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Robust Regulation for Infinite-Dimensional Systems and Signals in the Frequency Domain

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Abstract

In this thesis, the robust output regulation problem is studied both in the time domain and in the frequency domain. The problem to be addressed is to find a stabilizing controller for a given plant so that every signal generated by an exogenous system, or shortly exosystem, is asymptotically tracked despite perturbations in the plant or some external disturbances. The exosystem generating the reference and disturbance signals is assumed to be infinite-dimensional.

The main contribution of this thesis is to develop the robust regulation theory for an infinite-dimensional exosystem in the frequency domain framework. In order to do that, the time domain theory is studied in some detail and new results that emphasize the smoothness requirement on the reference and disturbance signals due to infinite-dimensionality of the exosystem are presented. Two types of controllers are studied, the feedforward controllers and the error feedback controllers, the latter of which facilitate robust regulation. These results exploit the structure at infinity of tha plant transfer function. In this thesis, a new definition of the structure at infinity suitable for infinite-dimensional systems is developed and its properties are studied.

The frequency domain theory developed is based on the insights into the corresponding time domain theory. By following some recent time domain ideas the type of robustness and stability types are chosen so that they facilitate the use of an infinite-dimensional exosystem. The robustness is understood in the sense that stability should imply regulation. The chosen stability types resemble the time domain polynomial and strong stabilities and allow robust regulation of signals that have an infinite number of unstable dynamics along with transfer functions vanishing at infinity.

The main contribution of this thesis is the formulation of the celebrated internal model principle in the frequency domain terms in a rather abstract algebraic setting. Unlike in the existing literature, no topological aspect of the problem is needed because of the adopted definition of robustness. The plant transfer function is only assumed to have a right or a left coprime factorization but not necessarily both. The internal model principle leads to a necessary and sufficient condition for the solvability of the robust regulation problem.

The second main contribution of the thesis is to design frequency domain

controllers for infinite-dimensional systems and exosystems. In this thesis, the Davison's simple controller design for stable plants is extended to infinite-dimensional systems and exosystems. Then a controller design procedure for unstable plants containing two phases is proposed. In the first phase, a stabilizing controller is constructed for a given plant. The second phase is to design a robustly regulating controller for a stable part of the plant. This design procedure nicely combines with the Davison's type controllers and is especially suitable for infinite-dimensional plants with transfer functions in the Callier-Desoer class of transfer functions.

Foreword

The work that finally took the form of this somewhat coherent piece of information you are currently reading was carried out during the years 2008-2013 at the Department of Mathematics at Tampere University of Technology. The past five years have been the most interesting – and hectic – time in my life, not least because of my precious children. It has not always been easy to balance between work and family, but I have been fortunate to have several people around me who have made it all a lot easier.

I wish to express my sincere gratitude to my supervisor Professor Seppo Pohjolainen for taking me as a part of his research group and for guiding me through my doctoral studies. I have been privileged to work as a full time researcher from the very beginning of my doctoral studies, for which I must thank Seppo, but also the nice people who decided to grant me a position in Doctoral Program TISE.

I want to thank my fellow research group members Timo Hämäläinen and especially Lassi Paunonen, whose advice I have asked more often than I care to admit, for their professional support. I am grateful to Professor Birgit Jacob and Professor Joseph Winkin who accepted to do the preliminary examination of the thesis and contributed to it by their valuable comments.

Finally, I would like to thank my parents, relatives, and friends for their support during my studies. Above all I am indebted to my wife Kaisa who have helped me in my doctoral studies in numerous different ways.

Tampere, September 25, 2013

Petteri Laakkonen

List of Notations and Abbrevitations

Notation	Description

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Notation	**Description**

Notation	Description

Notation	**Description**

Notation	Description
$\lvert\cdot\rvert_{\mathbf{r,c}}^{i}$	minor of a matrix 9
$\geq_H$	majorization relation 15, 20
$=_H$	relation of equivalence at infinity 15, 20
$[\cdot]_H$	equivalence class of a function with respect to $=_H$ 15, 20
$\geqq_H$	global majorization relation 36
$\equiv_H$	relation of global equivalence 36
$[[\cdot]]_H$	equivalence class of a function with respect to $\equiv_H$ 36
$\bar{\cdot}$	refers to the complex conjugate of a complex number or the closure of a complex set 8
$\cdot'$	refers to a perturbed operator or a perturbed transfer function
MIMO	Multiple-input and multiple-output
SISO	Single-input and single-output

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Chapter 1

Introduction

1.1 Introduction to Robust Regulation

This thesis deals with one of the central issues in the mathematical systems theory, the robust regulation. The mathematical background is in state space and frequency domain theories for linear distributed parameter systems. With distributed parameter systems one is able to model phenomena involving partial differential equations. Basic examples of partial differential equations are heat, wave, and delay equations, which naturally arise in the natural sciences and related engineering applications.
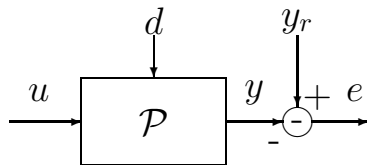
Figure 1.1: The basic control configuration.

On a general level the problem considered in this thesis is as in Figure 1.1. There $\mathcal{P}$ is a given system and y is an output or a measurement, which describes the data available from the systems behavior. Reference signal y_r is the desired behavior of the system or, to be more precise, of the measurement. Input u is a signal through which one can control the system. The aim is to zero out the error e between the reference and the measurement signals despite an external disturbance d.

