Tampere University

JUKKA-PEKKA HUMALOJA

Robust and Model Predictive Control for Boundary Control Systems

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ACADEMIC DISSERTATION
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ABSTRACT

In this thesis, robust and model predictive control are considered for boundary control systems. In terms of robust control, the existing results, especially the internal model principle, are generalized to cover this class of systems. The concept of approximate robust regulation for boundary control systems is presented, as, due to the internal model principle, in practice it is not possible to construct an exact robust regulating controller if the output space of the controlled system is infinite-dimensional. A practical controller design is presented to achieve robust regulation in this approximate sense.

Model predictive control (MPC) is considered for the class of regular linear systems which includes regular boundary control systems. The continuous-time system is approximated by a discrete-time one by using the Cayley-Tustin transform, and MPC is considered for the discrete-time system. Stability and optimality are proved for the proposed discrete-time MPC designs, which extends the corresponding finite-dimensional MPC designs to the class of regular linear systems.

TIIVISTELMÄ

Tässä väitöskirjassa tarkastellaan robustia ja mallia ennakoivaa säätöä reunasäätöjärjestelmien kannalta. Robustin säädön osalta tunnettuja tuloksia, erityisesti sisäisen mallin periaate, yleistetään tälle systeemiluokalle. Approksimatiivisen robustin reguloinnin käsite esitellään reunasäätöjärjestelmien viitekehyksessä, koska sisäisen mallin periaatteen nojalla tarkasti reguloivan robustin säätäjän konstruointi ei käytännössä ole mahdollista, jos säädettävän systeemin ulostulo on ääretönulotteinen. Lisäksi esitellään käytännöllinen säätäjärakenne, jota käyttämällä robusti regulointi voidaan saavuttaa tässä approksimatiivisessa mielessä.

Mallia ennakoivaa säätöä (MPC) tarkastellaan ääretönulotteisten systeemien luokalle, joka kattaa osan reunasäätöjärjestelmistä. Jatkuva-aikaista järjestelmää approksimoidaan diskreettiaikaisella käyttäen Cayley-Tustin muunnosta, ja MPC-ongelma muodostetaan diskreettiaikaiselle systeemille. Diskreettiaikaiselle MPC-ongelmalle todistetaan optimaalinen ja stabiloiva ratkeavuus, mikä yleistää vastaavan äärellisulotteisen MPC-tuloksen tarkasteltujen ääretönulotteisten systeemien luokalle.

PREFACE

This research was carried out in the Mathematics Laboratory of Tampere University of Technology (TUT) in the Systems Theory Research Group. After completing my Master's degree in 2014, I continued to work on the problem considered in my Master's thesis as a Project Researcher and was supposed to do this thesis on that topic as well. However, the funding for the project ended by the end of the same year, so there was no point in continuing the research. Instead, I was hired by Professor Seppo Pohjolainen, my then supervisor, to develop web-based exercises for mathematics courses at TUT. As I was not interested in doing my Ph.D. thesis based on that work, Seppo Pohjolainen introduced me to the class of port-Hamiltonian systems which would become a new class of systems to be considered in the framework of robust output regulation. Thus, I started this research while working on developing the web-based mathematics exercises. I would like to thank Seppo Pohjolainen for introducing me to the topic, for giving this opportunity, and for all the support he has provided.

When Seppo Pohjolainen officially retired, Assistant Professor Lassi Paunonen became my supervisor. He arranged me a full-time position to work on this research from the beginning of 2016, which both significantly expedited the thesis process and improved my well-being. Additionally, his mentoring throughout the thesis process has taught me a great deal about academic research and was essential for completing this thesis. For all the support, I would like to express my gratitude to Lassi Paunonen.

I would like to thank an old member Petteri Laakkonen and a former member Timo Hämäläinen of the Systems Theory Research Group for many interesting and helpful conversations, and for providing a pleasant working environment. Thanks for the pleasant working environment also go to the newer members of the group: Duy Phan-Duc, Konsta Huhtala, and Dmytro Baidiuk.

For their contributions to this research, I would like to thank my collaborators Mikael Kurula and Associate Professor Stevan Dubljevic. I owe my special thanks to Stevan Dubljevic for introducing me to the topic of Model Predictive Control and for hosting my two visits to University of Alberta in Edmonton, Canada, where I also finished writing the thesis.

I would also like to thank the pre-examiners Professor Lars Grüne and Senior Lecturer Mark Opmeer for their constructive and helpful comments on the original manuscript.

Finally, I would like to thank my parents, sister, family and friends for taking my mind off the research every now and then, and more importantly, for all the support they have provided throughout my studies.

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Jukka-Pekka Humaloja

CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

BCS	A Boundary Control System
MPC	Model Predictive Control
ODE	An Ordinary Differential Equation
PDE	A Partial Differential Equation
RORP	The Robust Output Regulation Problem

A	system operator, generator of a strongly continuous (C_0-)semigroup
$\mathcal{A}$	system operator associated with a boundary control system
A_d	discrete-time approximation of the system operator A corresponding to the Cayley-Tustin transform: $A_d := (d+A)(d-A)^{-1}$
B	control operator, right inverse of a boundary control operator
$\mathcal{B}$	boundary control operator
B_d	discrete-time approximation of the input operator B corresponding to the Cayley-Tustin transform: $B_d := \sqrt{2d}(d-A)^{-1}B$
C	observation operator
$\mathcal{C}$	boundary observation operator
C_d	discrete-time approximation of the output operator C corresponding to the Cayley-Tustin transform: $C_d := \sqrt{2d}C(d-A)^{-1}$
D	feedthrough operator
D_d	discrete-time approximation of the feedthrough operator D corresponding to the Cayley-Tustin transform: $D_d := C(d-A)^{-1}B + D$
$\dim X$	dimension of the space X
$\mathcal{D}(A)$	domain of a linear operator A
$e(t)$	regulation error: $e(t) := y(t) - y_{ref}(t)$
E	operator associated with a disturbance signal w
F	operator associated with a reference signal y_{ref}
$\mathcal{G}_1$	system operator of a dynamical error feedback controller
$\mathcal{G}_2$	input operator of a dynamical error feedback controller
K	output operator of a dynamical error feedback controller
L_Λ	Λ-extension of a linear operator L: $L_\Lambda x := \lim_{\lambda\to\infty} \lambda L(\lambda - A)^{-1}x$
$\mathcal{L}(X,Y)$	the set of bounded linear operators from the normed space X to the normed space Y
$\mathcal{N}(A)$	kernel of a linear operator A

$P(s)$	transfer function of a system at $s \in \mathbb{C}$
$\mathcal{O}$	class of admissible perturbations
$R(\lambda, A)$	resolvent operator of a linear operator A at λ: $R(\lambda, A) := (\lambda - A)^{-1}$
$\mathcal{R}(A)$	range of a linear operator A
$\rho(A)$	resolvent set of a linear operator A
S	signal generator
$\sigma(A)$	spectrum of a linear operator A
$T(t)$	strongly continuous semigroup
$u(t)$	input for a plant
U	input space
$v(t)$	state of the exosystem
$w(t)$	disturbance signal
W	exosystem state space
$x(t)$	state of a plant
X	state space
X_1	scaled space: $X_1 := (\mathcal{D}(A), \|(s_0 - A) \cdot\|), \quad s_0 \in \rho(A)$
X_{-1}	scaled space: $X_{-1} := \overline{(X, \|(s_0 - A)^{-1} \cdot\|)}, \quad s_0 \in \rho(A)$
$y(t)$	output of a plant
$y_{ref}(t)$	reference signal
Y	output space
$z(t)$	state of a controller
Z	controller state space

LIST OF PUBLICATIONS

I. J.-P. Humaloja, L. Paunonen, and S. Pohjolainen. Robust regulation for port-Hamiltonian systems of even order. *Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, pp. 152–156, 2016.

II. J.-P. Humaloja and L. Paunonen. Robust regulation of infinite-dimensional port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, vol. 63 issue 5, pp. 1480–1486, 2018.

III. J.-P. Humaloja, M. Kurula, and L. Paunonen. Approximate robust output regulation of boundary control systems. *IEEE Transactions on Automatic Control*, to appear, DOI: 10.1109/TAC.2018.2884676.

IV. J.-P. Humaloja and S. Dubljevic. Linear model predictive control for Schrödinger equation. *Proceedings of the 2018 Annual American Control Conference (ACC)*, pp. 2569–2574, 2018.

V. J.-P. Humaloja and S. Dubljevic. Model predictive control for regular linear systems. *Automatica* (submitted).

AUTHOR'S CONTRIBUTION

I. I wrote most of the paper and was the corresponding author. Seppo Pohjolainen wrote parts of the introduction. The theoretical results were discussed with Lassi Paunonen. Both coauthors gave comments on the presentation of the results and the structure of the manuscript.

II. I wrote the paper and the code for the simulation study. Lassi Paunonen gave comments on the presentation of the results and the structure of the manuscript, and assisted with the numerical aspects of the simulation.

III. I wrote Sections IV, V, parts of the other sections excluding Section II, and was the corresponding author. Mikael Kurula wrote Section II and the remaining parts of the other sections. Lassi Paunonen wrote the code for the numerical approximation of the wave equation considered in Section V. I wrote the remaining code and performed the simulations. Both coauthors gave comments on the presentation of the results and the structure of the manuscript.

IV. I wrote most of the paper and was the corresponding author. Stevan Dubljevic wrote parts of the introduction, gave comments on the presentation of the results, and provided code for model predictive control. I wrote the remaining code and performed the simulations.

V. I wrote most of the paper and was the corresponding author. Stevan Dubljevic wrote the abstract and most of the introduction, provided code for model predictive control and gave comments on the presentation of the results and the structure of the manuscript. I wrote the remaining code and performed the simulations.

1. INTRODUCTION

Mathematical control theory is the area of application-oriented mathematics that is concerned with the analysis and design of control systems. In this context, controlling a system means forcing it to behave in a desired way. The behavior of the system is usually assessed by measuring some observable properties (outputs) of the system, and the system should then be manipulated such that these measurements have desired values. This control objective is called output regulation. A simple example of output regulation would be cruise control in cars, where the velocity of the vehicle is kept at a constant value by a servomechanism controlling the throttle of the car.

The controlled systems are often modeled by ordinary differential equations (ODEs) or partial differential equations (PDEs). Many technological systems can be modeled by ODEs, but there are many important processes such as those involving diffusion, vibrations, and elasticity that are described by PDEs. In this thesis, control of PDEs is considered in the case that the control enters and the measurement is taken through the boundary of the system. Such approach is essentially relevant to systems that can be accessed solely via their boundaries.

Systems modeled by ODEs are *finite-dimensional*, which means that the state of the system can be expressed by a finite number of components. This is not possible for PDEs which are *infinite-dimensional* systems. While finite-dimensional linear systems can be considered by means of matrix algebra, the analysis of infinite-dimensional linear systems involves operator theory, functional analysis and semigroup theory, using which the infinite-dimensional linear systems can be expressed as abstract linear systems of the form (2.1), where (A, B, C, D) are general linear operators instead of matrices.

Robust output regulation combines the problem of output regulation with tolerating external and internal uncertainties and perturbations in the system. The theory of robust output regulation was developed for finite-dimensional systems in the 1970's, and since then it has been extended to several classes of infinite-dimensional systems (for details, see Section 2.1). The first main objective of the thesis is to continue this work by extending the theory to the class of boundary control systems.

Model predictive control (MPC) comprises a wide range of control methods which utilize the dynamical model of the controlled systems and optimization to obtain the control signal. The goal in the MPC setting is usually to steer the state and/or the output of the system to zero, but the methodology can be easily applied, e.g., to tracking of constant reference signals. Most of the existing MPC theory only covers finite-dimensional systems. The second main objective of the thesis is to extend some of the linear MPC results for finite-dimensional systems to a large class of infinite-dimensional linear systems.

1.1 Research objectives and scope of the thesis

The main objective of the thesis is to extend the theory of robust output regulation and model predictive control to cover more general classes of systems. More precisely, the objectives are:

1. **Robust output regulation for boundary control systems**

   The objective is to extend the theory of robust output regulation to the class of boundary control systems. These systems are modeled by partial differential equations where the control enters through the boundary of the spatial domain and often also the measurement is taken through the boundary. The prior theoretical results cover the class of regular linear systems which includes some boundary control systems, but there is no theory for the general boundary control system framework. Robust output regulation for boundary control systems involves design of regulating controllers, but more importantly the generalization of the internal model principle for this class of systems. These objectives are considered in Publications I – III.

2. **Model predictive control for regular linear systems**

   Model predictive control designates a wide range of control methods that utilize the process model to obtain the control moves by minimizing an objective function. These control methods have been especially popular in chemical process industries and they have been extensively developed for finite-dimensional systems, that is, systems governed by ordinary differential equations. Relatively recently the MPC methodology has been applied to infinite-dimensional systems as well. However, in most cases the PDE model is approximated by an ODE model by using an appropriate spatial discretization, after which the existing MPC theory for finite-dimensional systems can be utilized. Hence, extensive theory on model predictive control strictly for infinite-dimensional systems has not yet been developed . Here the objective is to extend results from the theory of linear model predictive control for finite-dimensional systems to regular linear systems. This objective is considered in Publications IV-V.

1.2 Thesis outline

The outline of the thesis is as follows. In Sections 2.1, 2.2 and 2.3, the theoretical background to robust output regulation, model predictive control and boundary control systems, respectively, is presented. Earlier results on the topics are presented as well. In Chapter 3, the results of the thesis on robust output regulation and model predictive control are presented and they are linked to the existing results in the research fields. Finally, in Chapter 4 the thesis is concluded with discussion on the future extensions of the presented results and further research topics.

1.3 Notation

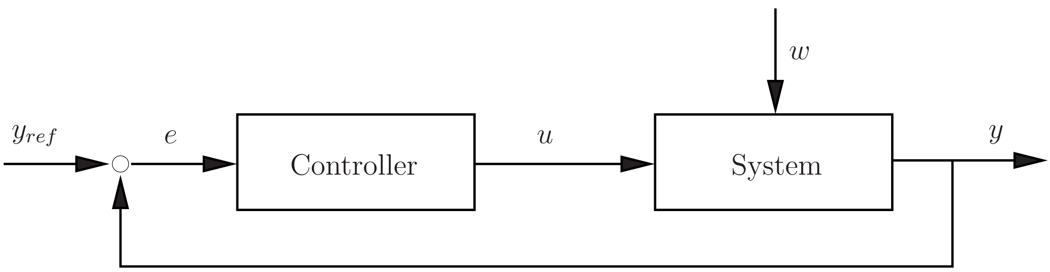
Here $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, kernel, spectrum and resolvent of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$ and $\rho(A)$, respectively. For a linear operator $A : \mathcal{D}(A) \subset X \to X$ and a fixed $s_0 \in \rho(A)$, define the scale spaces $X_1 := (\mathcal{D}(A), \|(s_0 - A)\cdot\|)$ and $X_{-1} = \overline{(X, \|(s_0 - A)^{-1}\cdot\|)}$ [40, Sec. 2.10]. The scale spaces are related by $X_1 \subset X \subset X_{-1}$, where the inclusions are dense and with continuous embeddings. The extension of A to X_{-1} is denoted by A_{-1} and the Λ-extension of a linear operator L is denoted by $L_\Lambda := \lim_{\lambda\to\infty} \lambda L(\lambda - A)^{-1}$.

2. THEORETICAL BACKGROUND

2.1 Robust output regulation

The aim in *output regulation* is to find such an *input* u that the *output* y of a system converges to a given *reference signal* y_{ref} in spite of possible external *disturbance signals* w. *Robustness* is the property that allows small perturbations, arising, e.g., from modeling errors or approximations, in the parameters of the plant.

Robust output regulation can be achieved, e.g., by constructing an *error feedback controller* that produces the input u based on the *regulation error* $e := y - y_{ref}$. The control setup is visualized below. The reference signal y_{ref} and the disturbance signal w are assumed to be generated by an *exosystem* (short for *exogenous system*) which will be presented later on.



The considered systems are usually described in the state-space form by

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \qquad x(0) = x_0 \tag{2.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{2.1b}$$

where one can see how u, y and w are related to the system. The state space $X \ni x(t)$ is an infinite-dimensional Hilbert space, and the input space U and output space Y are finite- or infinite-dimensional Hilbert spaces. For now, we assume that the operators B, C and D are bounded linear operators, i.e., $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$

and $D \in \mathcal{L}(U, Y)$. Furthermore, the operator A is assumed to be the generator of a strongly continuous semigroup $T(t)$.

The exosystem that generates the reference signal y_{ref} and the disturbance signal w is given by

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \tag{2.2a}$$

$$w(t) = Ev(t), \tag{2.2b}$$

$$y_{ref}(t) = -Fv(t). \tag{2.2c}$$

on a finite-dimensional space $W = \mathbb{C}^q$ for some $q \in \mathbb{N}$. We further assume that the signal generator S has purely imaginary eigenvalues $\sigma(S) = \{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$ with algebraic multiplicity one. The operators E and F are bounded, i.e., $E \in \mathcal{L}(W, X)$ and $F \in \mathcal{L}(W, Y)$. Such exosystems are capable of producing sinusoidal reference and disturbance signals. We note that more general exosystems could be considered as well, e.g., infinite-dimensional or S having eigenvalues with higher algebraic multiplicities, but in this thesis we have restricted to the exosystems described in the preceding.

Finally, the error feedback controller is another dynamical system

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \tag{2.3a}$$

$$u(t) = Kz(t), \tag{2.3b}$$

where the state space $Z \ni z(t)$ is a finite- or an infinite-dimensional Banach space and $\mathcal{G}_1$ is the generator of a C_0-semigroup on Z. The controller parameters $\mathcal{G}_1$, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$ are to be chosen such that robust output regulation is achieved where the *internal model principle* has to be utilized.

The internal model principle was originally stated for finite-dimensional linear systems by Francis and Wonham in [12, 13]. The principle essentially states that a controller may achieve robust output regulation if and only if it contains at least a $\dim Y$-fold reduplicated model of the dynamics of the signal generator S. Extension of the internal model principle for infinite-dimensional systems was at some extend investigated in the Ph.D. thesis by Bhat [6], where mainly time-delay systems were considered. Other early developments of the internal model principle for infinite-dimensional systems were done by Schumacher in [38], however, without considering robustness, and Yamamoto and Hara in [44], where an analogue of the internal model principle was stated in the frequency domain for periodic output regulation.

In the next subsection, we consider the more recent development of the internal model principle for infinite-dimensional systems in more detail. We note that even though robust regulating controllers have been designed based on the results of Francis and Wonham [12, 13] even for well-posed systems [37], in rather few occasions there has been discussion on why the designed controllers are robust in general. Here we will focus on these general characterizations of robust regulating controllers and omit the references considering merely a specifically designed controller.

2.1.1 Internal model principle for infinite-dimensional systems

General conditions for a controller to achieve output regulation were presented in [7] but without considering the robustness aspect. It was shown in [7, Thm IV.2] that for plants of the form (2.1) with $D = 0$, the output regulation problem is solvable if and only if there exist mappings $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, U)$ with $\mathcal{R}(\Pi) \subset \mathcal{D}(A)$ such that

$$\Pi S = A\Pi + B\Gamma + E \tag{2.4a}$$

$$0 = C\Pi + F. \tag{2.4b}$$

The idea of the proof was to show that the regulation error $e(t) = y(t) - y(t)_{ref} = Cx(t) + Fv(t)$ decays to zero exactly when (2.4) have solutions. Equations (2.4) are called the *regulator equations* and we will see different versions of them later on.

Theorem IV.2 of [7] in a sense characterizes all error feedback regulating controllers, even though interestingly it does not contain any information about the controller parameters per se. Regardless, having the solutions $\Pi \in \mathcal{L}(W, X)$ and $\Gamma \in \mathcal{L}(W, U)$, it is shown in [7, Thm. IV.2] that a regulating controller is given by choices $Z = X \times W$ and

$$\begin{aligned} \mathcal{G}_2 &= \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \qquad K = \begin{bmatrix} K_0 & \Gamma - K_0\Pi \end{bmatrix} \\ \mathcal{G}_1 &= \begin{bmatrix} A + BK_0 - G_1C & E + B(\Gamma - K_0\Pi) - G_1F \\ -G_2C & S - G_2F \end{bmatrix}, \end{aligned} \tag{2.5}$$

where $K_0 \in \mathcal{L}(X,U)$ and $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ are such that $A + BK_0$ and

$$\begin{bmatrix} A & E \\ 0 & S \end{bmatrix} - \mathcal{G}_2 \begin{bmatrix} C & F \end{bmatrix}$$

are the generators of exponentially stable semigroups. Here suitable operators K_0 and $\mathcal{G}_2$ are assumed to exist.

Even though robustness was not considered in [7], it has been shown in [21] by Immonen that the controller (2.5) is in fact robust in the single-input single-output case (every regulating controller is robust in that case). Immonen and Pohjolainen had generalized the internal model principle of Francis and Wonham in [22], the results of which and other related results can be found in Immonen's Ph.D. thesis [20] as well. The results were based on the definition of *internal model structure* as follows [22, Def. 3.1]:

Definition 1 *The dynamic controller* (2.3) *has the internal model structure if there exists* $\Gamma \in \mathcal{L}(W,Z)$ *with* $\mathcal{R}(\Gamma) \subset \mathcal{D}(\mathcal{G}_1)$ *such that for all* $\Delta \in \mathcal{L}(W,Y)$

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \tag{2.6}$$

implies $\Delta = 0$.

Before presenting the internal model result of [22], let us present the *closed-loop system* consisting of the plant and the controller. That is, setting u and y equal in (2.1) and (2.3), the system can be written in the extended state space $X_e = X \times Z$ with the extended state $x_e = [x,\ z]^T$ as

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = [x(0),\ z(0)]^T \tag{2.7a}$$

$$e(t) = C_e x_e(t) + D_e v(t) \tag{2.7b}$$

where the regulation error e is chosen as the output and

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \qquad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix},$$

$C_e = [C \quad DK]$ and $D_e = F$. Under the standing assumptions, the operator A_e is the generator of a C_0-semigroup.

Having the closed-loop system presented, the internal model result [22, Thm.3.3] can be expressed as follows:

Theorem 1 *Let the controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *be chosen such that the closed-loop operator* A_e *is the generator of an exponentially stable semigroup. Then for every* $E \in \mathcal{L}(W, X)$ *and every* $F \in \mathcal{L}(W, Y)$*, and for all* x_0, z_0, v_0*, the regulation error* $e(t)$ *decays to zero at an exponential rate if and only if the controller has the internal model structure. The controller is robust with respect to perturbations to* $A, B, C, D, \mathcal{G}_2, K$ *that preserve the exponential closed-loop stability and the internal model structure.*

As noted in [22, Rem. 3.5], the operators S and $\mathcal{G}_1$ do not allow arbitrary (no matter how small) perturbations according to Theorem 1, which is in accordance with the original internal model principle [12, 13]. However, some specific perturbations may in fact be tolerated in $\mathcal{G}_1$, provided that the exponential closed-loop stability and the internal model structure are preserved.

Note that as opposed to the internal model result of [7], the internal model structure of Definition 1 actually includes the controller parameters $\mathcal{G}_1$ and $\mathcal{G}_2$. The controller parameter K is not present in the internal model structure, but it contributes to the exponential stabilization of the closed-loop system, as required in Theorem 1.

The internal model structure of [22] has been further reformulated in [17] by Hämäläinen and Pohjolainen. It was first shown that for an exponentially stable closed-loop system, an error feedback controller solves the output regulation problem if there exists $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e)$ and

$$\Sigma S = A_e \Sigma + B_e \tag{2.8a}$$

$$0 = C_e \Sigma + D_e, \tag{2.8b}$$

which look somewhat similar to the regulator equations (2.4) of [7]. In [17], equations (2.8) were called the *constrained Sylvester equations*, but we will refer to them as regulator equations where (2.8a) is a *Sylvester equation.*

If the closed-loop system is assumed to be exponentially stable, then the solution Σ of the Sylvester equation (2.8a) exists and is unique by [35, Cor. 8]. In order to discuss robustness, consider arbitrary bounded perturbations $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}$ of the operators A, B, C, D, E, F such that the exponential closed-loop stability is preserved. Writing the solution of (2.8) as $\tilde{\Sigma} = [\tilde{\Pi} \quad \tilde{\Gamma}]^T$, the equations split into

$$\tilde{\Pi}S = \tilde{A}\tilde{\Pi} + \tilde{B}K\tilde{\Gamma} + \tilde{E} \tag{2.9a}$$

$$\tilde{\Gamma}S = \mathcal{G}_1\tilde{\Gamma} + \mathcal{G}_2\left(\tilde{C}\tilde{\Pi} + \tilde{D}K\tilde{\Gamma} + \tilde{F}\right) \tag{2.9b}$$

$$0 = \tilde{C}\tilde{\Pi} + \tilde{D}K\tilde{\Gamma} + \tilde{F}. \tag{2.9c}$$

Based on the preceding, we arrive to [17, Def. 8] which states that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is robustly regulating if

$$\tilde{\Gamma}S = \mathcal{G}_1\tilde{\Gamma} + \mathcal{G}_2\left(\tilde{C}\tilde{\Pi} + \tilde{D}K\tilde{\Gamma} + \tilde{F}\right) \tag{2.10}$$

splits into

$$\tilde{\Gamma}S = \mathcal{G}_1\tilde{\Gamma} \quad \text{and} \quad \tilde{C}\tilde{\Pi} + \tilde{D}K\tilde{\Gamma} + \tilde{F} = 0. \tag{2.11}$$

Note that (2.10) and (2.11) are equivalent to (2.6) with $\Delta = \tilde{C}\tilde{\Pi} + \tilde{D}K\tilde{\Gamma} + \tilde{F}$.

A rather simple condition for the controller to be robustly regulating in the sense of [17, Def. 8] is finally given in the form of the $\mathcal{G}$*-conditions* [17, Def. 10]:

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \forall k \in \{1, 2, \ldots, q\} \tag{2.12a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{2.12b}$$

where $\{i\omega_k\}_{k=1}^q = \sigma(S)$ are the eigenvalues of the signal generator S. In fact, [17, Def. 10] was given for a decomposable state-space $Z = Z_1 \times Z_2$ and controllers of the form

$$\mathcal{G}_1 = \begin{bmatrix} R_1 & R_2 \\ 0 & G_1 \end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} R_3 \\ G_2 \end{bmatrix},$$

where $R_1 : \mathcal{D}(R_1) \subset Z_1 \to Z_1, R_2 \in \mathcal{L}(Z_2, Z_1), R_3 \in \mathcal{L}(Y, Z_1), G_1 : \mathcal{D}(G_1) \subset Z_2 \to Z_2, G_2 \in \mathcal{L}(Y, Z_2)$, and G_1 and G_2 satisfy the $\mathcal{G}$-conditions (2.12). However, $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the $\mathcal{G}$-conditions in this case as well. Note that similar to the internal model structure of [22, Def. 3.1], the $\mathcal{G}$-conditions do not include any information on the parameter K which merely contributes to stabilizing the closed-loop system.

The $\mathcal{G}$-conditions were further extended in [29, Def. 5.1] based on the finite-dimensional results of Francis and Wonham. Assume temporarily that the eigenvalue $i\omega_k$ of S has algebraic multiplicity n_k. In this case, the $\mathcal{G}$- conditions are extended with

$$\mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k-1} \subset \mathcal{R}(i\omega_k - \mathcal{G}_1) \qquad \forall k \in \{1, 2, \ldots, q\}$$

which accounts for the multiple eigenvalues of S. The same $\mathcal{G}$-conditions are valid even if S had infinitely many eigenvalues.

The internal model structure was further simplified to a *p-copy internal model*, where p refers to the dimension of the output space Y. The definition is given as follows [29, Def. 6.1]:

Definition 2 *A controller* $(\mathcal{G}_1\mathcal{G}_2, K)$ *is said to incorporate a p-copy of the internal model of the exosystem if for all* $k \in \{1, 2, \ldots, q\}$ *it holds that*

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \dim Y$$

and $\mathcal{G}_1$ *has at least* $\dim Y$ *independent Jordan chains of length greater than or equal to* n_k *associated with the eigenvalue* $i\omega_k$*.*

Furthermore, it was shown [29, Thm. 6.2] that if $\sigma(A_e) \cap \sigma(S) = \emptyset$ and if $\dim Y < \infty$, a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ contains a p-copy of the internal model of the exosystem if and only if it satisfies the $\mathcal{G}$-conditions. Note that the spectrum condition is satisfied especially if A_e is the generator of an exponentially stable semigroup. The preceding internal model results are also presented in the Ph.D. thesis by Paunonen [28].

The internal model principle was most notably extended beyond the framework of bounded input and ouput operators in [30] where compatible regular linear systems were considered (see [39, Def.5.1.1, Def. 5.6.3]): Let A still be the generator of a semigroup and for a fixed $s_0 \in \rho(A)$, define the scale spaces $X_1 := (\mathcal{D}(A), \|(s_0 - A) \cdot \|)$ and $X_{-1} := \overline{(X, \|(s_0 - A)^{-1} \cdot \|)}$ [40, Sec. 2.10]. Assume that the input and output operators are such that $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ and that the feedthrough operator D is bounded. The extension of A to the space X_{-1} is denoted by A_{-1} and it is further assumed that B and C satisfy $\mathcal{R}\left((s_0 - A_{-1})^{-1} B\right) \subset \mathcal{D}(C)$ and $C(s_0 - A_{-1})^{-1} B \in \mathcal{L}(U, Y)$ for all $s_0 \in \rho(A)$. The example class considered in [30] was *regular linear systems* in the sense of [41], where the operator C is replaced by its Λ-extension [41, Eq. (5.8)]

$$C_\Lambda x := \lim_{\lambda \to \infty} \lambda C(\lambda - A)^{-1} x.$$

Under the preceding assumptions, it has been shown in [30, Thm 4.1] that the existence of the solution Σ of the regulator equations (2.8) is equivalent to the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ achieving output regulation, thus, extending the results of [17]. Furthermore, it has been shown in [30, Thm 7.2] that a controller achieves robust output regulation if and only if it stabilizes the closed-loop system and satisfies the $\mathcal{G}$-conditions (2.12), which extends the results of [17, 29]. Finally, the p-copy result of [29, Thm 6.2] was extended to this larger class of systems in [30, Thm 6.2].

2.2 Model predictive control

The history of model predictive control reaches back to the 1970's as well (see [8, Sect. 1.2] for a historical perspective). Since MPC is not simply a specific control strategy but essentially a wide range of control algorithms, for a detailed introduction on classical MPC we refer to the book [36] by Rawlings and Mayne. Here we will present a brief overview on the general aspects of MPC and review results that are the most relevant to our work.

In this thesis, the model predictive control considerations are related to minimization problem of the form

$$\min_{u(t)} J(x_0, u(t)) = \int_0^\infty \langle y(t), Qy(t)\rangle + \langle u(t), Ru(t)\rangle \, dt, \tag{2.13}$$

where $Q, R > 0$ are some positive (definite) weights. Here u and y come from the state-space model (2.1) (with $w = 0$), and some constraints, e.g., upper and lower bounds may be imposed on u and y. Without any input or output constraints, it is known (see [9, Sect. 6.2] for the case B, C bounded, $D = 0$) that the optimal control $u(t)$ is obtained based on the solutions $\bar{R}$ of the *Riccati equation*

$$K^*SK = A^*\bar{R} + \bar{R}A + C^*QC \tag{2.14}$$

on $\mathcal{D}(A)$, where $S := R + D^*QD$ and $K := -S^{-1}(B^*\bar{R} + D^*QC)$. Assuming that nonnegative solutions to (2.14) exist, the optimal control that also stabilizes the system is given by $u(t) = Kx(t)$ with $\bar{R}$ being the maximal nonnegative solution of (2.14) [9, Lem. 6.2.6]. This optimal control result has also been generalized to (weakly) regular linear systems in [42] and for well-posed linear systems in [24, Ch. 10] and [25]. The Riccati equation remains in the same form for regular linear systems, but in K we must replace B^* with its Λ-extension B^*_Λ and to S we must add $\lim\limits_{s\to\infty} B^*_\Lambda \bar{R}(s-A)^{-1}B$ [42, Rem. 12.9].

However, if input constraints are present, there is no guarantee that the optimal input $u = Kx$ would satisfy them. An advantage of the model predictive control methods is that treatment of constraints is conceptually simple and they can be systematically included in the control design process [8]. A typical feature in MPC is that the infinite-horizon objective function (2.13) is cast into a finite-horizon objective function by adding a suitable penalty term. For a given time instance T_i and horizon T, the finite-horizon objective function is of the form

$$J_T(x_{T_i}, u(t)) = \int_{T_i}^{T+T_i} \langle y(t), Qy(t)\rangle + \langle u(t), Ru(t)\rangle \, dt + P(x(T_i + T)) \tag{2.15}$$

where P denotes a state penalty function. When the finite-horizon problem is solved, the obtained control is implemented, after which the horizon is shifted to the future and the process is repeated. Instead of adding a terminal penalty, a *terminal constraint* can be imposed as well. That is, to require that the state (or output) can be steered to some *terminal constraint set* by the end of the finite horizon.

In [23], the terminal penalty approach was analyzed for infinite-dimensional systems with bounded distributed controls and under input constraints. For these systems, it was shown that MPC with a control Lyapunov functional as a terminal penalty cost is exponentially stabilizing. Existence of optimal control in this setting was proved later in [31–33].

As already noted in Section 1.1, most of the cases where MPC is considered for infinite-dimensional systems rely on some kind of spatial approximation of the underlying partial differential equations so that the original system is approximated by finite-dimensional ordinary differential equations and the classical MPC theory can be utilized. Approximating PDEs and implementing MPC for the approximated systems naturally pose challenges of their own, and on those topics one can see e.g. the Ph.D. thesis by Altmüller [1] and the references therein. However, as our interest is focused on the properties of the infinite-dimensional systems themselves, we will restrict our consideration to cases where MPC is analyzed for the actual PDE model.

There are some references in [1] where MPC schemes were analyzed for PDEs without spatial approximations. There the analysis was focused on minimal stabilizing horizons, so that with such a horizon, stability of MPC can be guaranteed for the finite-horizon problem without using any terminal constraints or costs, but a suitable choice of the objective function J_T is required. General analysis of these schemes was presented in [14] and they have been applied, e.g., to wave, heat and Fokker-Planck equation in [2–4, 11].

There are a few more cases where MPC has been considered for abstract infinite-dimensional systems [10,34,43]. In [10], MPC was formulated for semilinear parabolic PDEs with boundary control (but with bounded control and observation operators) with input and output constraints. Due to the nonlinear model and the handling of output constraints by an exterior penalty method, the resulting optimization problem was nonlinear and no results on optimality were provided. In [34], MPC was formulated for boundary controlled hyperbolic systems (but with bounded control and observation operators) with proofs of stability and optimality. However, [34] was formulated for a zero terminal constraint, i.e., that the state of the system can be steered to zero during the finite horizon $[0, T]$.

In [43], MPC was formulated for transport-reaction processes by utilizing a time discretization scheme for the original PDE. For a time discretization parameter $h > 0$, the discrete-time linear system operators are given by

$$\begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} = \begin{bmatrix} (\delta - A)^{-1}(\delta + A) & \sqrt{2\delta}(\delta - A_{-1})^{-1}B \\ \sqrt{2\delta}C(\delta - A)^{-1} & P(\delta) \end{bmatrix},$$

where $P(\delta) := C_\Lambda(\delta - A_{-1})^{-1}B + D$ is the transfer function of the system and $\delta = 2/h$. In [43], this temporal discretization is called the Cayley-Tustin transform. If $u(k)$ denotes an approximation of $u(t)/\sqrt{h}$ over the interval $t \in ((h-1)k, hk)$, then the continuous-time system (2.1) can be approximated by a discrete-time one of the form

$$x(k) = A_d x(k-1) + B_d u(k), \qquad x(0) = x_0 \tag{2.16a}$$

$$y(k) = C_d x(k-1) + D_d u(k), \tag{2.16b}$$

where $y(k)/\sqrt{h}$ is an approximation of $y(t)$ on the interval $t \in ((h-1)k, k)$. It follows from the results of [18] that if the continuous-time system is well-posed and has finite-dimensional input and output spaces, then $y(k)/\sqrt{h}$ converges to $y(t)$ as $h \to 0$.

In the discrete time setting, the objective function (2.13) takes the form

$$\min_{u(k)} J(x_0, u(k)) = \sum_{k=1}^{\infty} \langle y(k), Qy(k)\rangle + \langle u(k), Ru(k)\rangle \tag{2.17}$$

where u and y come from the discrete-time system (2.16). Similar to the continuous-time objective function, the discrete infinite-horizon objective function can be cast into a finite one by adding a terminal penalty term, so that at step n the objective function is of the form

$$J_N(x_n, u(k)) = \sum_{k=n+1}^{n+N} \langle y(k), Qy(k)\rangle + \langle u(k), Ru(k)\rangle + \left\langle x(n+N), \bar{Q}x(n+N)\right\rangle, \tag{2.18}$$

where we assumed that the terminal penalty can be expressed as a quadratic state penalty. For stable systems, it can be assumed that the input is zero beyond the horizon N, and the operator $\bar{Q}$ is obtained as the solution of the discrete-time Lyapunov equation

$$A_d^*\bar{Q}A_d + C_d^*\bar{Q}C = \bar{Q}. \tag{2.19}$$

This choice of the terminal penalty was originally presented in [27] for finite-dimensional systems and also utilized in [43] for transport-reaction processes. Due to the properties of the Cayley-Tustin transform, the solution(s) of the discrete-time Lyapunov equation (2.19) coincide with the solution(s) of the continuous-time Lyapunov equation

$$A^*\bar{Q} + \bar{Q}A + C^*QC = 0 \tag{2.20}$$

on the dual space of X_{-1}, which essentially follows from [9, Ex. 4.30]. In [43], the penalty operator $\bar{Q}$ was also derived in the case where the system has a finite number of unstable eigenvalues. The approach was the same as in [27] (also presented in the Ph.D. thesis [26] by Muske) for finite-dimensional discrete-time systems, but applied to transport-reaction processes. However, no stability or optimality proofs were presented in [43].

2.3 Boundary control systems

The main system class considered in this thesis are *boundary control systems* (BCS) which, as opposed to (2.1), are expressed as

$$\dot{x}(t) = \mathcal{A}x(t) \tag{2.21a}$$

$$\mathcal{B}x(t) = u(t) \tag{2.21b}$$

$$\mathcal{C}x(t) = y(t) \tag{2.21c}$$

where the input and output operators $\mathcal{B}$ and $\mathcal{C}$, respectively, act on the boundary of the spatial domain (hence the name). A precise definition of boundary control systems is given as follows [9, Def. 3.3.2]:

Definition 3 *The system* (2.21a)–(2.21b) *is a* boundary control system *if the following hold:*

1. *The operator* $A : \mathcal{D}(A) \subset X \to X$ *with* $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$ *and* $Ax = \mathcal{A}x$ *for* $x \in \mathcal{D}(A)$ *is the generator of a* C_0*-semigroup on* X*;*
2. *There exists a* $B_r \in \mathcal{L}(U, X)$ *such that* $\mathcal{R}(B_r) \subset \mathcal{D}(\mathcal{A})$*,* $\mathcal{A}B_r \in \mathcal{L}(U, X)$ *and* $\mathcal{B}B_r = I_U$*.*

The output equation (2.21c) can be added to the boundary control system by assuming that $\mathcal{C}$ is a linear operator defined on $\mathcal{D}(\mathcal{C}) \supset \mathcal{D}(\mathcal{A})$ and mapping to some Hilbert space Y with the additional property that $\mathcal{C}B_r \in \mathcal{L}(U, Y)$. This criterion is slightly different from the definition of boundary control and observation system [5, Def. 2.3.13] in Augner's Ph.D. thesis, where instead of $\mathcal{C}B_r \in \mathcal{L}(U, Y)$ the requirement is $\mathcal{C} \in \mathcal{L}(X_1, Y)$.

The *transfer function* of a boundary control system is given by

$$P(s) := \mathcal{C}(s - A)^{-1}(\mathcal{A}B_r - sB_r) + \mathcal{C}B_r.$$

If the limit $\lim_{s\to\infty} P(s)$ exists along the real line, then the BCS is a regular linear system and can be written in the state-space form (2.1), in which case the feedthrough term is given by $D := \lim_{s\to\infty} P(s)$. However, the input operator B in the state-space form (2.1) does not correspond to the right inverse B_r of $\mathcal{B}$ but to another (unique) operator $B \in \mathcal{L}(U, X_{-1})$ [40, Sect. 10.1].

3. RESULTS AND DISCUSSION

3.1 Robust output regulation

In Publication I, the controller design of [16] was utilized to construct a robust regulating controller for even-order port-Hamiltonian systems which are a special class of boundary control systems. Even though robust output regulation of boundary control systems was considered in [16], there the input and output operators were assumed to be bounded. However, the results of [16] also hold for observation operators in $\mathcal{L}(X_1, Y)$. The unboundedness of the boundary control operator is not an issue either provided that it has a bounded right inverse as required by the definition of boundary control systems (see Definition 3).

The controller design in Publication I was novel in the sense that a stabilizing output feedback term was added to the usual controller structure (2.3). Thus, even if the controlled plant is unstable, such as an impedance energy-preserving port-Hamiltonian system, by adding the stabilizing output feedback it is still possible to construct a finite-dimensional robust regulating controller. In previous works, an observer was added to the controller to account for the unstable plant, which then makes the controller infinite-dimensional, and thus, not implementable in practice.

In Publication II, the controller design of Publication I was extended to impedance passive port-Hamiltonian systems of arbitrary order. The controller design was additionally simplified slightly by instead of using a stabilizing output feedback, the regulation error was used as a feedthrough in the controller and it was shown to stabilize the plant as well. The proposed controlled was of the form (Publication II, Theorem 8)

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t) \tag{3.1a}$$

$$u(t) = Kz(t) - Qe(t), \tag{3.1b}$$

where we choose $Z = Y^q$ and the parameters can be chosen as

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_k I_Y)_{k=1}^q \in \mathcal{L}(Z) \tag{3.2a}$$

$$\mathcal{G}_2 = \begin{bmatrix} -I_Y \\ -I_Y \\ \vdots \\ -I_Y \end{bmatrix} \in \mathcal{L}(Y, Z) \tag{3.2b}$$

$$K = \epsilon(P_s(i\omega_k)^\dagger)_{k=1}^q \in \mathcal{L}(Z, U) \tag{3.2c}$$

$$Q > 0, \tag{3.2d}$$

where the tuning parameter $\epsilon > 0$ is chosen sufficiently small, $\{i\omega_k\}_{k=1}^q$ are the eigenvalues of the signal generator S in (2.2) and $P_s(i\omega_k)^\dagger$ is the Moore-Penrose pseudoinverse of the plant under output feedback at $i\omega_k$, i.e., $P_s(i\omega_k) = P(i\omega_k)(I + QP(i\omega_k))^{-1}$.

