_k are called hypermonogenic functions.

In this work we first study what is the corresponding function theory, if we assume that the Clifford algebra has a positive signature. This work is accomplished in papers I-IV. We deduce the basic theory very completely and then we prove the Cauchy type integral formulas. Especially we study the case where functions takes their values in the so called k-vector spaces.

In papers V and VI we study mean value properties for hypermonogenic

functions.

In the introduction part of this thesis i.e., pages before the appendix papers, we'd like to give a brief summary of topics which are important (in the author's point of view) in the hyperbolic function theory. It is not complete but gives some ideas for the further studies of the topic. Especially it is not a review of my research because one may found all the details from the papers included into this thesis.

Preface

This work is mainly carried out in Department of Mathematics at Tampere University of Technology during the years 2006-2010. First of all I'd like to thank my supervisor Sirkka-Liisa Eriksson for her guidance and support in the course of this thesis.

I am also grateful to the head of the department, professor Seppo Pohjolainen for excellent working environments and research conditions. Also I would like to thank all the people at the department of mathematics for the nice working atmosphere. Especially I am grateful to Osmo Kaleva for many TEXnical assistances.

During this job I have visited a few times in the Clifford Research group at Ghent university. I like to thank all the people at the Galglaan for the inspiring atmosphere and the superb research conditions. Especially I'd like to thank prof. Frank Sommen to showing for me how the professional mathematicians work. I learned to push the limits of what would concretely mean to have a clear and deep thinking, to take a huge distance from things and events so that the essence could be touched.

For the financial support I'd like to thank the Emil Aaltonen foundation, the Magnus Ehrnrooth foundation and the Vilho, Yrjö and Kalle Väisälä foundation.

Lastly I want to thank my family and friends for the encouragement and the support.

This thesis is dedicated to my dear son Viljami.

Hervanta, March 2010

Heikki Orelma

Contents

List of Publications

This thesis is based on the following publications.

(I) Eriksson S.–L. and Orelma H., *Hyperbolic Function Theory in the Clifford Algebra* $\mathcal{C}\ell_{n+1,0}$, Adv. appl. Clifford alg. 19 (2009), 283 – 301

(II) Eriksson S.–L. and Orelma H., *A Glance at Hyperbolic Function Theory in the Context of Geometric Algebras: Hypergenic Operators*, Proceedings of the 3rd Nordic EWM Summer School for PhD Students in Mathematics, TUCS General Publications, No 53, June 2009

(III) Eriksson S.–L. and Orelma H., *On Modified Dirac Operators in Geometric Algebras: Integration of Multivector Functions*, Clifford Algebras and Inverse Problems (Tampere 2008), Tampere Univ. of Tech. Department of Math. research Report No. 90 (2009), pp. 55 – 68

(IV) Eriksson S.–L. and Orelma H., *Topics on Hyperbolic Function Theory in Geometric Algebra with a Positive Signature*, Comput. Methods Funct. Theory Volume 10 (2010), No. 1, 249 – 263

(V) Eriksson S.–L. and Orelma H., *A Hyperbolic Interpretation of Cauchy Type Kernels in Hyperbolic function Theory*, submitted (preprint included).

(VI) Eriksson S.–L. and Orelma H., *A Mean-Value Theorem for Some Eigenfunctions of the Laplace-Beltrami Operator on the Upper-Half Space*, submitted (preprint included).

Brief Summary of Publications

In first four papers I-IV we are working with the Clifford algebra with the positive signature, that is, if $\mathbb{R}^{n+1}$ is the Euclidean space with an orthonormal basis $\{e_0, e_1, ..., e_n\}$, the Clifford algebra $\mathcal{C}\ell_{n+1,0}$ is an associative algebra with unit generated by the relations

$$e_i e_j + e_j e_i = 2\delta_{ij},$$

for $i, j = 0, ..., n$. As a geometric model we use the upper-half space

$$\mathbb{R}^{n+1}_+ = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_0 > 0\}$$

with the metric

$$h_k = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_0^{\frac{2k}{n-1}}}.$$

For each $k \in \mathbb{R}$ we define the operator H_k by

$$H_k f = \partial_{\mathbf{x}} f - \frac{k}{x_0} Q_0 f$$

where $\partial_{\mathbf{x}}$ is the Euclidean Dirac operator, i.e.,

$$\partial_{\mathbf{x}} = e_0 \partial_{x_0} + \cdots + e_n \partial_{x_n}$$

and Q_0 is a projection type mapping. Null-solutions of H_k are called hypergenic functions. We study their basic function theory and deduce the Cauchy integral formulea for them.

In papers V and VI we study mean-value properties. We are working with the Clifford algebra with the negative signature, that is, if $\mathbb{R}^n$ is the Euclidean space with a basis $\{e_1, ..., e_n\}$, the Clifford algebra $\mathcal{C}\ell_{0,n}$ is generated by the relations

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

for $i, j = 1, ..., n$. As a geometric model we use the upper-half space

$$\mathbb{R}^{n+1}_+ = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$$

with the metric

$$g_k = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^{\frac{2k}{n-1}}}.$$

On $\mathbb{R}^{n+1}_+$ we define the modified Dirac operator M_k by

$$M_k f = D_x f + \frac{k}{x_n} Q' f,$$

where the operator D_x is the Euclidean Cauchy-Riemann operator, i.e.,

$$D_x = \partial_{x_0} + e_1 \partial_{x_1} + \cdots + e_n \partial_{x_n}.$$

Solutions of the equation $M_k f = 0$ are called k-hypermonogenic functions.