The problem can be formulated in the time domain by using the state space formulation or in the frequency domain by using transfer functions. The most recent results related to the subject of this thesis are given in state space terms. While the main contribution of this thesis lies in developing the theory of robust
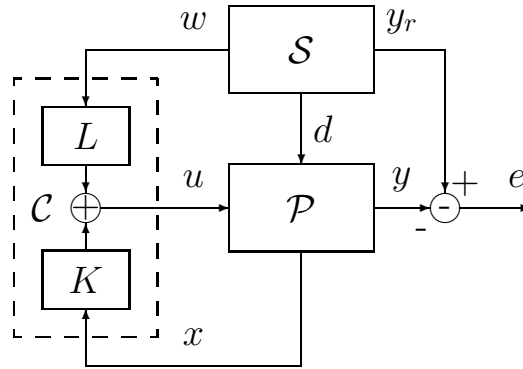
(a) The feedforward control configuration
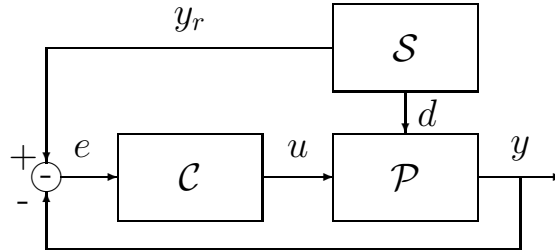
(b) The error feedback control configuration

Figure 1.2: Two control configurations. The feedforward control configuration 1.2(a) and the error feedback control configuration 1.2(b).

regulation in the frequency domain, the work is based on the insight about the time domain theory. In addition, the time domain formulation of the problem is more intuitive, so assume for now that the signals are functions of time. In the ideal case the control can be chosen so that the error is zero all the time, but, in practice, that is impossible. A more realistic aim would be to keep the error small or to try to get rid of the error asymptotically. In this thesis, the latter is the desired behavior of the error, and it is called asymptotic tracking, which mathematically means that $e(t) \to 0$, as $t \to \infty$. In this thesis, an appropriate control is provided by a controller $\mathcal{C}$. The two control configurations considered in this thesis are the feedforward control configuration shown in Figure 1.2(a) and the error feedback control configuration shown in Figure 1.2(b). The exogenous system, or the exosystem, $\mathcal{S}$ is generating the reference and the disturbance signals. The exosystem is assumed to be infinite-dimensional. One of the main problems of the thesis is to design controllers that have two properties; they should guarantee asymptotic tracking for all possible initial states of the plant, and the controller and the exosystem and they should stabilize the closed loop system. The problem of designing such controllers is

called the output regulation problem.

Any periodical signal can be generated by an infinite-dimensional exosystem, which is a major motivation for using them for regulation problems. A robot arm in a factory is a good example of a situation where a periodic reference signal is needed. Another motivating example is the noise damping in a ventilation duct by using a speaker attached on its wall, which have been studied, e.g., in [85]. In that case, the noise can be seen to be periodical since it is caused by a rotating fan at one end of the duct. The speaker should dampenn the disturbance noise at the open end of the ventilation duct, so the reference signal is just zero. Both of the control configurations presented above can be applied to the regulation problem at hand. It makes sense to assume that one has a good idea of the dynamics of the disturbance so that a feedforward controller is feasible. However, if there are other sources of disturbances, or the noise caused by the fan changes over time, one might want to use a feedback controller.

The controllers for infinite-dimensional exosystems typically have irrational transfer matrices. Such controllers are not implementable with the usual means, so the applicability such controllers in engineering applications is limited. However, there may exist very complex control systems, e.g., biological systems, which may be infinite-dimensional by nature or for which infinite-dimensional systems serve as a useful abstraction. For example, very long finite platoons of vehicles can be modelled as infinitely long platoons [43]. It is important study the related theory in order to understand the behavior and limitations of such systems. For example, [12] shows that the infinite-dimensionality of a platoon model can cause some undesired consequences. The designed controllers can serve as a target model that is approximated by a finite-dimensional physical system. By doing so, one accepts a small persistent error, but in principle this error can be made arbitrarily small. Measuring the error and finding optimal approximations are related to the topological side of the problem, which is not pursued in this thesis. Practical output regulation and approximation of infinite-dimensional controllers by finite-dimensional ones have been studied in [39].

A mathematical model of a real world phenomena unavoidably contains some errors. Small errors can stem from various sources, for example faulty parameter estimation, numerical errors, or model simplifications, such as linearization. Furthermore, the system modelled can change during the time due attrition of components or other causes. All these errors can be understood as perturbations in the mathematical model. A controller should solve the output regulation problem despite such perturbation in order to be fully functional. A controller is called structurally robust, or shortly robust, if it tolerates small perturbations in the model parameters. The problem of finding such controllers is called the robust regulation problem. Understanding robust control mechanisms and

their limitations is of fundamental importance since they can be encountered in biological systems in addition to engineering applications [46].

The robust regulation problem was solved for finite-dimensional linear systems during the 1970s, and the very nature of robustly regulating controllers was captured by a fundamental result called the internal model principle due to Francis and Wonham [28]. The internal model principle states, roughly speaking, that a robustly regulating controller should have the dynamics to be tracked build in it. In mathematical terms, this means that the controller should contain a suitably reduplicated copy of the exosystem. Since then, many authors have extended the results of finite-dimensional systems to infinite-dimensional systems. Recent development to this direction is the research on robust regulation problem with infinite-dimensional exosystems. The major obstacle with an infinite-dimensional exosystem is the trade off between the generality of the exosystem and the stabilizability of the closed loop system. It is known that if a controller contains an internal model of an infinite-dimensional exosystem, then it is often impossible to stabilize the closed loop exponentially [39], and one has to consider a weaker stability type, for example strong stability. Unfortunately, this leads to inferior robustness properties of stability. In addition, the reference and the disturbance signals need to be smooth enough in order for the problem to be solvable. The required smoothness is related to the high frequency behavior of the plant transfer function [34, 39, 72].

The robust regulation problem can be alternatively formulated in the frequency domain. In the frequency domain, the plant is a transfer function, which gives an input-output description, whereas a state space description in the time domain also models internal dynamics. If one wants the input and output spaces of all the transfer functions to be finite-dimensional, one has to modify the way the reference and disturbance signals are generated. The reference and the disturbance signals are generated by a transfer function Θ called the generator from stable signals $\hat{y}_0$ and $\hat{d}_0$, and the control configuration of Figure 1.2(b) is modified to the one depicted in Figure 1.3.