The most important contribution of Publication II was that the sufficiency of the internal model for achieving robust output regulation was proved for boundary control systems in general (Publication II, Theorem 4), not for mere port-Hamiltonian systems. Moreover, several technical details which were merely assumed in Publication I were shown to hold, such as admissibility of the observation operator.

In Publication III, the necessity of the internal model in robust output regulation of boundary control systems was proved, thus completing the internal model principle for boundary control systems (Publication III, Theorem IV.8). The result is given as follows:

Theorem 2 *Assume that the closed-loop system* (2.7) *is regular and exponentially stabilized by a controller of the form* (3.1)*. Then the controller solves the robust output regulation problem if and only if it satisfies the* $\mathcal{G}$*-conditions* (2.12)*.*

However, the internal model principle implies that if the output space of the plant is infinite-dimensional, which may occur, e.g., in boundary observation of a 2- or 3-dimensional system, any robust regulating controller is necessarily infinite-dimensional for such a system. Therefore, the novel concept of *approximate robust output regulation* was presented:

The Approximate Robust Output Regulation Problem. Let $\delta > 0$ be given.

Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in such a way that the following are satisfied:

1. The closed-loop system generated by A_e is exponentially stable.
2. For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 ds \leq Me^{-\alpha t}(\|x_{e0}\|^2 + \|v_0\|^2) + \delta\|v_0\|^2$$

   for some $M, \alpha > 0$ independent of $x_{e0} \in X_e, v_0 \in W$.

3. If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed in such a way that the closed-loop system remains regular and exponentially stable, then there exists a $\delta' > 0$ such that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 ds \leq M' e^{-\alpha' t}(\|x_{e0}\|^2 + \|v_0\|^2) + \delta'\|v_0\|^2$$

   for some $M', \alpha' > 0$ independent of x_{e0}, v_0.

Furthermore, a finite-dimensional controller that solves the robust output regulation in this approximate sense was designed. The controller is of the form (3.1) with the choice $Z = Y_N^q$, and the parameters can be chosen as (Publication III, Theorem IV.11)

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_k I_{Y_N})_{k=1}^q \in \mathcal{L}(Z)$$
$$\mathcal{G}_2 = \begin{bmatrix} -P_N \\ -P_N \\ \vdots \\ -P_N \end{bmatrix} \in \mathcal{L}(Y, Z)$$
$$K = \epsilon(P_N P_s(i\omega_k)^\dagger)_{k=1}^q \in \mathcal{L}(Z, U)$$

and Q is a stabilizing output feedback operator for the plant. The parameters are mostly as in (3.2) with the addition that Y_N is a closed finite-dimensional subspace of Y and P_N is a projection onto Y_N along Y. Note that if Y_N was chosen as Y, then the controller would be exactly the same as (3.2).

As a third contribution of the paper, necessary and sufficient conditions for the solvability of the output regulation problem (no robustness requirement) were also given in the framework of boundary control systems (Publication III, Theorem IV.5), and such a controller was designed as well.

3.2 Model predictive control

In Publication IV, the MPC design of [43] was extended to Schrödinger equation. Most of the contribution was to show that the MPC approach can be equivalently utilized to complex valued infinite-dimensional systems. Additionally, the time discretization and the MPC formulation for the discrete-time system were done rigorously as a tutorial for utilizing the methodology. However, no stability or optimality proofs were presented.

In Publication V, the MPC design of [43] was extended to the general class of regular linear systems. Terminal penalty functions were considered for both stable and exponentially stabilizable systems, and conditions for the existence of such functions were addressed, especially from the original continuous-time system point of view. Stability and optimality of the MPC design from [43] for stable systems was proved in Theorem 2, where the stability of the system and the assumed infinite-time admissibility of the observation operator C for A guarantees the existence of the unique positive solution of the Lyapunov equation (2.20) (equivalently (2.19)). In Theorem 4, the MPC design for exponentially stabilizable systems utilized an exponentially stabilizing feedback in the terminal penalty to guarantee stability of the design.

The advantages of the discrete-time MPC formulation are that it can be rather easily characterized for the large class of regular linear systems, and that the discrete-time MPC with a quadratic state terminal penalty can be written as a finite quadratic optimization problem, where upper and lower bounds for the inputs and outputs can be explicitly considered as linear inequality constraints. Such an optimization problem has a global minimum, and thus, optimality of the control moves follows.

A disadvantage of the proposed MPC approach is that while the output constraints can be explicitly taken into account in the discrete-time case, the discrete-time output is merely an approximation of the output of the actual continuous-time system, and therefore pointwise satisfaction of the output constraints cannot be guaranteed for the continuous-time system. However, the continuous-time output still satisfies the constraints in some approximate sense, which might be sufficient for some applications.

An additional minor disadvantage of the proposed MPC approach is that, when solving the control moves on-line, usually piecewise constant inputs are obtained. This is not a problem in general, but in the framework of boundary control systems there is a smoothness requirement for the inputs [9, Sect. 3.3]. Thus, the proposed MPC approach cannot be directly utilized for boundary control systems, which is essentially the reason why regular linear systems are considered in Publication V. Naturally, the approach is valid for regular boundary control systems as those can be expressed as regular linear systems.

4. CONCLUSIONS AND FUTURE OUTLOOK

4.1 Robust output regulation

The theory of robust output regulation was extended to boundary control systems, especially necessary and sufficient conditions for the solvability of the output regulation and robust output regulation problems were presented. In deriving necessary and sufficient conditions for achieving robust output regulation, the internal model principle was generalized to boundary control systems.

For future research objectives, there still are a few system classes, e.g., well-posed systems and system nodes, to which the internal model principle has not been extended. However, these systems are mostly of academic interest as in practical applications well-posed systems are often regular (as noted, e.g., in [25]). Regardless, there still are system classes for which the internal model theory could be extended.

A more practical research direction might arise from the approximate robust output regulation. As already noted in the previous chapter, the internal model principle implies that for a system with infinite-dimensional output space, any robust regulating controller is necessarily infinite-dimensional, and thus, not implementable in practice. An approximate robust regulating controller has already been designed in Publication III, but no general characterization of such controllers has been given. The concept of a partial internal model was presented for boundary control systems in [19], but there is still a great deal of research to be done on partial internal models and characterization of approximate robust regulating controllers in general.

One more addition that could be made to the theory of robust output regulation is the rejection of arbitrary disturbance signals. In the current framework, it is assumed that the frequencies of the disturbance signal are exactly known, which is not a rather practical assumption, even though no further knowledge of the disturbance signal is required. Regardless, using, e.g., the techniques of active disturbance rejection which have been recently considered also for PDEs [15], it might be possible to combine the rejection of virtually unknown disturbance signals with robust output regulation.

4.2 Model predictive control

Using the Cayley-Tustin time discretization, a continuous-time regular linear system was approximated by a discrete-time system, for which the model predictive control problem was formulated. The MPC problem resulted in a quadratic optimization problem with linear inequality constraints, for which a global minimum can be found, thus proving optimality of the obtained controls. When a suitable terminal penalty function is used, stability of the MPC procedure was shown as well.

As already noted in the previous chapter, even if the MPC formulation can be applied to a large class of systems via the time discretization, the approximate behavior of the discrete-time system with respect to the original continuous-time system may cause problems especially in the consideration of output constraints. It should be possible to consider bounds for the continuous output in the discrete-time MPC setting as well, but this might lead to nonlinear constraints which would make finding the global minimum more difficult. Regardless, the behavior of the continuous-time output should at least be inspected under the inputs obtained form the discrete-time MPC.

Disregarding the drawbacks arising from the time discretization, as the Cayley-Tustin transform maps even a well-posed continuous-time system into a discrete-time system with bounded operators, it could be possible to generalize several other results from discrete-time MPC of finite-dimensional systems (see e.g. [36]) to these discrete-time infinite-dimensional systems and hence to the continuous-time PDEs as well. This seems a potential topic of future research.

A fundamental aspect that should be noticed is that the MPC formulations presented in Publications IV and V require that the state of the system is known at the sampling times, which appears to be quite a common requirement in MPC in general. Since especially in the PDE framework measurements of the state are rarely available, an observer should be included in the MPC design to obtain the required information on the state. Using an observed state instead of the actual state should not affect the performance of the MPC too drastically, but it should be explicitly accounted for regardless.

REFERENCES

[1] N. Altmüller, "Model predictive control for partial differential equations," Ph.D. dissertation, University of Bayreuth, 2014, https://epub.uni-bayreuth.de/1823/.

[2] N. Altmüller and L. Grüne, "Receding horizon optimal control for the wave equation," in *IEEE Conference on Decision and Control*, 2010, pp. 3427–3432.

[3] ——, "Distributed and boundary model predictive control for the heat equation," *GAMM-Mitt.*, vol. 35, no. 2, pp. 131–145, 2012.

[4] ——, "A comparative stability analysis of neumann and dirichlet boundary MPC for the heat equation," in *IFAC Workshop on Control of Systems Governed by Partial Differential Equation*, 2013, pp. 133–138.

[5] B. Augner, "Stabilization of infinite-dimensional port-Hamiltonian systems via dissipative boundary feedback," Ph.D. dissertation, Bergische Universität Wuppertal, 2016, http://elpub.bib.uni-wuppertal.de/servlets/DocumentServlet?id=5990.

[6] M. K. P. Bhat, "Regulator theory for evolution equations," Ph.D. dissertation, University of Toronto, 1976.

[7] C. I. Byrnes, I. G. Laukó, D. S. Gilliam, and V. I. Shubov, "Output regulation for linear distributed parameter systems," *IEEE Trans. Automat. Control*, vol. 45, no. 12, pp. 2236–2252, 2000.

[8] E. F. Camacho and C. Bordons, *Model Predictive Control*, 2nd ed., ser. Advanced Textbooks in Control and Signal Processing, M. J. Grimble and M. A. Johnson, Eds. Springer, 2007.

[9] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, 1995.

[10] P. Dufour, D. J. Michaud, Y. Touré, and P. S. Dhurjati, "A partial differential equation model predictive control strategy: application to autoclave composite processing," *Computers and Chemical Engineering*, vol. 28, no. 4, pp. 545–556, 2004.

[11] A. Fleig and L. Grüne, "Estimates on the minimal stabilizing horizon length model predictive control of the Fokker-Planck equation," *IFAC-PapersOnLine*, vol. 49, no. 8, pp. 260–265, 2016.

[12] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *Appl. Math. Optim.*, vol. 2, no. 2, pp. 170–194, 1975.

[13] ——, "The internal model principle of control theory," *Automatica*, vol. 12, pp. 457–465, 1976.

[14] L. Grüne, "Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems," *SIAM J. Control Optim.*, vol. 48, no. 2, pp. 1206–1228, 2009.

[15] B.-Z. Guo, "Active disturbance rejection control: from ODEs to PDEs," *IFAC-PapersOnLine*, vol. 49, no. 8, pp. 278–283, 2016.

[16] T. Hämäläinen and S. Pohjolainen, "Robust regulation for exponentially stable boundary control systems in Hilbert space," *Proc. MMAR'02, Szczecin, Poland, September 2-5*, 2002.

[17] ——, "Robust regulation of distributed parameter systems with infinite-dimensional exosystems," *SIAM J. Control Optim.*, vol. 48, no. 8, pp. 4846–4873, 2010.

[18] V. Havu and J. Malinen, "The Cayley transform as a time discretization scheme," *NFAO*, vol. 28, no. 7-8, pp. 825–851, 2007.

[19] J.-P. Humaloja, L. Paunonen, and M. Kurula, "A partial internal model for approximate robust output regulation of boundary control systems," in *International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2018, pp. 586–591.

[20] E. Immonen, "State space output regulation theory for infinite-dimensional linear systems and bounded uniformly continuous exogenous signals," Ph.D. dissertation, Tampere University of Technology, 2006.

[21] ——, "On the internal model structure for infinite-dimensional systems: Two common controller tytpes and repetitive contol," *SIAM J. Control Optim.*, vol. 45, no. 6, pp. 2065–2093, 2007.

[22] E. Immonen and S. Pohjolainen, “The internal model principle for linear infinite-dimensional state space systems,” in *International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2006, pp. 652–656.

[23] K. Ito and K. Kunisch, “Receding horizon optimal control for infinite dimensional systems,” *ESAIM Control Optim. Calc. Var.*, vol. 8, pp. 741–760, 2002.

[24] K. Mikkola, “Infinite-dimensional linear systems, optimal control and algebraic Riccati equations,” Ph.D. dissertation, Helsinki University of Technology, 2002, http://lib.tkk.fi/Diss/2002/isbn9512260794/.

[25] ——, “State-feedback stabilization of well-posed linear systems,” *Integr. equ. oper. theory*, vol. 55, no. 2, pp. 249–271, 2006.

[26] K. R. Muske, “Linear model predictive control of chemical processes,” Ph.D. dissertation, The University of Texas at Austin, 1995.

[27] K. R. Muske and J. B. Rawlings, “Model predictive control with linear models,” *AIChE J.*, vol. 39, no. 2, pp. 262–287, 1993.

[28] L. Paunonen, “Output regulation theory for linear systems with infinite-dimensional and periodic exosystems,” Ph.D. dissertation, Tampere University of Technology, 2011.

[29] L. Paunonen and S. Pohjolainen, “Internal model theory for distributed parameter systems,” *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4753–4775, 2010.

[30] ——, “The internal model principle for systems with unbounded control and observation,” *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3967–4000, 2014.

[31] V. T. Pham, D. Georges, and G. Besançon, “Infinite-dimensional receding horizon optimal control for an open-channel system,” in *IFAC Symposium on Nonlinear Control Systems*, 2010.

[32] ——, “On the use of a global control Lyapunov functional in infinite-dimensional predictive control,” in *IFAC Symposium on System, Structure and Control*, 2010.

[33] ——, “Predictive control with guaranteed stability for hyperbolic systems of conservation laws,” in *IEEE Conference on Decision and Control*, 2010, pp. 6932–6937.

[34] ——, "Infinite-dimensional predictive control for hyperbolic systems," *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3592–3617, 2014.

[35] V. Q. Phóng, "The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems," *Math. Z.*, vol. 208, pp. 467–588, 1991.

[36] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*, 2nd ed. Nob Hill Publishing, 2016.

[37] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica*, vol. 39, pp. 1555–1569, 2003.

[38] J. M. Schumacher, "Finite-dimensional regulators for a class of infinite-dimensional systems," *Systems Control Lett.*, vol. 3, no. 1, pp. 7–12, 1983.

[39] O. J. Staffans, *Well-Posed Linear Systems.* Cambridge University Press, 2005.

[40] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups.* Birkhäuser Verlag AG, 2009.

[41] G. Weiss, "Transfer functions of regular linear systems. Part I: Characterzations of regularity," *Trans. Amer. Math. Soc.*, vol. 342, no. 2, pp. 827–854, 1994.

[42] M. Weiss and G. Weiss, "Optimal control of stable weakly regular linear systems," *Math. Control Signals Systems*, vol. 10, pp. 287–330, 1997.

[43] Q. Xu and S. Dubljevic, "Linear model predictive control for transport-reaction processes," *AIChE J.*, vol. 63, pp. 2644–2659, 2017.

[44] Y. Yamamoto and S. Hara, "Relationships between internal and external stability for infinite-dimensional systems with applications to a servo problem," *IEEE Trans. Automat. Control*, vol. 33, no. 11, pp. 1044–1052, 1988.

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Robust Regulation for Port-Hamiltonian Systems of Even Order

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***Abstract*— We present a controller that achieves robust regulation for a port-Hamiltonian system of even order. The controller is especially designed for impedance energy-preserving systems. By utilizing the stabilization results for port-Hamiltonian systems together with the theory of robust output regulation for exponentially stable systems, we construct a simple controller that solves the Robust Output Regulation Problem for an initially unstable system. The theory is illustrated on an example where we construct a controller for one-dimensional Schrödinger equation with boundary control and observation.**

I. Introduction

The class of port-Hamiltonian systems includes models of flexible structures, traveling waves, heat exchangers, bioreactors, and, in general, lossless and dissipative hyperbolic systems on one-dimensional spatial domain [7]. Due to this vast coverage of models, the stability and stabilization of port-Hamiltonian systems have been subjects of active research during the past decade [1], [12]. Stability and stabilization properties of systems are essential for robust output regulation problems, which ties robust regulation for port-Hamiltonian systems closely to this field of research.

The Internal Model Principle is the key to understanding how control systems can be robust, i.e., tolerate perturbations in the systems' parameters. The type of robust controller (low-gain controller) proposed by Davison [3] for stable systems has many practical advantages, as the structure of the controller is simple and it can be tuned with input-output measurements from the open loop system. The controller was generalized to infinite-dimensional systems and its tuning process was simplified in [5], [6]. The Internal Model Principle was generalized to regular linear systems in [9], [10]

In this paper, we construct a robust regulating controller for an impedance energy-preserving port-Hamiltonian system of even order. Even though the considered system is initially unstable, by combining output feedback with a typical controller structure we will be able to construct a simple controller that achieves robust output regulation on the system. By robust regulation we mean that the controller exponentially asymptotically tracks the reference signal y_{ref}, exponentially asymptotically rejects the boundary disturbance w and allows some perturbations in the plant [6].

As the main contribution of this paper we construct a simple robust regulating controller for an initially unstable system. Using the stability results presented in [1] we derive a sufficient criterion for exponential stability of port-Hamiltonian systems of even order. With the new criterion, we will show that the asymptotically stabilizing output feedback presented in [12] actually achieves exponential stability for impedance energy-preserving port-Hamiltonian systems of even order. When the system is exponentially stabilized, we can utilize the controller structure introduced in [5], [6] for exponentially stable systems, and construct a simple robust regulating controller for a system that was initially unstable.

The structure of this paper is as follows. In Section III we give some required background to port-Hamiltonian systems. Furthermore, we will present a sufficient condition for an even-order port-Hamiltonian system to be exponentially stable, and we will use the result to exponentially stabilize a port-Hamiltonian system. In Section IV we will introduce the control system, and in Section V we present the Robust Output Regulation Problem (RORP) and the Internal Model Principle. In Section VI we will present our main result which will be illustrated in Section VII where we construct a controller for one-dimensional Schrödinger equation. In Section VIII we conclude the paper.

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II. Notation

Here $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, null space and resolvent of a linear operator A are denoted by $D(A), \mathcal{R}(A), \mathcal{N}(A)$ and $\rho(A)$, respectively. A strongly continuous (C_0-) semigroup $T_A(t)$ generated by A is exponentially stable if there are positive constants M and α such that $||T_A(t)|| \leq Me^{-\alpha t}$.

III. Background on Port-Hamiltonian Systems

A linear port-Hamiltonian system of order N on the spatial interval $\zeta \in [a, b]$ is given by

$$\frac{\partial}{\partial t}x(\zeta,t) = \mathcal{A}x(\zeta,t), \quad x(0) = x_0, \tag{1a}$$

$$u(t) = \mathcal{B}x(\cdot,t), \tag{1b}$$

$$y(t) = \mathcal{C}x(\cdot,t), \tag{1c}$$

where $\mathcal{B}$ and $\mathcal{C}$ are linear operators, and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}x(\zeta,t) := \sum_{k=0}^{N} P_k \frac{\partial^k(\mathcal{H}(\zeta)x(t,\zeta))}{\partial\zeta^k}, \tag{2}$$

where the matrices $P_k \in \mathbb{C}^{n\times n}$ satisfy the condition $P_k^* = (-1)^{k+1}P_k$ for $k \geq 0$, and the matrix P_N is assumed to be invertible [12]. The Hamiltonian density matrix function $\mathcal{H} : [a, b] \to \mathbb{C}^{n\times n}$ is a measurable function such that there exists $0 < m \leq M$ such that for almost every $\zeta \in [a,b]$ we have $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^*$ and $m|\xi|^2 \leq \xi^*\mathcal{H}(\zeta)\xi \leq M|\xi|^2$ for $\xi \in \mathbb{C}^n$ [1]. The energy state space $X = L^2([a,b],\mathbb{C}^n)$ is equipped with the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^*\mathcal{H}(\zeta)f(\zeta)d\zeta, \tag{3}$$

and hence, X is a Hilbert space.

Let

$$\begin{aligned}&\Phi : H^N([a,b];\mathbb{C}^n) \to \mathbb{C}^{2nN},\\ &\Phi(x) := (x(b),\ldots,x^{(N-1)}(b), x(a),\ldots,x^{(N-1)}(a))\end{aligned} \tag{4}$$

be the boundary trace operator and introduce the boundary port variables f_∂, e_∂ defined by

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} := \frac{1}{\sqrt{2}}\begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}\Phi(\mathcal{H}x) := R_{ext}\Phi(\mathcal{H}x), \tag{5}$$

where $Q \in \mathbb{C}^{nN\times nN}$ is a block matrix given by

$$Q_{ij} := \begin{cases}(-1)^{j-1}P_{i+j-1}, & i+j \leq N+1\\ 0, & \text{else}\end{cases}. \tag{6}$$

Note that since P_N is assumed to be invertible, it follows that Q is invertible, and hence, R_{ext} is invertible as well. Using the boundary port variables we define the operators $\mathcal{B}$ and $\mathcal{C}$ as

$$\mathcal{B}x(t) := W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{7a}$$

$$\mathcal{C}x(t) := W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{7b}$$

where $W_B, W_C \in \mathbb{C}^{nN\times 2nN}$. [1]

Define the domain of the operator $\mathcal{A}$ as

$$D(\mathcal{A}) = \left\{\mathcal{H}x \in H^N([a,b],\mathbb{C}^n) \,\middle|\, W_B\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0\right\}. \tag{8}$$

Since we assumed P_0 to be skew-adjoint, it follows from [4, Thm. 4.1] that the operator A generates a contraction semigroup if and only if $W_B\Sigma W_B^* \geq 0$, where

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{9}$$

Furthermore, for $N = 1$ we have from [7, Lem. 9.1.4] that if $W_B\Sigma W_B^* > 0$ then $\mathcal{A}$ generates an exponentially stable semigroup. By utilizing the following proposition [1, Prop. 2.14], we will show that the result of [7, Lem. 9.1.4] holds for $N = 2$ as well.

Proposition 1: [1, Prop. 2.14] Let $N = 2$ and $\mathcal{H} \in W^1_\infty([a,b];\mathbb{C}^{n\times n})$, and assume

$$\mathrm{Re}\langle\mathcal{A}x, x\rangle_X \leq -\gamma\left(||(\mathcal{H}x)(a)||^2 + ||(\mathcal{H}x)'(a)||^2 + ||(\mathcal{H}x)(b)||^2\right) \tag{10}$$

for $x \in D(A)$ and for some $\gamma > 0$. Then $\mathcal{A}$ generates an exponentially stable and contractive C_0-semigroup.

□

Lemma 1: Let $N = 2$. If $W_B\Sigma W_B^* > 0$, then the operator $\mathcal{A}$ with domain (8) generates an exponentially stable C_0-semigroup.

Proof: Following the proof of [7, Lem. 9.1.4] we write $W_B = S\,[I+V, \quad I-V]$, where S is invertible and $VV^* < I$ (equivalently $V^*V < I$), and define a full rank matrix $W_C = [I+V^*, \quad -I+V^*]$. Thus, the matrix $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible.

Let $x \in D(\mathcal{A})$ be arbitrary. By definition of the domain of $\mathcal{A}$ we have that $\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \mathcal{N}(W_B)$. Following the proof of [7, Lem. 9.1.4], we may write

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I-V \\ -I-V \end{bmatrix} l$$

for some $l \in \mathbb{C}^{2n}$. Since $P_0^* = -P_0$, we have [1, Lem. 2.2] that

$$2\,\mathrm{Re}\langle\mathcal{A}x, x\rangle_X = \mathrm{Re}\langle f_\partial, e_\partial\rangle_{\mathbb{C}^{4n}} = l^*(-I+V^*V)l.$$

Furthermore, we have

$$\begin{aligned} y := W_C\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} &= [I+V^*, \quad -I+V^*]\begin{bmatrix} I-V \\ -I-V \end{bmatrix} l\\ &= 2(I-V^*V)l,\end{aligned}$$

from which we obtain

$$\mathrm{Re}\langle\mathcal{A}x, x\rangle_X = \frac{1}{8}y^*\left[-I+V^*V\right]^{-1}y \leq -m_1||y||^2 \tag{11}$$

for some $m_1 > 0$, where we used that $-I+V^*V < 0$.

Using (5), the definition of the domain of $\mathcal{A}$ and the definition of y we obtain

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} W_B \\ W_C \end{bmatrix}\begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}\Phi(\mathcal{H}x) := W\Phi(\mathcal{H}x).$$

Since $P_N = P_2$ is assumed to be invertible and $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible, the matrix W is invertible and $||Ww||^2 \geq m_2||w||^2$ for every $w \in \mathbb{C}^{4n}$ and some $m_2 > 0$. Taking norms on both sides we obtain

$$\begin{aligned}||y||^2 &= ||W\Phi(\mathcal{H}x)||^2\\ &\geq m_2||\Phi(\mathcal{H}x)||^2\\ &\geq m_2\left(||(\mathcal{H}x)(a)||^2 + ||(\mathcal{H}x)'(a)||^2 + ||(\mathcal{H}x)(b)||^2\right),\end{aligned} \tag{12}$$

and finally, by combining (11) and (12) we have reached relation (10), and thus, the operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup by Proposition 1. ■

It should be noted that the authors of [1] have also generalized the result of Proposition 1 to port-Hamiltonian systems of even order, where relation (10) becomes

$$\mathrm{Re}\langle\mathcal{A}x, x\rangle_X \leq -\gamma\sum_{\zeta=a,b}\sum_{k=0}^{N-1}\alpha_{\zeta,k}\left|\left|(\mathcal{H}x)^{(k)}(\zeta)\right|\right|^2 \tag{13}$$

for some $\gamma > 0$ and certain $\alpha_{\zeta,k} \geq 0$ [1, Prop. 2.16]. It is easy to see that in the general case $N \in 2\mathbb{N}$ the estimation in equation (12) can be done so that when combined with (11), we obtain relation (13). Furthermore, since the estimation in equation (12) is the only part of the proof of Lemma 1 that depends on the order N, the result of Lemma 1 can be generalized to port-Hamiltonian systems of even order as well. We then arrive at the generalization of Lemma 1:

Lemma 2: Let $N \in 2\mathbb{N}$. If $W_B \Sigma W_B^* > 0$, then the operator $\mathcal{A}$ with domain (8) generates an exponentially stable C_0-semigroup.

□

Using Lemma 2 we can now show that a certain class of port-Hamiltonian systems of even order can be exponentially stabilized by negative output feedback. Consider the class of impedance energy-preserving port-Hamiltonian systems that are systems satisfying the relation [12]

$$\frac{1}{2}\frac{d}{dt}||x(t)||_X^2 = u^*(t)y(t). \tag{14}$$

An impedance energy-preserving system can be identified based on the matrices W_B and W_C by [4, Thm. 4.4]. Essentially the matrices are given a certain structure such that they satisfy

$$W_B \Sigma W_B^* = W_C \Sigma W_C^* = 0, \tag{15a}$$
$$W_B \Sigma W_C^* = W_C \Sigma W_B^* = I, \tag{15b}$$

which can be checked very easily.

Stabilization of impedance energy-preserving systems is considered in [12], where it is shown that negative output feedback asymptotically stabilizes an impedance energy-preserving system. We will now show that, for systems of even order, exponential stability is actually achieved.

Lemma 3: Consider the system (1) with $N \in 2\mathbb{N}$ and assume that u and y are such that W_B and W_C satisfy equations (15a)–(15b). Then the system can be exponentially stabilized using negative output feedback.

Proof: Using negative output feedback to the system, i.e., $u(t) = r(t) - \kappa y(t)$ where $\kappa > 0$, the closed-loop system is described by [12]

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), \\ (W_B + \kappa W_C)\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= (\mathcal{B} + \kappa\mathcal{C})x(t) = r(t), \\ \mathcal{C}x(t) &= y(t). \end{aligned} \tag{16}$$

Now, consider the operator $\mathcal{A}_s = \mathcal{A}|_{D(\mathcal{A}_s)}$, where

$$D(\mathcal{A}_s) = \left\{ \mathcal{H}x \in H^N([a,b],\mathbb{C}^n) \,\middle|\, W_\kappa \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \tag{17}$$

where $W_\kappa = W_B + \kappa W_C$. It is shown in [12] that W_κ satisfies $W_\kappa \Sigma W_\kappa^* = 2\kappa I > 0$, and hence, if $N \in 2\mathbb{N}$, the operator $\mathcal{A}_s$ generates an exponentially stable C_0-semigroup due to Lemma 2. ■

IV. THE PLANT, EXOSYSTEM AND CONTROLLER

In this section we will present the plant, the exosystem and the controller. The plant is an impedance energy-preserving port-Hamiltonian system of even order given by

$$\dot{x}(t) = \mathcal{A}x(t), \qquad x(0) = x_0, \tag{18a}$$
$$\mathcal{B}x(t) = u(t) + w(t), \tag{18b}$$
$$\mathcal{C}x(t) = y(t), \tag{18c}$$

where $\mathcal{A}$ is given in (2) with $N \in 2\mathbb{N}$, $\mathcal{B}$ and $\mathcal{C}$ are given in (7a)–(7b) with W_B and W_C satisfying (15a)–(15b), and $w(t)$ is a bounded and differentiable disturbance signal.

Since W_B satisfies $W_B \Sigma W_B^* = 0$, the system (18) is a boundary control system [4, Thm. 4.2], and hence there are operators $A : D(A) \to X$ with $D(A) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$ and $Ax = \mathcal{A}x$ for $x \in D(A)$, and $B \in \mathcal{L}(U,X)$ such that $\mathcal{R}(B) \subset D(\mathcal{A})$ and $\mathcal{B}Bu = u$ [2, Def. 3.3.2]. Using these operators, the transfer function from u to y is given by [6]

$$P(s) = \mathcal{C}(sI - A)^{-1}(\mathcal{A}B - sB) + \mathcal{C}B. \tag{19}$$

The exosystem that generates the boundary disturbance signal $w(t)$ and the reference signal $y_{ref}(t)$ is given by

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \tag{20a}$$
$$w(t) = Ev(t), \tag{20b}$$
$$y_{ref}(t) = -Fv(t) \tag{20c}$$

on a finite-dimensional space $W = \mathbb{C}^q$. Here $S = \operatorname{diag}(i\omega_1, i\omega_2, \dots, i\omega_q)$ with $\{\omega_i\}_{i=1}^q \subset \mathbb{R}$ and $\omega_i \neq \omega_j$ for $i \neq j$, $E \in \mathcal{L}(W,U)$ and $F \in \mathcal{L}(W,Y)$. Furthermore, we assume that for every $k \in \{1,2,\dots,q\}$ the transfer function $P(i\omega_k) \in \mathcal{L}(U,Y)$ is surjective, which is crucial to the solvability of the robust output regulation problem.

The dynamic error feedback controller is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t) \qquad z(0) = z_0, \tag{21a}$$
$$u(t) = Kz(t) - \kappa y(t), \tag{21b}$$

where $e(t) = y(t) - y_{ref}(t)$ is the error signal, $\kappa > 0$, and the parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are to be chosen such that robust output regulation is achieved. Note that in the usual formulation of the controller we have $\kappa = 0$, and hence, the parameter κ is not included in the controller parameters. However, the extra term $-\kappa y(t)$ is required to exponentially stabilizing the plant (18). The controller (21) is an abstract linear system on Banach space Z. The operator $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$ [8].

In order to give the state-space presentation of the closed-loop control system, we define a new variable $\xi = x - B_s r - Gv$, where $r = Kz$, the operator $B_s \in \mathcal{L}(U,X)$ is such that $\mathcal{R}(B_s) \subset D(\mathcal{A})$ and $(\mathcal{B} + \kappa\mathcal{C})B_s r = r$, and the operator $G \in \mathcal{L}(W,X)$ is such that $\mathcal{R}(G) \subset D(\mathcal{A})$ and $(\mathcal{B} + \kappa\mathcal{C})Gv = Ev$. These operators exist as the plant (18) with input $u = Kz - \kappa y$ is a boundary control system [12]. Define now the extended state-space by $X_e := X \times \mathbb{C}^q$, and let $\xi_e(t) := (\xi(t), z(t))$ be the extended state. Following [6], the closed-loop control system can be written as

$$\dot{\xi}_e = A_e \xi_e + Hv + Dy_{ref}, \tag{22}$$

where $D(A_e) = D(A_s) \times \mathbb{C}^q$ and

$$A_e = \begin{bmatrix} A_s - B_s K \mathcal{G}_2 \mathcal{C} & \mathcal{A} B_s K - B_s K(\mathcal{G}_1 + \mathcal{G}_2 \mathcal{C} B_s K) \\ \mathcal{G}_2 \mathcal{C} & \mathcal{G}_1 + \mathcal{G}_2 \mathcal{C} B_s K \end{bmatrix},$$
$$H = \begin{bmatrix} \mathcal{A} G - B_s K \mathcal{G}_2 \mathcal{C} G - GS \\ \mathcal{G}_2 \mathcal{C} G \end{bmatrix},$$
$$D = \begin{bmatrix} B_s K \mathcal{G}_2 \\ -\mathcal{G}_2 \end{bmatrix}, \tag{23}$$

where the operator A_s is given by $A_s : D(A_s) \to X$ with $D(A_s) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B} + \kappa\mathcal{C})$ and $A_s x = \mathcal{A}x$ for $x \in D(A_s)$.

V. The Robust Output Regulation Problem

In this section we formulate the robust output regulation problem and present a few related concepts. We consider perturbations $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ of the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ where the operators in the class $\mathcal{O}$ of admissible perturbations are such that (i) the perturbed plant $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is a boundary control system and (ii) $i\omega_k \in \rho(\tilde{\mathcal{A}})$ for $k \in \{1, 2, \ldots, q\}$. It is easy to see that these conditions are satisfied for all bounded and sufficiently small perturbations to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and for arbitrary bounded perturbations to the operators E and F [10].

The following formulation of the robust output regulation problem is given in [10]:

The Robust Output Regulation Problem. Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\alpha > 0$.
3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\tilde{\alpha} \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\tilde{\alpha} > 0$.

We say that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy of the internal model of the exosystem S if for all $k \in \{1, 2, \ldots, q\}$ we have $\dim(\mathcal{N}(i\omega_k - \mathcal{G}_1)) \geq \dim(Y)$ [8]. Since we assumed that the eigenvalues of S are distinct and we have $\dim(Y) < \infty$, the controller incorporates a p-copy of the internal model of the exosystem if $\mathcal{G}_1 = \mathrm{diag}(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y)$. Furthermore, a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to satisfy the $\mathcal{G}$-conditions if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \tag{24a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{24b}$$

for all $k \in \{1, 2, \ldots, q\}$. [8]

VI. Construction of the Robust Controller

In this section we will prove that a controller of the form (21) with suitably chosen parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ and $\kappa > 0$ solves the Robust Output Regulation Problem for an impedance energy-preserving port-Hamiltonian system of even order.

Theorem 1: Let the control system be as described in Section IV. There is a controller of the form (21) such that for every $\kappa > 0$ there exists an $\epsilon_\kappa > 0$ such that for every $0 < \epsilon \leq \epsilon_\kappa$ the Robust Output Regulation Problem is solved.

Proof: Let us begin the proof from the stabilization of the plant (18). We denote temporarily $Kz(t) = r(t)$, and hence, the input for the plant (18) is of the form $u(t) = r(t) - \kappa y(t)$. Since the plant is an impedance energy-preserving port-Hamiltonian system of even order, such an input exponentially stabilizes the plant due to Lemma 3. Thus, there is an operator $A_s : D(A_s) \to X$ with $D(A_s) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B} + \kappa\mathcal{C})$ and $A_s x = \mathcal{A}x$ for $x \in D(A_s)$ that generates an exponentially stable C_0-semigroup.

Now that the plant is exponentially stabilized, we will choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the controller exponentially stabilizes the closed-loop system and solves the robust output regulation problem. We can utilize the controller parameter choices made in [6] where a robust regulating controller was constructed for an exponentially stable system. Essentially the controller parameters are chosen in such a way that the controller incorporates a p-copy of the internal model of the exosystem and satisfies the $\mathcal{G}$-conditions.

Following [6] and [10] we define $Z = Y^q$ and choose the controller parameters as

$$\mathcal{G}_1 = \mathrm{diag}\,(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y) \in \mathcal{L}(Z), \tag{25a}$$
$$K = \epsilon K_0 = \epsilon \left[K_0^1, K_0^2, \ldots, K_0^q\right] \in \mathcal{L}(Z, U), \tag{25b}$$
$$\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=1}^q = (-(P_\kappa(i\omega_k) K_0^k)^*)_{k=1}^q = \begin{bmatrix} -(P_\kappa(i\omega_1) K_0^1)^* \\ \vdots \\ -(P_\kappa(i\omega_q) K_0^q)^* \end{bmatrix} \in \mathcal{L}(Y, Z), \tag{25c}$$

where $P_\kappa(i\omega_k) = P(i\omega_k)(I + \kappa P(i\omega_k))^{-1}$ is the transfer function of the stabilized plant [11]. As we assumed that $P(i\omega_k)$ is surjective for every $k \in \{1, 2, \ldots, q\}$, it follows that $P_\kappa(i\omega_k)$ is surjective as well for every k.

Since the surjectivity assumption of the transfer function holds, we choose the components K_0^k of K_0 such that the operators $P_\kappa(i\omega_k) K_0^k$ are invertible, e.g., by choosing $K_0^k = P_\kappa(i\omega_k)^\dagger$ (the Moore-Penrose pseudoinverse of $P_\kappa(i\omega_k)$), in which case we have $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1, 2, \ldots, q\}$ [10].

It has been shown in [6] that, if the plant is exponentially stable, there exists an $\epsilon^* > 0$ such that the closed-loop system is exponentially stable for every $0 < \epsilon \leq \epsilon^*$ and that the proposed controller solves the robust output regulation problem. Since we choose the controller parameters according to [6] and exponentially stabilized the plant, it follows from the results of [6] that, when the plant is exponentially stabilized with output feedback $u(t) = Kz(t) - \kappa y(t)$, where $\kappa > 0$, there exists an $\epsilon_\kappa > 0$ such that for every $0 < \epsilon \leq \epsilon_\kappa$ the closed-loop system is exponentially stable and the robust output regulation problem is solved. ■

VII. Example

As an example we study Schrödinger equation on the spatial interval $\zeta \in [0,1]$ considered in [1], given by

$$\frac{\partial}{\partial t} w(\zeta,t) = i \frac{\partial^2}{\partial \zeta^2} w(\zeta,t), \quad t \geq 0, \tag{26a}$$

which is a second-order port-Hamiltonian system with $P_2 = i$, $P_1 = P_0 = 0$, $\mathcal{H}(\zeta) = 1$ and state $x(\zeta,t) = w(\zeta,t)$ [1]. The inputs are given by

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x'(0,t) \\ x(1,t) \end{bmatrix} + w(t), \tag{26b}$$

and the outputs are given by

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} ix(0,t) \\ ix'(1,t) \end{bmatrix} \tag{26c}$$

Using the boundary port variables f_∂ and e_∂ the inputs and outputs can be written as

$$\begin{aligned} u(t) = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \\ y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & i & 0 \\ 1 & 0 & 0 & i \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}. \end{aligned} \tag{27}$$

As the matrices W_B and W_C satisfy equations (15a)–(15b), the system (26) is an impedance energy-preserving port-Hamiltonian system of order two, and thus, we may use Theorem 1 to construct a robust regulating controller for the system.

Let us first consider the transfer function of the system (26), given by

$$P(s) = \begin{bmatrix} -\dfrac{\tanh(i\sqrt{is})}{\sqrt{is}} & \dfrac{i}{\cosh(i\sqrt{is})} \\ \dfrac{i}{\cosh(i\sqrt{is})} & -\sqrt{is}\tanh(i\sqrt{is}) \end{bmatrix}. \tag{28}$$

The transfer function is surjective for every $s \neq 0$ and $s \neq -i\left(\frac{(2m+1)\pi}{2}\right)^2$ where $m \in \mathbb{N}$, and thus, we cannot track signals including those frequencies.

Let the exosystem be given by $S = \mathrm{diag}(-4i\pi^2, -i\pi^2)$ and $E = F = I$. If we choose the output feedback parameter as $\kappa = 1$, the transfer function for the stabilized plant is given by $P_\kappa(s) = P(s)(I + P(s))^{-1}$, and thus, for the eigenvalues of the signal generator S we have

$$P_\kappa(-4i\pi^2) = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = P_\kappa(-i\pi^2)^*. \tag{29}$$

Thus, if we choose $K_0^k = P_\kappa(i\omega_k)^{-1}$, the controller parameters are given by

$$\begin{aligned} \mathcal{G}_1 &= \mathrm{diag}(-i4\pi^2, -i4\pi^2, -i\pi^2, -i\pi^2), \\ \mathcal{G}_2 &= \begin{bmatrix} -I_Y \\ -I_Y \end{bmatrix}, \\ K &= \epsilon \left[P_\kappa(-4i\pi^2)^{-1}, P_\kappa(-i\pi^2)^{-1} \right], \end{aligned} \tag{30}$$

and based on Theorem 1 there now exists an $\epsilon_\kappa > 0$ such that for every $0 < \epsilon \leq \epsilon_\kappa$ the closed-loop system is exponentially stable and the robust output regulation problem is solved for the system (26).

VIII. Conclusions

We presented a simple robust regulating controller for an unstable, impedance energy-preserving port-Hamiltonian system of even order. By deriving a new condition for exponential stability of even-order port-Hamiltonian systems we were able to stabilize the system, which allowed us to utilize the theory of robust output regulation for exponentially stable system. Thus, we constructed a simple controller for an unstable system that exponentially stabilizes the original plant and solves the Robust Output Regulation Problem for the stabilized plant.