We denote the Clifford algebra generted by the elements $e_1, ..., e_{n-1}$ by $\mathcal{C}\ell_{0,n-1}$. Then we have the split $\mathcal{C}\ell_{0,n} = \mathcal{C}\ell_{0,n-1} \oplus \mathcal{C}\ell_{0,n-1}\, e_n$, that is, for each $a \in \mathcal{C}\ell_{0,n}$ there exist Pa and Qa in $\mathcal{C}\ell_{0,n-1}$ satisfying

$$a = Pa + (Qa)e_n.$$

Let f be a hypermonogenic function. In paper V we study mean-value propertied for the functions Pf and in paper VI for the functions Qf.

Next we review the most important results paper by paper.

(I) In this paper the basics of hyperbolic function theory using the Clifford algebra $\mathcal{C}\ell_{n+1,0}$ is developed. The basic operator equalities are deduced. The main result is the Cauchy integral formula.

(II) This paper is a survey of papers I, III and IV.

(III) This paper deals with the multivector functions. The basic multivector calculus is studied. In the last part of the paper we study integration theory on multivector functions.

(IV) This is the sequel of paper I. We deduce the Borel-Pompeiu formula and study multivector calculus.

(V) In this paper we study first hyperbolic geometry. We apply it to the Cauchy type kernels and we obtain the hyperbolic interpretation to the kernels.

(VI) In this paper we study a mean value property for eigenfunctions of the Laplace-Beltrami operator.

Lastly we express the author's contribution to the papers. In all publications the author is the corresponding writer.

(I-IV) In these papers the author did the mathematical work.

(V-VI) The author wrote the manuscripts and contributed to the ideas.

Chapter 1

Clifford Algebras

In this chapter we will consider briefly Clifford algebras over the Euclidean space $\mathbb{R}^n$.

1.1 A General Definition of Clifford Algebras

We will study the Clifford algebras for $\mathbb{R}^n$ with the quadratic form

$$Q_{r,s}(x) = -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_{r+s}^2.$$

where $r + s = n$. We will denote the space $\mathbb{R}^n$ with the quadratic form $Q_{r,s}$ by $\mathbb{R}^{r,s}$.

The corresponding Clifford algebra is denoted by $\mathcal{C}\ell_{r,s}$. Let $e_1, ..., e_{r+s}$ be any Q-orthonormal (i.e., $Q_{r,s}(e_j) = \pm 1$) basis of $\mathbb{R}^{r,s} \subset \mathcal{C}\ell_{r,s}$. Then $\mathcal{C}\ell_{r,s}$ is generated by $e_1, ..., e_{r+s}$ subject to the relations

$$e_i e_j + e_j e_i = \begin{cases} 2\delta_{ij}, & \text{if } i \leq r, \\ -2\delta_{ij}, & \text{if } i > r. \end{cases}$$

A more general basis free definition is available for example in [6] or [1].

The most important Clifford algebras in our case are $\mathcal{C}\ell_{0,n}$ and $\mathcal{C}\ell_{n,0}$ where the squares of the generators are:

$$e_i^2 = \begin{cases} 1, & \text{if } e_i \in \mathcal{C}\ell_{n,0}, \\ -1, & \text{if } e_i \in \mathcal{C}\ell_{0,n}. \end{cases}$$

Consider the ordered sets $A = \{a_1, ..., a_k\} \subset M = \{1, ..., n\}$, where $1 \leq a_1 < \cdots < a_k \leq n$, and define

$$e_A = e_{a_1} \cdots e_{a_k}.$$

Especially $e_\emptyset = 1$ and $e_{\{j\}} = e_j$. Each $a \in \mathcal{C}\ell_{r,s}$ admits the representation

$$a = \sum_{A \subset M} a_A e_A,$$

where $a_A \in \mathbb{R}$ for each $A \subset M$. The number of elements in the set A is denoted by $|A|$. Elements of the form

$$a = \sum_{|A|=k} a_A e_A,$$

are called **k-(multi)vectors**. The space of k-vectors is denoted by $\mathcal{C}\ell^k_{r,s}$. Obviously $\mathcal{C}\ell^0_{r,s} = \mathbb{R}$ and $\mathcal{C}\ell^1_{r,s} = \mathbb{R}^{r,s}$. Thus we may decompose $\mathcal{C}\ell_{r,s}$ as a direct sum of subspaces by

$$\mathcal{C}\ell_{r,s} = \mathbb{R} \oplus \mathbb{R}^{r,s} \oplus \mathcal{C}\ell^2_{r,s} \oplus \cdots \oplus \mathcal{C}\ell^{r+s}_{r,s}.$$

The natural projection $\mathcal{C}\ell_{r,s} \to \mathcal{C}\ell^k_{r,s}$ is denoted by $[\cdot]_k$. Each $a \in \mathcal{C}\ell_{r,s}$ admits the **multivector decomposition** as

$$a = \sum_{k=0}^{n} [a]_k.$$

Using the projections $[\cdot]_k$ we define teh following products. The **exterior product** $\wedge$ is defined on multivectors by

$$[a]_j \wedge [b]_k = [[a]_j [b]_k]_{j+k}.$$

and extended to the whole $\mathcal{C}\ell_{r,s}$ by linearity:

$$a \wedge b = \sum_{j,k=0}^{n} [[a]_j [b]_k]_{j+k}.$$

The **inner product** $\cdot$ is defined on multivectors by

$$[a]_j \cdot [b]_k = \begin{cases} [[a]_j [b]_k]_{|j-k|}, & \text{if } j,k \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and extended to the whole $\mathcal{C}\ell_{r,s}$ by linearity:

$$a \cdot b = \sum_{j,k=1}^{n} [[a]_j [b]_k]_{|j-k|}.$$

Also we need to define a few involutions, namely reversion, conjugation and main-involution. Since all involutions are algebra (anti)automorphisms we need to fix their values only for vectors.