In the frequency domain, a transfer function is said to be stable if it is an element of a ring $\mathbf{R}$ of stable transfer functions. The choice of $\mathbf{R}$ depends on the problem to be solved; there are several commonly used rings available [56]. The asymptotic tracking means that the elements of the error vector $\hat{e}$ are in $\mathbf{R}$. The problem of finding a controller $\mathcal{C}$ that stabilizes a given plant $\mathcal{P}$ and guarantees asymptotic tracking for all the signals generated by the generator Θ from stable signals $\hat{y}_0$ and $\hat{d}_0$ for all plants near the nominal plant $\mathcal{P}$ is called the robust regulation problem. Being close is usually understood in the sense of some topology. It should be noted that the frequency domain problem is formulated purely in algebraic terms.

In this thesis, the main aim is to develop the frequency domain theory
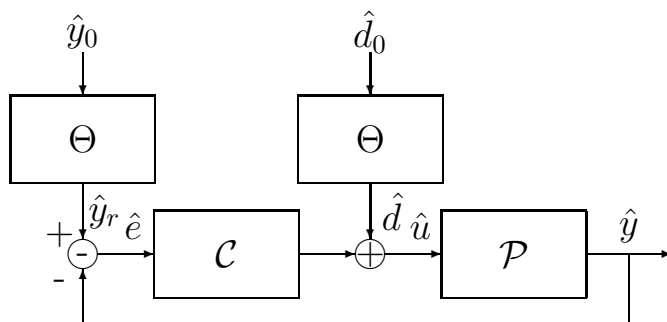
Figure 1.3: The control configuration in the frequency domain.

of the robust regulation problem for infinite-dimensional systems and signals. This means that the chosen transfer functions class should contain a variety of transfer functions of infinite-dimensional systems and the signal classes should contain Laplace transforms of signals generated by an infinite-dimensional exosystem. The ideas presented by Pohjolainen, Immonen, Hämäläinen, and Paunonen in [34, 39, 72] about the time domain robust regulation with infinite-dimensional exosystems are used to formulate and to solve the problem in the frequency domain.

The coprime factorization approach provides a simple framework to study stabilizability in the frequency domain. For the rational transfer functions and the fractions of H^∞ functions, it is not a restriction to assume that the plant transfer function has a left and a right coprime factorization, since all stabilizable transfer functions have them. In this thesis, the coprime factorization approach is used despite the fact that a plant does not necessarily have a coprime factorization in the algebraic structures to be considered. However, many of the transfer functions of interest have them, e.g., the stable transfer functions. In this thesis, only the existence of a left or a right coprime factorization is assumed, not necessarily both, in which sense the results in the thesis on robust regulation generalize those in the existing literature in which both coprime factorizations are invariably assumed to exist.

The exosystem is chosen so that the signals have an infinite number of poles approaching infinity on the imaginary axis. A similar problem has been studied in [35] by Hara et al., who designed controllers in the frequency domain with an infinite-dimensional internal model and discovered that it is impossible to find a state space realization that makes the closed loop system exponentially stable if the plant transfer function is strictly proper. This is a major restriction for the generality of the theory since many interesting systems are strictly proper. Robust regulation with an infinite-dimensional exosystem was later studied by Ylinen et al. in [101] in purely algebraic terms with H^∞-stability; a similar

restriction was encountered. In this thesis a way to overcome this restriction is found by defining two new rings of stable transfer functions resembling strong and polynomial stability in the time domain. The essential feature of the rings is that they allow transfer functions to be unbounded on the imaginary axis in contrast to H^∞-stability. Allowing the growth on the imaginary axis makes it possible for a controller to contain an infinite-dimensional internal model and to stabilize the closed loop system simultaneously.

A general algebraic approach to the robust regulation problem is due to Nett who studied the problem by using coprime factorizations [67]. Basically, Nett only assumed the ring $\mathbf{R}$ of stable transfer functions to be a general unitary ring with a topology induced by a metric. By using coprime fractions, Nett was able to formulate the internal model principle for the generators of form $\Theta = \theta^{-1} I$ where θ is an element of $\mathbf{R}$ under some topological assumptions. The frequency domain version of the internal model principle states, generally speaking, that a the denominator matrix of a coprime factorization of the controller should be divisible by θ in order the controller to be robustly regulating. Similar results for rational transfer functions can be found in [92] where the generator Θ was allowed to be a general rational function. In this case, θ appearing in the internal model principle is the largest invariant factor of the denominator in a coprime factorization of Θ. In [92], the solvability of the problem was characterized directly in terms of the coprime fractions of the plant and the generator, and it was noted that the results partially generalize to more general algebraic structures. The results in [67, 92] are generalized in the sense described below.

In this thesis, the robustness is understood in the sense that the closed loop stability should imply asymptotic tracking, whereas usually the robustness is understood in the sense of a topology. By defining the robustness this way one is able to consider the robust regulation without defining a topology in which sense the presented results generalize those in [67]. The treatment without a topological aspect is convenient since it is hard, if possible, to find a suitable topology for the class of transfer functions when the new rings of stable transfer functions to be presented in this thesis are used.

One of the main results of the thesis is that the internal model principle formulated in [67] is valid for the robustness type considered in this thesis. The signal class in this thesis is not initially generated by a frequency domain generator. To be able to show the internal model principle, a generator of form $\Theta = \theta^{-1} I$ that generates all the reference and the disturbance signals of interest is presented. Since the original signal class does not contain all of the signals generated by Θ, the results are more general to those in [67]. When showing that a robustly regulating controller necessarily contains an internal model, ideas from [92] are used. However, the results there are only for rational

matrices.

It is relatively easy to design controllers for rational transfer functions since coprime factorizations allow parametrization of all robustly regulating controllers [92]. An extremely simple controller for stable finite-dimensional systems consisting of an internal model and a feedback was proposed by Davison [17, 18]. This controller was later generalized for finite-dimensional exosystems and infinite-dimensional plants with their transfer functions in Callier-Desoer algebra in [33]. In the case of an infinite-dimensional exosystem, the controller design is not so straightforward. However, a modification of the Davison's robustly regulating controller for infinite-dimensional exosystems was presented in [101]. The drawback was that the plant transfer function was not allowed to be strictly proper. The approach in this thesis allows the generalization of the Davison's controller design for the infinite-dimensional exosystems even if the plant transfer function is strictly proper. It is also shown how the controllers for stable plants can be exploited if the transfer function is unstable. To the author's knowledge, this result is new even for finite-dimensional exosystems and plants. The proposed controller design is shown to be especially suitable for the functions in Callier-Desoer algebra.