References

[1] Augner, B. and Jacob, B. Stability and Stabilization of Infinite-dimensional Linear Port-Hamiltonian Systems. Evolution Equations and Control Theory 3(2): 207–229, 2014.
[2] Curtain, R. F. and Zwart, H. An Introduction to Infinite-Dimensional Linear Systems Theory. Texts in applied Mathematics. Springer-Verlag, 1995.
[3] Davison, E. J. Multivariable Tuning Regulators: The Feedforward and Robust Control of a General Servomechanism Problem. IEEE Trans. Automat. Control, 21(1): 35–47, 1976.
[4] Le Gorrec, Y., Zwart, H. and Maschke, B. Dirac structures and boundary control systems associated with skew-symmetric differential operators. SIAM J. Control Optim., 44(2): 1864–1892, 2005.
[5] Hämäläinen, T. and Pohjolainen, S. A Finite-Dimensional Robust Controller for Systems in the CD-Algebra. IEEE Trans. Automat. Control, 45(3): 421–431, 2000.
[6] Hämäläinen, T. and Pohjolainen, S. Robust Regulation for Exponentially Stable Boundary Control Systems in Hilbert Space. Proc. of the 8th IEEE Int. Conf. on Methods and Models on Automation and Control, September 2–5. pp. 171–178, 2002.
[7] Jacob, B. and Zwart, H. Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces, volume 223 of Operator Theory: Advances and Applications. Birkhäuser, Basel, Switzerland. 2012.
[8] Paunonen, L. and Pohjolainen, S. Internal model theory for distributed parameter systems. SIAM J. Control Optim. 48(7): 4753–4775, 2010.
[9] Paunonen, L. and Pohjolainen, S. The Internal model principle for systems with unbounded control and observation. SIAM J. Control Optim. 52(6): 3967–4000, 2014.
[10] Paunonen, L. Controller Design for Robust Output Regulation of Regular Linear Systems. IEEE Trans. Automat. Control. (to appear. http://dx.doi.org/10.1109/TAC.2015.2509439)
[11] Weiss, G. Regular linear systems with feedback. Mathematics of Control Signals and Systems. 7: 23–57, 1994.
[12] Villegas, J., Zwart, H., Le Gorrec, Y., Maschke, B. and van der Schaft, A. Stability and stabilization of a class of boundary control systems. Proc. of the 44th IEEE Conference on Decision and Control and the European Control Conference, December 12–15. pp. 3850–3855, 2005.

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Robust Regulation of Infinite-Dimensional Port-Hamiltonian Systems

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***Abstract*—We will give general sufficient conditions under which a controller achieves robust regulation for a boundary control and observation system. Utilizing these conditions we construct a minimal order robust controller for an arbitrary order impedance passive linear port-Hamiltonian system. The theoretical results are illustrated with a numerical example where we implement a controller for a one-dimensional Euler-Bernoulli beam with boundary controls and boundary observations.**

***Index Terms*—port-Hamiltonian systems, robust control, distributed parameter systems, linear systems.**

I. Introduction

The class of infinite-dimensional port-Hamiltonian systems includes models of flexible systems, traveling waves, heat exchangers, bioreactors, and in general, lossless and dissipative hyperbolic systems on one-dimensional spatial domains [1]–[3]. In this paper, we consider robust output regulation for port-Hamiltonian systems in general, and as an example we implement a robust controller for Euler-Bernoulli beam which can be formulated as a second-order port-Hamiltonian system. By robust regulation we mean that the controller asymptotically tracks the reference signal, rejects the disturbance signal and allows some perturbations in the plant.

The *internal model principle* is the key to understanding how control systems can be robust, i.e., tolerate perturbations in the parameters of the system. The principle indicates that the regulation problem can be solved by including in the controller a suitable *internal model* of the dynamics of the exosystem that generates the reference and disturbance signals. One of the first robust controllers that utilize the internal model principle is the low-gain controller proposed by Davison [4]. Davison's controller has many practical advantages as it has simple structure and it can be tuned with input-output measurements. The controller was generalized to infinite-dimensional systems and its tuning process was simplified in [5], [6].

The main contribution of this paper is that we present sufficient criteria for a controller to achieve robust output regulation for boundary control and observation systems. A corresponding result has already been shown for various system classes [7]–[9] but not for boundary control systems. As our second main result, we will construct a minimal order robust regulating controller for an arbitrary order impedance passive linear port-Hamiltonian system for which we can show certain assumptions to hold.

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Robust output regulation of port-Hamiltonian systems has been considered by the authors in [10], [11] where first- and even-order port-Hamiltonian systems were considered, respectively. Outside robust regulation, stability, stabilization and dynamic boundary control of port-Hamiltonian systems have been considered, e.g., in [12]–[15]. This paper generalizes the results of [10], [11] for port-Hamiltonian systems of arbitrary order N. Furthermore, as opposed to [10], [11] considering only impedance energy preserving systems, here we will be considering impedance passive systems as well. Additionally, here the observation operator is allowed to be unbounded, which is essential for true boundary observation. This is also an extension to the results of [6] where robust regulation of boundary control systems with bounded observations was considered.

Robust regulation has been considered for boundary control systems in [6] and for well-posed systems in general in [16]. In both references, the robust regulation result is formulated for a single controller structure, whereas our result (Theorem 4) holds for any controller that includes a suitable internal model of the exosystem and stabilizes the closed-loop system. Furthermore, both references assume that the controlled system is initially stable, which is not required here. We also note that in the proof of Theorem 8 we could utilize the frequency domain proof of [16, Thm. 1.1] to show that the minimal order controller stabilizes the closed-loop system, but we present an alternative time domain proof instead.

The structure of the paper is as follows. In Section II we present the control system consisting of the plant, exosystem and controller. In Section III we formulate the robust output regulation problem and present the robust regulation result for boundary control and observation systems. In Section IV we present the specific structure of linear port-Hamiltonian systems with stability and stabilization results, so that in Section V we can construct a robust regulating controller - that is also of minimal order - for these systems. The theoretical results are illustrated in Section VI where we construct a robust regulating controller for Euler-Bernoulli beam.

Here $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, kernel, spectrum and resolvent of a linear operator A are denoted by $\mathcal{D}(A), \mathcal{R}(A), \mathcal{N}(A), \sigma(A)$ and $\rho(A)$, respectively. The resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1}$, and it exists for all $\lambda \in \rho(A)$. The growth bound of the C_0-semigroup $T_A(t)$ generated by A is denoted by $\omega_0(T_A)$, and T_A is exponentially stable if $\omega_0(T_A) < 0$. In that case we also say that A is exponentially stable.

II. The plant, exosystem and controller

The plant is a boundary control system of the form

$$\dot{x}(t) = \mathcal{A}x(t), \quad x(0) = x_0, \tag{1a}$$
$$\mathcal{B}x(t) = u(t) + w(t), \tag{1b}$$
$$\mathcal{C}x(t) = y(t) \tag{1c}$$

where the disturbance signal $w(t)$ is generated by the exosystem that will be presented shortly. In Section IV, we will make an additional assumption that the plant is an impedance passive port-Hamiltonian system, but for now it is sufficient to consider the plant a *boundary control and observation system* given by the following definition:

Definition 1. [3, Def. 2.3.13] *Let X, U and Y be Hilbert spaces. The system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of linear operators $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$, $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset X \to U$ and $\mathcal{C} : \mathcal{D}(\mathcal{C}) \subset X \to Y$ is called a boundary control and observation system if the following hold:*

1) *$\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$.*
2) *The restriction $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ of $\mathcal{A}$ to the kernel of $\mathcal{B}$ generates a C_0-semigroup $(T_A(t))_{t\geq 0}$ on X.*
3) *There is a right inverse $B \in \mathcal{L}(U, X)$ of $\mathcal{B}$ such that $\mathcal{R}(B) \subset \mathcal{D}(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(U, X)$ and $\mathcal{B}B = I_U$.*
4) *The operator $\mathcal{C}$ is bounded from $\mathcal{D}(A)$ to Y, where $\mathcal{D}(A)$ is equipped with the graph norm of A.*

Let $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ be the generator of a C_0-semigroup $T_A(t)$ on X. We define the Λ-extension $\mathcal{C}_\Lambda$ of $\mathcal{C}$ by

$$\mathcal{C}_\Lambda x = \lim_{\lambda\to\infty} \lambda \mathcal{C} R(\lambda, A)x,$$

and its domain $\mathcal{D}(\mathcal{C}_\Lambda)$ consists of those $x \in X$ for which the limit exists. Throughout this paper, we also assume that $\mathcal{C}$ is *admissible* for A [17, Def. 4.3.1], i.e., for some $\tau > 0$ there exists a constant K_τ such that

$$\int_0^\tau ||\mathcal{C}T_A(t)x_0||_Y^2 dt \leq K_\tau^2 ||x_0||_X^2 \quad \forall x_0 \in \mathcal{D}(A).$$

Furthermore, if there exists a constant $K > 0$ such that $K_\tau \leq K$ for all $\tau > 0$, then we say that $\mathcal{C}$ is *infinite-time admissible* for A, for which we will give sufficient conditions in the port-Hamiltonian context later on.

The exosystem that generates the boundary disturbance signal $w(t)$ and the reference signal $y_{ref}(t)$ is a linear system

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0, \tag{2a}$$
$$w(t) = Ev(t), \tag{2b}$$
$$y_{ref}(t) = -Fv(t) \tag{2c}$$

on a finite-dimensional space $W = \mathbb{C}^q$ with some $q \in \mathbb{N}$. Here $S \in \mathcal{L}(W) = \mathbb{C}^{q\times q}$, $E \in \mathcal{L}(W, U)$ and $F \in \mathcal{L}(W, Y)$. Furthermore, we assume that S has purely imaginary eigenvalues $\sigma(S) = \{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$ with algebraic multiplicity one.

The transfer function of the plant (1) is given by

$$P(\lambda) = \mathcal{C}_\Lambda R(\lambda, A)(\mathcal{A}B - \lambda B) + \mathcal{C}_\Lambda B \in \mathcal{L}(U, Y), \tag{3}$$

and it is defined for every $\lambda \in \rho(A)$ as $\mathcal{R}(B) \subset \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$. Note that the boundedness of the transfer function implies that $\lambda\hat{u}$ must be bounded for every $\lambda \in \rho(A)$. Hence, by the Plancherel theorem we must have $u \in H^1$, which we will show to hold at the end of this section. Furthermore, we need to assume that $P(i\omega_k)$ is surjective for all $k \in \{1, 2, \ldots, q\}$, which is crucial to the solvability of the robust output regulation problem presented in Section III. Note that the surjectivity assumption implies that we must have $\dim(U) \geq \dim(Y)$.

Since the plant is a boundary control and observation system, it follows from Definition 1 that we can define an operator $G := BE \in \mathcal{L}(W, X)$ satisfying $\mathcal{A}G \in \mathcal{L}(W, X)$, $\mathcal{B}G = E$ and $\mathcal{R}(G) \subset \mathcal{D}(\mathcal{C})$. It is easily seen by following the proof of [18, Thm. 3.3.3] that if $u \in C^2(0, \tau; U)$ and $v \in C^2(0, \tau; W)$ for all $\tau > 0$, then the abstract differential equation

$$\dot{\xi}(t) = A\xi(t) + \mathcal{A}Bu(t) - B\dot{u}(t) + \mathcal{A}Gv(t) - G\dot{v}(t) \tag{4}$$

with $\xi(0) = \xi_0$ is well-posed. Furthermore, if $\xi_0 = x_0 - Bu_0 - Gv_0 \in \mathcal{D}(A)$, the classical solutions of (1) and (4) are related by $\xi(t) = x(t) - Bu(t) - Gv(t)$, and they are unique.

The plant (1) - as well as equation (4) - has a well-defined mild solution for $\dot{u} \in L^p(0, \tau; U)$, $\dot{v} \in L^p(0, \tau; W)$ for some $p \geq 1$ and $x_0 \in X$. In that case, the summary related to [18, Thm. 3.3.4] implies that the mild solution of (1) is given by

$$x(t) = T_A(t)(x_0 - Bu_0 - Gv_0) + Bu(t) + Gv(t) + \int_0^t T_A(t-s)(\mathcal{A}Bu(s) - B\dot{u}(s) + \mathcal{A}Gv(t) - G\dot{v}(t))ds.$$

Similarly for every $\xi_0 = x_0 - Bu_0 - Gv_0 \in X$, one obtains the mild solution of (4) using the above solution and the relation between $x(t)$ and $\xi(t)$. We will show at the end of this section that $\dot{u} \in L^2(0, \tau; U)$, which together with the fact that $v \in C^\infty(0, \tau; W)$ ensures that the mild solutions are well-defined.

The dynamic error feedback controller is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \qquad z(0) = z_0, \tag{5a}$$
$$u(t) = Kz(t) \tag{5b}$$

on a Banach space Z. The parameters $\mathcal{G}_1 \in \mathcal{L}(Z)$, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$ are to be chosen in such a way that robust output regulation is achieved for the plant (1).

We are finally in the position to give the formulation of the closed-loop system consisting of the plant (1) written as the abstract differential equation (4) and the controller (5). Furthermore, we will show that $\dot{u} \in L^2(0, \tau; U)$ for every $\tau > 0$. Using the above notation and definitions, the closed-loop system can be written on the extended state space $X_e = X \times Z$ with the extended state $\xi_e(t) = (\xi(t), z(t))^T$ as

$$\dot{\xi}_e(t) = A_e\xi_e(t) + B_e v(t), \quad \xi_e(0) = \xi_{e0}, \tag{6a}$$
$$e(t) = C_e\xi_e(t) + D_e v(t), \tag{6b}$$

where $e(t) := y(t) - y_{ref}(t)$ is the regulation error, $\xi_{e0} = (\xi_0, z_0)^T$, $C_e = [\mathcal{C}_\Lambda \quad \mathcal{C}_\Lambda BK]$, $D_e = \mathcal{C}_\Lambda G + F$ and

$$A_e = \begin{bmatrix} A - BK\mathcal{G}_2\mathcal{C}_\Lambda & \mathcal{A}BK - BK(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}_\Lambda BK) \\ \mathcal{G}_2\mathcal{C}_\Lambda & \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}_\Lambda BK \end{bmatrix},$$

$$B_e = \begin{bmatrix} \mathcal{A}G - GS - BK\mathcal{G}_2(\mathcal{C}_\Lambda G + F) \\ \mathcal{G}_2(\mathcal{C}_\Lambda G + F) \end{bmatrix}.$$

The operator A_e has domain $\mathcal{D}(A_e) = \mathcal{D}(A) \times Z$, and it can be written in the form

$$\begin{aligned} A_e = & \begin{bmatrix} A & 0 \\ 0 & \mathcal{G}_1 \end{bmatrix} + \begin{bmatrix} -BK\mathcal{G}_2 \\ \mathcal{G}_2 \end{bmatrix} [\mathcal{C}_\Lambda \quad \mathcal{C}_\Lambda BK] \\ & + \begin{bmatrix} 0 & \mathcal{A}BK - BK\mathcal{G}_1 \\ 0 & 0 \end{bmatrix} \\ := & A_1 + A_2 C_e + A_3. \end{aligned}$$

Since all the operators associated with the controller (5) are bounded and since $\mathcal{A}B, B \in \mathcal{L}(U, X)$ due to the plant (1) being a boundary control and observation system, it follows that the operators A_2 and A_3 are bounded. Furthermore, since $\mathcal{C}$ is admissible for A and $\mathcal{C}_\Lambda B \in \mathcal{L}(U, Y)$, it follows that C_e is admissible for A_1. Thus, since A_1 is clearly the generator of a C_0-semigroup, A_2 and A_3 are bounded, and C_e is admissible for A_1, it follows from [17, Thm. 5.4.2] and standard perturbation theory that the operator A_e is the generator of a C_0-semigroup, and that C_e is admissible for A_e as well. Finally, combining (5) and (6b) we obtain that $\dot{u} = K\mathcal{G}_1 z + K\mathcal{G}_2(C_e\xi_e + D_e v)$, which by the above reasoning shows that $\dot{u} \in L^2(0, \tau; U)$ for all $\tau > 0$, and thus, the mild solutions of (1) and (4) are well-defined.

III. The robust output regulation problem and the internal model principle

In this section, we formulate the *robust output regulation problem* and present the concept of the internal model via the *$\mathcal{G}$-conditions*. After that, we are in the position to present and prove the first main result of this paper.

In order to discuss robustness, we consider perturbations $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ of the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$. The class $\mathcal{O}$ of perturbations is defined such that the perturbed operators $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F})$ satisfy the following assumptions which the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are assumed to satisfy as well.

Assumption 2. *The operators $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F})$ satisfy the following:*

1) *The plant $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is a boundary control and observation system.*
2) *The operator $\tilde{\mathcal{C}}$ is admissible for $\tilde{A} = \tilde{\mathcal{A}}|_{\mathcal{N}(\tilde{\mathcal{B}})}$.*
3) *The transfer function of the plant $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is surjective and bounded for every eigenvalue of S.*
4) *$\tilde{E} \in \mathcal{L}(W, U)$ and $\tilde{F} \in \mathcal{L}(W, Y)$.*

It is easy to see that these conditions are satisfied for arbitrary bounded perturbations to E and F, whereas the boundary control and observation system requirement imposes stricter conditions on the perturbations on $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. However, at least sufficiently small bounded perturbations are acceptable. Note that the operators B and G associated with the boundary control and observation system will also change when the system is perturbed. We denote these operators by $\tilde{B}$ and $\tilde{G}$.

The Robust Output Regulation Problem. Choose a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$, the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2(0, \infty; Y)$ for some $\alpha > 0$.
3) If $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$, the regulation error satisfies $e^{\tilde{\alpha} \cdot} e(\cdot) \in L^2(0, \infty; Y)$ for some $\tilde{\alpha} > 0$.

We note that without the last item in the above list the problem is called *output regulation problem* which will be considered in the proof of our first main result in the next subsection.

The internal model principle states that the robust output regulation problem can be solved by including a suitable internal model of the dynamics of the exosystem in the controller. The internal model can be characterized using the definition of $\mathcal{G}$-conditions below. What follows is our first main result where we show that a controller satisfying the $\mathcal{G}$-conditions is robust.

Definition 3. [7, Def. 10] *A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to satisfy the $\mathcal{G}$-conditions if*

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \tag{7a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{7b}$$

for all $k \in \{1, 2, \ldots, q\}$, where $\sigma(S) = \{i\omega_k\}_{k=1}^q$.

A. Sufficient Robustness Criterion for a Controller

We will now show that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ that satisfies the $\mathcal{G}$-conditions solves the robust output regulation problem for a boundary control and observation system, provided that the controller exponentially stabilizes the closed-loop system.

Theorem 4. *Assume that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ exponentially stabilizes the closed-loop system. If the controller satisfies the $\mathcal{G}$-conditions, then it solves the robust output regulation problem. The controller is guaranteed to be robust with respect to all perturbations under which the closed-loop system remains exponentially stable and Assumption 2 is satisfied.*

Proof. Let $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F})$ be such arbitrary perturbations of class $\mathcal{O}$ that the perturbed closed-loop system generated by $\tilde{A}_e$ is exponentially stable. As the perturbations of the class $\mathcal{O}$ satisfy Assumption 2, it follows that $\tilde{B}_e$ and $\tilde{D}_e$ are bounded and $\tilde{C}_e$ is admissible for $\tilde{A}_e$. Thus, the closed-loop system is a regular linear system, and by [9, Thm. 4.1] we have that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the output regulation problem if and only if the regulator equations $\Sigma S = \tilde{A}_e \Sigma + \tilde{B}_e$ and $0 = \tilde{C}_e \Sigma + \tilde{D}_e$ have a solution $\Sigma := (\Pi, \Gamma)^T \in \mathcal{L}(W, X_e)$. Note that the result of [9, Thm. 4.1] only requires that the closed-loop system is regular, and therefore it can be used here. Further note that as $\tilde{A}_e$ is assumed to be exponentially stable and $\sigma(S) \subset i\mathbb{R}$, by [19] the Sylvester equation $\Sigma S = \tilde{A}_e \Sigma + \tilde{B}_e$ has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e)$. Thus, in order to show that the controller solves the output regulation problem, it remains to show that the bounded solution Σ of the Sylvester equation satisfies the second regulator equation as well. We will do this for the arbitrary perturbations $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$, which implies that the controller is robust under these perturbations, i.e., it solves the robust output regulation problem.

Let $k \in \{1, 2, \ldots, q\}$ be arbitrary and consider the eigenvector ϕ_k of S associated with the corresponding eigenvalue

$i\omega_k$ satisfying $S\phi_k = i\omega_k\phi_k$. Then $\Sigma S\phi_k = \tilde{A}_e\Sigma\phi_k + \tilde{B}_e\phi_k$ implies $(i\omega_k - \tilde{A}_e)\Sigma\phi_k = \tilde{B}_e\phi_k$, which yields

$$\begin{bmatrix} (i\omega_k - \tilde{A} + \tilde{B}K\mathcal{G}_2\tilde{\mathcal{C}}_\Lambda)\Pi\phi_k - (\tilde{\mathcal{A}}\tilde{B}K - \tilde{B}K(\mathcal{G}_1 + \mathcal{G}_2\tilde{\mathcal{C}}_\Lambda\tilde{B}K))\Gamma\phi_k \\ -\mathcal{G}_2\tilde{\mathcal{C}}_\Lambda\Pi\phi_k + (i\omega_k - \mathcal{G}_1)\Gamma\phi_k - \mathcal{G}_2\tilde{\mathcal{C}}_\Lambda\tilde{B}K\Gamma\phi_k \end{bmatrix}$$
$$= \begin{bmatrix} (\tilde{A}\tilde{G} - \tilde{G}S - \tilde{B}K\mathcal{G}_2(\tilde{\mathcal{C}}_\Lambda\tilde{G} + \tilde{F}))\phi_k \\ \mathcal{G}_2(\tilde{\mathcal{C}}_\Lambda\tilde{G} + \tilde{F})\phi_k \end{bmatrix}.$$

The second line implies $(i\omega_k - \mathcal{G}_1)\Gamma\phi_k = \mathcal{G}_2(\tilde{\mathcal{C}}_\Lambda\Pi\phi_k + \tilde{\mathcal{C}}_\Lambda\tilde{B}K\Gamma\phi_k + (\tilde{\mathcal{C}}_\Lambda\tilde{G} + \tilde{F})\phi_k)$, and now by the $\mathcal{G}$-conditions we have that $0 = \tilde{\mathcal{C}}_\Lambda\Pi\phi_k + \tilde{\mathcal{C}}_\Lambda\tilde{B}K\Gamma\phi_k + (\tilde{\mathcal{C}}_\Lambda\tilde{G} + \tilde{F})\phi_k = \tilde{C}_e\Sigma\phi_k + \tilde{D}_e\phi_k$. As the eigenvectors ϕ_k form an orthogonal basis on W and the choice of k was arbitrary, it follows that Σ satisfies the second regulator equation $\tilde{C}_e\Sigma + \tilde{D}_e = 0$ as well. Thus, [9, Thm. 4.1] implies that the controller solves the robust output regulation problem. □

IV. Background to port-Hamiltonian systems

In this section, we give some background to port-Hamiltonian systems. We note that while [1] is a classical reference paper regarding these systems, we use [3] as our main reference as it gives a slightly more general formulation for port-Hamiltonian systems than [1]. Therefore we will cite [3] for the base results as well, even though essentially the same results can be found in [1].

Define a linear port-Hamiltonian operator $\mathcal{A}$ of order N on the spatial interval $\zeta \in [a, b]$ as follows:

Definition 5. [3, Def. 3.2.1] *Let $N \in \mathbb{N}$ and $P_k \in \mathbb{C}^{n\times n}$ satisfying $P_k^* = (-1)^{k+1}P_k$ for $k \in \{1, 2 \dots, N\}$ with P_N invertible. Furthermore, let $P_0 \in L^\infty(a, b; \mathbb{C}^{n\times n})$ satisfying $\mathrm{Re}(P_0(\zeta)) := \frac{1}{2}(P_0(\zeta) + P_0^*(\zeta)) \le 0$ for a.e. $\zeta \in [a, b]$. Let the state space $X = L^2(a, b; \mathbb{C}^n)$ be equipped with the inner product $\langle\cdot,\cdot\rangle_X = \langle\cdot, \mathcal{H}\cdot\rangle_{L^2}$ where $\mathcal{H} : [a, b] \to \mathbb{C}^{n\times n}$ satisfies $m|\xi|^2 \le \langle\xi, \mathcal{H}(\zeta)\xi\rangle_{\mathbb{C}^n} \le M|\xi|^2$, $\xi \in \mathbb{C}^n$ a.e. $\zeta \in [a, b]$ for some constants $0 < m \le M < \infty$. Then the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ defined as*

$$\mathcal{A}x(\zeta, t) := \sum_{k=1}^{N} P_k \frac{\partial^k}{\partial\zeta^k}(\mathcal{H}(\zeta)x(\zeta, t)) + P_0(\zeta)\mathcal{H}(\zeta)x(\zeta, t),$$

with domain $\mathcal{D}(\mathcal{A}) = \{x \in X : \mathcal{H}x \in H^N(a, b; \mathbb{C}^n)\}$ is called a linear port-Hamiltonian operator of order N.

Let $\Phi : H^N(a, b; \mathbb{C}^n) \to \mathbb{C}^{2nN}$ defined by

$$\Phi(x) := (x(b), \dots, x^{(N-1)}(b), x(a), \dots, x^{(N-1)}(a))^T$$

be the boundary trace operator and define the boundary port variables f_∂, e_∂ by

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} := \frac{1}{\sqrt{2}} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix} \Phi(\mathcal{H}x) := R_{ext}\Phi(\mathcal{H}x) \quad (8)$$

where $Q \in \mathbb{C}^{nN\times nN}$ is a block matrix given by

$$Q_{ij} := \begin{cases} (-1)^{j-1}P_{i+j-1}, & i + j \le N + 1 \\ 0, & \text{else} \end{cases}.$$

Note that since P_N is assumed to be invertible, it follows that Q is invertible, and hence, R_{ext} is invertible as well.

Using the boundary port variables we can now define the boundary control and boundary observation operators $\mathcal{B}$ and $\mathcal{C}$, respectively. Their definitions are included in the following definition of port-Hamiltonian systems.

Definition 6. [3, Def. 3.2.10] *Let $\mathcal{A}$ be a port-Hamiltonian operator of order N with associated boundary port variables f_∂ and e_∂. Further let $W_B, W_C \in \mathbb{C}^{nN\times 2nN}$ be full rank matrices such that $\mathcal{N}(W_B) \cap \mathcal{N}(W_C) = \{0\}$. Then the input map $\mathcal{B} : \mathcal{D}(\mathcal{B}) = \mathcal{D}(A) \subset X \to U := \mathbb{C}^{nN}$ and the output map $\mathcal{C} : \mathcal{D}(\mathcal{C}) = \mathcal{D}(A) \subset X \to Y := \mathbb{C}^{nN}$ are defined as*

$$\mathcal{B}x(t) := W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \quad \mathcal{C}x(t) := W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad (9)$$

and the system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called a port-Hamiltonian system.

We note that the above definition implies that we have full control and measurements, which is not very common in practice. However, the exponential stability criterion given in part a) of Lemma 7 essentially requires that we have full control. If we were considering a less general class of port-Hamiltonian systems, e.g., first- or even order systems, we could utilize [15, Thm. III.2] or [12, Prop. 2.16], respectively, to obtain exponential stability with fewer controls. However, to our knowledge there are no weaker exponential stability criteria than the one given in part a) of Lemma 7 for arbitrary order port-Hamiltonian systems, and thus, we assume having full control and measurements.

We have by [3, Thm. 3.2.21] that a port-Hamiltonian system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a boundary control and observation system if and only if the operator $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ generates a C_0-semigroup on X. Furthermore, by [3, Thm. 3.3.6] the operator A generates a contractive C_0-semigroup if and only if $W_B\Sigma W_B^* \ge 0$ where

$$\Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (10)$$

Following [3, Def. 3.2.12], we define a system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *impedance passive* if it satisfies

$$\mathrm{Re}\langle\mathcal{A}x(t), x(t)\rangle_X \le \mathrm{Re}\langle\mathcal{B}x(t), \mathcal{C}x(t)\rangle_{\mathbb{C}^{nN}}, \quad x \in \mathcal{D}(\mathcal{A})$$

and *impedance energy preserving* if the above holds as an equality. These systems can be easily identified based on W_B, W_C and P_0. Define a matrix P_{W_B,W_C} such that

$$P_{W_B,W_C}^{-1} = \begin{bmatrix} W_B\Sigma W_B^* & W_B\Sigma W_C^* \\ W_C\Sigma W_B^* & W_C\Sigma W_C^* \end{bmatrix}.$$

By [3, Prop. 3.2.16], a port-Hamiltonian system is impedance energy preserving if and only if $P_0(\zeta) = -P_0(\zeta)^*$ for a.e. $\zeta \in [a, b]$ and $P_{W_B,W_C} = \Sigma$, and it is impedance passive if and only if $\mathrm{Re}\,P_0(\zeta) \le 0$ for a.e. $\zeta \in [a, b]$ and $P_{W_B,W_C} \le \Sigma$.

We consider impedance energy preserving and impedance passive port-Hamiltonian systems as they can be exponentially stabilized using output feedback. Stabilization of port-Hamiltonian systems with negative output feedback was first presented for first-order impedance energy preserving port-Hamiltonian systems in [20, Sec. IV], and we will next generalize the result for systems of arbitrary order N.

Lemma 7. *a) A port-Hamiltonian system that satisfies $W_B\Sigma W_B^* > 0$ is exponentially stable.*

b) *An impedance passive port-Hamiltonian system can be exponentially stabilized using negative output feedback $u(t) = -\kappa y(t)$ for any $\kappa > 0$.*

Proof. a) The claim can be proved similarly to [11, Lem. 2] by using the techniques utilized in the proof of [2, Lem. 9.1.4] and the estimate $\mathrm{Re}\langle Ax, x\rangle_X \le \mathrm{Re}\langle f_\partial, e_\partial\rangle_{\mathbb{C}^{nN}}$ which holds as $\mathrm{Re}\, P_0(\zeta) \le 0$ a.e. $\zeta \in [a, b]$. Eventually, we obtain

$$\mathrm{Re}\langle Ax, x\rangle_X \le -\gamma \sum_{k=0}^{N-1} \sum_{\zeta=a,b} |(\mathcal{H}x)^{(k)}(\zeta)|^2$$

for some $\gamma > 0$, which by [3, Thm. 4.3.24] is sufficient for the port-Hamiltonian system being exponentially stable.

b) Let W_B and W_C be such that the port-Hamiltonian system is impedance passive. It has been shown in [20, Sec. IV] that the closed-loop system with negative output feedback $u(t) = -\kappa y(t)$ can be written as

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), \\ (W_B + \kappa W_C) \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= (\mathcal{B} + \kappa\mathcal{C})x(t) \equiv 0, \\ \mathcal{C}x(t) &= y(t). \end{aligned}$$

By [3, Prop. 3.2.16, Lem. 3.2.18], it holds for impedance passive port-Hamiltonian systems that $W_B \Sigma W_B^* \ge 0$, $W_C \Sigma W_C^* \ge 0$ and $W_B \Sigma W_C^* = I = W_C \Sigma W_B^*$. Denote $W_\kappa := W_B + \kappa W_C$ which satisfies

$$W_\kappa \Sigma W_\kappa^* = W_B \Sigma W_B^* + 2\kappa I + \kappa^2 W_C \Sigma W_C^* \ge 2\kappa I > 0,$$

and now part a) completes the proof. □

V. Robust regulating controller for impedance passive port-Hamiltonian systems

In this section, we will construct a finite dimensional, minimal order controller for an impedance passive port-Hamiltonian system and a finite dimensional exosystem as given in (2). The choices of the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are adopted from [21, Sec. 4]. However, as an impedance passive port-Hamiltonian system is not necessarily exponentially stable to begin with, we will need to add an extra term to the controller in order to ensure the exponential stability of the closed-loop system. The controller that we will construct is of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0, \\ u(t) &= Kz(t) - \kappa e(t), \end{aligned}$$

where as opposed to the controller given in (5) we have the extra feedthrough term $-\kappa e(t)$. Here the control signal consists of two parts $u(t) = u_1(t) + u_2(t)$ where the second term contributes to exponentially stabilizing the plant and the first one provides the robust regulating control. Note that instead of $-\kappa y(t)$ we use $-\kappa e(t)$ which we will show to stabilize the plant as well. Furthermore, using $-\kappa e(t)$ simplifies the controller as $y(t)$ and $y_{ref}(t)$ are not needed separately.

We define $Z = Y^q$. The controller parameters are chosen as $\kappa > 0$ and

$$\begin{aligned} \mathcal{G}_1 &= \mathrm{diag}\,(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y) \in \mathcal{L}(Z), \\ K &= \epsilon K_0 = \epsilon \left[K_0^1, K_0^2, \ldots, K_0^q\right] \in \mathcal{L}(Z, U), \\ \mathcal{G}_2 &= (\mathcal{G}_2^k)_{k=1}^q = (-(P_\kappa(i\omega_k)K_0^k)^*)_{k=1}^q \in \mathcal{L}(Y, Z) \end{aligned}$$

where $\epsilon > 0$ is the tuning parameter and $P_\kappa(i\omega_k) = P(i\omega_k)(I + \kappa P(i\omega_k))^{-1}$ is the transfer function of the triplet $(\mathcal{A}, \mathcal{B} + \kappa\mathcal{C}, \mathcal{C})$. Note that since $P(i\omega_k)$ is assumed to be surjective for every $k \in \{1, 2, \ldots, q\}$, $P_\kappa(i\omega_k)$ is surjective as well. Further note that if we choose $K_0^k = P_\kappa(i\omega_k)^\dagger$ (the Moore-Penrose pseudoinverse of $P_\kappa(i\omega_k)$), then $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1, 2, \ldots q\}$.

Theorem 8. *Assume that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is an impedance passive port-Hamiltonian system of an arbitrary order N and (S, E, F) is a finite-dimensional exosystem such that Assumption 2 is satisfied. Then there exists an $\epsilon_\kappa^* > 0$ such that for any $0 < \epsilon < \epsilon_\kappa^*$ the controller with the above parameter choices solves the robust output regulation problem.*

Proof. Consider an input of the form $u(t) = Kz(t) - \kappa e(t) = u_1(t) - \kappa y(t) + \kappa y_{ref}(t)$. The plant with such an input can be written as

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), \\ (\mathcal{B} + \kappa\mathcal{C})x(t) &= u_1(t) + \kappa y_{ref}(t) + w(t), \\ \mathcal{C}x(t) &= y(t), \end{aligned}$$

where we also included the boundary disturbance signal $w(t)$. Note that as $w(t) = Ev(t)$ and $y_{ref}(t) = -Fv(t)$, the term $\kappa y_{ref}(t)$ can be considered an additional disturbance to the original system.

We know by Lemma 7 that the negative output feedback exponentially stabilizes the impedance passive port-Hamiltonian system, and thus, the operator $A_\kappa := \mathcal{A}|_{\mathcal{N}(\mathcal{B}+\kappa\mathcal{C})}$ generates an exponentially stable C_0-semigroup on X. Furthermore, as the stabilized plant is a boundary control and observation system, there exists an operator B_κ satisfying $(\mathcal{B} + \kappa\mathcal{C})B_\kappa = I_U$, and we can define an operator $G_\kappa := B_\kappa(E - \kappa F)$ that satisfies $(\mathcal{B} + \kappa\mathcal{C})G_\kappa = E - \kappa F$ and takes the reference signal $\kappa y_{ref}(t)$ into account.

The closed-loop system consisting of the plant and the controller is still given as in (6) with A, B and G replaced by A_κ, B_κ and G_κ, respectively, and the Λ-extension of $\mathcal{C}$ is given by $\mathcal{C}_\Lambda x = \lim_{\lambda\to\infty} \lambda\mathcal{C}R(\lambda, A_\kappa)x$. Note that since the plant is an impedance passive port-Hamiltonian system, we have by Lemma 7 that $(W_B + \kappa W_C)\Sigma(W_B + \kappa W_C)^* > 0$, and thus, by Lemma 9 presented in the Appendix the operator $\mathcal{C}$ is admissible for A_κ.

Now that the feedthrough term of the controller is associated with the plant, the remaining controller is of the standard form given in (5). Thus, we have by the proof of [21, Thm 4.1] that the controller satisfies the $\mathcal{G}$-conditions, and hence, by Theorem 4 the controller solves the robust output regulation problem, provided that the closed-loop system is exponentially stable.

To conclude the proof, we will show that the closed-loop operator A_e is similar to an exponentially stable operator and hence, exponentially stable. Choose a similarity transformation

$$Q = \begin{bmatrix} -I & \epsilon H \\ 0 & I \end{bmatrix} = Q^{-1} \in \mathcal{L}(X_e)$$

where the operator $H := (H_1, H_2, \ldots, H_q) \in \mathcal{L}(Z, \mathcal{D}(A_\kappa))$ is chosen as

$$H_k := R(i\omega_k, A_\kappa)(\mathcal{A}B_\kappa - i\omega_k B_\kappa)K_0^k$$

for all $k \in \{1, 2, \dots, q\}$. Let us define $\hat{A}_e := QA_eQ^{-1}$. We will next show that $\hat{A}_e$ is exponentially stable, which implies that A_e is exponentially stable as well.

By the choices of H_k, we have $(i\omega_k - A_\kappa)H_k = \mathcal{A}B_\kappa K_0^k - i\omega_k B_\kappa K_0^k$, i.e., $H_k i\omega_k = A_\kappa H_k + \mathcal{A}B_\kappa K_0^k - B_\kappa K_0^k i\omega_k$, and thus, $H\mathcal{G}_1 = A_\kappa H + \mathcal{A}B_\kappa K_0 - B_\kappa K_0 \mathcal{G}_1$ due to the diagonal structure of $\mathcal{G}_1$. Furthermore,

$$\begin{aligned}\mathcal{C}_\Lambda(H_k + B_\kappa K_0^k) =& \mathcal{C}_\Lambda R(i\omega_k, A_\kappa)(\mathcal{A}B_\kappa - i\omega_k B_\kappa)K_0^k \\ &+ \mathcal{C}_\Lambda B_\kappa K_0^k = P_\kappa(i\omega_k)K_0^k,\end{aligned}$$

and thus, $\mathcal{C}_\Lambda(H + B_\kappa K_0) = -\mathcal{G}_2^*$. Using the above identities $\hat{A}_e$ can be written as

$$\begin{aligned}\hat{A}_e = &\begin{bmatrix} A_\kappa - \epsilon(H + B_\kappa K_0)\mathcal{G}_2\mathcal{C}_\Lambda & 0 \\ -\mathcal{G}_2\mathcal{C}_\Lambda & \mathcal{G}_1 - \epsilon\mathcal{G}_2\mathcal{G}_2^* \end{bmatrix} \\ &+ \epsilon^2 \begin{bmatrix} 0 & -(H + B_\kappa K_0)\mathcal{G}_2\mathcal{G}_2^* \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Since $\mathcal{C}$ is admissible for A_κ and $(H+B_\kappa K_0)\mathcal{G}_2$ is bounded, there exists an $\epsilon_\kappa > 0$ such that for all $0 < \epsilon < \epsilon_\kappa$ the operator $A_\kappa - \epsilon(H + B_\kappa K_0)\mathcal{G}_2\mathcal{C}_\Lambda$ generates an exponentially stable semigroup. Furthermore, we have by [22, App. B] that the semigroup generated by $\mathcal{G}_1 - \epsilon\mathcal{G}_2\mathcal{G}_2^*$ is exponentially stable for every $\epsilon > 0$ and that there exists a constant $M > 0$ such that $||R(\lambda, \mathcal{G}_1 - \epsilon\mathcal{G}_2\mathcal{G}_2^*)|| \le M/\epsilon$ for $\lambda \in \mathbb{C}_+$. Consider the operator $\hat{A}_e$ in the form $A_1 + \epsilon^2 A_2$. Using the above upper bound for $||R(\lambda, \mathcal{G}_1 - \epsilon\mathcal{G}_2\mathcal{G}_2^*)||$ it can be shown that there exists an ϵ^* such that for all $0 < \epsilon < \epsilon^*$ and $\lambda \in \mathbb{C}_+$ we have $\left|\left|\epsilon^2 A_2 R(\lambda, A_1)\right|\right| < 1$. Thus, it follows that there exists an ϵ_κ^* such that for all $0 < \epsilon < \epsilon_\kappa^*$ the resolvent of $\hat{A}_e$ is bounded in the right half plane, i.e., $\hat{A}_e$ is exponentially stable.

Since the controller satisfies the $\mathcal{G}$-conditions and the closed-loop system is exponentially stable for every $0 < \epsilon < \epsilon_\kappa^*$, we have by Theorem 4 that the controller solves the robust output regulation problem for any $0 < \epsilon < \epsilon_\kappa^*$. □

VI. Robust control of a 1D Euler-Bernoulli beam

In this section, we construct a robust controller for Euler-Bernoulli beam which is an example of a port-Hamiltonian system of order two. The formulation of Euler-Bernoulli beam as a port-Hamiltonian system is adopted from [3, Ex. 3.1.6].

The Euler-Bernoulli beam equation is given on the spatial interval $\zeta \in [0, 1]$ by

$$\rho(\zeta)\frac{\partial^2}{\partial t^2}\nu(\zeta, t) = -\frac{\partial^2}{\partial \zeta^2}\left(EI(\zeta)\frac{\partial^2}{\partial \zeta^2}\nu(\zeta, t)\right)$$

where $\nu(\zeta, t)$ denotes the displacement at position ζ at time t, $\rho(\zeta)$ is the mass density times the cross sectional area, $E(\zeta)$ is the modulus of elasticity and $I(\zeta)$ is the area moment of the cross section. Due to their physical interpretations, the functions ρ, E and I are uniformly bounded and strictly positive for all $\zeta \in [0, 1]$.