The **main involution** is the algebra automorphism $' : \mathcal{C}\ell_{r,s} \to \mathcal{C}\ell_{r,s}$ defined by $\mathbf{x}' = -\mathbf{x}$.

The **reversion** is the algebra antiautomorphism $^* : \mathcal{C}\ell_{r,s} \to \mathcal{C}\ell_{r,s}$ defined by $\mathbf{x}^* = \mathbf{x}$.

The **conjugation** is the algebra antiautomorphism $^- : \mathcal{C}\ell_{r,s} \to \mathcal{C}\ell_{r,s}$ defined as a composition of the previous involutions, that is, if $a \in \mathcal{C}\ell_{r,s}$, then $\overline{a} = (a')^* = (a^*)'$.

It is an easy exercise to see that the conjugation is well defined.

1.2 Clifford Algebras with the signature $(0, n)$ and $(n, 0)$

The Clifford algebras $\mathcal{C}\ell_{0,n}$ and $\mathcal{C}\ell_{n,0}$ has the special role. The reason for it is that if $\mathbf{x} \in \mathbb{R}^n$ is a vector, then we may compute its Euclidean norm using the Clifford multiplication. In $\mathcal{C}\ell_{0,n}$ we have

$$\mathbf{x}\overline{\mathbf{x}} = \overline{\mathbf{x}}\mathbf{x} = |\mathbf{x}|^2$$

and in $\mathcal{C}\ell_{n,0}$ we have

$$\mathbf{x}\mathbf{x} = |\mathbf{x}|^2.$$

Consequently, if $\mathbf{x} \in \mathcal{C}\ell_{0,n}$ is non-zero we define its **inverse** by

$$\mathbf{x}^{-1} = \frac{\overline{\mathbf{x}}}{|\mathbf{x}|^2}.$$

and similarly, if $\mathbf{x} \in \mathcal{C}\ell_{n,0}$ is non-zero its **inverse** is defined by

$$\mathbf{x}^{-1} = \frac{\mathbf{x}}{|\mathbf{x}|^2}.$$

Let us abbreviate $e_0 = 1$. Then we may embed the vector space $\mathbb{R}^{n+1}$ into the Clifford algebra $\mathcal{C}\ell_{0,n}$ or $\mathcal{C}\ell_{n,0}$ by

$$(x_0, x_1, ..., x_n) \mapsto \sum_{j=0}^{n} x_j e_j.$$

Thus an element $x \in \mathbb{R}^{n+1}$ is identified with the Clifford number

$$x = x_0 + \mathbf{x}$$

and it is called a **paravector**. We may compute its Euclidean norm in $\mathcal{C}\ell_{0,n}$ by

$$x\overline{x} = \overline{x}x = |x|^2$$

and if $x \neq 0$ its inverse is defined by

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

Hence we see that a non-zero paravector admits an inverse. But in general Clifford algebras are not division algebras. If $n > 2$ then

$$(1 - e_1e_2e_3)(1 + e_1e_2e_3) = 0.$$

Hence some elements are zero divisors if $n > 2$.

Chapter 2

Geometric and Analytic Preliminaries

2.1 The Hyperbolic Upper-Half Space

In this section we will consider the certain Riemannian manifold, called the Poincaré upper-half space. Especially we are interested in computing distances on it.

In the next section the hyperbolic function theory is related to the Poincaré upper-half space $(\mathbb{R}^{n+1}_+, g)$, where the hyperbolic metric in canonical coordinates is defined by

$$g = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

In general, any oriented smooth Riemannian manifold with the metric

$$g = \sum_{i,j=0}^{n} g_{ij} dx_i dx_j$$

admits the volume form (see [7]):

$$dV_g(x) = \sqrt{\det(g_{ij})} dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n.$$

In the canonical coordinates on the upper-half space $\det(g_{ij}) = 1/x_n^{2(n+1)}$. Then the volume element is

$$dx_h := dV_g(x) = \frac{dx}{x_n^{n+1}},$$

where $dx = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n$ is the Euclidean volume element. We define the hyperbolic surface element on a smooth manifold-with-boundary U in $\mathbb{R}^{n+1}_+$ with the codimension 0 by

$$d\sigma_h = \frac{\nu dS}{x_n^n},$$

where ν is the unit normal field on U and dS the classical scalar surface element.

The metric g allows us to define a distance on $\mathbb{R}^{n+1}_+$. The geodesics are described more detailed in the following theorem.

Theorem 2.1.1 *On the Poincaré half-space $\mathbb{R}^{n+1}_+$ geodesics are circles or lines which meet the boundary orthogonally.*

Proof. See [15] p. 71 or [8] p. 38.■

The hyperbolic upper-half space is an immersed submanifold of $\mathbb{R}^{n+1}$. Its tangent space at any point $x \in \mathbb{R}^{n+1}_+$ can nonetheless be viewed as a subspace of $T_x\mathbb{R}^{n+1}$. In addition, by dimensional reasons $T_x\mathbb{R}^{n+1} = T_x\mathbb{R}^{n+1}_+$ for each $x \in \mathbb{R}^{n+1}_+$. Let $\iota : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ be the canonical immersion. Then we may identify $\iota(\mathbb{R}^{n+1}_+)$ and $\mathbb{R}^{n+1}_+$ as sets. That identification allow us to use two different geometric structures on $\mathbb{R}^{n+1}_+$ parallel: the hyperbolic and the Euclidean structures. Thus we may loosely speak of Euclidean distances and balls on $\mathbb{R}^{n+1}_+$, that is, we may compute the Euclidean distance $|x - y|$ for $x, y \in \mathbb{R}^{n+1}_+$.