1.2 Organization and Main Contributions

The theoretical work of the thesis is carried out in Chapters 2-4. An overview of the organization and the main results of this thesis is given below. Each of the research chapters contains a comprehensive introduction to its topic and a detailed list of research results.

Chapter 2 The topic of this chapter is the structure of transfer function matrices at infinity. A new definition for the structure at infinity of an infinite-dimensional plant is presented and its properties are studied. The need for defining the structure at infinity stems from the connection between the solvability of the output regulation problem and the structure at infinity which is to be established in Chapter 3. The problem has been that the structure at infinity of an infinite-dimensional plant has not been defined before. A definition for the structure of a transfer function is needed because the transfer functions of infinite-dimensional plants are not meromorphic at infinity, which results into a complicated high frequency behavior. A major part of the results of this chapter has already been published by the author in [52].

Chapter 3 The topic of this chapter is the (robust) output regulation problem in the time domain. The restrictions and the trade-offs caused by an

infinite-dimensional exosystem noticed in the recent literature, for example in [34, 39, 72], are emphasized. The chapter focuses on the required smoothness properties of the reference and the disturbance signals. The structure at infinity defined in Chapter 2 is exploited to characterize the solvability of the output regulation problem with feedforward and error feedback controllers. In particular, the required smoothness properties of the reference and the disturbance signals are made precise by using the structure of the plant transfer function at infinity. The results concerning the solvability of the output regulation problem with a feedforward controller can be found in articles [52, 53] by the author.

Chapter 4 The topic of this chapter is the robust regulation problem with an infinite-dimensional exosystem in the frequency domain. The ideas from the time domain robust regulation are used, i.e., by revising the definition of robustness and considering the weaker type of stability rather than the commonly used ones, the robust regulation problem is formulated so that the theoretical results for rational matrices in [92] can be generalized to a more general theoretical framework.

The major contributions of this chapter, and the whole thesis, are the following. First, the internal model principle is formulated and proved for the proposed problem. This generalizes the results in [67, 92]. Secondly, it is shown that the Davison's controller design in [17, 18] for stable plants can be used with an infinite-dimensional exosystem. Finally, it is shown how the controllers for stable plants can be exploited if the plant transfer function is unstable. To the authors knowledge, this result is new even for finite-dimensional exosystems and plants. The proposed controller design is shown to be suitable for the functions in Callier-Desoer algebra. The theory is based on the use of coprime factorizations, but only a left or a right, not necessarily both, coprime factorization is needed. This extends the earlier results on robust regulation appearing in the literature, which invariably assume the existence of both coprime factorizations.

Chapter 5 In this chapter the results of the thesis are summarized and compared. Directions for further research are discussed.

1.3   Notations and Definitions

The sets of complex numbers, real numbers, imaginary numbers, integers, and natural numbers are denoted by $\mathbb{C}$, $\mathbb{R}$, $i\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, respectively. Notation $\overline{\beta}$ is used for the complex conjugate of a $\beta \in \mathbb{C}$. The real and the imaginary parts of β are denoted by $\Re(\beta)$ and $\Im(\beta)$, respectively.

The extended complex plane $\mathbb{C} \cup \{\infty\}$ is denoted by $\mathbb{C}_\infty$. Notation $\mathbb{C}_\beta^+$ is used for the right half-plane $\{s \in \mathbb{C} \mid \Re(s) > \beta\}$ and $\mathbb{C}^+$ denotes the set $\mathbb{C}_0^+$. An s_0-centered disc $\{s \in \mathbb{C} \mid |s - s_0| < r\}$ with radius r is denoted by $B_r(s_0)$. The closure of $S \subseteq \mathbb{C}$ is denoted by $\overline{S}$.

Let X and Y be Hilbert spaces. The set of all linear operators from X to Y is denoted $\mathcal{L}(X, Y)$. The set of bounded linear operators from X to Y is denoted by $\mathcal{B}(X, Y)$. Shorthand notations $\mathcal{L}(X)$ and $\mathcal{B}(X)$ are used for $\mathcal{L}(X, X)$ and $\mathcal{B}(X, X)$. The domain, the range and the null space of an operator $G \in \mathcal{L}(X, Y)$ are denoted by $\mathcal{D}(G)$, $\mathcal{R}(G)$ and $\mathcal{N}(G)$, respectively.

If H is a set, then the set of all $n \times m$-matrices with elements in H is denoted by $H^{n \times m}$. The set of all matrices with elements in H is denoted by $\mathcal{M}(H)$. Notation $\|\cdot\|$ is used for vector and operator norms. An inner product in a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$.

Let $G : \mathbb{C} \to \mathbb{C}^{n \times m}$ be a matrix valued function. If $\operatorname{rank}(G(s))$, the rank of $G(s)$, is k almost everywhere, then it is said that *the normal rank of G is k* and is denoted by $\operatorname{nrank}(G(s)) = k$.

Let G be an $n \times m$-matrix. An $i \times i$-minor of G is denoted by $|A|_{\mathbf{r}, \mathbf{c}}^i$, where the multi-indices $\mathbf{r} = (r_1, r_2, \ldots, r_i)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_i)$, where $1 \leq r_j < r_{j+1} \leq n$ and $1 \leq c_j < c_{j+1} \leq m$, define the rows and columns selected. The determinant of a square matrix G is denoted by $\det(G)$.