In order to write the Euler-Bernoulli beam equation as a port-Hamiltonian system, let us define the state $x(\zeta, t)$ by

$$x(\zeta, t) = \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} := \begin{bmatrix} \rho\nu_t(\zeta, t) \\ \nu_{\zeta\zeta}(\zeta, t) \end{bmatrix}.$$

Now we can write the equation as $\partial_t x(\zeta, t) = \mathcal{A}x(\zeta, t)$ where

$$\mathcal{A}x(\zeta, t) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2}{\partial \zeta^2}\left(\begin{bmatrix} \rho(\zeta)^{-1} & 0 \\ 0 & EI(\zeta) \end{bmatrix} x(\zeta, t)\right),$$

which is a second-order port-Hamiltonian operator with $P_0 = P_1 = 0$,

$$P_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{H}(\zeta) = \begin{bmatrix} \rho(\zeta)^{-1} & 0 \\ 0 & EI(\zeta) \end{bmatrix}.$$

Using the new state variables, define the control and observation operators by $\mathcal{B}x(\cdot, t) := [x_1'(0, t), x_1(0, t), x_2'(1, t), x_2(1, t)]^T$ and $\mathcal{C}x(\cdot, t) := [-x_2(0, t), x_2'(0, t), -x_1(1, t), x_1'(1, t)]^T$, from which it can be seen that the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is an impedance energy preserving port-Hamiltonian system.

Let the reference signal y_{ref} and the disturbance signal d be given by

$$y_{ref}(t) := \begin{bmatrix} -\sin(\pi t) \\ -\cos(2\pi t) \\ \cos(\pi t) \\ \sin(2\pi t) \end{bmatrix} \quad \text{and} \quad d(t) := \begin{bmatrix} \sin(2\pi t) \\ \cos(\pi t) \\ \cos(2\pi t) \\ \sin(\pi t) \end{bmatrix},$$

so that we have $S := \operatorname{diag}(-2i\pi, -i\pi, i\pi, 2i\pi)$, and E and F are suitably chosen matrices.

The controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ are chosen according to the previous section, i.e., we choose

$$\begin{aligned}&\kappa = 1, \quad \epsilon = 0.17, \\ &\mathcal{G}_1 = \operatorname{diag}\left(-2i\pi I_Y, -i\pi I_Y, i\pi I_Y, 2i\pi I_Y\right), \\ &\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=1}^4, \quad \mathcal{G}_2^k = -I_Y \quad \forall k \in \{1, 2, 3, 4\}, \\ &K = \epsilon\left[P_\kappa(-2i\pi)^{-1}, P_\kappa(-i\pi)^{-1}, P_\kappa(i\pi)^{-1}, P_\kappa(2i\pi)^{-1}\right],\end{aligned}$$

where P_κ is the transfer function of the triplet $(\mathcal{A}, \mathcal{B} + \kappa\mathcal{C}, \mathcal{C})$ and ϵ is chosen such that the growth bound of the closed-loop system is close to its minimum. Note that as we chose $K_0^k = P_\kappa(i\omega_k)^{-1}$, each block of $\mathcal{G}_2$ is equal to $-I_Y$.

Figure 1 shows the numerical simulation of the Euler-Bernoulli beam with initial conditions $v_0 = 1$, $\xi_0 = 0$ and $z_0 = 0$. It can be seen that the regulation error diminishes very rapidly. In the simulation the spatial derivatives were approximated by finite differences with grid size 0.05.

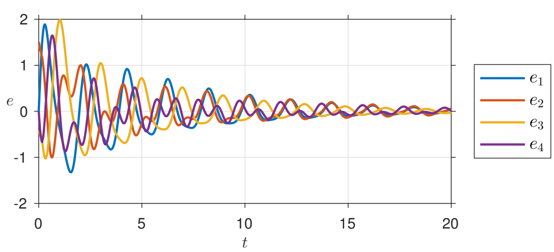

Figure 1. Regulation error on $t \in [0, 20]$.

VII. Conclusions

We considered robust regulation of impedance passive port-Hamiltonian systems of arbitrary order and showed that a controller satisfying the $\mathcal{G}$-conditions is robust. The robustness result not only holds for impedance passive port-Hamiltonian systems but for any boundary control and observation system satisfying Assumption 2. We also presented a simple, minimal order controller structure that satisfies the $\mathcal{G}$-conditions and showed that it stabilizes the closed-loop system, thus solving the robust output regulation problem. The theory was illustrated with an example where we implemented such a controller for a one-dimensional Euler-Bernoulli beam with boundary controls and boundary observations.

APPENDIX

Lemma 9. *Consider a port-Hamiltonian system $(\mathcal{A},\mathcal{B},\mathcal{C})$ as in Definition 6 and assume that the operator $\mathcal{B}$ is such that $W_B\Sigma W_B^* > 0$. Then the observation operator $\mathcal{C}$ is infinite-time admissible for the semigroup T_A generated by $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$.*

Proof. Consider the classical solution $x(t) = T_A(t)x_0$ of $\dot{x}(t) = Ax(t)$, $x(0) = x_0 \in \mathcal{D}(A)$ and recall the estimate that was mentioned in the proof of Lemma 7:

$$\mathrm{Re}\langle Ax, x\rangle_X \leq \mathrm{Re}\langle f_\partial, e_\partial\rangle_{\mathbb{C}^{nN}}. \tag{11}$$

Since $x \in \mathcal{D}(A)$, we have that $\mathcal{B}x = 0$, i.e., $(f_\partial, e_\partial)^T \in \mathcal{N}(W_B)$. As $W_B\Sigma W_B^* > 0$, [1, Lem. A.1] implies that we may write $W_B = S[I+V_B \quad I-V_B]$ where S is invertible and V_B is square satisfying $V_B^*V_B < I$. Furthermore, as $(f_\partial, e_\partial)^T \in \mathcal{N}(W_B)$, by [1, Lem. A.2] we may write

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I - V_B \\ -I - V_B \end{bmatrix} \ell \tag{12}$$

for some $\ell \in \mathbb{C}^{nN}$. Let us define the output as $y = \mathcal{C}x$ and write $W_C = [C_1, C_2]$ with $C_{1,2}$ square. We have

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} 0 \\ C_1(I - V_B) - C_2(I + V_B) \end{bmatrix} \ell$$

for some $\ell \in \mathbb{C}^{nN}$. Since $\mathcal{N}(W_B) \cap \mathcal{N}(W_C) = \{0\}$, it follows from the above that the square matrix $R := C_1(I - V_B) - C_2(I + V_B)$ is invertible. Now using the estimate (11) together with (12) we obtain

$$\begin{aligned} \frac{d}{dt}||x(t)||_X^2 &= 2\,\mathrm{Re}\langle Ax, x\rangle_X \leq 2\,\mathrm{Re}\langle f_\partial, e_\partial\rangle_{\mathbb{C}^{nN}}. \\ &= \ell^*(-2I + 2V_B^*V_B)\ell \\ &= y^*R^{-*}(-2I + 2V_B^*V_B)R^{-1}y \\ &\leq -m||y||_{\mathbb{C}^{nN}}^2, \end{aligned}$$

for some $m > 0$ as $V_B^*V_B < I$. Integrating both sides over $[0, \tau]$ and using $y(t) = \mathcal{C}T_A(t)x_0$ yields

$$||x(\tau)||_X^2 - ||x_0||_X^2 \leq -m \int_0^\tau ||\mathcal{C}T_A(t)x_0||_{\mathbb{C}^{nN}}^2 dt.$$

Letting $\tau \to \infty$, we have $||x(\tau)||_X^2 \to 0$ as T_A is exponentially stable by part a) of Lemma 7, and we obtain

$$\int_0^\infty ||\mathcal{C}T_A(t)x_0||_{\mathbb{C}^{nN}}^2 dt \leq \frac{1}{m}||x_0||_X^2,$$

which concludes the proof. □

REFERENCES

[1] Y. L. Gorrec, H. Zwart, and B. Maschke, “Dirac structures and boundary control systems associated with skew-symmetric differential operators,” *SIAM J. Control Optim.*, vol. 44, no. 5, pp. 1864–1892, 2005.

[2] B. Jacob and H. Zwart, *Linear Port-Hamiltonian Systems on Infinite dimensional Spaces*, ser. Operator Theory: Advances and Applications. Birkhäuser, 2012, vol. 223.

[3] B. Augner, “Stabilization of infinite-dimensional port-Hamiltonian systems via dissipative boundary feedback,” Ph.D. dissertation, Bergische Universität Wuppertal, 2016, http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20160719-090307-4.

[4] E. J. Davison, “Multivariable tuning regulators: The feedforward and robust control of a general servomechanism problem,” *Proc. CDC'75*, 1975.

[5] T. Hämäläinen and S. Pohjolainen, “A finite-dimensional robust controller for systems in the CD-algebra,” *IEEE Trans. Automat. Control*, vol. 45, no. 3, pp. 421–431, 2000.

[6] ——, “Robust regulation for exponentially stable boundary control systems in Hilbert space,” *Proc. MMAR'02, Szczecin, Poland, September 2-5*, 2002.

[7] ——, “Robust regulation of distributed parameter systems with infinite-dimensional exosystems,” *SIAM J. Control Optim.*, vol. 48, no. 8, pp. 4846–4873, 2010.

[8] L. Paunonen and S. Pohjolainen, “Internal model theory for distributed parameter systems,” *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4753–4775, 2010.

[9] ——, “The internal model principle for systems with unbounded control and observation,” *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3967–4000, 2014.

[10] J.-P. Humaloja, L. Paunonen, and S. Pohjolainen, “Robust regulation for first-order port-Hamiltonian systems,” *Proc. EEC'16, Aalborg, Denmark, June 29th - July 1st*, 2016.

[11] ——, “Robust regulation for port-Hamiltonian systems of even order,” *Proc. MTNS'16, MN, USA, July 12-15*, 2016.

[12] B. Augner and B. Jacob, “Stability and stabilization of infinite-dimensional linear port-Hamiltonian systems,” *Evolution Equations and Control Theory*, vol. 3, no. 2, pp. 207–229, 2014.

[13] A. Macchelli, Y. L. Gorrec, H. Ramirez, and H. Zwart, “On the synthesis of boundary control laws for distributed parameter port-Hamiltonian systems,” *IEEE Trans. Automat. Control*, vol. 62, no. 4, pp. 1700–1713, 2017.

[14] H. Ramirez, Y. L. Gorrec, A. Macchelli, and H. Zwart, “Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback,” *IEEE Trans. Automat. Control*, vol. 59, no. 10, pp. 2849–2855, 2014.

[15] J. Villegas, H. Zwart, Y. L. Gorrec, and B. Maschke, “Exponential stability of a class of boundary control systems,” *IEEE Trans. Automat. Control*, vol. 54, no. 1, pp. 142–147, 2009.

[16] R. Rebarber and G. Weiss, “Internal model based tracking and disturbance rejection for stable well-posed systems,” *Automatica*, vol. 39, pp. 1555–1569, 2003.

[17] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, ser. Advanced Texts. Birkhäuser Verlag AG, 2009.

[18] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, ser. Texts in Applied Mathematics. Springer-Verlag, 1995, vol. 21.

[19] V. Q. Phông, “The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems,” *Math. Z.*, vol. 208, pp. 467–588, 1991.

[20] J. Villegas, H. Zwart, Y. L. Gorrec, B. Maschke, and A. van der Schaft, “Stability and stabilization of a class of boundary control systems,” *Proc. CDC'05, Seville, Spain, December 12-15*, 2005.

[21] L. Paunonen, “Controller design for robust output regulation of regular linear systems,” *IEEE Trans. Automat. Control*, vol. 61, no. 10, pp. 2974–2986, 2016.

[22] T. Hämäläinen and S. Pohjolainen, “A self-tuning robust regulator for infinite-dimensional systems,” *IEEE Trans. Automat. Control*, vol. 56, no. 9, pp. 2116–2127, 2011.

PUBLICATION III

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Approximate robust output regulation of boundary control systems

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***Abstract*—We extend the internal model principle for systems with boundary control and boundary observation, and construct a robust controller for this class of systems. However, as a consequence of the internal model principle, any robust controller for a plant with infinite-dimensional output space necessarily has infinite-dimensional state space. We proceed to formulate the approximate robust output regulation problem and present a finite-dimensional controller structure to solve it. Our main motivating example is a wave equation on a bounded multidimensional spatial domain with force control and velocity observation at the boundary. In order to illustrate the theoretical results, we construct an approximate robust controller for the wave equation on an annular domain and demonstrate its performance with numerical simulations.**

***Index Terms*—Robust control, Distributed parameter systems, Linear systems, Controlled wave equation**

I. INTRODUCTION

Intuitively speaking, the problem of output regulation of a given plant amounts to designing an output feedback controller which stabilizes the plant, and in addition the output of the plant tracks a given reference signal in spite of a given disturbance signal. If a single controller solves the output regulation problem for the plant and also for small perturbations of the plant, and for more or less arbitrary reference and disturbance signals, then the controller is said to solve the *robust* output regulation problem. See the beginning of §IV for exact definitions of these concepts.

Output tracking and disturbance rejection have been studied actively in the literature for distributed parameter systems with bounded control and observation operators [1], [2], [3], [4], [5] and robust controllers have been constructed for classes of systems with unbounded control and observation operators, such as well-posed [6] and regular [7] systems, in [8], [9], [10], [11]. The key in designing robust controllers is the *internal model principle* which in its classical form states that a controller can solve the robust output regulation problem only if it contains p copies of the dynamics of the exosystem, where p is the dimension of the output space of the plant. The internal model principle was first presented for finite-dimensional linear plants by Francis and Wonham [12] and Davison [13]. The principle was later generalized to infinite-dimensional linear systems in [11], [14], [15] under the assumption that the plant is regular.

In this paper, we focus on output regulation for boundary controlled systems with boundary observation. Our motivating example is a wave equation on a multidimensional spatial domain, with force control and velocity observation on a part of the boundary. This n-D wave system is challenging from the robust control point of view since it is neither regular nor well-posed. Moreover, the output space of the wave system is infinite-dimensional and then the *internal model principle* implies that any robust controller must also be infinite-dimensional. However, as the main contribution of this paper, we demonstrate that it is possible to achieve *approximate* tracking of the reference signal in the sense that the difference between the output and the reference signal becomes small as $t \to \infty$. More precisely, we introduce a new finite-dimensional controller that solves the robust output regulation problem in this approximate sense, hence extending the recent results of [16] to continuous time. At the same time, we extend the class of reference signals that can be tracked. As a part of the construction of this controller, we present an upper bound for the regulation error.

The second main result of this paper is a generalization of the internal model principle presented in [14], [15] to boundary control systems that are not necessarily regular linear systems. The sufficiency of the internal model for achieving robust control has been presented in [17], albeit here our formulation is more general in terms of boundary controls and disturbances. The necessity of the internal model is a new result for boundary control systems.

As our third main contribution we characterize and construct a minimal finite dimensional controller to solve the output regulation problem. Due to the reduced size of the controller, it does not have any guaranteed robustness properties. The controller concept was presented for regular linear systems in [11], and here we will generalize such controllers for boundary control systems.

In §II, we present the wave equation and show how it fits into the abstract framework of the later sections. In §III, we present the abstract plant, the exosystem and the controller (which is to be constructed), and reformulate the interconnection of these three systems as a regular input/state/output system. In §IV, we present the output regulation, the robust output regulation and the approximate robust output regulation problems, and present controller structures to solve them. A regulating controller without the robustness requirement is presented in §IV-A, and an approximate robust regulating controller is presented in §IV-C. In §IV-B, we present the

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internal model principle for boundary control systems, following which we present a precise robust regulating controller in §IV-D. In §V, we construct an approximate robust regulating controller for the wave equation on an annular domain and demonstrate its performance with numerical simulation. The paper is concluded in §VI.

Here $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, kernel, spectrum and resolvent of a linear operator A are denoted by $\mathcal{D}(A), \mathcal{R}(A), \mathcal{N}(A), \sigma(A)$ and $\rho(A)$, respectively. The right pseudoinverse of a surjective operator P is denoted by $P^{[-1]}$.

II. The wave equation

In this section, we describe the example which motivates the robust output regulation theory in this paper, a wave equation (the plant) on a bounded domain $\Omega \subset \mathbb{R}^n$ with force control and velocity observation at a part of the boundary. We try to keep the exposition brief; more details can be found in [18], [19], [20].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (an open connected set) with a Lipschitz-continuous boundary $\partial\Omega$ split into two parts Γ_0, Γ_1 such that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\partial\Gamma_0, \partial\Gamma_1$ both have surface measure zero. We consider the wave equation

$$\begin{cases} \rho(\zeta)\dfrac{\partial^2 w}{\partial t^2}(\zeta,t) = \operatorname{div}\big(T(\zeta)\,\nabla w\,(\zeta,t)\big), \quad \zeta\in\Omega, \\ u(\zeta,t) = \nu\cdot T(\zeta)\,\nabla w(\zeta,t), \quad \zeta\in\Gamma_1, \\ y(\zeta,t) = \dfrac{\partial w}{\partial t}(\zeta,t), \qquad \zeta\in\Gamma_1, \\ 0 = \dfrac{\partial w}{\partial t}(\zeta,t), \qquad \zeta\in\Gamma_0,\ t>0 \\ w(\cdot,0)=w_0, \qquad \dfrac{\partial w}{\partial t}(\cdot,0) = w_1, \end{cases} \tag{II.1}$$

where $w(\zeta,t)$ is the displacement from the equilibrium at the point $\zeta\in\Omega$ and time $t\ge 0$, $\rho(\cdot)$ is the mass density, $T^*(\cdot) = T(\cdot) \in L^2(\Omega;\mathbb{R}^n)$ is Young's modulus and $\nu \in L^\infty(\partial\Omega;\mathbb{R}^n)$ is the unit outward normal at $\partial\Omega$. We require $\rho(\cdot)$ and $T(\cdot)$ to be essentially bounded from both above and below away from zero. Please note that the input u is the force perpendicular to Γ_1 and the output y is the velocity at Γ_1 while waves are reflected at the part Γ_0 of the boundary where the displacement is constant.

In order to solve the robust output regulation problem for the wave system, we shall need to stabilize (II.1) exponentially using a viscous damper on Γ_1, which corresponds to the output feedback

$$u(\zeta,t) = -b^2(\zeta)\,y(\zeta,t), \qquad \zeta\in\Gamma_1,\ t\ge 0.$$

This requires that we make some additional assumptions solely for the purpose of obtaining exponential stability (see §II-B below for more details). Additionally, to prove later on that the velocity observation on Γ_1 is admissible, we assume that

$$\delta := \inf_{\zeta\in\Gamma_1} b(\zeta)^2 > 0. \tag{II.2}$$

A. The wave equation as a formal boundary control system

Our first step is to show that the wave equation on a bounded domain in $\mathbb{R}^n$ can be written as a boundary control system (BCS) in the sense of [21]. To this end, we first write the wave equation

$$\rho(\zeta)\,\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \operatorname{div}\big(T(\zeta)\,\nabla w(\zeta,t)\big) \qquad \text{on} \quad \Omega\times\mathbb{R}_+$$

in the first-order form (as an equality in $L^2(\Omega)^{n+1}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\rho(\cdot)\,\dot w(\cdot,t)\\ \nabla w(\cdot,t)\end{bmatrix} = \begin{bmatrix}0 & \operatorname{div}\\ \nabla & 0\end{bmatrix}\begin{bmatrix}1/\rho(\cdot) & 0\\ 0 & T(\cdot)\end{bmatrix}\begin{bmatrix}\rho(\cdot)\,\dot w(\cdot,t)\\ \nabla w(\cdot,t)\end{bmatrix}, \tag{II.3}$$

where div denotes the (distribution) divergence operator and ∇ is the (distribution) gradient. Hence, the state at any time is the pair of momentum and strain densities on Ω.

Under the standing assumptions on ρ and T, the operator of multiplication by $\mathcal{H} := \left[\begin{smallmatrix}1/\rho(\cdot) & 0\\ 0 & T(\cdot)\end{smallmatrix}\right]$ defines an inner product on $L^2(\Omega)^{n+1}$ via

$$\langle x,z\rangle_{\mathcal{H}} := \langle \mathcal{H}x, z\rangle_{L^2(\Omega)^{n+1}}$$

and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is equivalent to $\langle\cdot,\cdot\rangle_{L^2(\Omega)^{n+1}}$. The space $L^2(\Omega)^{n+1}$ equipped with this equivalent inner product is denoted by $X_{\mathcal{H}}$ and will be used as the state space of the plant.

We next introduce some function spaces for the wave equation. The notation $H^1(\Omega)$ stands for the Sobolev space of all elements of $L^2(\Omega)$ whose distribution gradient lies in $L^2(\Omega)^n$ and $H^1(\Omega)$ is equipped with the graph norm of the gradient. Similarly $H^{\mathrm{div}}(\Omega)$ is the space of all elements of $L^2(\Omega)^n$ whose distribution divergence lies in $L^2(\Omega)$, equipped with the graph norm of div. In order for (II.3) to make sense as an equation in $L^2(\Omega)^{n+1}$, we need for every fixed $t\ge 0$ that $\dot w(\cdot,t)\in H^1(\Omega)$, $\nabla w(\cdot,t)\in L^2(\Omega)$, and $T(\cdot)\,\nabla w(\cdot,t)\in H^{\mathrm{div}}(\Omega)$, or equivalently

$$\begin{bmatrix}\rho\,\dot w(t)\\ \nabla w(t)\end{bmatrix} \in \mathcal{H}^{-1}\begin{bmatrix}H^1(\Omega)\\ H^{\mathrm{div}}(\Omega)\end{bmatrix}, \qquad t\ge 0.$$

If $\Gamma_0=\emptyset$, then the output y lives in the fractional-order space $H^{1/2}(\partial\Omega)$ on the boundary of Ω (see, e.g., [19, §13.5] or [20]). This space is important to us also when $\Gamma_0\neq\emptyset$, because the *Dirichlet trace* γ_0 maps $H^1(\Omega)$ continuously onto $H^{1/2}(\partial\Omega)$. Indeed, we set

$$\mathcal{W} := \left\{ w\in H^{1/2}(\partial\Omega) \;\middle|\; w\big|_{\Gamma_0} = 0\right\} \qquad \text{with}$$
$$\|w\|_{\mathcal{W}} := \left\|\gamma_0^{[-1]}\, w\right\|_{H^1(\Omega)},$$

where $|$ denotes the restriction to a given subdomain in the appropriate sense and

$$\gamma_0^{[-1]} := \gamma_0\big|_{\mathcal{N}(\gamma_0)^\perp}^{-1} \in \mathcal{L}\big(H^{1/2}(\partial\Omega); H^1(\Omega)\big).$$

Moreover, we introduce

$$H^1_{\Gamma_0}(\Omega) := \left\{ g\in H^1(\Omega) \mid g\big|_{\Gamma_0} = 0\right\},$$

with the norm inherited from $H^1(\Omega)$. This setup makes both $\mathcal{W}$ and $H^1_{\Gamma_0}(\Omega)$ Hilbert spaces; indeed, $H^{1/2}(\partial\Omega)$ is continuously embedded into $L^2(\partial\Omega)$ by [19, (13.5.3)], and so $H^1_{\Gamma_0}(\Omega)$ is the kernel of $P_{\Gamma_0}\gamma_0 \in \mathcal{L}\big(H^1(\Omega), L^2(\partial\Omega)\big)$, where

P_{Γ_0} is the orthogonal projection onto $L^2(\Gamma_0)$ in $L^2(\partial\Omega)$. This proves that $H^1_{\Gamma_0}(\Omega)$ is a Hilbert space, and moreover, γ_0 maps the Hilbert space $H^1_{\Gamma_0}(\Omega) \ominus \mathcal{N}(\gamma_0)$ unitarily onto $\mathcal{W}$ which is then also complete.

The embedding $\iota : \mathcal{W} \to L^2(\Gamma_1)$ is continuous, because $\iota = P_{\Gamma_1}\widetilde{\iota}\gamma_0\gamma_0^{[-1]}$, where $\widetilde{\iota}$ is the continuous embedding of $H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$. The embedding is also dense by [19, Thm 13.6.10], so that we may define $\mathcal{W}'$ as the dual of $\mathcal{W}$ with pivot space $L^2(\Gamma_1)$ (see [19, §2.9]). Then in particular

$$\langle \omega, w\rangle_{\mathcal{W}',\mathcal{W}} = \langle \omega, w\rangle_{L^2(\Gamma_1)}, \qquad \omega \in L^2(\Gamma_1),\, w \in \mathcal{W}.$$

Thm 1.8 in Appendix 1 of [18] states that the *restricted normal trace* $\gamma_\perp h := (\nu\cdot\gamma_0 h)\big|_{\Gamma_1}$, $h \in H^1(\Omega)^n$, has a unique extension to a continuous operator (still denoted by $\gamma_\perp$) that maps $H^{\mathrm{div}}(\Omega)$ *onto* $\mathcal{W}'$. Please note that $\gamma_\perp$ is *not* the Neumann trace γ_N: If $\Gamma_0 = \emptyset$, then $\mathcal{W} = H^{1/2}(\partial\Omega)$ and the relation between the two operators is $\gamma_N x = \gamma_\perp \nabla x$, for a sufficiently regular x, where the equality is in $H^{-1/2}(\partial\Omega)$. The space $H^{-1/2}(\partial\Omega)$ equals $\mathcal{W}'$ in the case where $\Gamma_0 = \emptyset$ (which is not the main case of interest to us, see (II.6) below).

Now we include the boundary condition at Γ_0 into the domain of $\left[\begin{smallmatrix} 0 & \mathrm{div} \\ \nabla & 0\end{smallmatrix}\right]\mathcal{H}$, see (II.3), by requiring that $\dot w \in H^1_{\Gamma_0}(\Omega)$ instead of the weaker $\dot w \in H^1(\Omega)$ which we motivated above. We can then write (II.1) as

$$\begin{cases} \dot x(t) = \mathfrak{A}\mathcal{H}x(t), \\ u(t) = \mathfrak{B}\mathcal{H}x(t), \quad t \geq 0, \qquad x(0) = \begin{bmatrix} \rho\, w_0' \\ \nabla w_0 \end{bmatrix}, \\ y(t) = \mathfrak{C}\mathcal{H}x(t), \end{cases} \tag{II.4}$$

where $x(t) = \left[\begin{smallmatrix} \rho\,\dot w(t) \\ \nabla w(t)\end{smallmatrix}\right]$ is the state at time t, $\mathfrak{A} = \left[\begin{smallmatrix} 0 & \mathrm{div} \\ \nabla & 0\end{smallmatrix}\right]$, $\mathfrak{B} = \begin{bmatrix} 0 & \gamma_\perp\end{bmatrix}$, and $\mathfrak{C} = \begin{bmatrix}\gamma_0 & 0\end{bmatrix}$, with domains

$$\mathcal{D}(\mathfrak{A}) := \mathcal{D}(\mathfrak{B}) := \mathcal{D}(\mathfrak{C}) := \begin{bmatrix} H^1_{\Gamma_0}(\Omega) \\ H^{\mathrm{div}}(\Omega)\end{bmatrix} \subset X_{\mathcal{H}},$$

which is Hilbert when equipped with the graph norm of $\mathfrak{A}$. Here $X_\mathcal{H}$ is the state space, $U = \mathcal{W}'$ the input space, and $Y = \mathcal{W}$ the output space.

In [18, Thm 3.2] it was shown that (II.4) has the structure of a *boundary triplet* (or abstract *boundary space* in the original terminology of [22, §3.1.4]). This easily implies that the undamped wave equation is a boundary control system in the sense of Curtain and Zwart [21, Def. 3.3.2]:

Definition II.1. Let the *state space* X and *input space* U be Hilbert spaces, and let $\mathcal{A} : X \supset \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B} : X \supset \mathcal{D}(\mathcal{B}) \to U$ be linear operators with $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$.

The control system $\dot x(t) = \mathcal{A}\,x(t)$, $\mathcal{B}\,x(t) = u(t)$, $t \geq 0$, $x(0) = x_0$, is called a *boundary control system (BCS)* if the following conditions are met:

1) The operator $A := \mathcal{A}\big|_{\mathcal{D}(A)}$ with domain
$$\mathcal{D}(A) := \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$$
generates a C_0-semigroup on X and
2) there exists a $B \in \mathcal{L}(U,X)$ such that $BU \subset \mathcal{D}(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(U,X)$, and $\mathcal{B}B = I_U$.

An output equation may be added to the BCS by setting $y(t) = \mathcal{C}x(t)$, where $\mathcal{C}$ is a linear operator defined on $\mathcal{D}(\mathcal{C}) \supset \mathcal{D}(\mathcal{A})$ and mapping into some Hilbert *output space* Y, with the additional property that $\mathcal{C}B \in \mathcal{L}(U,Y)$. We shall briefly say that $(\mathcal{B},\mathcal{A},\mathcal{C})$ is a BCS on (U,X,Y) if all of the above conditions are met. Finally, we say that a BCS on (U,X,Y) is *(impedance) passive* if the input space U can be identified with the dual Y' of the output space and

$$\mathrm{Re}\,\langle \mathcal{A}x, x\rangle_X \leq \mathrm{Re}\,\langle \mathcal{B}x, \mathcal{C}x\rangle_{Y',Y}, \qquad x \in \mathcal{D}(\mathcal{A}).$$

For more information on abstract passive BCS, we refer to [23], [24]. Unlike the setting of Malinen and Staffans, the original definition of Curtain and Zwart does not consider the observation operator $\mathcal{C}$ or passivity, and it is not assumed that $\mathcal{D}(\mathcal{A})$ is a Hilbert space. The robust output regulation theory presented in §IV below is formulated for the general, abstract systems in Definition II.1.

We now return to the particular case of the wave equation (II.4). However, later we shall need to use $L^2(\Gamma_1)$ as both input and output space rather than $\mathcal{W}'$ and $\mathcal{W}$. Fortunately, this can be achieved by restricting $(\mathfrak{B}\mathcal{H}, \mathfrak{A}\mathcal{H}, \mathfrak{C}\mathcal{H})$: Choose the new input space as $\mathcal{U} := L^2(\Gamma_1)$ and set

$$\mathcal{D}(\widetilde{\mathfrak{A}}) = \left\{x \in \mathcal{H}^{-1}\mathcal{D}(\mathfrak{A}) \;\middle|\; \mathfrak{B}\mathcal{H}x \in L^2(\Gamma_1)\right\}$$

with the norm given by

$$\|x\|^2_{\mathcal{D}(\widetilde{\mathfrak{A}})} := \|\mathcal{H}x\|^2_{X_\mathcal{H}} + \|\mathcal{A}\mathcal{H}x\|^2_{X_\mathcal{H}} + \|\mathcal{B}\mathcal{H}x\|^2_{\mathcal{U}}.$$

Furthermore, we define the restrictions

$$\widetilde{\mathfrak{A}} := \mathfrak{A}\mathcal{H}\big|_{\mathcal{D}(\widetilde{\mathfrak{A}})},\ \widetilde{\mathfrak{B}} := \mathfrak{B}\mathcal{H}\big|_{\mathcal{D}(\widetilde{\mathfrak{A}})},\ \widetilde{\mathfrak{C}} := \iota\mathfrak{C}\mathcal{H}\big|_{\mathcal{D}(\widetilde{\mathfrak{A}})},$$

where $\iota : \mathcal{W} \to L^2(\Gamma_1)$ is again the (continuous) injection.

Theorem II.2. *The triple* $(\widetilde{\mathfrak{B}}, \widetilde{\mathfrak{A}}, \widetilde{\mathfrak{C}})$ *is a passive BCS on* $(\mathcal{U}, X_\mathcal{H}, \mathcal{U})$.

Proof. We first note that $\mathcal{N}(\widetilde{\mathfrak{B}}) = \mathcal{H}^{-1}\mathcal{N}(\mathfrak{B}) \subset \mathcal{D}(\widetilde{\mathfrak{A}})$ and then we prove that $\widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}})}$ generates a unitary group on $X_\mathcal{H}$. It follows from [18, Cor. 3.4] that $\mathfrak{A}\big|_{\mathcal{N}(\mathfrak{B})}$ is a skew-adjoint, unbounded operator on $L^2(\Omega)^{n+1}$ and we will show that this implies that $\widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}})}$ is skew-adjoint on $X_\mathcal{H}$. Indeed, for an arbitrary fixed $z \in X_\mathcal{H}$, there exists $w \in X_\mathcal{H}$ such that for all $x \in \mathcal{N}(\widetilde{\mathfrak{B}}) = \mathcal{H}^{-1}\mathcal{N}(\mathfrak{B})$ we have

$$\langle x, w\rangle_{X_\mathcal{H}} = \left\langle \widetilde{\mathfrak{A}}x, z\right\rangle_{X_\mathcal{H}} = \langle \mathfrak{A}\mathcal{H}x, \mathcal{H}z\rangle_{L^2(\Omega)^{n+1}} \tag{II.5}$$

if and only if $\mathcal{H}z \in \mathcal{D}(\mathfrak{A}\big|^*_{\mathcal{N}(\mathfrak{B})}) = \mathcal{N}(\mathfrak{B})$, where the adjoint is computed with respect to the inner product in $L^2(\Omega)^{n+1}$. Hence, $\widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}})}$ has the same domain as its adjoint with respect to $X_\mathcal{H}$, and for every z in this common domain, (II.5) can be continued as

$$\left\langle \widetilde{\mathfrak{A}}x, z\right\rangle_{X_\mathcal{H}} = \langle \mathcal{H}x, -\mathfrak{A}\mathcal{H}z\rangle_{L^2(\Omega)^{n+1}} = \left\langle x, -\widetilde{\mathfrak{A}}z\right\rangle_{X_\mathcal{H}},$$

for all $x \in \mathcal{N}(\widetilde{\mathfrak{B}})$. By Stone's theorem, $\widetilde{\mathfrak{A}}$ generates a unitary group on $X_\mathcal{H}$.

As $\gamma_\perp$ maps $H^{\mathrm{div}}(\Omega)$ onto $\mathcal{W}'$, it is clear that $\widetilde{\mathfrak{B}}$ maps $\mathcal{D}(\widetilde{\mathfrak{A}})$ onto $\mathcal{U}$, and thus, $\widetilde{B} := \widetilde{\mathfrak{B}}^{[-1]} \in \mathcal{L}\big(\mathcal{U}, \mathcal{D}(\widetilde{\mathfrak{A}})\big)$ has the properties in Definition II.1.2. Moreover, the $\mathfrak{A}$-boundedness of $\mathfrak{C}$ and the fact that $\mathcal{H}\mathcal{D}(\widetilde{\mathfrak{A}})$ is continuously embedded in $\mathcal{D}(\mathfrak{A})$ imply that $\widetilde{\mathfrak{C}}\widetilde{B} \in \mathcal{L}(\mathcal{U}, \mathcal{W})$. Finally,

$$\mathrm{Re}\,\left\langle \widetilde{\mathfrak{A}}x, x\right\rangle_{X_\mathcal{H}} = \mathrm{Re}\,\left\langle \widetilde{\mathfrak{B}}x, \widetilde{\mathfrak{C}}x\right\rangle_{\mathcal{U}}, \qquad x \in \mathcal{D}\left(\widetilde{\mathfrak{A}}\right),$$

follows from the following integration by parts formula which was established in the appendix of [18], valid for all $h \in H^{\mathrm{div}}(\Omega)$ and $g \in H^1_{\Gamma_0}(\Omega)$:

$$\langle \operatorname{div} h, g\rangle_{L^2(\Omega)} + \langle h, \nabla g\rangle_{L^2(\Omega)^n} = \langle \gamma_\perp f, \gamma_0 g\rangle_{\mathcal{W}',\mathcal{W}};$$

recall that $\mathcal{W}'$ is the dual of $\mathcal{W}$ with pivot space $\mathcal{U}$ and that $\widetilde{\mathfrak{B}}x \in \mathcal{U}$ for $x \in \mathcal{D}(\widetilde{\mathfrak{A}})$. □

B. Exponential stabilization and admissible observation

The robust controller design in §IV involves exponential stabilization of the plant with output feedback, and in this section we will comment on this problem for the wave equation (II.1). We shall use a special case of a result by Guo and Yao [25] to obtain exponential stabilization using the so-called *multiplier method*. The case where all physical parameters are identity was covered also in [19], see [26], [27] for other related results.

In order to apply the multiplier method, we assume that the boundary $\partial\Omega$ is of class C^2 and that it is partitioned into the reflecting part Γ_0 and the input/output part Γ_1 in the following way (see [19, Chap. 7] for a longer discussion): Fix $\zeta^0 \in \mathbb{R}^n \setminus \overline{\Omega}$ arbitrarily and define $m(\zeta) := \zeta - \zeta^0$, $\zeta \in \mathbb{R}^n$. We assume that

$$\begin{aligned}\Gamma_0 &= \operatorname{int}\{\zeta \in \partial\Omega \mid m(\zeta)\cdot\nu(\zeta) \le 0\} \neq \emptyset \qquad \text{and}\\ \Gamma_1 &= \{\zeta \in \partial\Omega \mid m(\zeta)\cdot\nu(\zeta) > 0\} \neq \emptyset,\end{aligned} \tag{II.6}$$

and that the sets $\Gamma_0, \Gamma_1 \subset \partial\Omega$ form a partition of the boundary $\partial\Omega$ in the sense that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$. In our wave equation, we add a viscous damper $u = -b^2 y$ on Γ_1, where

$$b(\zeta)^2 := m(\zeta)\cdot\nu(\zeta), \qquad \zeta \in \Gamma_1. \tag{II.7}$$

This damper is rigorously interpreted as the following equation in $\mathcal{W}'$:

$$\gamma_\perp T\,\nabla w(t) = -b^2\,\gamma_0\,\dot w(t), \qquad t \ge 0.$$

In order to guarantee *exponential stability*, we do not need to explicitly make the common, but rather restrictive, assumption that $\overline{\Gamma_0}\cap\overline{\Gamma_1} = \emptyset$. However, combining the assumption that $\partial\Omega$ is of class C^2 with the assumption (II.2) that we need for the admissibility of velocity observation, we unfortunately seem to end up in a situation where necessarily $\overline{\Gamma_0}\cap\overline{\Gamma_1} = \emptyset$.

The total energy associated to a solution $x = \left[\begin{smallmatrix} g \\ h\end{smallmatrix}\right]$ of the wave equation in Thm II.2 at time t is

$$\frac12\left\|\begin{bmatrix} g(t)\\ h(t)\end{bmatrix}\right\|^2_{X_\mathcal{H}} := \frac12\int_\Omega \frac{1}{\rho(\zeta)}\, g(\zeta,t)^2 + h(\zeta,t)^* T(\zeta) h(\zeta,t)\,\mathrm{d}\zeta,$$

representing the sum of kinetic and potential energy.

Theorem II.3. *Assume that ρ and T are constant, that $\Omega \subset \mathbb{R}^n$ is a bounded C^2-domain with $n \le 3$, and that Γ_k satisfy* (II.6). *Then there exist $c > 1$ and $\omega > 0$, such that all $\left[\begin{smallmatrix} g \\ h\end{smallmatrix}\right] \in C^1(\mathbb{R}_+; X_\mathcal{H})$ with $\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{smallmatrix} g(t) \\ h(t)\end{smallmatrix}\right] = \mathcal{A}\mathcal{H}\left[\begin{smallmatrix} g(t) \\ h(t)\end{smallmatrix}\right]$ and $\gamma_\perp T\,h(t) = -(m\cdot\nu)\,\gamma_0\,g(t)$ for $t \ge 0$, and $h(0) \in \nabla H^1_{\Gamma_0}(\Omega)$, satisfy*

$$\left\|\begin{bmatrix} g(t)\\ h(t)\end{bmatrix}\right\|^2_{X_\mathcal{H}} \le c\,e^{-\omega t}\left\|\begin{bmatrix} g(0)\\ h(0)\end{bmatrix}\right\|^2_{X_\mathcal{H}}, \qquad t \ge 0. \tag{II.8}$$

Proof. Let $\left[\begin{smallmatrix} g \\ h\end{smallmatrix}\right]$ have the properties in the statement and let $\eta \in H^1_{\Gamma_0}(\Omega)$ be such that $\nabla\eta = h(0)$. Setting

$$w(t) := \eta + \frac{1}{\rho}\int_0^t g(s)\,\mathrm{d}s, \qquad t \ge 0, \tag{II.9}$$

we get that $\dot w(t) = g(t)/\rho$ and $\nabla w(t) = h(t)$ for all $t \ge 0$. Moreover, w is a classical solution of the wave equation since

$$\ddot w(t) = \operatorname{div}\left(\frac{T}{\rho}\nabla w\right)(t), \qquad t \ge 0, \tag{II.10}$$

with the left-hand side in $C\big(\mathbb{R}_+; L^2(\Omega)\big)$. Note that the constant matrix T/ρ is positive definite and hence invertible.

In [28, Ex. 3.1], a Riemannian manifold $(\mathbb{R}^n, g)$ is associated to (II.10), and it is concluded that the vector field $H := \sum_{k=1}^n (\zeta_k - \zeta_k^0)\,\partial/\partial\xi_k$ on this manifold satisfies the condition [25, (3.2)] with $a = 1$ (here ζ_k is coordinate number k of ζ). We further observe that $w(t) \in H^1_{\Gamma_0}(\Omega)$ and $\gamma_\perp T\nabla w(t) = -(m\cdot\nu)\,\gamma_0\,\dot w(t)$ for all $t \ge 0$, while $w(0) \in H^1_{\Gamma_0}(\Omega)$, and $\dot w(0) \in L^2(\Omega)$. By [25, Thm 1], we have (II.8). □

In general, a solution w of (II.3) is only required to be *constant* on Γ_0. The condition $h(0) \in \nabla H^1_{\Gamma_0}(\Omega)$ corresponds to the initial condition $w(0) \in H^1_{\Gamma_0}(\Omega)$ via (II.9), and this implies the stronger statement that w is constantly equal to *zero* on Γ_0. This is one way to guarantee that the potential energy decays to zero.