Lemma 2.1.2 (V, p. 3) *The distance $d_h(x, a)$ between the points $x = x_0 + e_1x_2 + \cdots + x_ne_n$ and $a = a_0 + e_1a_1 + \cdots + a_ne_n$ in $\mathbb{R}^{n+1}_+$ is*

$$d_h(x, a) = \operatorname{arcosh} \lambda(x, a),$$

where

$$\lambda(x, a) = \lambda(a, x) = \frac{|x - a|^2 + 2a_nx_n}{2x_na_n} = \frac{|x - a|^2}{2x_na_n} + 1.$$

Next we briefly review the connection between the hyperbolic and the Euclidean distance of two points. As a direct computation we obtain the following formulae.

Lemma 2.1.3 (V, p. 3) *If* $x = x_0 + e_1 x_2 + \cdots + x_n e_n$ *and* $a = a_0 + e_1 a_1 + \cdots + a_n e_n$ *are points in* $\mathbb{R}^{n+1}_+$*, then*

$$|x-a|^2 = 2x_n a_n \left(\lambda(x,a) - 1\right),$$
$$|x-\widehat{a}|^2 = 2x_n a_n \left(\lambda(x,a) + 1\right),$$
$$\frac{|x-a|^2}{|x-\hat{a}|^2} = \frac{\lambda(x,a)-1}{\lambda(x,a)+1} = \tanh^2\Big(\frac{d_h\,(x,a)}{2}\Big),$$

where $\widehat{a} = a_0 + e_1 a_1 + \cdots + a_{n-1} e_{n-1} - a_n e_n$.

Next we shall prove the connection between the hyperbolic and the Euclidean ball in $\mathbb{R}^{n+1}_+$.

Proposition 2.1.4 (V, p. 4) *Let* $\iota : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ *be the canonical immersion. Then*

$$\iota(B_h(a,R_h)) = B_e\big(\tau(a,R_h), R_e(a,R_h)\big),$$

where

$$\tau(a,R_h) = a_0 + a_1 e_1 + \cdots + a_{n-1}e_{n-1} + a_n e_n \cosh R_h$$

is the Euclidean center and

$$R_e(a,R_h) = a_n \sinh R_h$$

is the corresponding Euclidean radius.

The preceding proposition allow us to abbreviate briefly by

$$B_h(a,R_h) = B_e\big(\tau(a,R_h), R_e(a,R_h)\big).$$

Proposition 2.1.4 says that if x is a boundary point of the hyperbolic ball $B_h(a,R_h)$, that is $d_h(a,x) = R_h$ then the Euclidean distance between the points x and $\tau(a,R_h)$ is $|x-\tau(a,R_h)| = a_n \sinh R_h$. Putting these immediate consequences together we obtain the following corollary, which has an important role in the theory of mean-value properties.

Corollary 2.1.5 (V, p. 4) *If* $x \in \mathbb{R}^{n+1}_+$ *and* $\tau(a,x) = a_0 + a_1 e_1 + \cdots + a_{n-1}e_{n-1} + a_n e_n \cosh d_h(a,x)$ *then*

$$|x - \tau(a,x)| = a_n \sinh d_h(x,a).$$

Chapter 3

On Hyperbolic Function Theory

3.1 Operators on the Upper-Half Space

In this section we shall study some geometric operators on the Poincaré upper-half space. First we recall the Hodge $*$-operator and its basic properties, see e.g. [14]. Although the theory of the section is classical we'd like to give short proofs for the most important results. The most important references of the section are the book of Helgason [5], von Westenholz [14], and the paper of Leutwiler [9]. Note that the similar technique is also available in the more general setting, see [11].

The Poincaré half-space is a Riemannian manifold (N, g_k) such that $N = \mathbb{R}^{n+1}_+$ with the metric

$$g_k = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^{2k}}.$$

The manifold $\mathbb{R}^{n+1}_+ = \{(x_0, ..., x_n \in \mathbb{R}^{n+1} : x_n > 0)$ admits a three different type of geometries:

- If $k > 0$ we obtain similar geometry that in the hyperbolic space. Distances "decrease" when x_n increase and vice versa.
- If $k = 0$ we obtain the Euclidean geometry.
- If $k < 0$ we obtain the geometry with the "infinite distances" as $x_n \to \infty$, but distances "tends to zero" in the neighborhood of the x_n-axis.

The metric gives the inner product

$$\langle X, Y \rangle = g_k(X, Y)$$

for each tangent space. The norm is defined by $\|X\| = \sqrt{\langle X, X\rangle}$ for $X, Y \in T_xN$. Let $\partial_{x_i}|_x$ be a basis of T_xM and $dx_i|_x$ is its dual basis of T_x^*N, i.e. $dx_i(\partial_{x_i}) = \delta_{ij}$.