The Hardy space of all analytic functions that are bounded in $\mathbb{C}_\beta^+$ is denoted by H_β^∞. Shorthand notation H^∞ is used for H_0^∞ and the set of all functions that belong to H_β^∞ for some $\beta < 0$ is denoted by H_-^∞. The hardy space of analytic functions $f : \mathbb{C}_\beta^+ \to X$ where X is a Hilbert space such that

$$\|f\|_{2,\beta} = \left(\sup_{x > \beta} \int_{-\infty}^\infty \|f(x + y\mathrm{i})\|^2 dy \right)^{\frac{1}{2}} < \infty$$

is denoted by $H_\beta^2(X)$. The class of Lebesgue measurable and square integrable functions $f : [a, b] \to X$ is denoted by $\mathbf{L}_2((a, b), X)$ where X is a Hilbert space. If $X = \mathbb{C}$, then the shorthand notation $\mathbf{L}_2(a, b)$ is used. The class of Lebesgue measurable and absolutely integrable functions $f : [a, b] \to \mathbb{C}$ is denoted by $\mathbf{L}_1(a, b)$.

An ordered sequence with elements x_k is denoted by $(x_k)_{k \in \mathbb{Z}}$. The class of bounded complex sequences $x = (x_k)_{k \in \mathbb{Z}} \subset \mathbb{C}$ is denoted by ℓ^∞. Notation ℓ^2 is used for the class of complex sequences $x = (x_k)_{k \in \mathbb{Z}} \subset \mathbb{C}$ that are square summable.

Chapter 2

Structure at Infinity

2.1   Introduction to Structure at Infinity

There are many different zero types, both finite and infinite, in the literature, for example, transmission zeros, system zeros and different decoupling zeros. Theory of infinite zeros is closely related to that of finite zeros. Thus, it is not a surprise that results on finite zeros were soon after followed by corresponding results on infinite zeros. Most of the work done on different zeros was done in 1970s and 1980s. A review on this research can be found in [86]. Structure at infinity of finite-dimensional system is related to the infinite transmission zeros and is thoroughly studied.

Structure at infinity has a central role in many control theoretical problems of finite-dimensional systems. Solvability of model matching problems can be characterized in terms of the structure at infinity [58, 59]. Disturbance rejection and different decoupling problems can be formulated as a model matching problem, so it is natural that the structure at infinity appears in many related results [19, 20, 23, 64]. Furthermore, the asymptotic behavior of the root-loci is dependent on the structure at infinity [38, 47].

Structure at infinity describes the behavior of a rational matrix at high frequencies and there are many equivalent definitions available. Transfer functions of finite-dimensional systems are rational, so it follows that theory of rational matrices can be used to study their properties.

The first study to define the zeros and poles at infinity of multiple-input multiple-output (MIMO) systems was that of Rosenbrock in [81]. His definition was based on the Smith-McMillan form over the ring of polynomials. Let an principal ideal domain $\mathbf{R}$ be given and let the field of fractions associated with $\mathbf{R}$ be $\mathbf{F_R}$, for definition see for example [92]. Then for every $G \in \mathbf{F}_{\mathbf{R}}^{n \times m}$ there

exists such $\mathbf{R}$-invertible matrices U and V in $\mathcal{M}(\mathbf{R})$ that

$$UGV = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where Λ has non-zero diagonal elements $f_i = \frac{n_i}{d_i}$, $i = 1, 2, \ldots, \text{rank}(G) = l$. Furthermore, n_i and d_i are coprime, n_i divides n_{i+1} and d_{i+1} divides d_i for all $i = 1, 2, \ldots, l-1$ [92, p. 404]. The diagonal matrix is called the Smith-McMillan form of G. A degree function δ is always defined in a Euclidean domain, and an algorithm based on the degree function can be used to bring a matrix into the Smith-McMillan form. In the Euclidean domain of polynomials a degree function is given by the order of polynomials denoted by $\deg(\cdot)$. For proper rational functions a degree function is defined by $\delta(f) = \deg(d) - \deg(n)$, where $f = \frac{n}{d}$ and n and d are polynomials.

If $\mathbf{R}$ is chosen to be the set of polynomials, then $\mathbf{F_R}$ is the set of rational functions and U and V above are polynomial matrices. One says, that G has a zero of order k at $s_0 \in \mathbb{C}$ if n_i has a zero of order k there for some i. In this case one sets $q_i(s_0) = k$. There is a pole of order k at $s_0 \in \mathbb{C}$ if d_i has a zero there for some i. In this case, one sets $q_i(s_0) = -k$. If f_i has no poles nor zeros at s_0, then $q_i(s_0) = 0$. The defined l-tuple $(q_1(s_0), \ldots, q_l(s_0))$ is called the structure at s_0.

Rosenbrock defined the zeros of $G(s)$ at infinity to be zeros at $s = 1$ of $G(\alpha s / s - 1)$, where $\alpha \neq 0$ is not a finite zero of $G(s)$. He also gave another non-equivalent definition. These two definitions were discussed in [77] where the structure at infinity was defined to be the structure at 0 of $G(1/s)$. This definition is equivalent to the Rosenbrock's definition mentioned above.

An alternative way to define the structure at infinity is to choose $\mathbf{R}$ to be the set of proper rational functions $\mathbb{C}_{pr}(s)$, i.e., rational functions with the degree of the numerator polynomial less than or equal to the degree of the denominator polynomial. Again, $\mathbf{F_R}$ is the set of all rational functions. The structure at infinity is the l-tuple with $q_i = \deg(d_i) - \deg(n_i)$, where $\deg(\cdot)$ refers to the degree of a polynomial. The definitions above are equivalent [91].

Other definitions and approaches to define structure at infinity are available. Structure at infinity can be calculated by using minors [44, 90]. This method uses the degree function of an Euclidean domain. An alternative method uses Laurent expansion at infinity and certain Toeplitz matrices to calculate structure at infinity [76]. A geometric algorithm resulting in the same l-tuple as above was given in [10]. For a module theoretical definition see [11] and the references therein.

The above approach, which defines the structure at infinity, is not possible for infinite-dimensional systems because the transfer functions are not mero-morphic at infinity in general. Since systems with finite-dimensional input and

output spaces and bounded operators A, B, C, and D have a transfer function analytic at infinity the above definitions generalize for such systems [60]. For other systems the situation is far more complicated.

For time-delay systems, Malabre and Rabah provided two different structures at infinity, i.e., strong and weak structures, in a series of articles [61, 62, 63] and used them in model matching and decoupling problems. Restricting the way the infinity was approached allowed similar definition to those mentioned above for certain time-delay systems. For strong structure, the infinity was approached so that the real part approaches infinity. For weak structure, the infinity was approached along the positive real line.