Returning to the case of the general BCS, we will replace the multiplication by $-m\cdot\nu$ on $L^2(\Gamma_1)$ by an admissible output feedback operator $Q \in \mathcal{L}(\mathcal{Y},\mathcal{U})$ which stabilizes the given BCS exponentially: Let $(\mathcal{B},\mathcal{A},\mathcal{C})$ be a BCS on (U,X,Y). We call $Q \in \mathcal{L}(Y,U)$ an *admissible (static output) feedback operator* for $(\mathcal{B},\mathcal{A},\mathcal{C})$ if $(\mathcal{B}+Q\mathcal{C},\mathcal{A},\mathcal{C})$ is a also a BCS. Moreover, let the Hilbert spaces Y and Y' be duals with some pivot Hilbert space $\widetilde U$, and let $Q \in \mathcal{L}(Y,Y')$. We say that Q is *uniformly accretive* if there exists some $\delta > 0$ such that

$$\operatorname{Re}\,\langle Qy, y\rangle_{Y',Y} \ge \delta\,\|y\|^2_{\widetilde U}, \qquad y \in Y.$$

By an *admissible observation operator* for a C_0-semigroup $\mathbb{T}$ on X with generator A, we mean a linear operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$ for which there exist some $\tau > 0$ and $K_\tau \ge 0$ such that

$$\int_0^\tau \|C\,\mathbb{T}(t)\,x\|_Y^2\,\mathrm{d}t \le K_\tau^2\,\|x\|_X^2, \qquad \forall x \in \mathcal{D}(A). \tag{II.11}$$

If (II.11) holds for some $\tau > 0$ and $K_\tau \ge 0$, then for every $\tau > 0$ it is possible to choose a $K_\tau \ge 0$ such that (II.11) holds. The observation operator is *infinite-time admissible* if (II.11) holds for all $\tau > 0$ with K_τ replaced by some bound K which is independent of τ. In particular, if the semigroup $\mathbb{T}$ is exponentially stable, then every admissible observation operator is infinite-time admissible [19, Prop. 4.3.3].

Proposition II.4. *Let $(\mathcal{B},\mathcal{A},\mathcal{C})$ be a* passive *BCS on (Y',X,Y) and let $Q \in \mathcal{L}(Y,Y')$ be a uniformly accretive, admissible output feedback operator for $(\mathcal{B},\mathcal{A},\mathcal{C})$. The resulting BCS $(\mathcal{B}+Q\mathcal{C},\mathcal{A},\mathcal{C})$ is also passive and we denote its associated semigroup by $\mathbb{T}_Q$. The observation operator $\mathcal{C}$,* *interpreted as an operator mapping into the pivot space $\widetilde{Y}$ rather than into Y, is infinite-time admissible for $\mathbb{T}_Q$.*

Proof. By the definitions of admissible feedback operator and BCS, it follows that $(\mathcal{B}+Q\mathcal{C},\mathcal{A},\mathcal{C})$ is a BCS on $(Y',X,\widetilde{Y})$, and by definition the generator of $\mathbb{T}_Q$ is $A_Q := \mathcal{A}\big|_{\mathcal{N}(\mathcal{B}+Q\mathcal{C})}$.

For a fixed $x_0 \in \mathcal{D}(A_Q)$, the associated state trajectory $x(t) = \mathbb{T}_Q(t)\,x_0$ stays in $\mathcal{D}(A_Q)$, and by the assumed passivity, for all $t \geq 0$ we have

$$\begin{aligned}\operatorname{Re}\,\langle \mathcal{A}x(t),x(t)\rangle_X &\leq \operatorname{Re}\,\langle \mathcal{B}x(t),\mathcal{C}x(t)\rangle_{Y',Y}\\ &= -\operatorname{Re}\,\langle Q\mathcal{C}x(t),\mathcal{C}x(t)\rangle_{Y',Y}\,.\end{aligned}$$

Multiplying this by 2 and integrating over $[0,\tau]$, we get

$$\begin{aligned}\|x(\tau)\|_X^2-\|x(0)\|_X^2 &= \int_0^\tau 2\operatorname{Re}\,\langle A_Qx(t),x(t)\rangle_X\,\mathrm{d}t\\ &\leq -2\delta\int_0^\tau \|\mathcal{C}x(t)\|_{\widetilde{Y}}^2\,\mathrm{d}t.\end{aligned}$$

Letting $\tau\to+\infty$, we obtain that $\mathcal{C}$ is infinite-time admissible, since

$$\int_0^\infty \|\mathcal{C}\mathbb{T}_Q(t)x_0\|_{\widetilde{Y}}^2\,\mathrm{d}t \leq \frac{1}{2\delta}\|x_0\|_X^2,\quad x_0\in\mathcal{D}(A_Q).$$

□

We end the section by discussing the wave system as an example for the above abstract definitions. It is clear that the multiplication by $b^2 = m\cdot\nu$ in (II.7) is a bounded operator on $L^2(\Gamma_1)$, and hence it is also in $\mathcal{L}(\mathcal{W},\mathcal{W}')$ and it is uniformly accretive if (II.2) holds. Furthermore, multiplication by b^2 is an admissible feedback operator for the wave system in (II.4) and for its restriction in Thm II.2. Indeed, $\mathcal{N}(\mathfrak{B}\mathcal{H}+b^2\,\mathfrak{C}\mathcal{H}) = \mathcal{N}(\widetilde{\mathfrak{B}}+b^2\,\widetilde{\mathfrak{C}}) \subset \mathcal{D}(\widetilde{\mathfrak{A}})$, by [18, Thm 3.5] the operator $\mathfrak{A}\mathcal{H}\big|_{\mathcal{N}(\mathfrak{B}\mathcal{H}+b^2\,\mathfrak{C}\mathcal{H})} = \widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}}+b^2\,\widetilde{\mathfrak{C}})}$ generates a contraction semigroup on $X_{\mathcal{H}}$, and the operators

$$\mathfrak{B}\mathcal{H}+b^2\mathfrak{C}\mathcal{H} = \begin{bmatrix} b^2\gamma_0 & \gamma_\perp\end{bmatrix}\mathcal{H}\qquad\text{and}\qquad \widetilde{\mathfrak{B}}+b^2\,\widetilde{\mathfrak{C}}$$

are continuous and surjective; hence they have right-inverses with the properties required in Definition II.1.

III. The plant, the controller, and the exosystem

In the next section, we solve the robust output regulation problem for a general BCS $(\mathcal{B},\mathcal{A},\mathcal{C})$ on the Hilbert spaces (U,X,Y); the system is not necessarily related to the wave equation. In the following we assume that the whole boundary $\partial\Omega$ is accessible via $\mathcal{B}$ and R_1, R_2 are arbitrary restrictions to certain parts of $\partial\Omega$. We first add an external disturbance w to the BCS, thus obtaining the plant

$$\begin{cases}\dot{x}(t)=\mathcal{A}x(t), & x(0)=x_0,\\ \mathcal{B}x(t)=R_1u(t)+R_2w(t), & t\geq 0,\\ \mathcal{C}x(t)=y(t),\end{cases}\tag{III.1}$$

where u and w may act on different parts of the boundary depending on R_1 and R_2.

In what follows, Q is such that R_1Q is an admissible static output feedback operator for (III.1) such that the semigroup $\mathbb{T}_s$ generated by $A_s := \mathcal{A}\big|_{\mathcal{D}(\mathcal{A})\cap\mathcal{N}(\mathcal{B}+R_1Q\mathcal{C})}$ is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$ (here the subscript 's' stands for "stabilized plant").

We will connect the plant to the dynamic controller

$$\begin{cases}\dot{z}(t)=\mathcal{G}_1z(t)+\mathcal{G}_2(y(t)-y_{ref}(t)), & z(0)=z_0\\ u(t)=Kz(t)-Q(y(t)-y_{ref}(t)), & t\geq 0,\end{cases}\tag{III.2}$$

where y_{ref} is an external reference signal and the state space Z of the controller is a Hilbert space, but $\mathcal{G}_1\in\mathcal{L}(Z)$ is bounded. Moreover, we assume that $\mathcal{G}_2\in\mathcal{L}(Y,Z)$, $K\in\mathcal{L}(Z,U)$ and $Q\in\mathcal{L}(Y,U)$. The disturbance signal w and the reference signal y_{ref} are assumed to be generated by an exosystem

$$\begin{cases}\dot{v}(t)=Sv(t), & v(0)=v_0,\\ w(t)=Ev(t), & t\geq 0,\\ y_{ref}(t)=-Fv(t),\end{cases}\tag{III.3}$$

which is a linear system on a finite-dimensional space $W=\mathbb{C}^q$, $q\in\mathbb{N}$. We assume that $S=\operatorname{diag}(i\omega_1,i\omega_2,\ldots,i\omega_q)$ with $\omega_i\neq\omega_j$ for $i\neq j$, $E\in\mathcal{L}(W,U)$ and $F\in\mathcal{L}(W,Y)$.

Setting u and y equal in (III.1) and (III.2), and using (III.3), we obtain

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x\\ z\end{bmatrix}=\begin{bmatrix}\mathcal{A} & 0\\ \mathcal{G}_2\mathcal{C} & \mathcal{G}_1\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{G}_2F\end{bmatrix}v,\\ (R_2E-R_1QF)v=\begin{bmatrix}\mathcal{B}+R_1Q\mathcal{C} & -R_1K\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix},\\ e=\begin{bmatrix}\mathcal{C} & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}+Fv,\end{cases}\tag{III.4}$$

where we chose the regulation error $e(t) =: y(t)-y_{ref}(t)$ as the output and the state-space is $X_e := X\times Z$. This system is no longer a BCS and we now proceed to write it in the standard input/state/output form. First we observe that we may interpret the feedthrough Q of the controller as a part of the plant without changing (III.4). This amounts to pre-stabilizing the plant via replacing the input equation of (III.1) by $(\mathcal{B}+R_1Q\mathcal{C})x(t)=R_1u(t)+(R_2E-R_1QF)v(t)$ and simultaneously removing the term $-Q(y(t)-y_{ref}(t))$ from the output equation of (III.2).

As R_1Q is assumed to be an admissible feedback operator, the pre-stabilized plant $(\mathcal{B}+R_1Q\mathcal{C},\mathcal{A},\mathcal{C})$ is a BCS and by Def. II.1.2, we can choose a right inverse $B_s\in\mathcal{L}(U,X)$ of $\mathcal{B}+R_1Q\mathcal{C}$ such that

$$B_sR_1U\subset\mathcal{D}(\mathcal{A}),\ \mathcal{A}B_sR_1\in\mathcal{L}(U,X),\ \mathcal{C}B_sR_1\in\mathcal{L}(U,Y).\tag{III.5}$$

In order to present the transfer function of $(\mathcal{B}+R_1Q\mathcal{C},\mathcal{A},\mathcal{C})$, consider the auxiliary function

$$P_0(\lambda) := \mathcal{C}(\lambda-A_s)^{-1}(\mathcal{A}B_s-\lambda B_s)+\mathcal{C}B_s,\quad \lambda\in\rho\,(A_s)\,.$$

Now, define the transfer function by

$$P_s(\lambda) := P_0(\lambda)R_1,\qquad \lambda\in\rho\,(A_s)\,.\tag{III.6}$$

The auxiliary function P_0 becomes useful later on in describing the mapping from v to y.

Now let $\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right]$ be a classical state trajectory of (III.4), i.e., $\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right]\in C^1(\mathbb{R}_+;X_e)$, $\mathcal{G}_2y_{ref}\in C(\mathbb{R}_+;Z)$, $(\mathcal{B}+R_1Q\mathcal{C})x\in C(\mathbb{R}_+;U)$, $w\in C^1(\mathbb{R}_+;U)$, and the first two lines of (III.4) hold for all $t \geq 0$. Next introduce a new state variable for (III.4) by

$$x_e := \begin{bmatrix} 1 & -B_sR_1K \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} B_sE_sv \\ 0 \end{bmatrix} \in C^1(\mathbb{R}_+; X_e),$$

where we denote $E_s := R_2E - R_1QF$ for brevity. This transformation can be inverted as

$$\begin{bmatrix} x \\ z \end{bmatrix} := \begin{bmatrix} 1 & B_sR_1K \\ 0 & 1 \end{bmatrix} x_e + \begin{bmatrix} B_sE_sv \\ 0 \end{bmatrix}. \tag{III.7}$$

Differentiating x_e and using the first line of (III.4), we get

$$\dot{x}_e = \begin{bmatrix} \mathcal{A} - B_sR_1K\mathcal{G}_2\mathcal{C} & \mathcal{A}B_sR_1K - B_sR_1K\tilde{\mathcal{G}}_1 \\ \mathcal{G}_2\mathcal{C} & \tilde{\mathcal{G}}_1 \end{bmatrix} x_e + \begin{bmatrix} \mathcal{A}B_sE_s - B_sE_sS - B_sR_1K\mathcal{G}_2(\mathcal{C}B_sE_s + F) \\ \mathcal{G}_2(\mathcal{C}B_sE_s + F) \end{bmatrix} v,$$

where we denote $\tilde{\mathcal{G}}_1 := \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sR_1K$ for brevity.

With the new state variable, the input equation of (III.4) becomes

$$E_sv = \begin{bmatrix} \mathcal{B} + R_1Q\mathcal{C} & -K \end{bmatrix} \left(x_e + \begin{bmatrix} B_s \\ 0 \end{bmatrix} (R_1Kz + E_sv) \right)$$

which simplifies to $x_e \in \mathcal{N}\left(\begin{bmatrix} \mathcal{B} + R_1Q\mathcal{C} & 0 \end{bmatrix}\right)$. Hence recalling that $A_s = \mathcal{A}|_{\mathcal{D}(\mathcal{A})\cap\mathcal{N}(\mathcal{B}+R_1Q\mathcal{C})}$ and defining

$$A_e := \begin{bmatrix} A_s - B_sR_1K\mathcal{G}_2\mathcal{C} & \mathcal{A}B_sR_1K - B_sR_1K\tilde{\mathcal{G}}_1 \\ \mathcal{G}_2\mathcal{C} & \tilde{\mathcal{G}}_1 \end{bmatrix}\Bigg|_{\mathcal{D}(A_e)},$$
$$\mathcal{D}(A_e) := \mathcal{N}(\mathcal{B} + R_1Q\mathcal{C}) \times Z, \tag{III.8}$$

we get that every classical solution of (III.4) satisfies $x_e(t) \in \mathcal{D}(A_e)$ for all $t \geq 0$ and $\dot{x}_e = A_ex_e + B_ev$, where the control operator $B_e \in \mathcal{L}(W, X_e)$ is

$$B_e := \begin{bmatrix} \mathcal{A}B_sE_s - B_sE_sS - B_sR_1K\mathcal{G}_2(\mathcal{C}B_sE_s + F) \\ \mathcal{G}_2(\mathcal{C}B_sE_s + F) \end{bmatrix}.$$

Finally, using (III.7) the output for (III.4) becomes

$$e = \begin{bmatrix} \mathcal{C} & \mathcal{C}B_sR_1K \end{bmatrix} x_e + (\mathcal{C}B_sE_s + F)v.$$

Thus, the closed-loop system is of the form

$$\begin{cases} \dot{x}_e = A_ex_e + B_ev, \\ e = C_ex_e + D_ev, \end{cases} \tag{III.9}$$

where

$$C_e := \begin{bmatrix} \mathcal{C} & \mathcal{C}B_sR_1K \end{bmatrix}, \qquad \mathcal{D}(C_e) := \begin{bmatrix} \mathcal{D}(\mathcal{C}) \\ Z \end{bmatrix}, \qquad \text{and}$$
$$D_e := \mathcal{C}B_sE_s + F \in \mathcal{L}(W, Y).$$

We denote the transfer function of (III.9) from v to e with

$$P_e(\lambda) = C_e(\lambda - A_e)^{-1}B_e + D_e.$$

The above calculations show that every classical solution of (III.4) with $v \in C(\mathbb{R}_+; W)$ is also a classical solution of (III.9). Conversely, assume that $x_e \in C^1(\mathbb{R}_+; X_e)$ with $x_e(t) \in \mathcal{D}(A_e)$, $v \in C(\mathbb{R}_+; W)$ and (III.9) holds on $\mathbb{R}_+$. Then v, $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$ in (III.7) and e satisfy (III.4). We conclude that (III.4) and (III.9) are equivalent systems in the sense that they have the same classical solutions.

The following result forms the basis for the output regulation theory in the next section. Note that we do not assume that the original plant (III.1) is well-posed or regular, but the closed-loop system (III.9) nevertheless has these properties.

Theorem III.1. *The operator A_e in* (III.8) *generates a C_0-semigroup $\mathbb{T}_e$ on X_e and C_e is an admissible observation operator for $\mathbb{T}_e$. The closed-loop system* (III.9) *is well-posed and regular such that $P_e(\lambda) \to D_e$ as $\operatorname{Re}\lambda \to \infty$.*

Proof. We begin by splitting $A_e = A_1 + A_2 + A_3$, where

$$A_1 = \begin{bmatrix} A_s & 0 \\ 0 & \mathcal{G}_1 \end{bmatrix}, \qquad \mathcal{D}(A_1) = \mathcal{D}(A_e),$$
$$A_2 = \begin{bmatrix} -B_sR_1K\mathcal{G}_2\mathcal{C} & 0 \\ \mathcal{G}_2\mathcal{C} & 0 \end{bmatrix}, \qquad \mathcal{D}(A_2) = \mathcal{D}(A_e),$$
$$A_3 = \begin{bmatrix} 0 & \mathcal{A}B_sR_1K - B_sR_1K(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sR_1K) \\ 0 & \mathcal{G}_2\mathcal{C}B_sR_1K \end{bmatrix},$$
$$\mathcal{D}(A_3) = X_e.$$

Here A_1 generates a C_0-semigroup $\mathbb{T}_1$ on X_e. The operator A_2 can be factored as

$$A_2 = \begin{bmatrix} -B_sR_1K\mathcal{G}_2 \\ \mathcal{G}_2 \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix},$$

where the first factor is bounded from Y into X_e. Our assumption that $\mathcal{C}$ is admissible for $\mathbb{T}_s$ implies that $\begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} : X_e \supset \mathcal{D}(A_e) \to Y$ is an admissible observation operator for $\mathbb{T}_1$, and by [19, Thm 5.4.2], $A_1 + A_2$ generates a C_0-semigroup $\mathbb{T}_2$ on X_e and $\begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$ is admissible for $\mathbb{T}_2$. Since A_3 is bounded, A_e generates a C_0-semigroup by [19, Thm 5.4.2] and due to the boundedness of $\mathcal{C}B_sR_1K$, C_e is admissible for $\mathbb{T}_e$. As in addition B_e and D_e are bounded, the well-posedness and regularity of the closed-loop system follow immediately from [19, Thm 4.3.7] ☐

IV. Output Regulation

We begin this section by presenting the three output regulation problems considered in this paper. The structure for the remainder of this section will be presented after the problem definitions.

The Output Regulation Problem. For a given plant (III.1), choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in (III.2) in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha\cdot}e(\cdot) \in L^2([0,\infty); Y)$ for some $\alpha > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$.

Furthermore, if the controller solves the output regulation problem despite perturbations in the parameters of the plant or the exosystem, then we say that the controller solves the *robust* output regulation problem with respect to this class of perturbations. To make this precise, we first define the class of admissible perturbations:

Definition IV.1. A quintuple $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F')$ of linear operators belongs to the *class $\mathcal{O}$ of admissible perturbations* if it has the following properties:

1) The triple $(\mathcal{B}' + R_1 Q\mathcal{C}', \mathcal{A}', \mathcal{C}')$ is a BCS on (U, X, Y).
2) The observation operator $\mathcal{C}'$ is admissible for the semigroup generated by $A'_s := \mathcal{A}'|_{\mathcal{N}(\mathcal{B}'+R_1 Q\mathcal{C}')}$.
3) The eigenvalues of S are in the resolvent set of the perturbed pre-stabilized plant, i.e., $\{i\omega_k\}_{k=1}^q \subset \rho(A'_s)$.
4) $E' \in \mathcal{L}(W, U)$ and $F' \in \mathcal{L}(W, Y)$.

In the above definition it would appear that the class $\mathcal{O}$ of perturbations depends on Q. However, as Q only contributes to stabilizing the plant, we have much more freedom choosing Q than choosing the other controller parameters (as seen later on). For example, in the wave equation considered in Section II, any uniformly accretive operator can be chosen as Q. Therefore, in Definition IV.1, one could think of Q being chosen such that the class $\mathcal{O}$ is as large as possible. Moreover, if $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F') \in \mathcal{O}$ then the transfer function (III.6) of the triple $(\mathcal{B}' + R_1 Q\mathcal{C}', \mathcal{A}', \mathcal{C}')$ is well-defined and bounded at the frequencies of the exosystem.

We make the natural assumption that the unperturbed system is in class $\mathcal{O}$ as well, that is, $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F) \in \mathcal{O}$. Note that this does not include the assumption that the semigroup generated by A_s is exponentially stable. Further note that even though $(\mathcal{B}, \mathcal{A}, \mathcal{C})$ is assumed to be a BCS, that is not required from $(\mathcal{B}', \mathcal{A}, \mathcal{C}')$ but only from $(\mathcal{B}' + R_1 Q\mathcal{C}', \mathcal{A}', \mathcal{C}')$.

From Definition IV.1 it follows that the perturbed closed-loop system is well-posed and regular. Please note that while no perturbations are allowed in the eigenvalues of the generator S of the exosystem or in the controller parameter $\mathcal{G}_1$, the parameters $(\mathcal{G}_2, K, Q)$ would in fact allow certain bounded perturbations. We will comment on this more thoroughly in Remark IV.9.

The Robust Output Regulation Problem. For a given plant, choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in such a way that the following are satisfied:

1) The controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ solves the output regulation problem.
2) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F') \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha' \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\alpha' > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$.

In Section IV-C, we will construct a controller that solves the robust output regulation problem *approximately*. That is, the regulation error does not decay asymptotically to zero but can be made small. For this purpose, we introduce the following new control problem:

The Approximate Robust Output Regulation Problem. Let $\delta > 0$ be given. Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies
$$\int_t^{t+1} \|e(s)\|^2 \, \mathrm{d}s \leq M e^{-\alpha t} (\|x_{e0}\|^2 + \|v_0\|^2) + \delta \|v_0\|^2$$
for some $M, \alpha > 0$ independent of $x_{e0} \in X_e, v_0 \in W$.
3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F') \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then there exists a $\delta' > 0$ such that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies
$$\int_t^{t+1} \|e(s)\|^2 \, \mathrm{d}s \leq M' e^{-\alpha' t} (\|x_{e0}\|^2 + \|v_0\|^2) + \delta' \|v_0\|^2$$
for some $M', \alpha' > 0$ independent of x_{e0}, v_0.

Remark IV.2. The approximate robust output regulation problem formulation implies that, in the absence of perturbations, the asymptotic regulation error must be smaller than $\delta \|v_0\|^2$ for any given (or in practice chosen) $\delta > 0$. However, when perturbations are present, the asymptotic regulation error is merely bounded by $\delta' \|v_0\|^2$. For details, see Theorem IV.11, (IV.14)–(IV.15) and the discussion therein.

Now that we have presented the different output regulation problems to be considered, the structure of the remaining section is as follows. Before proceeding to constructing the controllers, we will present two auxiliary results to be used throughout the remainder of this section. In §IV-A we present a regulating controller without the robustness requirement, in §IV-B we present the internal model principle for boundary control systems, in §IV-C we present an approximate robust controller, and finally in §IV-D we present a precise robust controller.

The following auxiliary result is a consequence of [15, Thm 4.1] under the assumption that the closed-loop system (III.9) is a regular linear system. The result states that the solvability of the *regulator equations*

$$\Sigma S = A_e \Sigma + B_e \tag{IV.1a}$$
$$0 = C_e \Sigma + D_e \tag{IV.1b}$$

is equivalent to the solvability of the output regulation problem. The result of [15, Thm 4.1] essentially follows from [15, Lem. 4.3] by which the regulation error can be written as

$$e(t) = C_e \mathbb{T}_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(t),$$

where the first part decays to zero at an exponential rate provided that $\mathbb{T}_e$ is exponentially stable, C_e is an admissible observation operator for $\mathbb{T}_e$ and Σ is the solution of (IV.1a).

Theorem IV.3. *Assume that the closed-loop system is regular and exponentially stabilized by a controller* $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$*. Then the controller solves the output regulation problem if and only if the regulator equations* (IV.1) *have a solution* $\Sigma \in \mathcal{L}(W, X_e)$*. The solution* Σ *is unique when it exists.*

Proof. We first note that the feedthrough term $-Qe(t)$ in the controller is not part of the controller in [15, Thm 4.1]. However, as in (III.4) we can interpret the feedthrough Q as a part of the plant (III.1) and simultaneously remove it from the controller (III.2), so that the input equation becomes $(\mathcal{B} + R_1 Q\mathcal{C})x(t) = R_1 u(t) + R_1 Q y_{ref}(t) + R_2 w(t)$. The closed-loop system is unaffected by this algebraic trick, and hence, we may continue with a pre-stabilized plant and the same controller structure as in [15, Thm 4.1].

Now the result follows from [15, Thm 4.1] as an exponentially stable semigroup is also strongly stable, and for A_e being the generator of an exponentially stable semigroup and $\sigma(S) \subset i\mathbb{R}$ the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ always has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ by [29, Cor. 8]. Furthermore, the exponential decay of the regulation error follows from the assumed exponential stability of the closed-loop system. □

Theorem IV.3 assumes that the controller exponentially stabilizes the closed-loop system. We will therefore need to show that the controllers we present in Proposition IV.6, Theorem IV.11 and Corollary IV.14 have this property. For this, we present the following tool which uses the notation of §III. Here we need to assume that there exists an operator Q as described in the following:

Lemma IV.4. *Let $Z = Y_N^q$, where Y_N is equal to $\mathbb{C}$ or a closed subspace of Y. Choose the controller parameter $Q \in \mathcal{L}(Y, U)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$. Choose the remaining parameters as*

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I, i\omega_2 I, \ldots, i\omega_q I) \in \mathcal{L}(Z),$$
$$K = \epsilon K_0 = \epsilon[K_0^1, K_0^2, \ldots, K_0^q] \in \mathcal{L}(Z, U),$$
$$\mathcal{G}_2 = (\mathcal{G}_2^k P_N)_{k=1}^q \in \mathcal{L}(Y, Z),$$

where I is the identity in Y_N, and P_N is a projection onto Y_N in Y if $Y_N \subset Y$ or the identity on Y otherwise. Additionally, assume that $\mathcal{G}_2^k$ and K_0^k satisfy $\sigma(\mathcal{G}_2^k P_N P_s(i\omega_k) K_0^k) \subset \mathbb{C}_-$ for all $k \in \{1, 2, \ldots, q\}$.

Then there exits an $\epsilon^* > 0$ such that the closed-loop system (III.9) *is exponentially stable for all $0 < \epsilon < \epsilon^*$.*

Proof. Define the operator $H = (H_1, H_2, \ldots, H_q) \in \mathcal{L}(Z, X)$ by choosing

$$H_k := (i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_k B_s)R_1 K_0^k$$

for all $k \in \{1, 2, \ldots, q\}$. By the choice of H_k we have $(i\omega_k - A_s)H_k = \mathcal{A}B_s R_1 K_0^k - i\omega_k B_s R_1 K_0^k$, i.e., $H_k i\omega_k = A_s H_k + \mathcal{A}B_s R_1 K_0^k - B_s R_1 K_0^k i\omega_k$, and thus, $H\mathcal{G}_1 = A_s H + \mathcal{A}B_s R_1 K_0 - B_s R_1 K_0 \mathcal{G}_1$ due to the diagonal structure of $\mathcal{G}_1$. Define

$$R = \begin{bmatrix} -1 & \epsilon H \\ 0 & 1 \end{bmatrix} = R^{-1} \in \mathcal{L}(X_e)$$

and denote $\hat{A}_e = RA_eR^{-1}$. Note that as $\mathcal{R}(H) \subset \mathcal{N}(\mathcal{B} + R_1 Q\mathcal{C})$, it follows that $\mathcal{D}(\hat{A}_e) = \mathcal{D}(A_e)$. Using the above identity we can write $\hat{A}_e$ as

$$\hat{A}_e = \begin{bmatrix} A_s - \epsilon\tilde{H}\mathcal{G}_2\mathcal{C} & 0 \\ -\mathcal{G}_2\mathcal{C} & \mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H} \end{bmatrix} + \epsilon^2 \begin{bmatrix} 0 & \tilde{H}\mathcal{G}_2\mathcal{C}\tilde{H} \\ 0 & 0 \end{bmatrix}.$$

where we denote $\tilde{H} := H + B_s R_1 K_0$ for brevity.

In the remaining part of the proof we apply the Gearhart-Prüss-Greiner Theorem in [30, Thm V.1.11]. More precisely, we will show that the resolvent of $\hat{A}_e$ is uniformly bounded on the closed right-half plane. We first note that since $\mathcal{C}$ is admissible for $\mathbb{T}_s$ which is exponentially stable, we have by [19, Thm 4.3.7] that $\mathcal{C}(\lambda - A_s)^{-1}$ is uniformly bounded for all $\lambda \in \overline{\mathbb{C}}_+$. Thus, as $\tilde{H}\mathcal{G}_2$ is bounded, there exists an $M_0 > 0$ such that $\|\tilde{H}\mathcal{G}_2\mathcal{C}(\lambda - A_s)^{-1}\| \le M_0$, and for $0 < \epsilon < M_0^{-1}$ a Neumann series expansion implies that $1 + \epsilon\tilde{H}\mathcal{G}_2\mathcal{C}(\lambda - A_s)^{-1}$ is invertible. Thus, we obtain that

$$(\lambda - A_s + \epsilon\tilde{H}\mathcal{G}_2\mathcal{C})^{-1} = (\lambda - A_s)^{-1}(1 + \epsilon\tilde{H}\mathcal{G}_2\mathcal{C}(\lambda - A_s)^{-1})^{-1}$$

is uniformly bounded in the right half plane. Hence, the semigroup generated by $A_s - \epsilon\tilde{H}\mathcal{G}_2\mathcal{C}$ is exponentially stable by [30, Thm V.1.11].

Note that by the choice of H_k we have

$$\begin{aligned} &\mathcal{C}(H_k + B_s R_1 K_0^k) \\ &= \mathcal{C}(i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_k B_s)R_1 K_0^k + \mathcal{C}B_s R_1 K_0^k \\ &= P_s(i\omega_k)K_0^k, \end{aligned}$$

and thus $\sigma(\mathcal{G}_2^k P_N \mathcal{C}(H_k + B_s R_1 K_0^k)) \subset \mathbb{C}_-$ by the assumption made on $\mathcal{G}_2^k$ and K_0^k. Furthermore, since $\sigma(\mathcal{G}_1) = \{i\omega_k\}_{k=1}^q$, the operator $\mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H}$ satisfies the stability conditions of the operator $A_c - \epsilon\tilde{P}K$ in [31, Appendix B]. Hence, by [31, Appendix B] there exist constants $M_1, \beta > 0$ such that for all $\epsilon > 0$ sufficiently small we have $\|\mathbb{T}_2(t)\| \le M_1 e^{-\epsilon\beta t}$ for $t \ge 0$, where $\mathbb{T}_2$ is the semigroup generated by $\mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H}$. This further implies that

$$\|(\lambda - \mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H})^{-1}\| \le \frac{M_1}{\epsilon\beta}, \qquad \lambda \in \overline{\mathbb{C}}_+.$$

Consider the operator $\hat{A}_e$ in the form $A_1 + \epsilon^2 A_2$. Since we have shown that the diagonal operators of A_1 generate exponentially stable semigroups and since $\mathcal{C}$ is admissible for A_s, it follows that A_1 is the generator of an exponentially stable semigroup. Furthermore, there exists an $M_2 > 0$ such that for all $\epsilon > 0$ sufficiently small, the estimate $\|(\lambda - A_1)^{-1}\| \le M_2/\epsilon$ holds for all $\lambda \in \overline{\mathbb{C}}_+$. Since A_2 is bounded, this implies that

$$\|\epsilon^2 A_2(\lambda - A_1)^{-1}\| \le \epsilon\|A_2\|M_2, \qquad \lambda \in \overline{\mathbb{C}}_+,$$

so that for $\epsilon < (\|A_2\|M_2)^{-1}$ we have $\|\epsilon^2 A_2(\lambda - A_1)^{-1}\| < 1$ on the closed right half plane. Using another Neumann series expansion, we obtain that

$$(\lambda - \hat{A}_e)^{-1} = (\lambda - A_1)^{-1}(1 - \epsilon^2 A_2(\lambda - A_1)^{-1})^{-1}$$

is uniformly bounded on $\overline{\mathbb{C}}_+$.

Thus, by the preceding argument there exists an $\epsilon^* > 0$ such that the resolvent of $\hat{A}_e$ is uniformly bounded on $\overline{\mathbb{C}}_+$ for all $0 < \epsilon < \epsilon^*$. By the Gearhart-Prüss-Greiner theorem, the semigroup $\hat{\mathbb{T}}_e$ generated by $\hat{A}_e$ is exponentially stable, and therefore, the semigroup $R\hat{\mathbb{T}}_e R^{-1}$ generated by A_e is exponentially stable as well, for all $0 < \epsilon < \epsilon^*$. □

A. A regulating controller

The following theorem gives necessary and sufficient conditions for a controller to achieve output regulation for the plant (III.1), i.e., a criterion equivalent to the solvability of the regulator equations. The result extends [15, Thm 5.1] to boundary control systems.

Theorem IV.5. *Assume that the closed-loop system is regular and exponentially stabilized by the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$.*

Then the controller solves the output regulation problem if and only if the equations

$$P_s(i\omega_k)Kz_k = -P_0(i\omega_k)E_s\phi_k - F\phi_k \tag{IV.2a}$$

$$(i\omega_k - \mathcal{G}_1)z_k = 0 \tag{IV.2b}$$

have solutions $z_k \in Z$ *for all* $k \in \{1, 2, \ldots, q\}$*, where* $\{\phi_k\}_{k=1}^q$ *is the Euclidean basis of* $\mathbb{C}^q$*. Furthermore, the solutions* z_k *are unique when they exist.*

Proof. Let us first assume that the controller solves the output regulation problem, i.e., by Theorem IV.3 the regulator equations have a solution $\Sigma = (\Pi, \Gamma)^T \in \mathcal{L}(W, X_e)$. Let $k \in \{1, 2, \ldots, q\}$ be arbitrary. As ϕ_k is an eigenvector of S, applying the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ to ϕ_k yields $(i\omega_k - A_e)\Sigma\phi_k = B_e\phi_k$, i.e.,

$$\begin{bmatrix}(i\omega_k - A_s + B_sR_1K\mathcal{G}_2\mathcal{C})\Pi\phi_k - (\mathcal{A}B_sR_1K - B_sR_1K\tilde{\mathcal{G}}_1)\Gamma\phi_k \\ -\mathcal{G}_2\mathcal{C}\Pi\phi_k + (i\omega_k - \tilde{\mathcal{G}}_1)\Gamma\phi_k\end{bmatrix} = \begin{bmatrix}(\mathcal{A}B_sE_s - B_sE_sS - B_sR_1K\mathcal{G}_2(\mathcal{C}B_sE_s + F))\phi_k \\ \mathcal{G}_2(\mathcal{C}B_sE_s + F)\phi_k\end{bmatrix}.$$

where we again denote $\tilde{\mathcal{G}}_1 := \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sR_1K$. The second line implies

$$(i\omega_k - \mathcal{G}_1)\Gamma\phi_k = \mathcal{G}_2(\mathcal{C}\Pi + \mathcal{C}B_sR_1K\Gamma + (\mathcal{C}B_sE_s + F))\phi_k. \tag{IV.3}$$

Now, as applying the second regulator equation to ϕ_k yields

$$0 = C_e\Sigma\phi + D_e\phi_k = \mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + (\mathcal{C}B_sE_s + F)\phi_k, \tag{IV.4}$$

it follows from (IV.4) and (IV.3) that $(i\omega_k - \mathcal{G}_1)\Gamma\phi_k = 0$. If we choose $z_k = \Gamma\phi_k$, then (IV.2b) follows immediately. Furthermore, from (IV.4) we obtain

$$\mathcal{C}\Pi\phi_k = -\mathcal{C}B_sR_1K\Gamma\phi_k - (\mathcal{C}B_sE_s + F)\phi_k. \tag{IV.5}$$

Substituting $\mathcal{C}\Pi\phi_k$ for (IV.5) in the first line of the Sylvester equation yields

$$\begin{aligned}&(i\omega_k - A_s)\Pi\phi_k - \mathcal{A}B_sR_1K\Gamma\phi_k + B_sR_1K\mathcal{G}_1\Gamma\phi_k \\ &= (\mathcal{A}B_sE_s - B_sE_sS)\phi_k,\end{aligned} \tag{IV.6}$$

and utilizing $S\phi_k = i\omega_k\phi_k$ and $\mathcal{G}_1\Gamma\phi_k = i\omega\Gamma\phi_k$, we obtain from (IV.6) that

$$\Pi\phi_k = (i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_kB_s)(R_1K\Gamma\phi_k + E_s\phi_k). \tag{IV.7}$$

Finally, substituting $\Pi\phi_k$ for (IV.7) in (IV.4) yields

$$0 = P_s(i\omega_k)K\Gamma\phi_k + P_0(i\omega_k)E_s\phi_k + F\phi_k,$$

from which (IV.2a) follows as we chose $z_k = \Gamma\phi_k$.

Now assume that equations (IV.2a)–(IV.2b) have solutions $z_k \in Z$. Define $\Pi \in \mathcal{L}(W, X), \Gamma \in \mathcal{L}(W, Z)$ and $\Sigma = (\Pi, \Gamma)^T$ by

$$\begin{aligned}\Gamma &:= \sum_{k=1}^{q}\langle\cdot, \phi_k\rangle z_k, \\ \Pi &:= \sum_{k=1}^{q}\langle\cdot, \phi_k\rangle(i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_kB_s)(R_1Kz_k + E_s\phi_k).\end{aligned} \tag{IV.8}$$

The definitions imply that $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e) \subset \mathcal{D}(C_e)$, and we will show that Σ is the solution of the regulator equations.

Let $k \in \{1, 2, \ldots, q\}$ be arbitrary. Considering the first line of $(i\omega_k - A_e)\Sigma\phi_k - B_e\phi_k$, we obtain using (IV.2b), $S\phi_k = i\omega_k$, the definition of Π, and (IV.2a) that

$$\begin{aligned}&(i\omega_k - A_s)\Pi\phi_k + B_sR_1K\mathcal{G}_2\mathcal{C}\Pi\phi_k \\ &\quad - (\mathcal{A}B_sR_1K - B_sR_1K(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sR_1K))\Gamma\phi_k \\ &\quad - (\mathcal{A}B_sE_s - B_sE_sS - B_sR_1K\mathcal{G}_2(\mathcal{C}B_sE_s + F))\phi_k \\ &= B_sR_1K\mathcal{G}_2(\mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + \mathcal{C}B_sE_s\phi_k + F\phi_k) \\ &= B_sR_1K\mathcal{G}_2(P_s(i\omega_k)K\Gamma\phi_k + P_0(i\omega_k)E_s\phi_k + F\phi_k) = 0.\end{aligned}$$

Note that by (IV.2a) we also have

$$\begin{aligned}C_e\Sigma\phi_k + D_e\phi_k &= \mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + \mathcal{C}B_sE_s\phi_k + F\phi_k \\ &= P_s(i\omega_k)K\Gamma\phi_k + P_0(i\omega_k)E_s\phi_k + F\phi_k = 0,\end{aligned}$$

i.e., Σ solves the second regulator equation. Finally, the second line of $(i\omega_k - A_e)\Sigma\phi_k - B_e\phi_k$ yields

$$\begin{aligned}&- \mathcal{G}_2\mathcal{C}\Pi\phi_k + (i\omega_k - \mathcal{G}_1)\Gamma\phi_k - \mathcal{G}_2\mathcal{C}B_sR_1K\Gamma\phi_k \\ &- \mathcal{G}_2(\mathcal{C}B_sE_s + F)\phi_k \\ &= - \mathcal{G}_2(\mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + \mathcal{C}B_sE_s\phi_k + F\phi_k) = 0.\end{aligned}$$

Thus, as $\{\phi_k\}_{k=1}^q$ is a basis of $\mathbb{C}^q$ and the choice of k was arbitrary, Σ is the solution of the regulator equations $\Sigma S = A_e\Sigma + B_e$ and $C_e\Sigma + D_e = 0$. Now, by Theorem IV.3, the controller solves the output regulation problem.

It yet remains to prove the uniqueness of the solutions z_k of (IV.2a)–(IV.2b). Let z_k and z_k' be two solutions of (IV.2a)–(IV.2b), and use (IV.8) to define $\Sigma = (\Pi, \Gamma)^T$ and $\Sigma' = (\Pi', \Gamma')^T$ corresponding to z_k and z_k', respectively. It now follows from the above proof that both Σ and Σ' satisfy the Sylvester equation, and by the uniqueness of the solution of the Sylvester equation we must have $\Sigma = \Sigma'$. In particular, $z_k = \Gamma\phi_k = \Gamma'\phi_k = z_k'$, i.e., the solutions z_k of (IV.2a)–(IV.2b) are unique. □

Based on Theorem IV.5, we can now construct a regulating controller for the plant (III.1). Choose $Z = W$ and choose the controller parameter $Q \in \mathcal{L}(Y, U)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for A_s. Choose the remaining parameters as

$$\mathcal{G}_1 = S = \operatorname{diag}(i\omega_1, i\omega_2, \ldots, i\omega_q), \tag{IV.9a}$$

$$K = \epsilon K_0 = \epsilon\,[u_1, u_2, \ldots, u_q], \tag{IV.9b}$$

$$\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=1}^q = (-(P_s(i\omega_k)u_k)^*)_{k=1}^q, \tag{IV.9c}$$

where $\epsilon > 0$ is called the tuning parameter and $u_k \in U$ are chosen such that [32, Sec. 4.2]

$$\begin{cases}P_s(i\omega_k)u_k = y_k, & y_k \neq 0, \\ u_k \notin \mathcal{N}(P_s(i\omega_k)) \text{ arbitrary}, & y_k = 0,\end{cases} \tag{IV.10}$$

where we denote $y_k = -P_0(i\omega_k)E_s\phi_k - F\phi_k$. For this to be possible, we need to assume that $P_s(i\omega_k) \neq 0$ and $y_k \in \mathcal{R}(P_s(i\omega_k))$ for all $k \in \{1, 2, \ldots, q\}$, so that there exist some $u_k \in U$ satisfying (IV.10). However, this assumption is also necessary for the solvability of the output regulation problem by Theorem IV.5.