Next we want to find orthonormal frame and coframe with respect to the inner product $\langle\cdot,\cdot\rangle$. We search the frame in the form $X_j = c_j\partial_{x_j}$. Since

$$\|X_j\|^2 = \frac{c_j^2}{x_n^{2k}}$$

we obtain the orthonormal frame field on (N, g):

$$X_j = x_n^k\partial_{x_j}$$

for $j = 0, 1, ..., n$. For a coframe we denote $\omega_j = b_j dx_j$. Then

$$1 = \omega_j(X_j) = b_j x_n^k dx_j(\partial_{x_j}) = b_j x_n^k$$

and thus $b_j = 1/x_n^k$, that is

$$\omega_j = \frac{dx_j}{x_n^k}.$$

Above discussion allows us to compute Hodge duals. We define the n-forms

$$d\breve{x}_j = dx_0 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n$$

and then

$$\breve{\omega}_j = \frac{d\breve{x}_j}{x_n^{kn}}$$

where $j = 0, 1, ..., n$. Then

$$*\omega_j = (-1)^j\breve{\omega}_j = (-1)^j\frac{d\breve{x}_j}{x_n^{kn}}.$$

Since $*\omega_j = \frac{*dx_i}{x_n^k}$ we obtain

$$*dx_j = (-1)^j\frac{d\breve{x}_j}{x_n^{k(n-1)}}$$

A well known fact is that $** = (-1)^{k(n-k+1)}$ for k-forms on $\mathbb{R}^{n+1}_+$. Thus we have

$$*d\breve{x}_j = (-1)^{n+j}x_n^{k(n-1)}dx_j.$$

A 1-form η is called **harmonic** if it is a solution of the system

$$d\eta = 0,\ \ d^*\eta = 0,$$

where d^* is the formal adjoint of d (cf. [13]) with respect to the above inner product. It can be shown that $d^* = *d*$.

Above discussion allows us to prove the following application of the general result.

Proposition 3.1.1 *If $\eta = u_0 dx_0 + u_1 dx_1 + \cdots + u_n dx_n$ is a 1-form on the upper-half space $(\mathbb{R}^{n+1}_+, g)$. Then*

$$d\eta = \sum_{k<j} \Big(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \Big) dx_k \wedge dx_j.$$

and

$$d^*\eta = x_n^{2k} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} - k(n-1) x_n^{2k+1} u_n.$$

Especially, η is harmonic if and only if its component functions satisfy the (M. Riesz) system

$$\begin{aligned} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} - \frac{k(n-1)}{x_n} u_n = 0, \\ \frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k}, \end{aligned}$$

for $k < j$.

Proof. First we compute

$$d\eta = \sum_{k<j} \Big(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \Big) dx_k \wedge dx_j.$$

On the other hand, since

$$*\eta = \sum_{j=0}^{n} (-1)^j u_j \frac{d\breve{x}_j}{x_n^{k(n-1)}}$$

we obtain

$$\begin{aligned} d * \eta &= \sum_{j=0}^{n} (-1)^j \frac{\partial u_j}{\partial x_j} \frac{dx_j \wedge d\breve{x}_j}{x_n^{k(n-1)}} - k(n-1)(-1)^n \frac{u_n}{x_n^{k(n-1)+1}} dx_n \wedge d\breve{x}_n \\ &= \Big(x_n^{2k} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} - k(n-1) x_n^{2k+1} u_n \Big) dV_{g_k} \end{aligned}$$

where

$$dV_{g_k}(x) = \frac{dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n}{x_n^{k(n+1)}}.$$

is the volume form. Since $1 = *dV_{g_k}$ we obtain that

$$d^*\eta = x_n^{2k} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} - (n-1)x_n^{2k+1} u_n.$$

The proof is complete.■

Trying to avoid messy formulae we see that the one of the best possible metric is described in the following corollary.

Corollary 3.1.2 *Let $(\mathbb{R}^{n+1}, h_k)$ be the hyperbolic space with the metric*

$$h_k = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n}{x_n^{\frac{2k}{n-1}}}$$

Then η is harmonic if and only if its component functions satisfy the (M. Riesz) system

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} - \frac{k}{x_n} u_n = 0,$$
$$\frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k},$$

for $k < j$.

Next we recall an isomorphism between the space of 1-forms and the space of paravectors. For the construction of an isomorphism we need to recall the paravector differential (see [12]):

$$dx = dx_0 + d\mathbf{x}.$$

where

$$d\mathbf{x} = \sum_{j=1}^{n} e_j dx_j$$

If f is a paravector we define the mapping φ by

$$\varphi : f \mapsto [dxf]_0.$$

If $f = f_0 + f_1e_1 + \cdots + f_ne_n$ then $\varphi(f) = f_0dx_0 - f_1dx_1 - \cdots - f_ndx_n$ and we see that φ is an isomorphism.

The first detailed study of the system (in the case $k = n-1$) represented in the previous corollary is due to Heinz Leutwiler in his paper [9]. He gave the following definition.

Definition 3.1.3 *A paravector valued function $f = f_0 + f_1e_1 + \cdots + f_ne_n$ is an H-**solution** if $\varphi(f)$ is a harmonic 1-form.*

An important geometric operator on a Riemannian manifold (N, h_k) is the so called **Laplace-Beltrami operator**. If the metric is expressed as $h_k = \sum_{i,j=1}^{n+1} h_{ij}dx_idx_j$ then the Laplace-Beltrami operator is defined by (cf. [5])

$$\Delta_{lb}f = \frac{1}{\sqrt{|\det(h_{ij})|}}\sum_{k=0}^{n}\frac{\partial}{\partial x_k}\Big(\sum_{j=0}^{n} h^{jk}\sqrt{|\det(h_{ij})|}\frac{\partial f}{\partial x_j}\Big)$$

where $f \in C^\infty(N)$ and h^{ij} are the elements of the inverse matrix of (h_{ij}).

The reader should notice that the Laplace-Beltrami operator is only a one of the geometric operators on a Riemannian manifold. For example, on n-dimensional Riemannian manifold there exists $(n+1)$-Laplace operators acting on differential forms.