2.1.1 Organization and Contributions of the Chapter

Section 2.2: Besides the definitions due to Malabre and Rabah, there are no other definitions of the structure at infinity suitable for infinite-dimensional systems available. In addition, the existing definitions are only suitable for infinite-dimensional systems with bounded operators [60] or for time-delay systems [61, 62, 63], so the class of transfer functions they cover is not very general. In Section 2.2.4, a new definition of the structure at infinity suitable for all infinite-dimensional systems with finite-dimensional input and output spaces is given. The trade-off is that the definition only describes the size of the transfer function and does not provide strong enough algebraic properties needed in many related problems. However, as seen in Chapters 3 and 4, the defined structure at infinity can be applied to give solvability conditions for certain output regulation problems.

The definition of the structure of a transfer function given in this thesis is based on a diagonal form similar to the Smith-McMillan form. An algorithm diagonalizing a given transfer function is given in Section 2.2.2. The algorithm resembles to the one proposed in [91] based on the use of the degree function on the Euclidean domain of the proper transfer functions. In the algorithm presented in this section, the degree function is replaced by a majorization relation, which allows a comparison of the sizes of the elements in the matrix. The majorization relation is presented and its properties are discussed in Section 2.2.1. In order to simplify the considerations, the way the infinity is approached is restricted to a set of paths, which leads to so called directed structure. Examples of the structure at infinity of infinite-dimensional systems are provided. The results in this section have been published by the author in the research article [52], see also [51].

Section 2.3: In this section, calculation of the structural functions is consid-

ered. The method using minors for finding the structure of a rational matrix at infinity presented in [44, 90] is extended to the infinite-dimensional systems. In addition, a relation between the structural functions and the singular values is established. To the author's knowledge this relation is new even for finite-dimensional systems. The method for calculating the structure at infinity of a transfer function has been published by the author in [52].

Section 2.4: In this section, invariance of the structure at infinity with respect to state and output feedbacks is considered. The results of this section generalize the well-known properties of the finite-dimensional systems stated in [24] to infinite-dimensional systems. The main results show that state or output feedbacks do not change the structure of a system at infinity of exponentially stabilizable or exponentially detectable systems. These properties are important since many controllers use different types of feedback loops.

Section 2.5: Robustness of structure at infinity of linear systems is discussed in this section. Later in Chapter 3, robustness of structure at infinity is shown to be related to robustness of output regulation. Examples show that the structure at infinity has in general poor robustness properties and that there can be some physically important perturbations that do not change the structure at infinity.

Section 2.6: In the output regulation problems considered in Chapter 3, one needs to consider both the finite transmission zeros and the high frequency behavior of the plant transfer functions. For this, it is convenient to define the global structure of systems that captures the structure at finite points as well as the structure at infinity, which is the purpose of this section. The definition of the global structure is an easy modification of the structure at infinity defined in Section 2.2.

2.2   Structure at Infinity

2.2.1   The Majorization Relation

The structure at infinity of rational matrices is defined by using the Smith-McMillan form over the Euclidean domain of all proper rational functions. The degree function δ of $\mathbb{C}_{pr}(s)$ is used in an algorithm that constructs the Smith-McMillan form at infinity [91]. The degree function's role in the algorithm is to compare the rates at which the transfer function's elements vanish at high frequencies.

The aim of this section is to present a majorization relation that enables comparison of high frequency behavior in the class of all complex functions much like the degree function δ of $\mathbb{C}_{pr}(s)$ enables the comparison in $\mathbb{C}_{pr}(s)$. This relation enables one to find a structure algorithm that produces a diagonal form for transfer functions of infinite dimensional systems. The diagonal form resembles the Smith-McMillan form and defines the structure at infinity for infinite-dimensional systems.

The set of all complex functions $f : \mathbb{C} \to \mathbb{C}_{\infty}$ is denoted by $\mathbf{C}$. The set of all paths in complex plane approaching infinity is denoted by

$$\mathrm{P} = \{p : [0, \infty) \to \mathbb{C} \,|\, \forall M > 0 : \exists \rho \geq 0 : \forall \alpha \geq \rho : |p(\alpha)| > M\}.$$

For the rest of the thesis the following conventions are used: $\frac{\infty}{\infty} = 1 = \frac{0}{0}$, $\frac{a}{0} = \infty$ for $a \neq 0$ and $\frac{a}{\infty} = 0$ for $a \neq \infty$.

Definition 2.2.1. Let $f, g \in \mathbf{C}$, and $H \subseteq \mathrm{P}$. If

$$\forall p \in H : \exists \rho \geq 0 : \sup_{\alpha \geq \rho} \left| \frac{g(p(\alpha))}{f(p(\alpha))} \right| < \infty,$$

it is said that f majorizes g with respect to H. This is denoted by $f \geq_H g$. If $f \geq_H g$ and $g \geq_H f$, notation $f =_H g$ is used.

Some direct consequences of Definition 2.2.1 are listed in the following lemma.

Lemma 2.2.2. Let $H \subseteq \mathrm{P}$, and let $f_1, f_2, f_3, f_4 \in \mathbf{C}$. The relations $\geq_H$ and $=_H$ have the following properties:

1. If $f_1 \geq_H f_2$ and $f_1 \geq_H f_3$, then $f_1 \geq_H f_2 + f_3$.

2. If $f_1 \geq_H f_2$ and $f_3 \geq_H f_4$, then $f_1 f_3 \geq_H f_2 f_4$.

3. If $f_1 =_H 1$, then $f_1 f_2 =_H f_2$.

4. Relation $=_H$ is an equivalence relation in the set of all complex functions.

Notation $[f]_H$ is used for the equivalence class of a function f. Setting $[f]_H \geq_H [g]_H$, if $f \geq_H g$, defines a partial order in $\mathbf{C}/=_H$. In the next section the fact that there exists the supremum of a finite set of equivalence classes is needed.