Proposition IV.6. *There exists an $\epsilon^* > 0$ such that the controller with the parameter choices* (IV.9a)–(IV.9c) *solves the output regulation problem for all $0 < \epsilon < \epsilon^*$.*

Proof. First of all we note that the choices of $\mathcal{G}_1$ and K imply that the equations (IV.2a)–(IV.2b) have the solutions $z_k = \epsilon^{-1}\phi_k$ if $P_0(i\omega_k)E_s\phi_k + F\phi_k \neq 0$ or $z_k = 0$ otherwise. Now, as Q exponentially stabilizes the plant and $\mathcal{C}$ is admissible for A_s, $\sigma(\mathcal{G}_1) = \{i\omega_k\}_{k=1}^q$, and

$$\sigma(\mathcal{G}_2^k P_s(i\omega_k)K_0^k) = \sigma(-(P_s(i\omega_k)u_k)^* P_s(i\omega_k)u_k) \subset \mathbb{C}_-$$

as $P_s(i\omega_k)u_k \neq 0$ for $k \in \{1,2,\dots,q\}$, we have by Lemma IV.4 that there exists an $\epsilon^* > 0$ such that the closed-loop system is exponentially stable for all $0 < \epsilon < \epsilon^*$. Thus, by Theorem IV.5 the controller solves the output regulation problem. □

B. The Internal Model Principle

Before presenting an approximate robust controller in §IV-C and a robust controller in §IV-D, we will present a general result that characterizes robust controllers. That is, we will show that in order for a controller to achieve robust output regulation, it has to contain an internal model of the dynamics of the exosystem. We will express this using the following $\mathcal{G}$-conditions [33, Def. 10].

Definition IV.7. A quadruple of bounded operators $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ is said to satisfy the $\mathcal{G}$-conditions if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \quad \forall k \in \{1,2,\dots,q\} \tag{IV.11a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\}. \tag{IV.11b}$$

Note that while the parameters K and Q are not present in the $\mathcal{G}$-conditions, they contribute to exponentially stabilizing the closed-loop system. The sufficiency part of the following result has been presented in the case $R_1 = R_2 = I$ in [17, Thm 4] and the necessity part extends the results of [14, Thm 5.2] and [11, Thm 7] to boundary control systems.

Theorem IV.8. *Assume that the closed-loop system is regular and exponentially stabilized by the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$. Then the controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions.*

Proof. Let us assume that the controller solves the robust output regulation problem and show that (IV.11) hold starting with (IV.11a). Let $k \in \{1,2,\dots,q\}$ be arbitrary and $w \in \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $z \in Z$ and $y \in Y$ such that $w = (i\omega_k - \mathcal{G}_1)z = \mathcal{G}_2 y$. Let us leave the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ unperturbed and choose such perturbations from $\mathcal{O}$ that $E_s' = 0$ and $F' = \langle \cdot, \phi_k \rangle (y - P_s(i\omega_k)Kz)$. Choose $\Sigma = (\Gamma, \Pi)^T \in \mathcal{L}(W, X_e)$ such that

$$\Gamma = \langle \cdot, \phi_k \rangle z, \quad \Pi = \langle \cdot, \phi_k \rangle (i\omega_k - A_s)(\mathcal{A}B_s - i\omega_k B_s)R_1 K z,$$

which can be shown to be the solution of the Sylvester equation by a direct computation. As $C_e\Sigma\phi_k + D_e'\phi_k = 0$ by the controller solving the robust output regulation problem, we obtain

$$\begin{aligned} w &= (i\omega_k - \mathcal{G}_1)z = \mathcal{G}_2 y = \mathcal{G}_2(P_s(i\omega_k)Kz + F'\phi_k) \\ &= \mathcal{G}_2(\mathcal{C}\Pi\phi_k + \mathcal{C}B_s R_1 K\Gamma\phi_k + F'\phi_k) \\ &= \mathcal{G}_2(C_e\Sigma\phi_k + D_e'\phi_k) = 0, \end{aligned}$$

and thus $w = 0$, which concludes the first part of the necessity proof.

Let us now show that (IV.11b) holds. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and let $\phi \in W$ be such that $\|\phi\| = 1$. Leave the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ unperturbed and choose $E' = 0$ and $F' = \langle \cdot, \phi \rangle y \in \mathcal{L}(W, Y)$. If we choose $\Sigma = 0 \in \mathcal{L}(W, X_e)$, for all $v \in W$ we have $\Sigma S v = 0$ and

$$\begin{aligned} A_e\Sigma v + B_e' v &= \begin{bmatrix} -B_s R_1 K \mathcal{G}_2 F' v \\ \mathcal{G}_2 F' v \end{bmatrix} = \begin{bmatrix} -\langle v, \phi \rangle B_s R_1 K \mathcal{G}_2 y \\ \langle v, \phi \rangle \mathcal{G}_2 y \end{bmatrix} \\ &= 0, \end{aligned}$$

and thus, $\Sigma = 0$ is the unique solution of the Sylvester equation. As the controller solves the robust output regulation problem, we have by Theorem IV.3 that

$$0 = C_e\Sigma\phi + D_e'\phi = F'\phi = \langle \phi, \phi \rangle y = y,$$

which concludes the necessity proof. The sufficiency part follows by simple modifications from [17, Thm. 4]. □

Remark IV.9. Theorem IV.8 states that any controller that stabilizes a regular closed-loop system exponentially and satisfies the $\mathcal{G}$-conditions solves the robust output regulation problem. In particular, this implies that if a robust regulating controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ is constructed, then every controller $(\mathcal{G}_1, \mathcal{G}_2', K', Q')$, where $(\mathcal{G}_2', K', Q')$ are boundedly perturbed $(\mathcal{G}_2, K, Q)$, solves the robust output regulation problem, provided that the closed-loop system remains exponentially stable and $(\mathcal{G}_1, \mathcal{G}_2')$ satisfy the $\mathcal{G}$-conditions. Note that only rather specific perturbations would be allowed in $\mathcal{G}_1$ as it has to include an exact internal model of the dynamics of the exosystem.

Note that the rank-nullity theorem and the second $\mathcal{G}$-condition imply that $\dim Z \geq \dim \mathcal{R}(\mathcal{G}_2) = \dim Y$. Thus, if the output space of the system is infinite-dimensional as, e.g., in the wave equation of §II, Theorem IV.8 implies that robust controllers for such systems are necessarily infinite-dimensional. However, we can construct a finite-dimensional controller that solves the robust output regulation problem *approximately*. We will construct such a controller in the next section. Finally, in §IV-D we will construct an infinite-dimensional controller that achieves exact robust output regulation. The following assumption is required for the remaining sections:

Assumption IV.10. The transfer function $P_s(\lambda)$ is surjective at all the eigenvalues $\{i\omega_k\}_{k=1}^q$ of S.

C. An approximate robust controller

In this section, we consider approximate robust output regulation on Y. We will solve the control problem by choosing a subspace Y_N of Y and constructing a controller that robustly tracks the reference signal projected onto Y_N. If Y_N is chosen to be finite-dimensional, we can construct a finite-dimensional robust regulating controller even if the output space of the system is infinite-dimensional. Furthermore, we derive an upper bound for the asymptotic regulation error. Our result generalizes the controller structure presented in [16, Thm. 3.5] where discrete-time systems with constant reference signals were considered.

Let Y_N be a closed subspace of Y and choose $Z := Y_N^q$. Choose the controller parameter $Q \in \mathcal{L}(Y,U)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$. Choose the remaining parameters as

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I_{Y_N}, i\omega_2 I_{Y_N}, \dots, i\omega_q I_{Y_N}), \tag{IV.12a}$$

$$K = \epsilon K_0 = \epsilon[K_0^1, K_0^2, \dots, K_0^q] \in \mathcal{L}(Z,U), \tag{IV.12b}$$

$$\mathcal{G}_2 = (\mathcal{G}_{20}^k P_N)_{k=1}^q \in \mathcal{L}(Y,Z), \tag{IV.12c}$$

where P_N is a projection onto Y_N, and $\mathcal{G}_{20}^k$ and K_0^k are such that

$$\sigma(\mathcal{G}_{20}^k P_N P_s(i\omega_k) K_0^k) \subset \mathbb{C}_- \tag{IV.13}$$

for all $k \in \{1,2,\dots,q\}$. We can choose, e.g., $\mathcal{G}_{20}^k = -I_{Y_N}$ and $K_0^k = (P_N P_s(i\omega_k))^{[-1]}$ for $k \in \{1,2,\dots,q\}$, and conversely, the spectrum condition implies that $\mathcal{G}_{20}^k$ and $P_N P_s(i\omega_k) K_0^k$ are boundedly invertible.

In the following theorem, we show that a controller with the aforementioned structure solves the approximate robust output regulation problem. Furthermore, we will show that for some constants $M, \alpha > 0$ and all $t \ge 0$ the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \le M e^{-\alpha t}(\|x_{e0}\|^2 + \|v_0\|^2) + \delta \|v_0\|^2, \tag{IV.14}$$

where x_{e0} and v_0 are the initial states of the closed-loop system and the exosystem, respectively, and δ is given by

$$\delta = \left\|(I-P_N)\textstyle\sum_{k=1}^q \left(P_s(i\omega_k)Kz_k + P_0(i\omega_k)E_s v_k + F v_k\right)\right\|^2, \tag{IV.15}$$

where v_k are the components of the unit vector $v_{\max} \in W$ satisfying $\|C_e\Sigma + D_e\| = \|C_e\Sigma v_{\max} + D_e v_{\max}\|_Y$ and $z_k = \Gamma v_k$ where Γ is given in (IV.16). Note that since W is finite dimensional, $v_{\max}$ is well-defined. Further note that we cannot guarantee pointwise convergence for the regulation error, and therefore the upper bound is presented in the integral form. Finally, since $\sum_{k=1}^q \left(P_s(i\omega_k)Kz_k + P_0(i\omega_k)E_s v_k + F v_k\right) \in Y$, the projection P_N (or rather the space Y_N) can be chosen such that δ becomes arbitrarily small. We will demonstrate this procedure in §V for the wave equation.

Theorem IV.11. *There exists an* $\epsilon^* > 0$ *such that for all* $0 < \epsilon < \epsilon^*$ *the controller with the parameter choices* (IV.12a)–(IV.12c) *solves the approximate robust output regulation problem and there exist some constants* $M, \alpha > 0$ *such that for all* $t \ge 0$ *the regulation error satisfies* (IV.14).

Furthermore, the controller is robust with respect to those perturbations of class $\mathcal{O}$ *that give rise to an exponentially stable perturbed closed-loop system, and the regulation error behaves as in* (IV.14) *for the perturbed parameters of the plant and the exosystem.*

Proof. By Lemma IV.4, the closed-loop system is exponentially stable for all sufficiently small $\epsilon > 0$. Thus, as $\sigma(S) \subset i\mathbb{R}$, the Sylvester equation has a unique solution $\Sigma = (\Pi, \Gamma)^T$, and a direct computation using (III.6) verifies that

$$\begin{aligned}\Gamma &= (\Gamma_k)_{k=1}^q \\ &= -\epsilon^{-1}\left(\langle \cdot, \phi_k\rangle (P_N P_s(i\omega_k)K_0^k)^{-1} P_N (P_0(i\omega_k)E_s + F)\phi_k\right)_{k=1}^q, \\ \Pi &= \sum_{k=1}^q \langle \cdot, \phi_k \rangle (i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_k B_s)(R_1 K\Gamma + E_s)\phi_k\end{aligned} \tag{IV.16}$$

solves $\Sigma S \phi_k = A_e \Sigma \phi_k + B_e \phi_k$, i.e., $(i\omega_k - A_e)\Sigma\phi_k = B_e\phi_k$ for all $k \in \{1,2,\dots,q\}$. Here one also uses that our Γ satisfies

$$\begin{aligned}P_N P_s(i\omega_k)K\Gamma\phi_k &= \epsilon P_N P_s(i\omega_k)K_0^k \Gamma_k \phi_k \\ &= -P_N P_0(i\omega_k)E_s\phi_k - P_N F\phi_k.\end{aligned} \tag{IV.17}$$

Note that (IV.16) is well-defined and bounded since $P_N P_s(i\omega_k)K_0^k$ are boundedly invertible by (IV.13).

Let us now consider the behavior of the regulation error. By [15, Lem. 4.3], we may write

$$e(t) = C_e \mathbb{T}_e(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(t),$$

and we obtain that for all $t \ge 0$

$$\begin{aligned}&\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \\ &= \int_t^{t+1} \|C_e\mathbb{T}_e(s)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(s)\|^2 \,\mathrm{d}s \\ &\le M e^{-\alpha t}(\|x_{e0}\|^2 + \|v_0\|^2) + \|C_e\Sigma + D_e\|^2 \|v_0\|^2\end{aligned}$$

for some $M, \alpha > 0$ as Σ is bounded, $\mathbb{T}_e$ is exponentially stable, C_e is admissible for $\mathbb{T}_e$, and due to the structure of the signal generator $\|v(t)\| = \|e^{St}v_0\| = \|v_0\|$.

We will show that

$$\begin{aligned}&C_e\Sigma v_{\max} + D_e v_{\max} \\ &= (I - P_N)\sum_{k=1}^q \left(P_s(i\omega_k)Kz_k + P_0(i\omega_k)E_s v_k + F v_k\right).\end{aligned}$$

A direct computation using (IV.16) shows that

$$\begin{aligned}&C_e\Sigma v_{\max} + D_e v_{\max} \\ &= \sum_{k=1}^q \left(P_s(i\omega_k)K\Gamma v_k + P_0(i\omega_k)E_s v_k + F v_k\right),\end{aligned} \tag{IV.18}$$

Denoting $z_k = \Gamma v_k$, we have by (IV.17) that

$$P_N P_s(i\omega_k)Kz_k = -P_N P_0(i\omega_k)E_s v_k - P_N F v_k, \tag{IV.19}$$

and now, combining (IV.19) with (IV.18) yields

$$\begin{aligned}&C_e\Sigma v_{\max} + D_e v_{\max} \\ &= (I - P_N)\sum_{k=1}^q \left(P_s(i\omega_k)Kz_k + P_0(i\omega_k)E_s v_k + F v_k\right),\end{aligned}$$

which implies (IV.15), and thus, (IV.14).

If the parameters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed in such a way that the closed-loop system remains exponentially stable, then the regulation error asymptotically satisfies $\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \le M' e^{-\alpha' t}(\|x_{e0}\|^2 + \|v_0\|^2) + \|C_e'\Sigma' +$

$D_e'\|^2\|v_0\|^2$ for all $t \geq 0$, where $M', \alpha' > 0$, and C_e', D_e' and Σ' are related to the the perturbed closed-loop system. By repeating the above computations with the perturbed parameters we clearly obtain $C_e'\Sigma' v_{\max}' + D_e' v_{\max}' = (I - P_N)\sum_{k=1}^q P_s'(i\omega_k)Kz_k' + P_0'(i\omega_k)E_s'v_k' + F'v_k')$, where z_k' is the unique solution of $P_N P_s'(i\omega_k)Kz_k' = -P_N P_0'(i\omega_k)E_s'v_k' - P_N F'v_k'$ in $\mathcal{N}(i\omega_k - \mathcal{G}_1)$. Thus, the controller approximately solves the robust output regulation problem. □

Remark IV.12. As an alternative to the error estimate given in (IV.14), one can make a coarser choice for δ that does not require $v_{\max}$:

$$\delta = \sum_{k=1}^q \left|\left|(I - P_N)\big(P_s(i\omega_k)Kz_k + P_0(i\omega_k)E_s\phi_k + F\phi_k\big)\right|\right|^2,$$

where $\{\phi_k\}_{k=1}^q$ is the Euclidean basis of W and $z_k = \Gamma\phi_k$.

Corollary IV.13. *In Theorem IV.11, the regulation error satisfies $e^{\beta\cdot}P_N e(\cdot) \in L^2([0,\infty);Y)$ for some $\beta > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$. Under perturbations of class $\mathcal{O}$ that give rise to an exponentially stable closed-loop system, the regulation error satisfies $e^{\beta'\cdot}P_N e(\cdot) \in L^2([0,\infty);Y)$ for some $\beta' > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$.*

Proof. Let us first show that $P_N C_e\Sigma + P_N D_e = 0$. A direct computation using (IV.16) together with (IV.17) shows that for all $k \in \{1, 2, \ldots, q\}$:

$$\begin{aligned} &P_N C_e\Sigma\phi_k + P_N D_e\phi_k \\ &= P_N P_s(i\omega_k)K\Gamma\phi_k + P_N P_0(i\omega_k)E_s\phi_k + P_N F\phi_k = 0, \end{aligned}$$

and as $\{\phi_k\}_{k=1}^q$ form a basis of $\mathbb{C}^q$, we have that $P_N C_e\Sigma + P_N D_e = 0$. By the proof of Theorem IV.11 we now have for some $\beta > 0$ that

$$\begin{aligned} \int_t^{t+1} \|e^{\beta s}P_N e(s)\|^2\, \mathrm{d}s &\leq e^{\beta(t+1)}\int_t^{t+1} \|P_N e(s)\|^2\, \mathrm{d}s \\ &\leq e^{\beta} M e^{(\beta-\alpha)t}(\|x_{e0}\|^2 + \|v_0\|^2), \end{aligned}$$

so for any $0 < \beta < \alpha$ we obtain

$$\begin{aligned} \int_0^{\infty} \|e^{\beta s}P_N e(s)\|^2\, \mathrm{d}s &\leq e^{\beta} M(\|x_{e0}\|^2 + \|v_0\|^2)\sum_{t=0}^{\infty} e^{(\beta-\alpha)t} \\ &= \frac{e^{\beta}M(\|x_{e0}\|^2 + \|v_0\|^2)}{1 - e^{\beta-\alpha}}, \end{aligned}$$

by which $e^{\beta\cdot}P_N e(\cdot) \in L^2([0,\infty))$ for any $0 < \beta < \alpha$. By the robustness part of Theorem IV.11, the same holds for some $0 < \beta' < \alpha'$ under perturbations of class $\mathcal{O}$ that give rise to an exponentially stable closed-loop system. □

D. A robust controller

In this section, we utilize the approximate controller structure of the previous section to construct an exact robust controller which, however, necessarily has infinite-dimensional state space if the output space of the plant is infinite-dimensional. Thus, we choose $Z = Y^q$ and choose the controller parameter $Q \in \mathcal{L}(U, Y)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$. Following [11, Sec. IV] or [17, Thm. 8], we choose the remaining parameters as

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y) \in \mathcal{L}(Z), \quad \text{(IV.20a)}$$

$$K = \epsilon K_0 = \epsilon\left[K_0^1, K_0^2, \ldots, K_0^q\right] \in \mathcal{L}(Z, U), \quad \text{(IV.20b)}$$

$$\mathcal{G}_2 = (-(P_s(i\omega_k)K_0^k)^*)_{k=1}^q \in \mathcal{L}(Y, Z). \quad \text{(IV.20c)}$$

Above the components K_0^k can be chosen freely provided that $P_s(i\omega_k)K_0^k$ are invertible. If we choose $K_0^k = P_s(i\omega_k)^{[-1]}$, then $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1, 2, \ldots q\}$, then the controller is the same as the approximate controller for the choice $Y_N = Y$. The following result follows immediately from Corollary IV.13.

Corollary IV.14. *There exists an $\epsilon^* > 0$ such that a controller with the parameter choices given in* (IV.20a)–(IV.20c) *solves the robust output regulation problem for all $0 < \epsilon < \epsilon^*$.*

Remark IV.15. The above result also follows from Lemma IV.4 and Theorem IV.8 as the choice $K_0^k = P_s(i\omega_k)^{[-1]}$ yields $\sigma(\mathcal{G}_2^k P_s(i\omega_k)K_0^k) = \sigma(-I_Y) \subset \mathbb{C}_-$, which together with the choice of Q completes the assumptions of Lemma IV.4, by which the closed-loop system is exponentially stable. Furthermore, it has been shown in the proof of [11, Thm 8] that $\mathcal{G}_1$ and $\mathcal{G}_2$ in (IV.20a)–(IV.20c) satisfy the $\mathcal{G}$-conditions, and thus, the controller solves the robust output regulation problem by Theorem IV.8.

V. Approximate robust regulation of the wave equation

Consider the wave equation as given in (II.1) with the spatial domain $\Omega := \{\zeta \in \mathbb{R}^2 \mid 1 < \|\zeta\| < 2\}$. Choose the partition $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0 = \{\zeta \in \partial\Omega \mid \|\zeta\| = 1\}$ and $\Gamma_1 = \{\zeta \in \partial\Omega \mid \|\zeta\| = 2\}$ which satisfies the assumption in (II.6), e.g., for $\zeta_0 = 0$, and thus the results presented in Section II-B are applicable.

For the approximate robust output regulation problem, let $\delta = 0.01$ be given. We choose the output space as $Y := L^2(\Gamma_1)$ which is equivalent to $L^2([0, 2\pi])$. Thus, for the finite-dimensional closed subspace Y_N we may choose, e.g.,

$$Y_N := \operatorname{span}\{1, \cos(k\cdot), \sin(k\cdot) \mid k = 1, \ldots, N\},$$

and the projection P_N from Y onto Y_N is then given by

$$P_N y := \frac{1}{\sqrt{2\pi}}\langle y, 1\rangle + \frac{1}{\sqrt{\pi}}\sum_{k=1}^N \left(\langle y, \cos(k\cdot)\rangle + \langle y, \sin(k\cdot)\rangle\right). \quad \text{(V.1)}$$

By standard Fourier analysis, it holds that for all $f \in L^2([0, 2\pi])$, we have $\lim_{N\to\infty} \|(1 - P_N)f\| = 0$, and thus, by Theorem IV.11, for a given reference y_{ref}, we can choose N in (V.1) sufficiently large such that asymptotically the regulation error becomes smaller than $\delta\|v_0\|^2$ (in the L^2-sense).

Let the reference and disturbance signals be given by

$$\begin{aligned} y_{ref}(\theta, t) &= -\frac{1}{2\pi^2}(\pi - \theta)^2\sin(\pi t) - \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos(2\pi t) \\ d(\theta, t) &= \cos(\theta)\sin(2\pi t) + \sin(\theta)\sin(\pi t) \end{aligned}$$

and the disturbance d acts on Γ_1. Thus, we choose $S = \operatorname{diag}(-2i\pi, -i\pi, i\pi, 2i\pi)$, and the operators E and F are chosen such that $y_{ref} = -Fv$ and $d = Ev$ for $v_0 = 1$. The controller parameter Q is chosen as $Q(\theta) = 3$, and according to §IV-C we choose

$$\mathcal{G}_1 = \mathrm{diag}(-2i\pi I_{Y_N}, -i\pi I_{Y_N}, i\pi I_{Y_N}, 2i\pi I_{Y_N}), \tag{V.2a}$$

$$K = \epsilon \left[K_0^1, K_0^2, K_0^3, K_0^4\right] \tag{V.2b}$$

$$\mathcal{G}_2 = (-P_N)_{k=1}^4 \tag{V.2c}$$

where $K_0^k = (P_N P_s(i\omega_k))^{[-1]}$, $N = 5$ and $\epsilon = 0.15$.

For simulation, the operators related to the wave equation are approximated by the orthonormal eigenfunctions of the Laplacian Δ with homogeneous boundary conditions. In polar coordinates, these are of the form

$$\phi_{n0}^1(r) = \frac{1}{\sqrt{2\pi}}\varphi_{n0}(r), \qquad n \in \mathbb{N}$$

$$\phi_{nm}^1(r,\theta) = \frac{1}{\sqrt{\pi}}\varphi_{nm}(r)\cos(m\theta), \qquad m, n \in \mathbb{N}$$

$$\phi_{nm}^2(r,\theta) = \frac{1}{\sqrt{\pi}}\varphi_{nm}(r)\sin(m\theta), \qquad m, n \in \mathbb{N},$$

where $\varphi_{nm}(r)$ are the appropriately normalized Bessel functions corresponding to the radial part of the Laplacian such that the functions $\{\phi_{nm}^{1,2}\}$ form an orthonormal basis of $L^2(\Omega)$. The eigenvalues are computed numerically and in the simulation we use $n = 8$ radial and $m + 1 = 12$ angular eigenfunctions corresponding to the eigenvalues. The transfer function P_s is computed using the approximated operators, and the initial conditions are given by $x_0 = 0$ and $z_0 = 0$.

In Figure 1, the output profile y of the controlled wave equation and the reference profile y_{ref} are displayed for $t \in [0, 10]$. It can be seen that the output starts to follow the reference signal rather soon, even though some undershooting can be observed throughout the simulation.

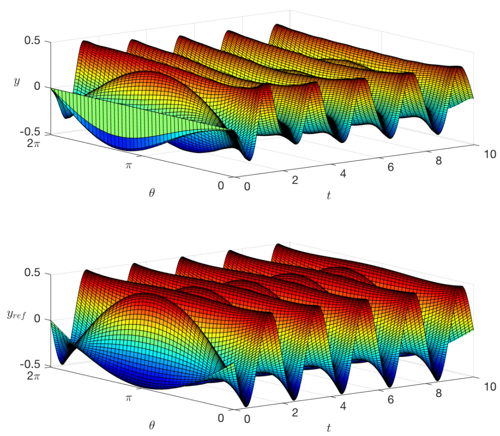

Fig. 1. The output profile y of the controlled wave equation and the reference profile y_{ref} for $t \in [0, 10]$ and in the same scales.

In Figure 2, the time average of the norm of the regulation error is displayed for $t \in [0, 20]$. Here it can be seen that, apart from the oscillations and initial errors, the regulation error decays at an exponential rate and that asymptotically it decays beyond the given $\delta\|v_0\|^2$. In Figure 3, the wave profile of the controlled system is displayed at time $t = 9$ and in Figure 4, the disturbance signal is displayed for $t \in [0, 6]$.

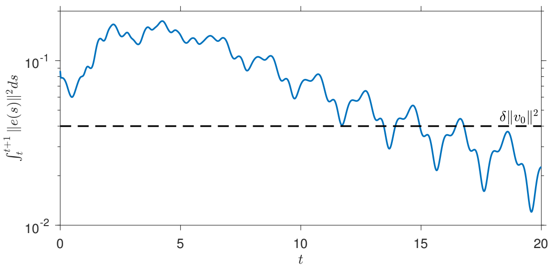

Fig. 2. The regulation error $\int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds$ for $t \in [0, 20]$.

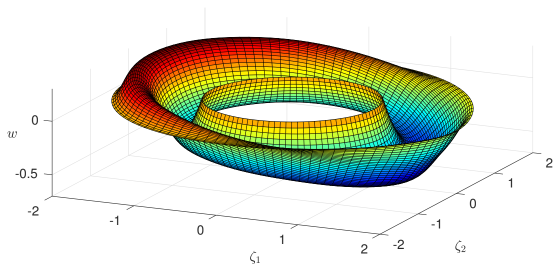

Fig. 3. The wave profile of the controlled system at $t = 9$.

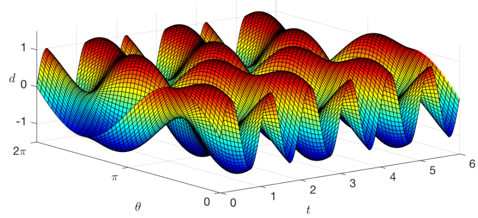

Fig. 4. The disturbance signal d for $t \in [0, 6]$.

VI. Conclusions

We developed output regulation for abstract boundary control systems, parametrizing all regulating and robust regulating controllers, and also suggesting some particular choices of such controllers. Since the internal model principle implies that the state space of any robust controller for a system with infinite-dimensional output space has infinite dimension, we extended the concept of approximate robust output regulation to boundary control systems. We demonstrated that approximate robust regulation can be achieved with a finite-dimensional controller by constructing such a controller for the two-dimensional wave equation and demonstrating its performance with numerical simulations.

REFERENCES

[1] C. I. Byrnes, I. G. Laukó, D. S. Gilliam, and V. I. Shubov, "Output regulation problem for linear distributed parameter systems," *IEEE Trans. Automat. Control*, vol. 45, no. 12, pp. 2236–2252, 2000.

[2] E. Immonen and S. Pohjolainen, "Feedback and feedforward output regulation of bounded uniformly continuous signals for infinite-dimensional systems," *SIAM J. Control Optim.*, vol. 45, no. 5, pp. 1714–1735, 2006.

[3] V. Natarajan, D. Gilliam, and G. Weiss, "The state feedback regulator problem for regular linear systems," *IEEE Trans. Automat. Control*, vol. 59, no. 10, pp. 2708–2723, 2014.

[4] J. Deutscher, "A backstepping approach to the output regulation of boundary controlled parabolic PDEs," *Automatica*, vol. 57, pp. 56 – 64, 2015.

[5] X. Xu and S. Dubljevic, "Output regulation problem for a class of regular hyperbolic systems," *International Journal of Control*, vol. 89, no. 1, pp. 113–127, 2016.

[6] O. J. Staffans, *Well-Posed Linear Systems*. Cambridge and New York: Cambridge University Press, 2005.

[7] G. Weiss, "Regular linear systems with feedback," *Math. Control Signals Systems*, vol. 7, pp. 23–57, 1994.

[8] T. Hämäläinen and S. Pohjolainen, "A finite-dimensional robust controller for systems in the CD-algebra," vol. 45, no. 3, pp. 421–431, 2000.

[9] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica*, vol. 39, pp. 1555–1569, 2003.

[10] S. Boulite, S. Hadd, H. Nounou, and M. Nounou, "The PI-controller for infinite dimensional linear systems in Banach state spaces," in *Proceedings of the 2009 American Control Conference*, St. Louis, Missouri, June 10–12 2009.

[11] L. Paunonen, "Controller Design for Robust Output Regulation of Regular Linear Systems," *IEEE Trans. Automat. Control*, vol. 61, no. 10, pp. 2974–2986, 2016.

[12] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," vol. 2, no. 2, pp. 170–194, 1975.

[13] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," vol. 21, no. 1, pp. 25–34, 1976.

[14] L. Paunonen and S. Pohjolainen, "Internal model theory for distributed parameter systems," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4753–4775, 2010.

[15] ——, "The internal model principle for systems with unbounded control and observation," *SIAM Journal on Control and Optimization*, vol. 52, no. 6, pp. 3967–4000, 2014.

[16] L. Paunonen, "Robust output regulation for continuous-time periodic systems," *IEEE Trans. Automat. Control*, vol. 62, no. 9, pp. 4363–4375, 2017.

[17] J.-P. Humaloja and L. Paunonen, "Robust regulation of infinite-dimensional port-Hamiltonian systems," *IEEE Trans. Automat. Control*, vol. 63, no. 5, pp. 1480–1486, 2018.

[18] M. Kurula and H. Zwart, "Linear wave systems on n-D spatial domains," *Internat. J. Control*, vol. 88, no. 5, pp. 1063–1077, 2015.

[19] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*. Basel: Birkhäuser Verlag, 2009, (electronic version).

[20] M. Kurula and H. Zwart, "The duality between the gradient and divergence operators on bounded Lipschitz domains," http://eprints.eemcs.utwente.nl/22373/, Department of Applied Mathematics, University of Twente, Enschede, Memorandum 1994, October 2012.

[21] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

[22] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator differential equations*, ser. Mathematics and its Applications (Soviet Series). Dordrecht: Kluwer Academic Publishers Group, 1991, vol. 48, translation and revised from the 1984 Russian original.

[23] J. Malinen and O. J. Staffans, "Conservative boundary control systems," *J. Differential Equations*, vol. 231, pp. 290–312, 2006.

[24] ——, "Impedance passive and conservative boundary control systems," *Complex Anal. Oper. Theory*, vol. 1, pp. 279–30, 2007.

[25] Y.-X. Guo and P.-F. Yao, "On boundary stability of wave equations with variable coefficients," *Acta Math. Appl. Sin. Engl. Ser.*, vol. 18, no. 4, pp. 589–598, 2002.

[26] R. F. Curtain and G. Weiss, "Exponential stabilization of well-posed systems by colocated feedback," *SIAM J. Control Optim.*, vol. 45, no. 1, pp. 273–297, 2006.

[27] C. Bardos, G. Lebeau, and J. Rauch, "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary," *SIAM J. Control Optim.*, vol. 30, no. 5, pp. 1024–1065, 1992.

[28] P.-F. Yao, "On the observability inequalities for exact controllability of wave equations with variable coefficients," *SIAM J. Control Optim.*, vol. 37, no. 5, pp. 1568–1599, 1999.

[29] V. Q. Phóng, "The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems," *Mathematische Zeitschrift*, vol. 208, no. 1, pp. 567–588, 1991.

[30] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 2000, vol. 194.

[31] T. Hämäläinen and S. Pohjolainen, "A self-tuning robust regulator for infinite-dimensional systems," *IEEE Trans. Automat. Control*, vol. 56, no. 9, pp. 2116–2127, 2011.

[32] L. Paunonen, "Robust controllers for regular linear systems with infinite-dimensional exosystems," *SIAM J. Control Optim.*, vol. 55, no. 3, pp. 1567–1597, 2017.

[33] T. Hämäläinen and S. Pohjolainen, "Robust regulation of distributed parameter systems with infinite-dimenional exosystems," *SIAM J. Control Optim.*, vol. 48, no. 9, pp. 4846–4873, 2010.

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PUBLICATION IV

J.-P. Humaloja and S. Dubljevic
Linear model predictive control for Schrödinger equation
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Linear Model Predictive Control for Schrödinger Equation*

Jukka-Pekka Humaloja**,1 and Stevan Dubljevic[2]

***Abstract*— The paper considers the finite-horizon constrained optimal control problem for Schrödinger equation with boundary controls and boundary observations. The plant is mapped from continuous to discrete time using the Cayley-Tustin transform, which preserves input-output–stability of the plant. The proposed transformation is structure and energy preserving and does not induce order reduction associated with the spatial discretization. The controller design setting leads to the finite horizon constrained quadratic regulator problem, which is easily realized and accounts in explicit manner for input and output/state constraints. The model predictive control (MPC) design is realized for Schrödinger equation and the results are illustrated with numerical simulations showing successful stabilization of Schrödinger equation with simultaneous satisfaction of input and output/state constraints.**

I. Introduction

A central concern in modern chemistry is controlled making and breaking of chemical molecular bonds. The state-of-the-art laser technology provides foundation for laser control in a favorable manner to alter the molecular dynamics phenomena. In particular, laboratory implementation and design are focused on successful laser field realizations capable of altering constructive and destructive interferences of the underlying molecular wave function [12].

In molecular control one seeks to achieve the best possible solution, and therefore it is natural to consider optimal control design methodologies as a starting point. Along this line, optimal control of quantum-mechanical systems was first considered by Dahleh, Peirce and Rabitz in [2], [9] where the finite-dimensional Schrödinger equation was considered under different circumstances. Later on, controller design problems for the finite-dimensional Schrödinger equation were considered in [7] by Mirrahimi and Rouchon and in [8] by Mirrahimi, Rouchon and Turinici. Recently, control of the infinite-dimensional Schrödinger equation has been considered, e.g., in [4], [10], [11]. The important notions of boundary applied actuation and observation in the context of Schrödinger equation have beed addressed in detailed manner in [14], due to the importance of accurate steering a system from initial to final observable state in the finite time.

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In reality, the key to the laser control realization is accurate description of the Hamiltonian. In some cases for simple molecular species [2], [9] one can describe Hamiltonian with high accuracy and successfully apply design, while for polyatomic molecules and the subsequent design achieving acceptable accuracy is a very complex task. In addition, to this complexity, the control realizations with the presence of complex external fields contribute to the higher level of difficulty in realizing and implementing molecular control [13]. Besides these difficulties, one might need to account for the generation of undesirable chemical products (reflected in breaking the wrong chemical bonds). Along this line in [13], a theoretical method for optimization based control has been presented with application of multiple constraints and with guaranteed convergence to desired physical objectives. Motivated by this notion, we explore another design methodology that is optimal, explicitly accounts for constraints and is already well-known in control practice, a model predictive control [6], [17].

In particular, we consider a linear model predictive control design which has been successfully applied for similar types of distributive parameter systems [16]. It will be shown that one can extend the well-known design of the MPC to the setting of complex distributed parameter systems described by the Schrödinger equation and incorporate input and output/state constraints - as well as optimality - in the computationally fast and numerically realizable design setting. An additional benefit to our MPC design is that continuous Schrödinger equation model of the underlying plant is not subjected to any type of order reduction by spatial discretization and the issue of boundary applied actuation is realized by applying an appropriate exact boundary transformation [14]. Generalization to infinite-dimensional systems requires taking several theoretical aspects into account, even though here we omit some of the technical details.

The structure of this paper is as follows. In Section II, we present the general Cayley-Tustin time discretization scheme for distributed parameter systems which is symplectic and structure preserving [3]. In Section III, we present Schrödinger equation with boundary controls and boundary observations and apply the Cayley-Tustin discretization to the system. In Section IV, the model predictive control problem is presented and solved for the Schrödinger equation. Numerical simulations are presented as well. Finally, conclusions are presented in Section V.

Here $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, kernel and resolvent of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively. For a linear operator $A : \mathcal{D}(A) \subset X \to X$ and a fixed $s_0 \in \rho(A)$, define the scale spaces $X_1 := \overline{(\mathcal{D}(A), \|(s_0 - A)\cdot\|)}$ and $X_{-1} := \overline{(X, \|(s_0 - A)^{-1}\cdot\|)}$ [14, Sec. 2.10]. The scale spaces are related by $X_1 \subset X \subset X_{-1}$ where the inclusions are dense and with continuous embeddings. The extension of A to X_{-1} is denoted by A_{-1}.

II. Cayley-Tustin Time Discretization

Consider a linear infinite-dimensional system described by the following equations:

$$\dot{x}(\zeta, t) = Ax(\zeta, t) + Bu(t), \quad x(\zeta, 0) = x_0(\zeta) \tag{1a}$$
$$y(t) = Cx(\zeta, t) + Du(t). \tag{1b}$$

The state-space X, the input space U and the output space Y are assumed to be Hilbert spaces. The linear operator $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup and for the other operators we assume that $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$ and $D \in \mathcal{L}(U, Y)$.

Given a discretization parameter $h > 0$, a Crank-Nicolson type time discretization of (1) is given by

$$\frac{x(\cdot, ih) - x(\cdot, (i-1)h)}{h} \approx A\frac{x(\cdot, ih) + x(\cdot, (i-1)h)}{2} + Bu(ih)$$
$$y(ih) \approx C\frac{x(\cdot, ih) + x(\cdot, (i-1)h)}{2} + Du(ih)$$

for $i \geq 1$. Approximating $u(ih)$ by $u_i^h/\sqrt{h}$ (using a chosen sampling), it has been shown in [5] that the Cayley-Tustin discretization is a convergent time discretization scheme for a general class of input-output stable systems satisfying $\dim U = \dim Y = 1$ such that $y_i^h/\sqrt{h}$ converges to $y(ih)$ in several different ways. A straightforward manipulation yields the Cayley-Tustin transform $(A, B, C, D) \to (A_d, B_d, C_d, D_d)$ by

$$S = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} := \begin{bmatrix} (\delta + A)(\delta - A)^{-1} & \sqrt{2\delta}(\delta - A_{-1})^{-1}B \\ \sqrt{2\delta}C(A - \delta)^{-1} & G(\delta) \end{bmatrix}$$

where $G(\delta) := C(\delta - A_{-1})^{-1}B + D$ denotes the transfer function of the system and $\delta = 2/h$ which needs to be in $\rho(A)$. It is easy to see that the operator A_d can be equivalently expressed as $A_d = -I + 2\delta(\delta - A)^{-1}$.

III. Schrödinger equation

In this section, we apply the Cayley-Tustin time discretization to the boundary controlled Schrödinger equation on the unit interval $\zeta \in [0, 1]$. The system is given for $x(\zeta, 0) = x_0(\zeta)$ by

$$\frac{\partial}{\partial t}x(\zeta, t) = j\frac{\hbar}{2m}\frac{\partial^2}{\partial \zeta^2}x(\zeta, t) - vx(\zeta, t) \tag{2a}$$
$$\frac{\partial}{\partial \zeta}x(0, t) = 0 \tag{2b}$$
$$\frac{\partial}{\partial \zeta}x(1, t) = u(t) \tag{2c}$$
$$x(0, t) = y(t) \tag{2d}$$

where $\hbar$ is the reduced Planck constant, m is the mass of the particle, $v > 0$ accounts for the potential energy of the particle and $u \in U, y \in Y$ are boundary control and boundary observation signals, respectively, where $U = Y := \mathbb{C}$.

In order to write the system (2) in the usual state-space form (1), we define an operator $\mathcal{A}$ by

$$\mathcal{A}x := j\frac{\hbar}{2m}\frac{\partial^2}{\partial \zeta^2}x - vx$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{x \in L_2(0, 1; \mathbb{C}) : \tfrac{\hbar}{2m}x \in H^2(0, 1; \mathbb{C}),\ \tfrac{\partial x}{\partial \zeta}(0) = 0\right\}.$$

Furthermore, we define a boundary control operator $\mathcal{B}$ by

$$\mathcal{B}x(\cdot, t) := \frac{\partial}{\partial \zeta}x(1, t)$$

with domain $\mathcal{D}(\mathcal{B}) := \mathcal{D}(\mathcal{A})$. The operator $\mathcal{A}$ corresponds to the port-Hamiltonian formulation of Schrödinger equation (see, e.g., [1, Ex. 2.18]), and [1, Thm. 2.3] implies that the operator $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ with domain $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$, i.e., the restriction of $\mathcal{A}$ to the kernel of $\mathcal{B}$, generates a C_0-semigroup.

The aforementioned implies that the pair $(\mathcal{A}, \mathcal{B})$ is a boundary control system in the sense of [14, Def. 10.1.1]. Thus, by [14, Prop. 10.1.2, Rem. 10.1.4] there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that the system (2) can be equivalently written as

$$\dot{x}(\zeta, t) = Ax(\zeta, t) + Bu(t), \quad x(0) = x_0 \tag{3a}$$
$$y(t) = \mathcal{C}x(\zeta, t) \tag{3b}$$

where $\mathcal{C} \in \mathcal{L}(X_1, Y)$ with domain $\mathcal{D}(\mathcal{C}) := \mathcal{D}(\mathcal{A})$ is defined as $\mathcal{C}x(\cdot, t) := x(0, t)$ so that (3b) corresponds to (2d).