Theorem 3.1.4 *In the upper-half space the Laplace-Beltrami operator is*

$$\Delta_{lb}f = x_n^{\frac{2k}{n-1}}\Big(\sum_{k=0}^{n}\frac{\partial^2 f}{\partial x_k^2} - k\frac{1}{x_n}\frac{\partial f}{\partial x_n}\Big),$$

where $\Delta = \sum_{k=0}^{n}\frac{\partial^2}{\partial x_k^{2k}}$ is the Euclidean Laplacian.

Proof. Since $h_{ij} = \frac{\delta_{ij}}{x_n^{\frac{2k}{n-1}}}$ we have $\sqrt{|\det(h_{ij})|} = \frac{1}{x_n^{k\frac{n+1}{n-1}}}$. Since $h^{ij} = x_n^{\frac{2k}{n-1}}\delta_{ij}$ we obtain

$$\Delta_{lb}f = x_n^{\frac{2k}{n-1}}\sum_{k=0}^{n}\frac{\partial}{\partial x_k}\Big(\sum_{j=0}^{n}\frac{x_n^{\frac{2k}{n-1}}\delta_{jk}}{x_n^{k\frac{n+1}{n-1}}}\frac{\partial f}{\partial x_j}\Big).$$

Thus

$$\Delta_{lb}f = x_n^{\frac{2k}{n-1}}\sum_{k=0}^{n}\frac{\partial}{\partial x_k}\Big(\frac{1}{x_n^k}\frac{\partial f}{\partial x_k}\Big) = x_n^{\frac{2k}{n-1}}\Big(\sum_{k=0}^{n}\frac{\partial^2 f}{\partial x_k^2} - k\frac{1}{x_n}\frac{\partial f}{\partial x_n}\Big)$$

and the proof is complete.■

Let us define the so called modified Dirac operator by

$$\mathcal{D}f = D_x f + \frac{k}{x_n}(e_n, f)$$

for $f = f_0 + f_1 e_1 + \cdots + f_n e_n$, where (e_n, f) is the Euclidean inner product such that $(e_n, 1) = 0$ and D_x is the Cauchy-Riemann operator defined by

$$D_x = \partial_{x_0} + e_1 \partial_{x_1} + \cdots + e_n \partial_{x_n}.$$

Sometimes D_x is also called the Dirac operator or the paravector derivative.

Proposition 3.1.5 *A function f is an H-solution if and only if $\mathcal{D}f = 0$.*

Proof. Let $f = f_0 + f_1 e_1 + \cdots + f_n e_n$. Then

$$[\mathcal{D}f]_0 = \frac{\partial f_0}{\partial x_0} - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} + \frac{k}{x_n} f_n$$

and

$$[\mathcal{D}f]_2 = \sum_{\substack{i,j=1 \\ i \neq j}} e_i e_j \frac{\partial f_j}{\partial x_i} = \sum_{i<j} e_i e_j \Big(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \Big).$$

If f is an H-solution then $\varphi(f) = f_0 dx_0 - f_1 dx_1 - \cdots - f_n dx_n$ is a solution of the system

$$\begin{aligned} &\frac{\partial f_0}{\partial x_0} - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} + \frac{k}{x_n} u_n = 0, \\ &\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \end{aligned}$$

for $i, j = 0, 1, ..., n$ such that $i < j$. The proof is complete.■

3.2 Modified Dirac Operators

Next we shall consider an operator defined on the set of Clifford algebra-valued functions. It is an extension of the previous modified Dirac operator. Thus our aim is to find the operator $M_k : C^\infty(\Omega, \mathcal{C}\ell_{0,n}) \to C^\infty(\Omega, \mathcal{C}\ell_{0,n})$ such that $M_k f = \mathcal{D}f$ for each paravector f. If $F \in C^\infty(\Omega, \mathcal{C}\ell_{0,n})$ then the generalization of the Dirac operator D_x is obvious. The inner product may be generalized using the Clifford inner multiplication since

$$e_n \cdot f = -(e_n, f)$$

for each paravector valued function f. Using that idea we may define the operator M_k by

$$M_k F = D_x F - \frac{k}{x_n} e_n \cdot F.$$

Of course the generalization is not the only possible one, but it gives us a quite fruitful theory. The preceding operator is called the **(left)-modified Dirac operator** (cf. [3]). We see that the element e_n has the special role. Hence we may also give the following representation for an arbitrary $a \in \mathcal{C}\ell_{0,n}$

$$a = Pa + Qae_n,$$

where Pa and Qa are elements of the Clifford algebra generated by the elements $\{e_1, ..., e_{n-1}\}$. It is easy to see that $Qae_n = e_n(Qa)'$. If we abbreviate $Q'a = (Qa)'$ we have the following result.

Proposition 3.2.1 *Using the preceding representation we have*

$$M_k F = D_x F + \frac{k}{x_n} Q' F$$

for $F \in C^1(\Omega, \mathcal{C}\ell_{0,n})$.