Lemma 2.2.3. Let $H \subseteq \mathrm{P}$. The supremum of $\{[f]_H, [g]_H\}$ exists with respect to the partial order $\geq_H$. Furthermore, the supremum is $[|f| + |g|]_H$.

Proof. Definition 2.2.1 implies that $[|f| + |g|]_H \geq_H [f]_H$ and $[|f| + |g|]_H \geq_H$ $[g]_H$, so $[|f| + |g|]_H$ is an upper bound. Let $[h]_H$ be an upper bound. By Definition 2.2.1, there exists $\rho \geq 0$ such that

$$\sup_{\alpha \geq \rho} \left| \frac{|f(p(\alpha))| + |g(p(\alpha))|}{h(p(\alpha))} \right| \leq \sup_{\alpha \geq \rho} \left| \frac{f(p(\alpha))}{h(p(\alpha))} \right| + \sup_{\alpha \geq \rho} \left| \frac{g(p(\alpha))}{h(p(\alpha))} \right| < \infty,$$

for all $p \in H$. This shows that $[h]_H \geq_H [|f| + |g|]_H$, so $[|f| + |g|]_H$ is the supremum. $\qquad\square$

Related to an $f \in \mathbf{C}$ function $\operatorname{sgn}(f) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is defined by setting

$$\operatorname{sgn}(f)(s) = \frac{|f(s)|}{f(s)}.$$

Recall that conventions $\frac{0}{0} = 1 = \frac{\infty}{\infty}$ were made, so the above functions is well-defined for all $f \in \mathbf{C}$. This function provides a convenient way to represent the supremum of $\{[f]_H, [g]_H\}$.

Lemma 2.2.4. *One has* $|f| + |g| =_H g + f \frac{\operatorname{sgn}(f)}{\operatorname{sgn}(g)}$ *for all* $H \subseteq \mathrm{P}$ *and* $f, g \in \mathbf{C}$.

Proof. The claim follows by the definitions of $\operatorname{sgn}(\cdot)$ and $=_H$, since

$$\left| g + f \frac{\operatorname{sgn}(f)}{\operatorname{sgn}(g)} \right| = \left| g + g \frac{|f|}{|g|} \right| = |f| + |g|.$$

$\qquad\square$

2.2.2   The Structure Algorithm

In this section, an algorithm for constructing a diagonal that generalizes the Smith-McMillan form to the class of matrices with complex function elements is presented. The algorithm produces two matrices that diagonalize a given matrix. The diagonalizing matrices satisfy the following boundedness property.

Definition 2.2.5. It is said that $M : \mathcal{D}(M) \subseteq \mathbb{C} \to \mathbb{C}^{n \times n}$ satisfies the uniform boundedness property with respect to $H \subseteq \mathrm{P}$ if for all $p \in H$ there exists such a $\rho > 0$ that $M(p(\alpha))$ is invertible for all $\alpha \geq \rho$ and $\|M(p(\alpha))\|$, $\|M^{-1}(p(\alpha))\|$ are uniformly bounded in $[\rho, \infty)$.

Besides the uniform boundedness property, no other requirements for the elements of the diagonalizing matrices are set. One can always find the diagonal form no matter how the infinity is approached because of the minimalistic requirements for the diagonalizing matrices. On the other hand, restricting the

way the infinity can be approached generally leads to simpler functions in the diagonal form.

Unlike with rational functions, one cannot generally compare the rate of convergence or divergence at high frequencies of two complex functions. For example, one cannot say which one of the functions $\sin(s)$ and $1/s$ is smaller at high frequencies, because $\sin(s)$ has infinitely many finite zeros diverging to infinity but it does not converge to zero, whereas $1/s$ converges to zero as $s \to \infty$ but it does not have zeros at finite frequencies. However, one can say that in some sense there exists a smallest function majorizing a given set of functions by Lemma 2.2.3. It is shown in the next theorem that a given matrix can be brought to a form with an element majorizing all the other elements.

Lemma 2.2.6. *Let $H \subseteq$ P. A matrix $G(s) \in \mathbf{C}^{n \times m}$ can be written in the form*

$$G(s) = V(s)F(s)U(s)$$

where $F(s) \in \mathbf{C}^{n \times m}$ has an element that majorizes all the other elements, and U and V satisfy the uniform boundedness property of Definition 2.2.5.

Proof. The matrix received by using elementary matrix operations to G is denoted by $F = (f_{ij})$. Note that F changes after each operation.

It is first shown how to bring a majorizing element to kth column. Add the first row multiplied by $h = \frac{sgn(f_{1k})}{sgn(f_{2k})}$ to the second one. The element at $(2,k)$-position now is $f_{2k} + hf_{1k}$. It majorizes f_{1k} by Lemma 2.2.4.

Add the second row multiplied by $h = \frac{sgn(f_{2k})}{sgn(f_{3k})}$ to the third one. The element at $(3,k)$-position is $f_{3k} + \frac{|f_{2k}(s)|}{sgn(f_{3k})}$, and it majorizes both f_{1k} and f_{2k} by the transitivity of relation $\geq_H$. Continuing this way one finds an element f_{nk} that majorizes all the elements in the kth column.

Use the procedure described above to bring a majorizing element to the first column, add the first column multiplied by $h = \frac{sgn(f_{n1})}{sgn(f_{n2})}$ to the second one, and interchange the first and the last row. The element at $(1,2)$-position majorizes all the elements in the first column. Repeat the process to find a majorizing element to the second column. Note that the elements in the first column may change. However, they are of the form $f_{l1} + hf_{l+1,1}$ where $h =_H 1$, so the found majorizing element majorizes the elements in the first and the second column by the first and the third properties of Lemma 2.2.2.

Continuing this way one finally finds an element f_{nm} that majorizes all the other elements. Interchanging rows and adding row or column multiplied by h to another corresponds to a multiplication by a square matrix. These square matrices are invertible. They have the uniform boundedness property because in every multiplication $h =_H 1$. Thus, one finds the desired F, U and V by multiplying the matrices corresponding to the elementary row and column operations appropriately. $\qquad \square$

Theorem 2.2.7. *Let $H \subseteq \mathrm{P}$. A matrix $G(s) \in \mathbf{C}^{n \times m}$ can be written in the form*

$$G(s) = V(s) \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} U(s) \qquad (2.1)$$

where $\Lambda(s) = \mathrm{diag}\,(q_1(s), \ldots, q_r(s))$ is a diagonal matrix with non-zero elements, $q_1 \geq_H q_2 \geq_H \cdots \geq_H q_r$, the zero blocks may be non-existent, and U and V satisfy the uniform boundedness property of Definition 2.2.5.