The aforementioned operator B can be found by solving the abstract elliptic problem [14, Rem. 10.1.5] $\mathcal{A}f = sf$, $\mathcal{B}f = u$ for any $s \in \rho(A)$ and $u \in U$. The solution is unique and satisfies $f = (s - A_{-1})^{-1}Bu$. A direct computation shows that for any $s \in \rho(A)$ and $u \in U$, the solution is given by

$$f_s(\zeta) = \frac{1}{c_s}\frac{\cosh(c_s\zeta)}{\sinh(c_s)}u \tag{4}$$

where $c_s = \frac{1-j}{\sqrt{2}}\sqrt{\frac{2m}{\hbar}(s + v)}$, and thus, the operator B is obtained by $Bu = (s - A_{-1})f_s(\zeta)$.

A. Discretized Operators

In this section, we will compute the discrete time linear system operators (A_d, B_d, C_d, D_d). In order to do that, let us find the resolvent of the operator A by considering the homogeneous PDE

$$\dot{x}(\zeta, t) = Ax(\zeta, t), \quad x(\zeta, 0) = x_0(\zeta) \tag{5}$$

where A is the same as in (3). Applying Laplace transform to (5) yields

$$sx(\zeta, s) - x(\zeta, 0) = j\frac{\hbar}{2m}\frac{\partial^2}{\partial \zeta^2}x(\zeta, t) - vx(\zeta, t), \tag{6}$$

that is,

$$\frac{\partial^2}{\partial \zeta^2}x(\zeta, t) = -j\frac{2m}{\hbar}sx(\zeta, s) - jv\frac{2m}{\hbar}x(\zeta, s) + j\frac{2m}{\hbar}x(\zeta, 0),$$

which can be equivalently written as

$$\frac{\partial}{\partial \zeta}\begin{bmatrix} x(\zeta,t) \\ \partial_\zeta x(\zeta,t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -j\frac{2m}{\hbar}(s+v) & 0\end{bmatrix}\begin{bmatrix} x(\zeta,t) \\ \partial_\zeta x(\zeta,t)\end{bmatrix} + \begin{bmatrix} 0 \\ j\frac{2m}{\hbar}x(\zeta,0)\end{bmatrix}.$$

The above system is an ODE of the form

$$\partial_\zeta \overline{X}(\zeta,s) = \overline{AX}(\zeta,s) + \overline{B}(\zeta),$$

the solution of which is given by

$$\overline{X}(\zeta,s) = e^{\overline{A}\zeta}\overline{X}(0,s) + \int_0^\zeta e^{\overline{A}(\zeta-\eta)}\overline{B}(\eta)d\eta. \qquad (7)$$

A direct computation shows that

$$e^{\overline{A}\zeta} = \begin{bmatrix} \cosh(c_s\zeta) & \frac{1}{c_s}\sinh(c_s\zeta) \\ c_s\sinh(c_s\zeta) & \cosh(c_s\zeta)\end{bmatrix}$$

where again $c_s = \frac{1-j}{\sqrt{2}}\sqrt{\frac{2m}{\hbar}(s+v)}$.

By the definition of $\mathcal{D}(A)$, we must have $\partial_\zeta x(0,s) = \partial_\zeta x(1,s) = 0$, which yields that in (7), $\overline{X}(0,s)$ is given by

$$\overline{X}(0,s) = \begin{bmatrix} -\frac{1}{c_s\sinh(c_s)}\int_0^1 j\frac{2m}{\hbar}\cosh(c_s(1-\eta))x(\eta,0)d\eta \\ 0\end{bmatrix}.$$

Finally, the solution of (6) is given by

$$\begin{aligned} x(\zeta,s) &= -\frac{j}{c_s}\frac{2m}{\hbar}\frac{\cosh(c_s\zeta)}{\sinh(c_s)}\int_0^1 \cosh(c_s(1-\eta))x(\eta,0)d\eta \\ &\quad + \frac{j}{c_s}\frac{2m}{\hbar}\int_0^\zeta \sinh(c_s(\zeta-\eta))x(\eta,0)d\eta \\ &:= (s-A)^{-1}x(\zeta,0) \end{aligned} \qquad (8)$$

which yields the expression for the resolvent operator.

Now that we have derived an expression for the resolvent of A, a direct computation shows that

$$\begin{aligned} A_d x(\zeta) &= -x(\zeta) + \frac{2\delta j}{c_\delta}\frac{2m}{\hbar}\int_0^\zeta \sinh(c_\delta(\zeta-\eta))x(\eta)d\eta \\ &\quad - \frac{2\delta j}{c_\delta}\frac{2m}{\hbar}\frac{\cosh(c_\delta\zeta)}{\sinh(c_\delta)}\int_0^1 \cosh(c_\delta(1-\eta))x(\eta)d\eta. \end{aligned} \qquad (9)$$

In order to compute B_d, we choose $s=\delta$ in (4) so that $Bu = (\delta - A_{-1})f_\delta(\zeta)$, and we obtain $(\delta - A_{-1})^{-1}Bu = f_\delta(\zeta)$, and thus,

$$B_d = \frac{\sqrt{2\delta}}{c_\delta}\frac{\cosh(c_\delta\zeta)}{\sinh(c_\delta)}. \qquad (10)$$

The operator C_d is simply given by

$$C_d x(\zeta) = -\frac{\sqrt{2\delta}j}{c_\delta}\frac{2m}{\hbar}\frac{1}{\sinh(c_\delta)}\int_0^1 \cosh(c_\delta(1-\eta))x(\eta)d\eta \qquad (11)$$

and finally, the operator $D_d = G(\delta) = C(\delta - A_{-1})^{-1}B$ is given by

$$D_d = \frac{1}{c_\delta}\frac{1}{\sinh(c_\delta)}. \qquad (12)$$

We note that

$$\lim_{s\to\infty} G(s) = 0,$$

which implies that the system (3) is in fact a *regular linear system* (see, e.g., [15]) and, in particular, well-posed.

B. Adjoint Operators

In this section, we will compute the adjoints of the operators (A_d, B_d, C_d, D_d). We note that the state space $X := L^2(0,1;\mathbb{C})$ is equipped with the inner product

$$\langle f,g\rangle_X = \int_0^1 f^*(\zeta)g(\zeta)d\zeta,$$

and the input and output spaces $U = Y := \mathbb{C}$ are equipped with the usual complex inner product $\langle u_1,u_2\rangle_{\mathbb{C}} = u_1^*u_2$.

By definition, the adjoint P^* of an operator P satisfies $\langle Px,y\rangle = \langle x,P^*y\rangle$ with respect to the corresponding inner products. Now for A_d, we obtain

$$\begin{aligned} \langle A_d x, z\rangle_X &= -\int_0^1 x^*(\zeta)z(\zeta)d\zeta \\ &\quad + \int_0^1\int_0^1 \frac{4m\delta j}{c_\delta^*\hbar}\frac{\cosh(c_\delta^*\zeta)}{\sinh(c_\delta^*)}\cosh(c_\delta^*(1-\eta))x^*(\eta)z(\zeta)d\eta d\zeta \\ &\quad - \int_0^1\int_0^\zeta \frac{4m\delta j}{c_\delta^*\hbar}\sinh(c_\delta^*(\zeta-\eta))x^*(\eta)z(\zeta)d\eta d\zeta \\ &= -\int_0^1 x^*(\zeta)z(\zeta)d\zeta \\ &\quad + \int_0^1 x^*(\zeta)\int_0^1 \frac{4m\delta j}{c_\delta^*\hbar}\frac{\cosh(c_\delta^*\eta)}{\sinh(c_\delta^*)}\cosh(c_\delta^*(1-\zeta))z(\eta)d\eta d\zeta \\ &\quad - \int_0^1 x^*(\zeta)\int_\zeta^1 \frac{4m\delta j}{c_\delta^*\hbar}\sinh(c_\delta^*(\eta-\zeta))z(\eta)d\eta d\zeta \\ &= \langle x, A_d^* z\rangle_X, \end{aligned}$$

so we have

$$\begin{aligned} A_d^* x(\zeta) &= -x(\zeta) - \frac{2\delta j}{c_\delta^*}\frac{2m}{\hbar}\int_\zeta^1 \sinh(c_\delta^*(\eta-\zeta))x(\eta)d\eta \\ &\quad + \frac{2\delta j}{c_\delta^*}\frac{2m}{\hbar}\frac{\cosh(c_\delta^*(1-\zeta))}{\sinh(c_\delta^*)}\int_0^1 \cosh(c_\delta^*\eta)x(\eta)d\eta. \end{aligned} \qquad (13)$$

For B_d we obtain

$$\langle B_d u, x\rangle_X = \langle u, \langle B_d, x\rangle_X\rangle_{\mathbb{C}} = \langle u, B_d^* x\rangle_{\mathbb{C}},$$

that is,

$$B_d^* x = \langle B_d, x\rangle_X = \frac{\sqrt{2\delta}}{c_\delta^*}\int_0^1 \frac{\cosh(c_\delta^*\zeta)}{\sinh(c_\delta^*)}x(\zeta)d\zeta. \qquad (14)$$

Similarly for C_d we have

$$\begin{aligned} \langle C_d x, y\rangle_{\mathbb{C}} &= \int_0^1 y\frac{\sqrt{2\delta}j}{c_\delta^*}\frac{2m}{\hbar}\frac{\cosh(c_\delta^*(1-\eta))}{\sinh(c_\delta^*)}x^*(\eta)d\eta \\ &= \langle x, C_d^* y\rangle_X, \end{aligned}$$

where

$$C_d^* = \frac{\sqrt{2\delta}j}{c_\delta^*}\frac{2m}{\hbar}\frac{\cosh(c_\delta^*(1-\zeta))}{\sinh(c_\delta^*)}. \qquad (15)$$

Finally, the adjoint D_d^* of D_d is simply given by

$$D_d^* = \frac{1}{c_\delta^*}\frac{1}{\sinh(c_\delta^*)}. \qquad (16)$$

IV. The Model Predictive Control Problem

In the case of complex scalar input and output spaces, the objective function with constraints at a given sampling time k is given by

$$\begin{aligned}\min_u \sum_{i=0}^{\infty} & y^*(k+i)Qy(k+i) + u^*(k+i+1)Ru(k+i+1) \\ s.t.\quad & x(\zeta,k+i) = A_d x(\zeta,k+i-1) + B_d u(k+i) \\ & y(k+i) = C_d x(\zeta,k+i) + D_d u(k+i) \\ & \operatorname{Re} u_{\min} \le \operatorname{Re} u(k+i) \le \operatorname{Re} u_{\max} \\ & \operatorname{Im} u_{\min} \le \operatorname{Im} u(k+i) \le \operatorname{Im} u_{\max} \\ & \operatorname{Re} y_{\min} \le \operatorname{Re} y(k+i) \le \operatorname{Re} y_{\max} \\ & \operatorname{Im} y_{\min} \le \operatorname{Im} y(k+i) \le \operatorname{Im} y_{\max}\end{aligned}$$

where Q and R are positive constants. Note that as the input and output spaces are complex, we need to consider lower and upper bounds separately for the real and imaginary parts of u and y. However, in the following we will restrict to considering only real inputs as complex inputs are not implementable in practice. Thus, in the following we treat the input space U as $\mathbb{R}$.

The aforementioned infinite-horizon open-loop objective function can be cast as a finite-horizon open-loop objective function under the assumption that the input u is zero beyond the control horizon N, i.e., $u(k+N) = 0$. Additionally, an output penalty term needs to be included. Under the assumption of observability, the output terminal penalty can be expressed as a terminal state penalty term $\langle x(k+N-1), \bar{Q}x(k+N-1)\rangle$, and the finite horizon open-loop objective function can the be written as

$$\min_{U_k} Y_k^* Q Y_k + U_k^T R U_k + \langle x(\zeta,k+N-1), \bar{Q}x(\zeta,k+N-1)\rangle_X \tag{17}$$

where $U_k \in \mathbb{R}^{N-1}, Y_k \in \mathbb{C}^{N-1}$ are given by

$$\begin{aligned} U_k &= \begin{bmatrix} u(k+1) & u(k+2) & \dots & u(k+N-1)\end{bmatrix} \\ Y_k &= \begin{bmatrix} y(k) & y(k+1) & \dots & y(k+N-2)\end{bmatrix}.\end{aligned}$$

In the preceding, the operator $\bar{Q}$ can be calculated from a self-adjoint solution of the following discrete Lyapunov equation (see [16])

$$A_d^* \bar{Q} A_d - \bar{Q} = -C_d^* Q C_d. \tag{18}$$

We will address solving (18) in more detail in Section IV-A.

A straightforward manipulation of the objective function given in (17) yields the following finite-dimensional quadratic optimization problem

$$\begin{aligned}\min_{U_k}\quad & U_k^T H U_k + 2\operatorname{Re}\left(U_k^T P x(\zeta,k)\right) \\ & + \langle x(\zeta,k), \bar{Q}x(\zeta,k)\rangle_X + \langle y(k), Qy(k)\rangle_{\mathbb{C}}\end{aligned} \tag{19}$$

where $H \in \mathbb{C}^{N-1\times N-1}$ is self-adjoint given by

$$h_{m,n} = \begin{cases} D_d^* Q D_d + B_d^* \bar{Q} B_d + R & \text{for } m = n \\ D_d^* Q C_d A_d^{m-n-1} B_d + B_d^* \bar{Q} A_d^{m-n} B_d & \text{for } m > n \\ h_{n,m}^* & \text{for } m < n \end{cases}$$

and $P \in \mathcal{L}(X, \mathbb{C}^{N-1})$ is given by

$$P = \begin{bmatrix} D_d^* Q C_d + B_d^* \bar{Q} A_d \\ D_d^* Q C_d A_d + B_d^* \bar{Q} A_d^2 \\ \vdots \\ D_d^* Q C_d A_d^{N-2} + B_d^* \bar{Q} A_d^{N-1}\end{bmatrix}.$$

Note that since we are restricted to real inputs, we only need to consider the real parts of H and $Px(\zeta,k)$. The objective function given in (19) is subjected to constraints

$$\begin{aligned} U_{\min} &\le U_k \le U_{\max} \\ \operatorname{Re} Y_{\min} &\le \operatorname{Re}\left(SU + Tx(\zeta,k)\right) \le \operatorname{Re} Y_{\max} \\ \operatorname{Im} Y_{\min} &\le \operatorname{Im}\left(SU + Tx(\zeta,k)\right) \le \operatorname{Im} Y_{\max},\end{aligned}$$

which can be written in the form

$$\begin{bmatrix} I \\ -I \\ \operatorname{Re} S \\ \operatorname{Im} S \\ -\operatorname{Re} S \\ -\operatorname{Im} S\end{bmatrix} U_k \le \begin{bmatrix} U_{\max} \\ -U_{\min} \\ \operatorname{Re}\left(Y_{\max} - Tx(\zeta,k)\right) \\ \operatorname{Im}\left(Y_{\max} - Tx(\zeta,k)\right) \\ -\operatorname{Re}\left(Y_{\max} - Tx(\zeta,k)\right) \\ -\operatorname{Im}\left(Y_{\max} - Tx(\zeta,k)\right)\end{bmatrix}$$

where $S \in \mathbb{C}^{N-1\times N-1}$ is lower triangular given by

$$s_{m,n} = \begin{cases} D_d & \text{for } m = n \\ C_d A_d^{m-n-1} B_d & \text{for } m > n \\ 0 & \text{for } m < n \end{cases}$$

and $T \in \mathcal{L}(X, \mathbb{C}^{N-1})$ is given by

$$T = \begin{bmatrix} C_d \\ C_d A_d \\ \vdots \\ C_d A_d^{N-2}\end{bmatrix}.$$

A. A Solution of the Lyapunov Equation

Before going into simulations regarding model predictive control of Schrödinger equation, we will derive a self-adjoint solution for the discrete time Lyapunov equation (18). It has been shown in [16] that the solutions of the discrete time Lyapunov equation coincide with the solutions of the continuous time Lyapunov equation

$$A^* \bar{Q} + \bar{Q} A = -\mathcal{C}^* Q \mathcal{C}.$$

We will find a solution of the continuous time Lyapunov equation by utilizing the spectral presentation of A.

Consider the eigenvalue equation

$$A\phi_k = \lambda_k \phi_k$$

for Schrödinger equation considered in Section III. A direct computation shows that the eigenvectors ϕ_k are of the form

$$\phi_k = \alpha \cosh\left(\frac{1-j}{\sqrt{2}}\sqrt{\frac{2m}{\hbar}(v+\lambda_k)}\zeta\right)$$

which satisfy $\partial_\zeta \phi_k(0) = 0$. Since $\phi_k \in \mathcal{D}(A)$, ϕ_k must also satisfy $\partial_\zeta \phi_k(1) = 0$, which yields

$$0 = \alpha \frac{1-j}{\sqrt{2}}\sqrt{\frac{2m}{\hbar}(v+\lambda_k)} \sinh\left(\frac{1-j}{\sqrt{2}}\sqrt{\frac{2m}{\hbar}(v+\lambda_k)}\right).$$

Since $\sinh(z) = 0$ holds for $z = jk\pi$, $n \in \mathbb{Z}$, we obtain that the eigenvalues of A are given by

$$\lambda_k = -j\frac{\hbar}{2m}(k\pi)^2 - v$$

for $k \in \mathbb{N}_0$, which implies that A is the generator of an exponentially stable C_0-semigroup. The eigenvectors ϕ_k are now given by

$$\phi_k = \alpha\cosh(jk\pi) = \alpha\cos(k\pi)$$

which form an orthonormal basis of X with the choices $\alpha = 1$ for $k = 0$ and $\alpha = \sqrt{2}$ otherwise.

Let us now apply the continuous Lyapunov equation to an arbitrary $x \in \mathcal{D}(A)$:

$$A^*\bar{Q}x + \bar{Q}Ax = -\mathcal{C}^*Q\mathcal{C}x.$$

Representing x in the basis formed by the eigenvectors of A yields

$$\sum_{k=0}^{\infty}\left(A^*\bar{Q}\langle x,\phi_k\rangle\phi_k + \bar{Q}A\langle x,\phi_k\rangle\phi_k + \mathcal{C}^*Q\mathcal{C}\langle x,\phi_k\rangle\phi_k\right) = 0,$$

that is,

$$\sum_{k=0}^{\infty}\left((A^*+\lambda_k)\bar{Q}\langle x,\phi_k\rangle\phi_k + \mathcal{C}^*Q\mathcal{C}\langle x,\phi_k\rangle\phi_k\right) = 0,$$

which especially holds if

$$\bar{Q}\langle x,\phi_k\rangle\phi_k = (-\lambda_k - A^*)^{-1}\mathcal{C}^*Q\mathcal{C}\langle x,\phi_k\rangle\phi_k \tag{20}$$

for all $k \in \mathbb{N}_0$. We note that as A is densely defined and $-\lambda_k^* \in \rho(A)$, we have by [14, Prop. 2.8.4] that $(-\lambda_k - A^*)^{-1} = \left((-\lambda_k^* - A)^{-1}\right)^*$. Now summation over k in (20) yields a solution:

$$\bar{Q}x = \sum_{k=0}^{\infty}\langle x,\phi_k\rangle\left(\mathcal{C}(-\lambda_k^* - A)^{-1}\right)^* Q\mathcal{C}\phi_k \tag{21}$$

where we have based on C_d^* that

$$\left(\mathcal{C}(s-A)^{-1}\right)^* = \frac{2mj}{c_s^*\hbar}\frac{\cosh(c_s^*(1-\zeta))}{\sinh(c_s^*)}$$

for all $s \in \rho(A)$.

We note that as $\mathcal{C}\phi_k = \alpha$ and $\mathcal{C}(-\lambda_k^* - A)^{-1}$ is uniformly bounded for all $k \in \mathbb{N}_0$, (21) is a convergent series. Thus, denoting the Mth partial sum of (21) by $\bar{Q}_M$, we obtain for every $x \in \mathcal{D}(A)$ that $\lim_{M\to\infty}\|(\bar{Q} - \bar{Q}_M)x\| \to 0$, which implies that the solution $\bar{Q}x$ can be evaluated to arbitrary precision $\epsilon > 0$ by choosing a sufficiently large (finite) M. A sufficiently large value for M can be determined, e.g., by numerical experiments, as done in the simulation of the following section.

B. Simulation Results for Schrödinger Equation

In this section, we present simulation results for Schrödinger equation considered in Section III under the model predictive control law (19). For the simulation, we consider Schrödinger equation for a free electron, so in atomic units the parameters in (2) are given by $m = 1$, $\hbar = 1$ and we choose $v = 1$.

The input and output weights are chosen as $R = 10$ and $Q = 5$, respectively. For the Cayley-Tustin time discretization, we choose $h = 0.05$, so $\delta = 40$. Furthermore, $d\zeta = 2^{-9}$ is chosen for numerical integration. The initial condition is $x_0(\zeta) = \cos(\pi\zeta)$ and the model predictive control horizon is $N = 10$. For computation of the function $\bar{Q}$, the series in (21) is approximated by summing the first $M = 101$ terms. The input and output constraints are given as $u_{\min} = -0.3$, $u_{\max} = 0.03$, $y_{\min} = -0.1 - 0.2j$ and $y_{\max} = 0.2 + 0.05j$.

The input profile of the simulation and the input constraints are shown in Figure 1. Figure 2 shows the comparison between the output profiles of the open- and closed-loop systems under model predictive control, along with output constraints. One can see from these figures that a maximal control effort is required near the beginning to keep the real and imaginary parts of the output signal within the allowed limits. Thereafter virtually no control is imposed nor required.

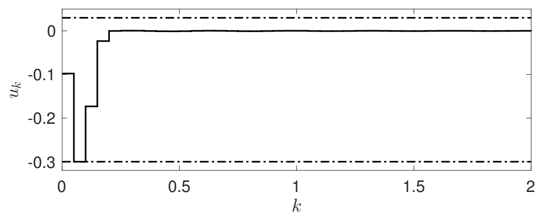

Fig. 1. Input profile model predictive control law under input and output constraints (solid line) and input constraints (dash-dot-line).

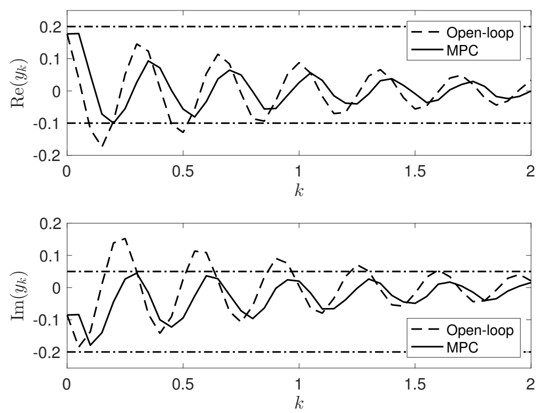

Fig. 2. Comparison between the profile of the closed-loop system under model predictive control (solid line) and the profile of the open-loop system (dashed line). Output constraints are shown in dash-dot-line

In Figures 3 and 4 the state profiles of the open- and closed-loop systems, respectively, are presented for comparison. The effect of control can be seen here as well, as the state under the model predictive control law in Figure 4 decays in the beginning faster than the state of the open-loop system. Even thought both the MPC and the open-loop states decay asymptotically to zero due to the system being exponentially stable, the most substantial difference between the open-loop and the MPC behaviors – as seen in Figure 2 – is that MPC keeps the output within the given constraints while the open-loop output violates them.

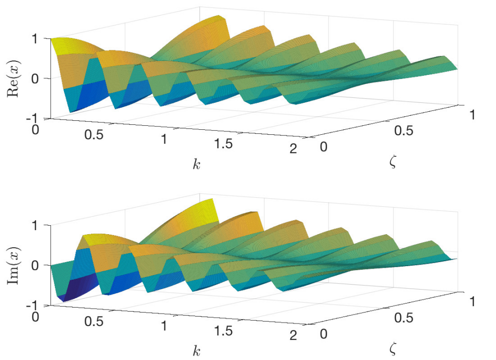

Fig. 3. The evolution of the state profile of the open-loop system.

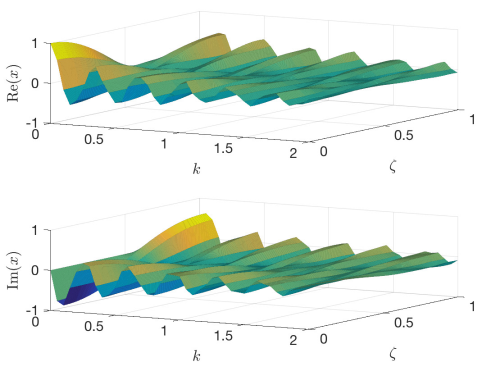

Fig. 4. The evolution of the state profile under the model predictive control law with input and output constraints.

V. Conclusions

We considered the finite-horizon constrained optimal control problem for the Schrödinger equation with boundary controls and boundary observations. The plant was mapped from continuous to discrete time using the Cayley-Tustin transformation, which is a convergent time discretization scheme for a rather general class of systems. No spatial approximations were required in the process. The control problem was solved for Schrödinger equation and the results were illustrated with numerical simulations.

References

[1] B. Augner and B. Jacob. Stability and stabilization of infinite-dimensional linear port-Hamiltonian systems. *Evolution Equations and Control Theory*, 3(2):207–229, 2014.

[2] M. A. Dahleh, A. P. Peirce, and H. Rabitz. Optimal control of uncertain quantum systems. *Phys. Rev. A*, 42(3):1065–1079, 1990.

[3] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, volume 31 of *Springer Series in Computational Mathematics*. Springer, Berlin, 2002.

[4] S. W. Hansen. Exact boundary controllability of a Schrödinger equation with an internal point mass. In *American Control Conference (ACC)*, pages 3809–3814, 2017.

[5] V. Havu and J. Malinen. The Cayley transform as a time discretization scheme. *NFAO*, 28(7-8):825–851, 2007.

[6] D. Q. Mayne. Model predictive control: Recent developments and future promise. *Automatica*, 50(12):2967 – 2986, 2014.

[7] M. Mirrahimi and P. Rouchon. Controllability of quantum harmonic oscillators. *IEEE Trans. Automat. Control*, 49(5):745–747, 2004.

[8] M. Mirrahimi, P. Rouchon, and G. Turinici. Lyapunov control of bilinear Schrödinger equations. *Automatica*, 41(11):1987 – 1994, 2005.

[9] A. P. Peirce, M. A. Dahleh, and H. Rabitz. Optimal control of quantum-mechanical systems: Existence, numerical approximation, and applications. *Phys. Rev. A*, 37(12):4950–4964, 1988.

[10] Y. Privat, E. Trélat, and E. Zuazua. Optimal shape and location of sensors or actuators in PDE models. In *American Control Conference (ACC)*, pages 4063–4068, 2014.

[11] Y. Privat, E. Trélat, and E. Zuazua. Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains. *JEMS*, 18(5):1043–1111, 2016.

[12] H. Rabitz and W. Zhu. Optimal control of molecular motion: Design, implementation, and inversion. *Accounts of Chemical Research*, 33(8):572–578, 2000. PMID: 10955988.

[13] C.-C. Shu, T.-S. Ho, X. Xing, and H. Rabitz. Frequency domain quantum optimal control under multiple constraints. *Phys. Rev. A*, 93:033417, Mar 2016.

[14] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Verlag AG, 2009.

[15] G. Weiss. Transfer functions of regular linear systems. Part I: Characterzations of regularity. *Trans. Amer. Math. Soc.*, 342(2):827–854, 1994.

[16] Q. Xu and S. Dubljevic. Linear model predictive control for transport-reaction processes. *AIChE J.*, 63:2644–2659, 2017.

[17] M. N. Zeilinger, D. M. Raimondo, A. Domahidi, M. Morari, and C. N. Jones. On real-time robust model predictive control. *Automatica*, 50(3):683 – 694, 2014.

PUBLICATION V

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Model predictive control for regular linear systems
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Model Predictive Control for Regular Linear Systems

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Abstract

The present work extends known finite-dimensional constrained optimal control realizations to the realm of well-posed regular linear infinite-dimensional systems modelled by partial differential equations. The structure-preserving Cayley-Tustin transformation is utilized to approximate the continuous-time system by a discrete-time model representation without using any spatial discretization or model reduction. The discrete-time model is utilized in the design of model predictive controller accounting for optimality, stabilization, and input and output/state constraints in an explicit way. The proposed model predictive controller is dual-mode in the sense that predictive controller steers the state to a set where exponentially stabilizing unconstrained feedback can be utilized without violating the constraints. The construction of the model predictive controller leads to a finite-dimensional constrained quadratic optimization problem easily solvable by standard numerical methods. Two representative examples of partial differential equations are considered.

Key words: infinite-dimensional systems, modeling and control optimization, controller constraints and structure, model predictive control, regular linear systems, Cayley-Tustin transform

1 Introduction

The concept of *regular linear systems* came about at the turn of 1990's by the work of George Weiss [34–36]. This subclass of abstract linear systems is essentially the Hilbert space counterpart of the finite-dimensional systems described by the state-space equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where, however, the operators A, B and C may be unbounded. Regular linear systems are often encountered in the study of partial differential equations (PDEs) with boundary controls and boundary observations, and they cover a large class of abstract systems of practical interest.

The control of linear distributed parameter systems (DPS) is a mature control field with seminal contributions given in [5, 13, 27, 31, 32]. The system theoretic properties and controller designs were explored in these contributions with the emphasis on full state feedback, boundary and/or in-domain stabilization, optimality and robustness. In addition, classical control problems such as state feedback regulation [21] and robust output regulation [22, 23] have been considered, and regulator theory has been developed for regular linear systems. Above contributions fully explored the functional space setting of the continuous-time system representation and only minor considerations have been devoted to the discrete-time counterparts. In addition, despite the myriad of work on unconstrained stabilization, the design of low order constrained optimal/suboptimal controllers for DPS which accounts for input and state/output constraints remained elusive.

Over the past decade, there have been several attempts to address control of distributed parameter systems within an input and/or state constrained optimal control setting. There are several works on dynamical analysis and optimal control of hyperbolic PDEs, most notably the work of Aksikas et al. on optimal linear quadratic feedback controller design for hyperbolic DPS. [1, 2, 30]. Other contributions considered optimal and model predictive control applied to Riesz spectral systems (parabolic and higher order dissipative PDEs) with a separable eigenspectrum of the underlying dissipative spectral operator and successfully designed algorithms that account for the input and state con-

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straints [6, 14, 41]. In prior contributions, some type of spatial approximation is applied to the PDE models to arrive at finite-dimensional models utilized in the controller design. As it will be claimed and demonstrated in the subsequent sections, the linear distributed parameter system can be treated intact and controller design can be accomplished without any spatial model approximation or reduction.

The research area of model predictive control (MPC) and contributions associated with this design methodology has flourished over past two decades [8, 15, 16, 24]. The appealing nature of applying to the state the first control input in a finite sequence of control inputs obtained as a solution of an online constrained, discrete-time, optimal control problem with explicit account for the control and state constraints, and achieving stability by adding a terminal cost or terminal constraints, or by extending the horizon of the the optimal control problem, is well understood and explored [15,16,26] but could not be easily extended to the DPS setting. Apart from the aforementioned contributions [6, 14, 41] where some type of model approximation has been applied, other contributions explored unconstrained MPC with emphasis on the computational complexity of the optimization problem [7]. However, the clear link between the discrete constrained optimization based MPC design, the well-understood modelling of distributed parameter systems described by PDEs, and the well-established control theory of linear DPS has not yet been established apart from the recent work by the authors [11, 40].

Motivated by the preceding, in this contribution, the model predictive control for regular linear systems is developed. In particular, the essential feature of the discrete-time infinite-dimensional representation necessary in the MPC design preserving the continuous-time system properties is established by applying the Caley-Tustin (CT) [10] time discretization, implying that no spatial discretization or model reduction is required. At the core of the CT transformation, one can find the application of a Crank-Nicolson type time discretization scheme which is a well-know implicit midpoint integration rule that is symmetric, symplectic (Hamiltonian preserving) [9], and guarantees structure preserving numerical integration so that stability and controllability are not altered by the discrete-time infinite-dimensional model representation. Furthermore, boundary and/or point actuation transformed to the discrete-setting yields bounded operators.

As the first main contribution, the MPC design utilized in [40] is generalized for stable regular linear systems (Theorem 2). Under the assumption of infinite-time admissibility of the observation operator, optimality and stability of the proposed design is proved. The design is demonstrated on a numerical example of the one-dimensional wave equation.

As the second main contribution, an MPC-based control design is presented to achieve constrained stabilization of exponentially stabilizable systems (Theorem 3) and the design is demonstrated on a simulation study of a tubular reactor. The proposed design belongs to the class of dual-mode control [17, 29] implying that the model predictive controller steers the state to the neighborhood of the origin where local unconstrained stabilizing feedback can be applied without violating the input constraints. A stabilizing terminal penalty is added to the MPC formulation to guarantee stabilizability while no terminal constraints are imposed. Stabilization of a finite-number of unstable eigenvalues is considered in the MPC setting in [40], but here the proposed methodology can be applied to arbitrary exponentially stabilizable systems. Finally, the proposed work provides a foundation to link regular linear systems to the well-established area of linear model predictive control designs.

The structure of the paper is as follows. In Section 2, we present the notation, the mathematical preliminaries concerning regular linear systems and the Cayley-Tustin time discretization scheme. In Section 3, we present the MPC problem, and in Sections 3.1 and 3.2, stability and optimality results of the proposed MPC and dual-mode control designs are presented. In Section 4, we present, as an example of a stable system, the wave equation on a one-dimensional spatial domain and compute the operators corresponding to the Cayley-Tustin transform and their adjoints. Furthermore, in Section 4.3, we derive a solution of the Lyapunov equation for the wave equation as required by the proposed MPC design. The performance of the MPC is demonstrated by numerical simulations of the controlled wave equation in Section 4.4. In Section 5, the dual-mode controller design is demonstrated on an unstable tubular reactor which is successfully stabilized by the proposed control strategy. Finally, the paper is concluded in Section 6.

2 Mathematical Preliminaries

2.1 Notation

Here $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, kernel and resolvent of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively. For a linear operator $A : \mathcal{D}(A) \subset X \to X$ and a fixed $s_0 \in \rho(A)$, define the scale spaces $X_1 := (\mathcal{D}(A), \|(s_0 - A)\cdot\|)$ and $X_{-1} = \overline{(X, \|(s_0 - A)^{-1} \cdot \|)}$ [32, Sec. 2.10]. The scale spaces are related by $X_1 \subset X \subset X_{-1}$ where the inclusions are dense and with continuous embeddings. The extension of A to X_{-1} is denoted by A_{-1}. The Λ-extension of an operator P is denoted by P_Λ (see (1)).

2.2 Regular Linear Systems

Consider a well-posed linear system (A, B, C, D), where $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup, $B \in \mathcal{L}(U, X_{-1})$ is the control operator, $C \in \mathcal{L}(X_1, Y)$ is the observation operator, and $D \in \mathcal{L}(U, Y)$. We assume that the spaces X, U, and Y are separable Hilbert spaces and that U and Y are finite-dimensional.

The operator B is called an *admissible input operator* for A if for some $\tau > 0$, the operator $\Phi_\tau \in \mathcal{L}(L^2(0, \infty; U), X_{-1})$ defined as [32, Sec. 4.2]:

$$\Phi_\tau u = \int_0^\tau T(\tau - s)Bu(s)ds,$$

satisfies $\mathcal{R}(\Phi_\tau) \subset X$. Correspondingly, the operator C is called an *admissible output operator* for A if for some $\tau > 0$, there exists a K_τ such that [32, Sec. 4.3]:

$$\int_0^\tau \|CT(s)x\|^2 ds \leq K_\tau \|x\|^2, \qquad \forall x \in \mathcal{D}(A).$$

Furthermore, if there exists a K such that $K_\tau \leq K$ for all $\tau > 0$, then C is called *infinite-time admissible.* The Λ-*extension* of the operator C is defined as [35]:

$$C_\Lambda x = \lim_{\lambda \to \infty} \lambda C(\lambda - A)^{-1}x, \tag{1}$$

and the domain of C_Λ consists of those elements $x \in X$ for which the limit exists.

Let G denote the transfer function of the system (A, B, C, D). The transfer function is called *regular* if $\lim_{\lambda \to \infty} G(\lambda)u = Du$ $(\lambda \in \mathbb{R})$ for all $u \in U$ [36, Thm. 1.3], in which case (A, B, C, D) is called a *regular linear system.*

The transfer function G of a regular system is given by:

$$G(s) := C_\Lambda(s - A_{-1})^{-1}B + D, \tag{2}$$

and in the time domain the system is described by the following equations:

$$\dot{x}(\zeta, t) = Ax(\zeta, t) + Bu(t), \qquad x(\zeta, 0) = x_0(\zeta) \tag{3a}$$
$$y(t) = C_\Lambda x(\zeta, t) + Du(t). \tag{3b}$$

Throughout this paper, we assume that we are dealing with regular linear systems with admissible B and C.

2.3 Cayley-Tustin Time Discretization

Consider a system given in (3). Given a time discretization parameter $h > 0$, the Tustin time discretization of (3) is given by

$$\frac{x(jh) - x((j-1)h)}{h} \approx A\frac{x(jh) - x((j-1)h)}{2} + Bu(jh)$$
$$y(jh) \approx C\frac{x(jh) - x((j-1)h)}{2} + Du(jh)$$

for $j \geq 1$, where we omitted the spatial dependence of x for brevity. Let $u_j^{(h)}/\sqrt{h}$ be the approximation of $u(t)$ on the interval $t \in ((j-1)h, jh)$, e.g., by the mean value sampling used in [10]:

$$\frac{u_j^{(h)}}{\sqrt{h}} = \frac{1}{h}\int_{(j-1)h}^{jh} u(t)dt.$$

It has been shown in [10] that the Cayley-Tustin discretization is a convergent time discretization scheme for input-output stable system nodes satisfying $\dim U = \dim Y = 1$ in the sense that $y_j^{(h)}/\sqrt{h}$ converges to $y(t)$ in several different ways as $h \to 0$. The discussion in [10, Sec. 6] further implies that the same holds for any finite dimensional U and Y. Thus, writing $y_j^{(h)}/\sqrt{h}$ and $u_j^{(h)}/\sqrt{h}$ in place of $y(jh)$ and $u(jh)$, respectively, simple computations yield the *Cayley-Tustin discretization* of (3) as:

$$x(\zeta, k) = A_d x(\zeta, k-1) + B_d u(k), \quad x(\zeta, 0) = x_0(\zeta)$$
$$y(k) = C_d x(\zeta, k-1) + D_d u(k),$$

where:

$$\begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} := \begin{bmatrix} -I + 2\delta(\delta - A)^{-1} & \sqrt{2\delta}(\delta - A_{-1})^{-1}B \\ \sqrt{2\delta}C(\delta - A)^{-1} & G(\delta) \end{bmatrix}$$

and $\delta := 2/h$. Clearly one must have $\delta \in \rho(A)$, so that the resolvent operator is well-defined. Thus, for a large enough δ, the discretization can be applied to unstable systems as well.

Remark 1 *Due to the standing assumptions it is easy to see that the discretized operators are bounded. In fact, the boundedness of B_d and C_d already follows from $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, respectively, and for D_d being bounded it would suffice that the system* (3) *is well-posed rather than regular.*

3 Model Predictive Control

The moving horizon regulator is based on a similar formulation emerging from the finite-dimensional system theory (see e.g. [20]). A corresponding controller in the infinite-dimensional case is presented, e.g., in [40]. At a given sampling time k, the objective function with constraints is given by:

$$\begin{aligned}\min_{u} &\sum_{j=k+1}^{\infty} \langle y_{k+j}, Qy_{k+j}\rangle_Y + \langle u_{k+j}, Ru_{k+j}\rangle_U \\ \text{s.t.} \quad & x_j = A_d x_{j-1} + B_d u_j \\ & y_j = C_d x_{j-1} + D_d u_j \\ & u_{\min} \le u_j \le u_{\max} \\ & y_{\min} \le y_j \le y_{\max},\end{aligned} \tag{4}$$

where Q and R are positive self-adjoint weights on the outputs y_j and inputs u_j, respectively. Here it is assumed for simplicity that U and Y are (finite-dimensional) real-valued spaces. For consideration of the MPC with complex input and output spaces, see [11], where the authors considered MPC for the Schrödinger equation.

The infinite-horizon objective function (4) can be cast into a finite-horizon objective function under certain assumptions on the inputs beyond the control horizon. Furthermore, a penalty term needs to be added to the objective function to account for the inputs and outputs beyond the horizon. We will present two approaches on this depending on the stability of the original plant.

3.1 *Stable systems*

If A is the generator of a (strongly) stable C_0-semigroup, we may assume that the input is zero beyond the control horizon N, i.e., $u_{k+N+i} = 0, \forall i \in \mathbb{N}$, and add a corresponding output penalty term. Under the assumption that C is infinite-time admissible for A, the terminal output penalty term can be written as a state penalty term, so that the finite-horizon objective function is given by:

$$\min_{u^N} \sum_{j=k+1}^{k+N} \langle y_j, Qy_j\rangle_Y + \langle u_j, Ru_j\rangle_U + \langle x_{k+N}, \bar{Q}x_{k+N}\rangle_X \tag{5}$$

with the same constraints as in (4), and where N is the length of the control horizon.

The operator $\bar{Q}$ can be calculated from the positive self-adjoint solution of the following discrete-time Lyapunov equation:

$$A_d^* \bar{Q} A_d - \bar{Q} = -C_d^* Q C_d, \tag{6}$$

or equivalently (see e.g. [5, Ex. 4.30]) the continuous-time Lyapunov equation:

$$A^*\bar{Q} + \bar{Q}A = -C^*QC \tag{7}$$

on the dual space of X_{-1}. The assumption of C being infinite-time admissible for A is required as it is equivalent to the continuous-time Lyapunov equation having solutions [32, Thm. 5.1.1]. Furthermore, as A is assumed to be stable, we have that the operator $\bar{Q} \in \mathcal{L}(X)$ given by:

$$\bar{Q}x = \lim_{\tau\to\infty} \int_0^{\tau} T^*(t)C^*QCT(t)x dt, \quad \forall x \in \mathcal{D}(A), \tag{8}$$

is the unique positive self-adjoint solution of the continuous-time Lyapunov equation (7) (equivalently (6)).