Proof. Assume $F = PF + e_n Q' F$. Since $e_n \cdot e_n = -1$ the proof follows.■

Sometimes (e.g. in papers I-IV) we also use the hyperbolic space $\mathbb{R}^{n+1}_+ = \{(x_0, ..., x_m) : x_0 > 0\}$ with the metric $h = \frac{dx_0^2 + \cdots + dx_n^2}{x_0^2}$. Then we use the split

$$a = P_0 a + e_0 Q_0 a$$

where e_0 is a non-scalar Clifford number since we are working with the vector variables and the Clifford algebra $\mathcal{C}\ell_{n+1,0}$. The corresponding operator (abbreviated by H_k) is defined by

$$H_k F = \partial_{\mathbf{x}} F - \frac{k}{x_0} Q_0 F$$

for $F \in C^\infty(\Omega, \mathcal{C}\ell_{n+1,0})$ where $\partial_{\mathbf{x}}$ is the Dirac operator

$$\partial_{\mathbf{x}} = e_0 \partial_{x_0} + e_1 \partial_{x_1} + \cdots + e_n \partial_{x_n}.$$

3.3 On Hypermonogenic Functions

If f is a solution of the equation $M_k f = 0$ it is called k-hypermonogenic. If $k = n-1$ solution are called briefly hypermonogenic. In this section we study some properties of the hypermonogenic functions. First we study their structure.

The adjoint operator of M_k is defined by

$$\overline{M}_k F = \overline{D}_x F - \frac{k}{x_n} Q' F$$

where $\overline{D}_x = \partial_{x_0} - e_1 \partial_{x_1} - \cdots - e_n \partial_{x_n}$. Let as abbreviate $M = M_{(n-1)}$ and $\overline{M} = \overline{M}_{(n-1)}$.

Theorem 3.3.1 ([3]) *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be an open subset and let $f : \Omega \to \mathcal{C}\ell_{0,n}$ be a twice differentiable function. Then*

$$P(\overline{M} M f) = \Delta P f - \frac{n-1}{x_n} \frac{\partial P f}{\partial x_n}$$

and

$$Q(\overline{M} M f) = \Delta Q f - \frac{n-1}{x_n} \frac{\partial Q f}{\partial x_n} + (n-1) \frac{Q f}{x_n^2}.$$

If f is hypermonogenic, then Pf satisfies the equation

$$\Delta P f - \frac{n-1}{x_n} \frac{\partial P f}{\partial x_n} = 0$$

and Qf satisfies the equation

$$\Delta Q f - \frac{n-1}{x_n} \frac{\partial Q f}{\partial x_n} + (n-1) \frac{Q f}{x_n^2} = 0.$$

We see that the P-part of a hypermonogenic function is hyperbolic harmonic and Q part satisfies the eigenvalue equation

$$\Delta_{lb} Q f = -(n-1) Q f.$$

Also we have the following important result.

Theorem 3.3.2 ([3]) *Let Ω be an open subset of $\mathbb{R}^{n+1}$. Then $f : \Omega \to \mathcal{C}\ell_{0,n}$ is hypermonogenic if and only if for only $a \in \Omega$ and only ball $B(a, r) \subset \mathbb{R}^{n+1}$ there exists a mapping $H : B(a, r) \to \mathcal{C}\ell_{0,n-1}$ satisfying the equations*

$$f = \overline{D}_x H$$

and

$$\Delta_{lb} H = 0$$

on $B(a, r)$.

As an example of a hypermonogenic function we express the following results. The first one is a motivation for the hyperbolic function theory in general.

Theorem 3.3.3 ([3]) *A mapping* $x \mapsto x^m$, *where* $m \in \mathbb{Z}$, *is hypermonogenic.*

Corollary 3.3.4 ([3]) *Functions* e^x, $\sin x$ *and* $\cos x$ *(with usual definitions as series) are hypermonogenic.*

3.4 Integral Representations

In this section we study integral representations for hypermonogenic functions. Especially we are interested Cauchy type formulas and mean-value properties.

Let $a \in \mathcal{C}\ell_{0,n}$. Then we have

$$a = Pa + (Qa)e_n.$$

The hat-involution is an algebra automorphism $\widehat{\ }: \mathcal{C}\ell_{0,n} \to \mathcal{C}\ell_{0,n}$ defined by

$$\widehat{a} = Pa - (Qa)e_n.$$

It is an easy exercise to see that $\widehat{e_j} = (-1)^{\delta_{jn}} e_j$ and if $a, b \in \mathcal{C}\ell_{0,n}$ then $\widehat{ab} = \widehat{a}\widehat{b}$.

In this section we denote the surface area of the unit sphere in $\mathbb{R}^{n+1}$ by ω_{n+1}.

The Cauchy formula for the P-part of a hypermonogenic function is then:

Proposition 3.4.1 ([2]) *If* f *is a hypermonogenic function on* Ω *and* $K \subset \Omega$ *is an oriented* $(n+1)$*-dimensional manifold-with-boundary. Then for each* $a \in K$ *we have*

$$Pf(a) = \frac{2^n a_n^n}{\omega_{n+1}} \int_{\partial K} P(p(x, a)\nu(x)f(x)) \frac{dS(x)}{x_n^{n-1}}$$

where dS is the scalar surface element, ν is the outer unit normal vector field, and

$$\begin{aligned} p(x,a) &= -\frac{1}{2^{2n-1}a_n^n}\overline{D}_x \int_{\frac{|a-x|}{|x-\widehat{a}|}}^{1} \frac{(1-s)^{n-1}}{s^n} ds \\ &= \frac{x_n^{n-1}}{2a_n} \frac{(x-a)^{-1}-(x-\widehat{a})^{-1}}{|x-a|^{n-1}|x-\widehat{a}|^{n-1}}. \end{aligned}$$

Similarly we obtain the Cauchy formula for Q-part of a hypermonogenic function.

Proposition 3.4.2 ([2]) *If f is a hypermonogenic function on Ω and $K \subset \Omega$ is an oriented $(n+1)$-dimensional manifold-with-boundary. Then for each $a \in K$ we have*

$$Qf(a) = \frac{2^n a_n^{n-1}}{\omega_{n+1}} \int_{\partial K} Q(q(x,a)\nu(x)f(x))dS(x)$$

where dS is the scalar surface element, ν is the outer unit normal vector field, and

$$\begin{aligned} q(x,a) &= -\frac{1}{2(n-1)}\overline{D}_x \frac{1}{|x-a|^{n-1}|x-\widehat{a}|^{n-1}} \\ &= \frac{1}{2} \frac{(x-a)^{-1}+(x-\widehat{a})^{-1}}{|x-a|^{n-1}|x-\widehat{a}|^{n-1}}. \end{aligned}$$

Using the hyperbolic geometry we may express the preceding kernels in the following form.

Theorem 3.4.3 (V, p. 8) *If $d_h(x,a)$ is the hyperbolic distance between the points x and a in $\mathbb{R}_+^{n+1}$ then*

$$\begin{aligned} p(x,a) &= \frac{\overline{x-\tau(a,x)}}{2^n x_n a_n^{n+1} \sinh^{n+1} d_h(x,a)} \\ &= \frac{1}{2^n x_n} \frac{\overline{x-\tau(a,x)}}{|x-\tau(a,x)|^{n+1}}, \end{aligned}$$

where

$$\tau(a,x) = a_0 + a_1 e_1 + \cdots + a_{n-1}e_{n-1} + a_n \cosh d_h(x,a) e_n.$$

The preceding theorem gives us an interpretation to the p-kernel. In the classical Clifford analysis the Cauchy's kernel is the mapping $x \mapsto \frac{\overline{x-a}}{|x-a|^{n+1}}$ (of course up to constant). In the hyperbolic case the p-kernel is just the Euclidean Cauchy's kernel, but we compute it in the different center. Also there is the coefficient $1/x_n$, what is something what we expected.

Theorem 3.4.4 (V, p. 10) *If $d_h(x,a)$ is the hyperbolic distance between the points x and a in $\mathbb{R}^{n+1}_+$ then*

$$
\begin{aligned}
q(x,a) &= \frac{\overline{(x-\tau(a,x))}\cosh d_h(x,a) - a_n \sinh^2 d_h(x,a) e_n}{(2a_n x_n)^n \sinh^{n+1} d_h(x,a)} \\
&= \frac{1}{(2x_n)^n} \frac{\overline{x-\tau(a,x)}}{|x-\tau(a,x)|^{n+1}} Q\tau(a,x) - \frac{1}{(2x_n)^n} \frac{1}{|x-\tau(a,x)|^{n-1}} e_n,
\end{aligned}
$$

where

$$
\tau(a,x) = a_0 + a_1 e_1 + \cdots + a_{n-1}e_{n-1} + a_n \cosh d_h(x,a) e_n.
$$

The preceding theorem gives us an interpretation to the q-kernel. Recall that the Newton's kernel in the theory of harmonic function is (up to constant) the mapping $x \mapsto \frac{1}{|x-a|^{n-1}}$. Thus we see that the q-kernel is a linear combination of the Cauchy's and the Newton's kernels with the center $\tau(a,x)$. Moreover, since we consider the kernel of the Cauchy's formula for the Q-part of a hypermonogenic function, it can be expected that the coefficient e_n is in the special role.

Using the hyperbolic kernels we may prove the following mean-value properties.

Theorem 3.4.5 ([4];V, p. 12) *Let $U \subset \mathbb{R}^{n+1}_+$ be open. The following properties are equivalent:*

(a) h is hyperbolically harmonic on U.

(b) h is smooth and

$$
h(a) = \frac{1}{\omega_n \sinh^n R_h} \int_{\partial B_h(a,R_h)} h(x) d\sigma_h(x)
$$

for all $\overline{B_h(a,R_h)} \subset U$.

(c) h is smooth and

$$h(a) = \frac{1}{V(B_h(a,R_h))} \int_{B_h(a,R_h)} h(x) dx_h(x)$$

for all $\overline{B_h(a,R_h)} \subset U$ *where* $V(B_h(a,R_h)) = \omega_n \int_0^{R_h} \sinh^n t dt$ *is the hyperbolic volume of the ball* $B_h(a,R_h)$.

Recall that the Q-part of a hypermonogenic function is a solution of the Laplace-Beltrami equation, i.e.,

$$\Delta_{lb} Qf = -(n-1)Qf.$$

Hence we may study their mean value properties:

Theorem 3.4.6 (VI, p. 6) *Let* $\Omega \subset \mathbb{R}^{n+1}_+$ *be an open subset and let* $h : \Omega \to \mathcal{C}\ell_{0,n-1}$ *be a smooth function. The following properties are equivalent.*

(1) h is an eigenfunction, i.e, is a solution of

$$\Delta_{lb} h(x) = -(n-1)h(x)$$

for $x \in \Omega$.

(2)

$$h(a) = \frac{1}{\omega_{n+1}\psi(R_h)} \int_{\partial B_h(a,R_h)} h(x) d\sigma_h(x)$$

where

$$\psi(R_h) = \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) dt$$

whenever $\overline{B(a,R_h)} \subset \Omega$.

(3)

$$h(a) = \frac{n-1}{\omega_{n+1}\phi(R_h)} \int_{B_h(a,R_h)} h(x) dx_h$$

where ω_{n+1} *is the surface area of the* $(n+1)$*-unit sphere and*

$$\phi(R_h) = (n-1)\cosh R_h \int_0^{R_h} \sinh^{n-2}(t) dt - \sinh^{n-1} R_h.$$

whenever $\overline{B(a,R_h)} \subset \Omega$.

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Chapter 4

Included Publications I-VI