Now that we have established that the finite-horizon objective function (5) is well-defined, to further manipulate the objective function (5) we introduce the notation $Y_k := (y_{k+n})_{n=1}^N \in Y^N$ and $U_k := (u_{k+n})_{n=1}^N \in U^N$. Hence, a manipulation of the objective function (5) leads to the following quadratic optimization problem:

$$\min_{U_k} \langle U_k, HU_k\rangle_{U^N} + 2\langle U_k, Px_k\rangle_{U^N} + \langle x_k, \bar{Q}x_k\rangle_X, \tag{9}$$

where $H \in \mathcal{L}(U^N)$ is positive and self-adjoint given by:

$$h_{i,j} = \begin{cases} D_d^*QD_d + B_d^*\bar{Q}B_d + R, & \text{for } i = j \\ D_d^*QC_dA_d^{i-j-1}B_d + B_d^*\bar{Q}A_d^{i-j}B_d, & \text{for } i > j \\ h_{j,i}^*, & \text{for } i < j \end{cases}$$

and $P \in \mathcal{L}(X, U^N)$ is given by $P = (D_d^*QC_dA_d^{k-1} + B_d^*\bar{Q}A_d^k)_{k=1}^N$

The objective function (9) is subjected to constrains $U_{\min} \le U_k \le U_{\max}$ and $Y_{\min} \le (SU_k + Tx_k) \le Y_{\max}$ which can be written in the form:

$$\begin{bmatrix} I \\ -I \\ S \\ -S \end{bmatrix} U_k \le \begin{bmatrix} U_{\max} \\ -U_{\min} \\ Y_{\max} - Tx_k \\ -Y_{\min} + Tx_k \end{bmatrix}, \tag{10}$$

where $S \in \mathcal{L}(U^N, Y^N)$ is given by:

$$s_{i,j} = \begin{cases} D_d, & \text{for } i = j \\ C_dA_d^{i-j-1}B_d, & \text{for } i > j \\ 0, & \text{for } i < j \end{cases}$$

and $T \in \mathcal{L}(X, Y^N)$ is given by $T = (C_dA_d^{k-1})_{k=1}^N$.

Considering a finite-dimensional output space $U = \mathbb{R}^m$, the inner products in the objective function given in (9) are simply vector products, and we have a finite dimensional quadratic optimization problem:

$$\min_{U_k} J(U_k, x_k) = U_k^T HU_k + 2U_k^T(Px_k). \tag{11}$$

Note that the term $\langle x_k, \bar{Q}x_k \rangle_X$ can be neglected as x_k is the initial condition for step $k+1$ and cannot be affected by the control input. Furthermore, as all the operators related to the objective function and the linear constraints are bounded under the standing assumptions, the quadratic optimization problem is exactly of the same form as the ones obtained for finite-dimensional systems. Thus, we obtain the convergence and stability results for free by the MPC theory on finite-dimensional systems. To highlight this observation, we present the following result:

Theorem 2 *Assume that A is the generator of a strongly stable C_0-semigroup and that C is an infinite-time admissible observation operator for A. Then, the input sequence (U_k) (and hence the sequence (u_k)) obtained as the solution of the feasible quadratic optimization problem* (11) *with constraints* (10) *converges to zero along with the states x_k, hence yielding asymptotic stability.*

PROOF. *As the MPC problem is equivalent to a finite-dimensional one, assuming that the problem is feasible at step $k=0$, the claim follows directly from [25, Thm. 3].*

3.2 Exponentially stabilizable systems

Let us now assume that the pair (A,B) is exponentially stablizable, i.e., there exists an admissible feedback operator $K \in \mathcal{L}(X_1, U)$ such that $A + BK_\Lambda$ is the generator of an exponentially stable C_0-semigroup [37, Def. 3.1]. Optimal (in terms of minimizing the continuous version of (4)) state feedback operator is obtained using the maximal solution $\bar{R} \in \mathcal{L}(X)$ of the continuous-time Riccati equation [18, Def. 10.1.2] (see also [38]) :

$$K^*SK = A^*\bar{R} + \bar{R}A + C^*QC \tag{12}$$

on $\mathcal{D}(A)$, where $S := R + D^*QD$ and $K := -S^{-1}(B_\Lambda^*\bar{R} + D^*QC)$ yields the optimal feedback operator. Moreover, it follows from the proof of [4, Thm. 9] that the solutions of (12) are equivalent to the solutions of the discrete-time Riccati equation:

$$K_d^*S_dK_d = A_d^*\bar{R}A_d - \bar{R} + C_d^*QC_d, \tag{13}$$

where $S_d := B_d^*\bar{R}B_d + R + D_d^*QD_d$ and $K_d := -S_d^{-1}(A_d\bar{R}B_d + D_d^*QC_d)$ yields the optimal state feedback for the discrete-time system with the maximal $\bar{R}$. Furthermore, $A_d + B_dK_d$ corresponds to the Cayley-Tustin discretization of $A + BK_\Lambda$, and thus, the feedback $u_k = K_dx_{k-1}$ asymptotically stabilizes the pair (A_d, B_d).

Returning to the MPC problem, we assume that the optimal state feedback is utilized beyond the control horizon, i.e., $u_{k+N+i} = K_dx_{k+N+i-1}, \forall i \in \mathbb{N}$. Thus, the input and output terminal penalties can be expressed as state terminal penalties by solving the discrete-time Lyapunov equations:

$$\begin{aligned} A_{K_d}^*\bar{Q}_1A_{K_d} - \bar{Q}_1 &= -K_d^*RK_d \\ A_{K_d}^*\bar{Q}_2A_{K_d} - \bar{Q}_2 &= -(C_d + K_dD_d)^*QC_d + K_dD_d) \end{aligned}$$

or equivalently their continuous-time counterparts:

$$\begin{aligned} A_K^*\bar{Q}_1 + \bar{Q}_1A_K &= -K^*RK \\ A_K^*\bar{Q}_2 + \bar{Q}_2A_K &= -(C + DK)^*Q(C + DK), \end{aligned}$$

where $A_K := A + BK_\Lambda$. Note that as A_K is the generator of an exponentially stable semigroup and K and C are admissible for A_K by their admissibility for A and [32, Thm 5.4.2], the positive self-adjoint solutions of the Lyapunov equations are unique by [32, Thm. 5.1.1] and obtained similar to (8).

Finally, the input and output terminal penalties are given by $\langle x_{k+N}, \bar{Q}_1x_{k+N} \rangle$ and $\langle x_{k+N}, \bar{Q}_2x_{k+N} \rangle$, respectively. Thus, the quadratic formulation of the MPC problem is given as in the stable case, except that in H and P the operator $\bar{Q}$ must be replaced with $\bar{Q}_1 + \bar{Q}_2$.

Note that the exponentially stabilizing full state feedback $u = Kx$ optimally solves the *unconstrained* minimization problem (4). Thus, in order to utilize it in the constrained setting, we need to first assume that the system is stabilizable by a sequence of inputs satisfying the input constraints. Under this assumption, MPC is utilized to steer the system into a region where $u_{\min} \le Kx \le u_{\max}$, at which point we can switch from MPC to the state feedback control. The existence of a constrained stabilizing input sequence can be guaranteed by allowing sufficiently high-gain inputs to cancel output the unstable dynamics of the system.

Theorem 3 *Assume that the system* (3) *is stabilizable by a sequence of inputs satisfying the input constraints. Then, the dual-mode control consisting of MPC and optimal state feedback asymptotically stabilizes the system while satisfying the input constraints.*

PROOF. *Due to the choice of the terminal penalty function $\bar{Q} = \bar{Q}_1 + \bar{Q}_2$, the finite-horizon objective function in* (11) *satisfies the assumptions of the function $V(\cdot)$ in [29, Thm. 1]. Thus, assuming that the problem is feasible at the step $k=0$, the controls obtained by solving the minimization problem* (11) *asymptotically steer the state of the system towards zero by [29, Thm. 1]. When the state reaches the region where the optimal state feedback can be used without violating the input constraints, the stabilization can be finalized with the feedback control.*

In practice, finding the optimal feedback K is rather challenging as the Riccati equation (12) can rarely be solved in analytic closed-form. Instead, some other stabilizing feedback can be used as a terminal penalty and stabilizing feedback as well. One possible option is to use output feedback $u_k = K_y y_k$. This is a valid choice as well as regularity of the system is preserved under output feedback (see [35]), and rather straightforward computations using Sherman-Morrison-Woodbury formula show that $A_d + B_d K_y (I - D_d K_y)^{-1} C_d$ corresponds to the Cayley-Tustin discretization of $A + BK_y(I - DK_y)^{-1}C$, i.e., A after output feedback. The result of Theorem 3 holds for any stabilizing feedback.

Remark 4 *Note that even though the results of Theorems 2 and 3 were presented for the discrete-time system, it follows from the input/output convergence and the asymptotic stability preserving property of the Cayley-Tustin discretization that for, e.g., piecewise constant inputs $u(t) = u_k/\sqrt{h}$ for $t \in [kt, (k+1)h]$, the output $y(t)$ of the continuous time system behaves on every interval $t \in [kh, (k+1)h]$ as $y_k/\sqrt{h}$ in some approximate sense, where h is the discretization parameter. Thus, the continuous-time output goes asymptotically approximately to zero along with the discrete-time outputs y_k. At that point, the assumed stability or the existence of a stabilizing feedback yield that the state of the continuous-time system decays asymptotically to zero, that is, the discrete-time control laws can be used to asymptotically stabilize the continuous-time system as well.*

4 Wave Equation

As an example of a stable system, consider the wave equation on a 1-D spatial domain $\zeta \in [0, 1]$ with viscous damping at one end and boundary control u and boundary observation y at the other end given by:

$$\frac{\partial^2}{\partial t^2} w(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial}{\partial \zeta} w(\zeta, t) \right) \tag{14a}$$

$$0 = T(\zeta) \frac{\partial}{\partial \zeta} w(1, t) + \frac{\kappa}{\rho} \frac{\partial}{\partial t} w(1, t) \tag{14b}$$

$$u(t) = \frac{\partial}{\partial \zeta} w(0, t) \tag{14c}$$

$$y(t) = \frac{\partial}{\partial t} w(0, t), \tag{14d}$$

where $\kappa > 0$. For simplicity we assume that the mass density ρ and the Young's modulus T are constants. We further assume that $\kappa \neq \sqrt{\rho T}$, which will be needed in Section 4.3.

In order to write (14) in a more compact form, let us first define a new state variable $x = [x_1,\, x_2]^T := [\rho \partial_t w,\, \partial_\zeta w]^T$ with state space $X = L_2(0, 1; \mathbb{R}^2)$ and an auxiliary matrix operator $\mathcal{H}(\zeta) := \mathrm{diag}(\rho(\zeta)^{-1},\ T(\zeta))$. Now define the operator $\mathcal{A}$ by:

$$\mathcal{A}x(\zeta, t) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta) x(\zeta, t) \right)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ x \in X : \mathcal{H}x \in H^1(0, 1; \mathbb{R}^2),\ Tx_2(1) = -\tfrac{\kappa}{\rho} x_1(1) \right\},$$

so that the first two lines of (14) can be equivalently written as $\dot{x} = \mathcal{A}x$. Finally, by defining operators $\mathcal{B}$ and $\mathcal{C}$ as $\mathcal{B}x := Tx_2(0)$ and $\mathcal{C}x := \rho^{-1} x_1(0)$, the system (14) can be equivalently written as:

$$\dot{x}(t) = \mathcal{A}x(t) \tag{15a}$$

$$u(t) = \mathcal{B}x(t) \tag{15b}$$

$$y(t) = \mathcal{C}x(t), \tag{15c}$$

which corresponds to the port-Hamiltonian formulation of the wave equation (see, e.g., [12, Ex. 9.2.1]).

In order to further write the system (15) in the usual state-space form, define the operator A as the restriction of $\mathcal{A}$ to the kernel of $\mathcal{B}$, i.e., $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ with domain $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$. Due to the definitions of $\mathcal{A}$ and $\mathcal{B}$, it can be shown using [33, Thm. III.2] that A is the generator of an exponentially stable C_0-semigroup. Consequently, the double $(\mathcal{A}, \mathcal{B})$ is a boundary control system in the sense of [32, Def. 10.1.1]. Thus, by [32, Prop. 10.1.2, Rem. 10.1.4], there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that (15) can be equivalently written as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{16a}$$

$$y(t) = C_\Lambda x(t) + Du(t) \tag{16b}$$

where $C := \mathcal{C}|_{\mathcal{D}(A)}$ and $D := \lim_{s \to \infty} G(s)$ which is well-defined assuming that the system is regular [12, Def. 13.1.11]. Note that by the general theory of regular linear systems, the transfer function of the system (16) can be equivalently expressed as $G(s) = \mathcal{C}(s - A_{-1})^{-1}B$ which is easier to evaluate than (2).

By [32, Rem. 10.1.5], the operator B can be found by solving the abstract elliptic problem $\mathcal{A}f = sf$, $\mathcal{B}f = u$ for any $u \in U$ and $s \in \rho(A)$, the unique solution of which satisfies $f = (s - A_{-1})^{-1}Bu$. Since here A is the generator of an exponentially stable C_0-semigroup, we can choose $s = 0$ and obtain the solution $f = (\rho/\kappa,\, -T^{-1})^T u$, and finally, the operator B is defined as:

$$Bu := A_{-1} \begin{bmatrix} \frac{\rho}{\kappa} \\ -\frac{1}{T} \end{bmatrix} u. \tag{17}$$

4.1 Discretized Operators

Assume that ρ and T are constants and consider the equation $\dot{x}(t) = Ax(t)$. Using the Laplace transform yields

$$sx(\zeta,s) - x(\zeta,0) = \frac{\partial}{\partial\zeta}\left(\begin{bmatrix} 0 & T \\ \rho^{-1} & 0 \end{bmatrix} x(\zeta,s)\right),$$

that is,

$$\frac{\partial}{\partial\zeta}x(\zeta,s) = \begin{bmatrix} 0 & \rho s \\ T^{-1}s & 0 \end{bmatrix} x(\zeta,s) - \begin{bmatrix} 0 & \rho \\ T^{-1} & 0 \end{bmatrix} x(\zeta,0).$$

The above is an ordinary differential equation of the form:

$$\frac{\partial}{\partial\zeta}x(\zeta,s) = \overline{A}x(\zeta,s) - \overline{B}x(\zeta,0),$$

the solution of which is given by:

$$x(\zeta,s) = e^{\overline{A}\zeta}x(0,s) - \int_0^\zeta e^{\overline{A}(\zeta-\eta)}\overline{B}x(\eta,0)d\eta \tag{18}$$

where:

$$e^{\overline{A}\zeta} = \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) & \sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \\ \left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) & \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \end{bmatrix}.$$

Recall that $\mathcal{D}(A)$ has the boundary conditions $Tx_2(1) + \frac{\kappa}{\rho}x_1(1) = 0$ and $Tx_2(0) = 0$, based on which $x(0,s)$ in (18) can be solved. Eventually, (18) is given by:

$$\begin{aligned} x(\zeta,s) = & \frac{\rho}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}s\right)} \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \\ \left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \end{bmatrix} \times \\ & \int_0^1 \left(\frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right) + \cosh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right)\right) x_1(\eta,0) \\ & \quad + \left(\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right)\right) x_2(\eta,0)d\eta \\ & - \int_0^\zeta \begin{bmatrix} \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) & \rho\cosh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) \\ T^{-1}\cosh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) & \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) \end{bmatrix} x(\eta,0)d\eta \\ := & (s-A)^{-1}x(\zeta,0), \end{aligned}$$

which yields the expression for the resolvent operator, from which we also obtain the operator $A_d = -I + 2\delta(\delta - A)^{-1}$.

Based on the expression we derived for the operator B in (17), we have:

$$(\delta - A_{-1})^{-1}B = -\begin{bmatrix} \frac{\rho}{\kappa} \\ -\frac{1}{T} \end{bmatrix} + \delta(\delta - A_{-1})^{-1}\begin{bmatrix} \frac{\rho}{\kappa} \\ -\frac{1}{T} \end{bmatrix},$$

and a direct calculation yields that:

$$\begin{aligned} B_d = & \sqrt{\frac{\rho}{T}} \frac{\sqrt{2\delta}\left(\frac{\rho T}{\kappa} - \kappa\right)\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right)}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)} \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \\ \left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \end{bmatrix} \\ & + \sqrt{2\delta}\begin{bmatrix} \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) - \frac{\rho}{\kappa}\cosh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \\ \frac{1}{T}\cosh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) - \frac{1}{\kappa}\sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \end{bmatrix}, \end{aligned}$$

which can be further simplified using the properties of hyperbolic: functions to

$$\begin{aligned} B_d = & \frac{-\sqrt{2\delta}}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)} \times \\ & \begin{bmatrix} \rho\cosh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta-1)\right) + \kappa\sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta+1)\right) \\ \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta-1)\right) - \frac{\kappa}{T}\cosh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta-1)\right) \end{bmatrix}. \end{aligned}$$

Furthermore, we obtain:

$$\begin{aligned} C_d x(\zeta) = & \frac{\sqrt{2\delta}}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right)\kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)} \times \\ & \int_0^1 \left(\frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right) + \cosh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)\right) x_1(\eta) \\ & \quad + \left(\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)\right) x_2(\eta)d\eta. \end{aligned}$$

Finally, based on the expression of B_d it is easy to see that the operator $D_d = G(\delta) = \mathcal{C}(\delta - A_{-1})^{-1}B$ is given by:

$$D_d = -\frac{1}{\sqrt{\rho T}} \frac{\kappa\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \sqrt{\rho T}\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)}. \tag{19}$$

We note that $\lim_{\delta\to\infty} G(\delta) = -(\rho T)^{-1/2}$ to verify that (15) indeed is a regular linear system.

4.2 Adjoint Operators

In order to find the adjoints of the discretized operators computed in the previous section, we equip the state-space X with the L^2 inner product, and the input and output spaces are equipped with the real scalar product. In order to find A_d^*, we find the adjoint of the resolvent operator $(s-A)^{-1}$:

$$
\begin{aligned}
&\langle(\delta-A)^{-1}x,z\rangle_X \\
&=\int_0^1 \frac{\rho z^*(\zeta)}{\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}s\right)+\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}s\right)}\begin{bmatrix}\cosh\left(\sqrt{\tfrac{\rho}{T}}s\zeta\right)\\ \left(\sqrt{\rho T}\right)^1\sinh\left(\sqrt{\tfrac{\rho}{T}}s\zeta\right)\end{bmatrix}\times \\
&\qquad\int_0^1\left(\tfrac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right)+\cosh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right)\right)x_1(\eta) \\
&\qquad\quad+\left(\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right)+\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(1-\eta)\right)\right)x_2(\eta)d\eta d\zeta \\
&\quad-\int_0^1 z^*(\zeta)\int_0^\zeta\begin{bmatrix}\sqrt{\tfrac{\rho}{T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(\zeta-\eta)\right) & \rho\cosh\left(\sqrt{\tfrac{\rho}{T}}s(\zeta-\eta)\right)\\ T^{-1}\cosh\left(\sqrt{\tfrac{\rho}{T}}s(\zeta-\eta)\right) & \sqrt{\tfrac{\rho}{T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(\zeta-\eta)\right)\end{bmatrix}x(\eta)d\eta d\zeta \\
&=\int_0^1\int_0^1\rho\frac{z_1^*(\eta)\cosh\left(\sqrt{\tfrac{\rho}{T}}s\eta\right)+z_2^*(\eta)\left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\tfrac{\rho}{T}}s\eta\right)}{\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}s\right)+\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}s\right)}d\eta\times \\
&\qquad\begin{bmatrix}\tfrac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(1-\zeta)\right)+\cosh\left(\sqrt{\tfrac{\rho}{T}}s(1-\zeta)\right)\\ \kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}s(1-\zeta)\right)+\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(1-\zeta)\right)\end{bmatrix}^* x(\zeta)d\zeta \\
&\quad-\int_0^1\int_\zeta^1 z^*(\eta)\begin{bmatrix}\sqrt{\tfrac{\rho}{T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(\eta-\zeta)\right) & \rho\cosh\left(\sqrt{\tfrac{\rho}{T}}s(\eta-\zeta)\right)\\ T^{-1}\cosh\left(\sqrt{\tfrac{\rho}{T}}s(\eta-\zeta)\right) & \sqrt{\tfrac{\rho}{T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}s(\eta-\zeta)\right)\end{bmatrix}d\eta x(\zeta)d\zeta \\
&=\langle x,(s-A)^{-*}z\rangle_X,
\end{aligned}
$$

and now, A_d^* is given by $A_d^*=-I+2\delta(\delta-A)^{-*}$.

For B_d we have $\langle B_d u,x\rangle_X=u\langle B_d,x\rangle_X=uB_d^*x$, and in a similar manner, we obtain for C_d that:

$$
\begin{aligned}
yC_dx=&\frac{y\sqrt{2\delta}}{\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta\right)+\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}\delta\right)}\times \\
&\int_0^1\left(\tfrac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)+\cosh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)\right)x_1(\eta,0) \\
&\quad+\left(\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)+\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)\right)x_2(\eta,0)d\eta. \\
=&\langle C_d^*y,x\rangle_X.
\end{aligned}
$$

Finally, D_d is self-adjoint.

4.3 Solution of the Lyapunov equation

In this section, we derive the positive solution for the continuous Lyapunov equation (7), which is realized by utilizing the spectral representation of A. Let us at first find the eigenvalues and eigenvectors of the operator A. A direct computation shows that the solution of the eigenvalue equation $A\phi_k=\lambda_k\phi_k$ is of the form:

$$
\begin{aligned}
\phi_{1,k}(\zeta)&=\alpha\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right)+\beta\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right)\\
\phi_{2,k}(\zeta)&=\frac{\alpha}{\sqrt{\rho T}}\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right)-\frac{\beta}{\sqrt{\rho T}}\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right).
\end{aligned}
$$

Since $\phi_k\in\mathcal{D}(\mathcal{A})$, we must have $\phi_{2,k}(0)=0$, which yields $\alpha=\beta$. Thus, the eigenvectors of A are of the form:

$$
\phi_k(\zeta)=\begin{bmatrix}\cosh\left(\sqrt{\tfrac{\rho}{T}}\lambda_k\zeta\right)\\ \frac{1}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}\lambda_k\zeta\right)\end{bmatrix},
$$

and the eigenvalues λ_k are determined from the condition $T\phi_{2,k}(1)=-\frac{\kappa}{\rho}\phi_{1,k}(1)$, i.e.,

$$
\sqrt{\frac{T}{\rho}}\sinh\left(\sqrt{\frac{\rho}{T}}\lambda_k\right)+\frac{\kappa}{\rho}\cosh\left(\sqrt{\frac{\rho}{T}}\lambda_k\right)=0.
$$

Using the exponential form of the hyperbolic functions we obtain that one of the eigenvalues is given by:

$$
\lambda_0=\frac{1}{2}\sqrt{\frac{T}{\rho}}\log\left(\frac{\sqrt{\rho T}-\kappa}{\sqrt{\rho T}+\kappa}\right), \tag{20}
$$

which is real if $\kappa<\sqrt{\rho T}$. Finally, by the periodicity of the exponential function along the imaginary axis, we obtain that in general the eigenvalues are given by $\lambda_k=\lambda_0+\sqrt{T/\rho}k\pi i$ for $k\in\mathbb{Z}$.

We note that damped wave equations have been considered, e.g., in [3] and [39, Sect. 4] - both referring to the original work by Rideau [28] - where similar spectra were obtained. Furthermore, it can be seen from (20) that the assumption $\kappa\neq\sqrt{\rho T}$ is required to ensure $\sigma(A)\neq\emptyset$, which is further required by [3, Thm. 3.5] to ensure that the eigenvectors of A constitute a Riesz basis for X. Indeed, we can define an invertible operator:

$$
M:=\begin{bmatrix}\cosh\left(\sqrt{\tfrac{\rho}{T}}\lambda_0\zeta\right) & -\sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}\lambda_0\zeta\right)\\ i\sinh\left(\sqrt{\tfrac{\rho}{T}}\lambda_0\zeta\right) & -i\sqrt{\rho T}\cosh\left(\sqrt{\tfrac{\rho}{T}}\lambda_0\zeta\right)\end{bmatrix},
$$

so that

$$
M\phi_k=\begin{bmatrix}\cos(k\pi\zeta)\\ \sin(k\pi\zeta)\end{bmatrix}
$$

is an orthonormal basis in X, and the biorthogonal sequence [32, Def. 2.5.1] $\left(\bar{\phi}_k\right)$ to (ϕ_k) is given by $\bar{\phi}_k=M^*M\phi_k$.

Let us now return to the Lyapunov equation and apply it to an arbitrary $x\in\mathcal{D}(A)$:

$$
A^*\bar{Q}x+\bar{Q}Ax+\mathcal{C}^*Q\mathcal{C}x=0.
$$

By [32, Prop. 2.5.2], we can write every $x\in X$ as:

$$
x=\sum_{k\in\mathbb{Z}}\left\langle z,\bar{\phi}_k\right\rangle\phi_k,
$$

which yields:

$$
\sum_{k\in\mathbb{Z}}\left(A^*\bar{Q}\left\langle x,\bar{\phi}_k\right\rangle\phi_k+\bar{Q}A\left\langle x,\bar{\phi}_k\right\rangle\phi_k+\mathcal{C}^*Q\mathcal{C}\left\langle x,\bar{\phi}_k\right\rangle\phi_k\right)=0,
$$

which by utilizing [32, Prop. 2.6.3] further yields:

$$
\sum_{k\in\mathbb{Z}}\left((A^*+\lambda_k)\bar{Q}\left\langle x,\bar{\phi}_k\right\rangle\phi_k+\mathcal{C}^*Q\mathcal{C}\left\langle x,\bar{\phi}_k\right\rangle\phi_k\right)=0.
$$

The above especially holds if $(A^* + \lambda_k)\bar{Q}\langle x, \bar{\phi}_k\rangle \phi_k = -\mathcal{C}^* Q\mathcal{C}\langle x, \bar{\phi}_k\rangle \phi_k$ for all $k \in \mathbb{Z}$. Thus, for an arbitrary $k \in \mathbb{Z}$, we obtain:

$$\bar{Q}\langle x, \bar{\phi}_k\rangle \phi_k = (-\lambda_k - A^*)^{-1}\mathcal{C}^* Q\mathcal{C}\langle x, \bar{\phi}_k\rangle \phi_k.$$

As A is densely defined and $-\bar{\lambda}_k \in \rho(A)$ since $\lambda_k \in \sigma(A)$, we have by [32, Prop. 2.8.4] that $(-\lambda_k - A^*)^{-1} = \left((-\bar{\lambda}_k - A)^{-1}\right)^*$, so we obtain :

$$\begin{aligned}\bar{Q}\langle x, \bar{\phi}_k\rangle \phi_k &= \left((-\bar{\lambda}_k - A)^{-1}\right)^* \mathcal{C}^* Q\mathcal{C}\langle x, \bar{\phi}_k\rangle \phi_k \\ &= \left(\mathcal{C}(-\bar{\lambda}_k - A)^{-1}\right)^* Q\mathcal{C}\langle x, \bar{\phi}_k\rangle \phi_k.\end{aligned}$$

Finally, summation over $k \in \mathbb{Z}$ yields the solution:

$$\bar{Q}x = \sum_{k\in\mathbb{Z}} \langle x, \bar{\phi}_k\rangle \left(\mathcal{C}(-\bar{\lambda}_k - A)^{-1}\right)^* Q\mathcal{C}\phi_k. \qquad (21)$$

Note that as $\mathcal{C}\phi_k = 1$ and $\mathcal{C}(-\lambda_k^* - A)^{-1}$ is uniformly bounded for all $k \in \mathbb{Z}$, the series in (21) is convergent (as it should since $\bar{Q} \in \mathcal{L}(X)$). Thus, for any $x \in X$ we may approximate:

$$\bar{Q}x \approx \bar{Q}_M x := \sum_{k=-M}^{M} \langle x, \bar{\phi}_k\rangle \left(\mathcal{C}(-\bar{\lambda}_k - A)^{-1}\right)^* Q\mathcal{C}\phi_k,$$

and it holds that $\lim_{M\to\infty} \|\bar{Q}x - \bar{Q}_M x\| = 0$, by which we can evaluate (21) to an arbitrary precision $\epsilon > 0$ by choosing a sufficiently large M. A suitable value for M can determined, e.g., by numerical experiments.

4.4 Simulation results for the wave equation

Consider the wave equation (14) with the parameter choices $\rho = T = 1$ and $\kappa = 0.75$. For the MPC, choose the optimization horizon as $N = 15$ and choose the input and output weights as $R = 10$ and $Q = 0.5$, respectively. For the Cayley-Tustin discretization, choose $h = 0.075$ so that $\delta \approx 26.67$. For numerical integration, an adaptive approximation of $d\zeta$ is used with 519 nodal points. To approximate the solution of the Lyapunov equation (21), we choose $M = 100$. The initial conditions for the wave equation in the port-Hamiltonian framework are given by $\partial_t w(\zeta) = \cos(\pi\zeta)$ and $\partial_\zeta w(\zeta) = \sin(\frac{1}{2}\pi\zeta)$.

The input and output constraints $-0.05 \leq u_k \leq 0.05$ and $-0.025 \leq y_k \leq 0.3$ are displayed in Figure 1 along with the control inputs $u(k)$ obtained from the MPC problem. The outputs of the system under the MPC and under no control are displayed as well. It can be seen that the MPC makes the output decay slightly faster in the beginning. Then control is imposed to satisfy the output constraints while the uncontrolled output violates them. Finally, a minor stabilizing control effort is imposed before both the MPC input and the output decay to zero. Naturally the uncontrolled output decays to zero as well due to the exponential stability of the considered system.

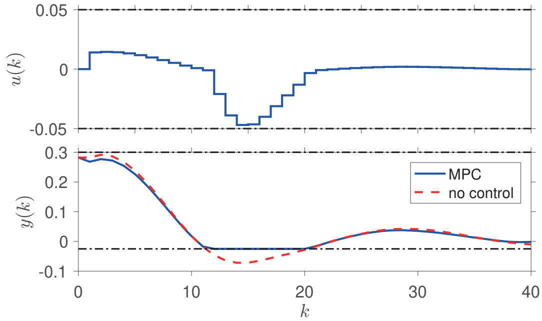

Figure 1. Above: MPC inputs $u(k)$ and the input constraints. Below: MPC and uncontrolled outputs and the output constraints.

Figure 2 displays the velocity profiles of the system under the model predictive control law and without control. No substantial differences can be observed in the velocity profiles, which is rather expected as the outputs in Figure 1 were rather close to one another. Relatively small differences in the outputs are natural as well, since the control inputs were constrained to rather small gain.

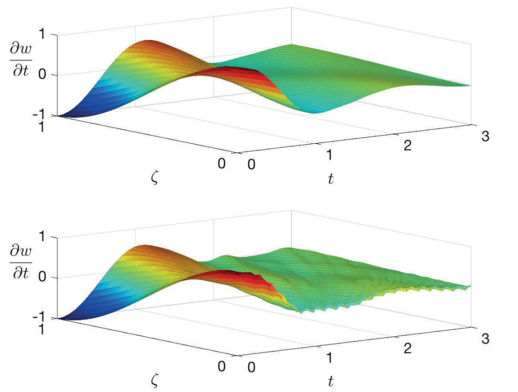

Figure 2. Above: the velocity profile of the wave equation without control. Below: the velocity profile under the model predicting control law.

5 Tubular reactor with recycle

As an example of an unstable system, consider a tubular reactor with recycle given as:

$$\frac{\partial}{\partial t}x(\zeta, t) = -v\frac{\partial}{\partial \zeta}x(\zeta, t) + \alpha x(\zeta, t) \qquad (22a)$$

$$x(0, t) = rx(1, t) + (r - 1)u(t) \qquad (22b)$$

$$y(t) = x(1, t) \qquad (22c)$$

on $\zeta \in [0,1]$, where the parameters are chosen as $v = 1, \alpha = 1/2$ and $r = 1/3$ so that the system has its spectrum in the right half plane but is exponentially stabilizable, e.g., by output feedback $u(t) = -y(t)$. Under this feedback, (22b) changes to $x(0,t) = (2r-1)x(1,t)$ but otherwise the system remains the same.

Similar to the wave equation in Section 4, we can compute the resolvent operator and find the discretized operator (A_d, B_d, C_d, D_d) and their adjoints. Since output feedback is used as a stabilizing terminal cost and in this case $D = 0$, for the terminal penalty one needs to solve the Lyapunov equation $A_s^*\bar{Q} + \bar{Q}A_s = -\mathcal{C}^*(Q+R)\mathcal{C}$, where A_s is the generator of the exponentially stable C_0-semigroup corresponding to the boundary control system (22) under output feedback $u(t) = -y(t)$. This can be done as in Section 4.3, except that the normalized eigenvectors of A_s already form an orthonormal basis in $X = L^2(0,1;\mathbb{R})$.

For the MPC problem formulation, the weights are chosen as $Q = 2$ and $R = 10$, and the input constraints are given by $-0.15 \leq u_k \leq 0.05$ while no output constraints are imposed. The optimization horizon is chosen as $N = 10$, and for approximation of the solution of the Lyapunov equation, 201 eigenvectors of A_s are used. For the Cayley-Tustin discretization, we choose $h = 0.1$ so that $\delta = 20$. The initial condition is given by $x_0(\zeta) = \frac{1}{2}\sin(\pi\zeta)$. For numerical integration, an adaptive approximation of $d\zeta$ is used with 510 nodal points.

In Figure 3, the dual-mode inputs and the outputs of the system under the dual-mode control are presented. For comparison, the output feedback control and the output under the feedback control are also presented. It can be seen that while the output feedback stabilizes the system faster, it does not satisfy the input constraints early on in the simulation. In the dual-model control, the MPC inputs first steer the output close to zero while satisfying the input constraints, and then at $k = 80$ it is switched to output feedback $u = -y$ which completes the stabilization.

In Figure 4, the state profiles of the tubular reactor are displayed under the dual-mode and the feedback controls. The states behave according to what could be expected based on the outputs, that is, both states decay asymptotically to zero and the state under output feedback decays faster.

6 Conclusions

In this work, a linear model predictive controller for regular linear systems was designed, and it was shown that for stable systems, stability of the zero output regulator follows from the finite-dimensional MPC theory. For stabilizable systems, constrained stabilization was achieved

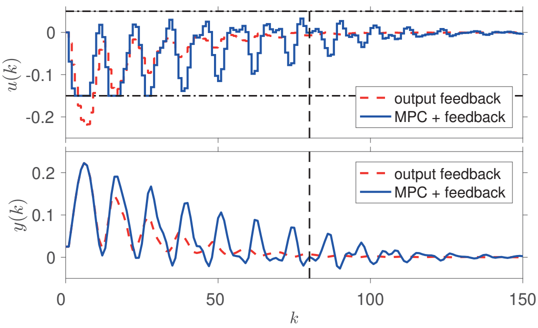

Figure 3. Above: dual-mode inputs, the input constraints and the output feedback. Below: outputs of the system under the dual-mode control and output feedback.

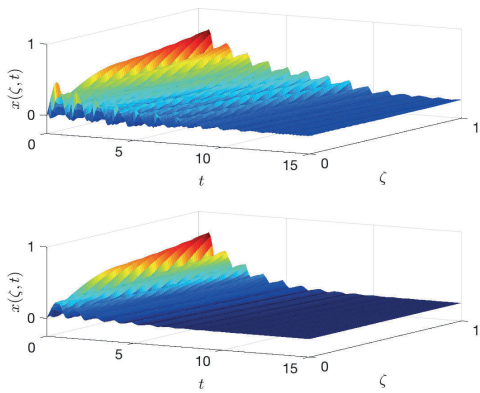

Figure 4. Above: the state profile of the tubular reactor under the-dual mode control. Below: the state profile under the output feedback.

by dual-mode control consisting of MPC and stabilizing feedback. The MPC design was demonstrated on an illustrative example where it was implemented for the boundary controlled wave equation. Constrained stabilization was demonstrated on a tubular reactor which had solely unstable eigenvalues. The performances of the control strategies were illustrated with numerical simulations.

It should be noted that the assumption of regularity was not in fact needed at any point when considering stable systems, but it was merely done for the convenience of the state-space presentation of the systems. Thus, the result of Theorem 2 can equivalently be formulated for well-posed instead of regular linear systems. Furthermore, by the obtained stability result, tracking of constant reference signals could be incorporated for MPC of regular linear systems by the classical MPC theory of finite-dimensional systems (see [24]). The result of Theorem 3 could be extended to well-posed linear systems as well, although state feedback stabilization and Riccati equations are much more involved concepts for these systems (see [18, 19]).

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References

[1] I. Aksikas, A. Fuxman, J. F. Forbes, and J. J. Winkin. LQ control design of a class of hyperbolic PDE systems: Application to fixed-bed reactor. *Automatica*, 45(6):1542–1548, 2009.

[2] I. Aksikas, J. J. Winkin, and D. Dochain. Optimal lq-feedback control for a class of first-order hyperbolic distributed parameter systems. *ESAIM: Control, Optimization and Calculus of Variations*, 14(04):897–908, 2008.

[3] S. Cox and E. Zuazua. The rate at which energy decays in a string damped at one end. *Indiana Univ. Math. J.*, 44(2):545–573, 1995.

[4] R. Curtain. Regular linear systems and their reciprocals: applications to Riccati equations. *Systems Control Lett.*, 49:81–89, 2003.

[5] R. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, 1995.

[6] S. Dubljevic, N. H. El-Farra, P. Mhaskar, and P. D. Christofides. Predictive control of parabolic PDEs with state and control constraints. *International Journal of robust and nonlinear control*, 16(16):749–772, 2006.

[7] L. Grüne. Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems. *SIAM J. Control Optim.*, 48(2):1206–1228, 2009.

[8] L. Grüne and J. Pannek. *Nonlinear model predictive control. Theory and algorithms.* London: Springer, 2011.

[9] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, volume 31 of *Springer Series in Computational Mathematics*. Springer, Berlin, 2002.

[10] V. Havu and J. Malinen. The Cayley transform as a time discretization scheme. *NFAO*, 28(7-8):825–851, 2007.

[11] J.-P. Humaloja and S. Dubljevic. Linear model predictive control for schrodinger equation. In *2018 Annual American Control Conference (ACC)*, pages 2569–2574, June 2018.

[12] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite dimensional Spaces*. Birkhäuser, 2012.

[13] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*. SIAM, Philadelphia, 2008.

[14] Liu Liu, Biao Huang, and S. Dubljevic. Model predictive control of axial dispersion chemical reactor. *Journal of Process Control*, 24(11):1671 – 1690, 2014.

[15] D. Q. Mayne. Model predictive control: recent developments and future promise. *Automatica*, 50(12):2967–2986, 2014.

[16] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789 – 814, 2000.

[17] H. Michalska and D. Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38(11):1623–1633, Nov 1993.

[18] K. Mikkola. *Infinite-dimensional linear systems, optimal control and algebraic Riccati equations*. PhD thesis, Helsinki University of Technology, 2002, `http://lib.tkk.fi/Diss/2002/isbn9512260794/`.

[19] K. Mikkola. State-feedback stabilization of well-posed linear systems. *Integral Equations Operator Theory*, 55(2):249–271, 2006.

[20] K. R. Muske and J. B. Rawlings. Model predictive control with linear models. *AIChE J.*, 39(2):262–287, 1993.

[21] V. Natarajan, D. S. Gilliam, and G. Weiss. The state feedback regulator problem for regular linear systems. *IEEE Trans. Automat. Control*, 59(10):2708–2723, 2014.

[22] L. Paunonen. Controller design for robust output regulation of regular linear systems. *IEEE Trans. Automat. Control*, 61(10):2974–2986, 2016.

[23] L. Paunonen and S. Pohjolainen. The internal model principle for systems with unbounded control and observation. *SIAM J. Control Optim.*, 52(6):3967–4000, 2014.

[24] J. B. Rawlings, D. Q. Mayne, and M. Diehl. *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, 2017.

[25] J. B. Rawlings and K. R. Muske. The stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38(10):1512–1516, Oct 1993.

[26] J. B. Rawlings and K. R. Muske. The stability of constrained receding horizon control. *IEEE Trans. Autom. Control*, 38(10):1512–1516, 1993.

[27] W.H. Ray. *Advanced Process Control*. McGraw-Hill, New York, 1981.

[28] P. Rideau. *Contrôle d'un assemblage de poutres flexibles par des capteurs-actionneurs ponctuels: étude du spectre du système*. PhD thesis, Ecole nationale supérieure des Mines de Paris, 1985.

[29] P. O. M. Scokaert, D. Q. Mayne, and J. B. Rawlings. Suboptimal model predictive control. *IEEE Transactions on Automatic Control*, 44(3):648–654, March 1999.

[30] Huilan Shang, J. F. Frobes, and M. Guay. Model predictive control for quasilinear hyperbolic distributed parameter systems. *Ind. & Eng. Chem. Res.*, 43:2140–2149, 2004.

[31] O. J. Staffans. Passive and conservative continuous-time impedance and scattering systems. part I: Well-posed systems. *Math. Control Signals Systems*, 15:291–315, 2002.

[32] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Verlag AG, 2009.

[33] J. A. Villegas, H. Zwart, Y. Le Gorrec, and B. Maschke. Exponential stability of a class of boundary control systems. *IEEE Trans. Automat. Control*, 54(1):142–147, 2009.

[34] G. Weiss. *Control and estimation of distributed parameter systems*, chapter The representation of regular linear systems on Hilbert spaces. Number 91 in Internat. Ser. Numer. Math. Birkhäuser, 1989.

[35] G. Weiss. Regular linear systems with feedback. *Math. Control Signals Systems*, 7:23–57, 1994.

[36] G. Weiss. Transfer functions of regular linear systems. Part I: Characterzations of regularity. *Trans. Amer. Math. Soc.*, 342(2):827–854, 1994.

[37] G. Weiss and R. Curtain. Dynamic stabilization of regular linear systems. *IEEE Trans. Automat. Control*, 42(1):4–21, 1997.

[38] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. *Math. Control Signals Systems*, 10:287–330, 1997.

[39] Cheng-Zong. Xu and G. Weiss. Eigenvalues and eigenvectors of semigroup generators obtained from diagonal generators by feedback. *Commun. Inf. Syst.*, 11(1):71–104, 2011.

[40] Qingqing Xu and S. Dubljevic. Linear model predictive control for transport-reaction processes. *AIChE J.*, 63:2644–2659, 2017.

[41] Yu Yang and S. Dubljevic. Boundary model predictive control of thin film thickness modelled by the Kuramoto-Sivashinsky equation with input and state constraints. *Journal of Process Control*, 23(9):1362 – 1379, 2013.