Proof. The algorithm presented here is similar to the one in [91]. Notation $F = (f_{ij})$ denotes the matrix during the algorithm received by using some elementary matrix operations to G. Note that F changes after each operation.

Step 1: At the beginning one has $F = G$. If there is no elements majorizing all the other elements, then use the elementary matrix operations described in the proof of Lemma 2.2.6 to bring F to a form with a majorizing element. Bring the majorizing element to (1,1)-position by interchanging rows and columns.

The element f_{11} now majorizes all the elements in the first row and column. It is possible to subtract the first row multiplied by $\frac{f_{i1}}{f_{11}}$ from the ith one because $1 \geq_H \frac{f_{i1}}{f_{11}}$. After this operation, $f_{i1} = 0$. In this way, it is possible to zero out all the elements in the first row except the first one. Similarly, all the elements in the first column can be zeroed out except the first one.

After zeroing out the off-diagonal elements in the first row and column the matrix F is of the form

$$F = \begin{bmatrix} f_{11} & 0 \\ 0 & F_1 \end{bmatrix}.$$

Lemma 2.2.2 implies that the elements in F_1 are majorized by f_{11} since f_{11} majorized all the elements in the matrix before zeroing out the of diagonal elements.

Step $k + 1$: After k steps the following matrix is obtained

$$F = \begin{bmatrix} f_{11} & 0 & \cdots & 0 & 0 \\ 0 & f_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{kk} & 0 \\ 0 & 0 & \cdots & 0 & F_k \end{bmatrix}.$$

If there are no majorizing elements in F_k, then use Lemma 2.2.6 to find one. By Lemma 2.2.2, f_{kk} still majorizes the newly found majorizing element of F_k. Bring the majorizing element to $(k + 1, k + 1)$-position. Zero out the first row and column of the submatrix F_k as described in the first step. It is easy to see that f_{kk} majorizes $f_{k+1,k+1}$.

End of the algorithm: If F_k at the beginning of the $(k+1)$th step is a zero matrix or $k + 1 > \min\{n, m\}$, then the algorithm ends.

Matrices corresponding to an interchange of rows or columns are constant matrices. The assumption that one of the elements of the submatrix F_k majorizes all the other elements implies that every matrix corresponding to an elementary row or column operation that adds a row or a column multiplied by an element to another satisfies the uniform boundedness property. It is therefore clear that there exist the matrices U and V such that G is of the form (2.1). $\square$

It follows from the uniform boundedness property of $U(s)$ and $V(s)$ that the elements of $\Lambda(s)$ describe the behavior of $G(s)$ at infinity. It can happen that $U(s)$ and $V(s)$ are not continuous. In particular, they are not in any ring of transfer functions commonly encountered in system theory. Despite this inconvenience, the diagonal elements describe the rate of convergence to zero or the rate of divergence to infinity, which is the property needed in Chapter 3. The next theorem establishes a uniqueness property of the diagonal entries of Λ.

Theorem 2.2.8. *The elements of Λ in (2.1) are unique up to multiplication by element h that satisfies $h =_H 1$.*

Proof. Let the matrices

$$\Lambda_1(s) = \operatorname{diag}\left(\lambda_1(s), \ldots, \lambda_r(s)\right)$$

and

$$\Lambda_2 = \operatorname{diag}\left(\sigma_1(s), \ldots, \sigma_r(s)\right)$$

be as in (2.1). For $i = 1, 2$

$$G(s) = V_i(s)L_i(s)U_i(s) = V_i(s) \begin{bmatrix} \Lambda_i(s) & 0 \\ 0 & 0 \end{bmatrix} U_i(s),$$

where $U_i(s)$ and $V_i(s)$ satisfy the uniform boundedness property. Write $V(s) = V_1^{-1}(s)V_2(s)$ and $U(s) = U_2(s)U_1^{-1}(s)$. Matrices $V(s)$ and $U(s)$ satisfy the uniform boundedness property and $L_1(s) = V(s)L_2(s)U(s)$.

By using the Binet-Cauchy formula (A.2) and noting that $L_2(s)$ have nonzero elements only on the diagonal, it can be seen that

$$
\begin{aligned}
\left.|L_1(s)|_{\mathbf{r},\mathbf{r}}^i \right. &= |V(s)L_2(s)U(s)|_{\mathbf{r},\mathbf{r}}^i \\
&= \sum_{\mathbf{h},\mathbf{v}} |V(s)|_{\mathbf{r},\mathbf{h}}^i \, |L_2(s)|_{\mathbf{h},\mathbf{v}}^i \, |U(s)|_{\mathbf{v},\mathbf{r}}^i \\
&= \sum_{\mathbf{h}} |V(s)|_{\mathbf{r},\mathbf{h}}^i \, |L_2(s)|_{\mathbf{h},\mathbf{h}}^i \, |U(s)|_{\mathbf{h},\mathbf{r}}^i .
\end{aligned}
\tag{2.2}
$$

Since all the elements of $U(s)$ and $V(s)$ are majorized by 1, Lemma 2.2.2 shows that $1 \geq_H |V(s)|^i_{\mathbf{r},\mathbf{h}} |U(s)|^i_{\mathbf{h},\mathbf{r}}$ for all $\mathbf{h}$ and $\mathbf{r}$. Set $r = (1, 2, \ldots, k)$, where $k \leq r$. By Lemma 2.2.2, the property $\sigma_j \geq_H \sigma_{j+1}$ for all $j = 1, \ldots, r-1$, and the equation (2.2),

$$\prod_{j=1}^k \sigma_j(s) \geq_H \prod_{j=1}^k \lambda_j(s). \tag{2.3}$$

Similarly one shows that

$$\prod_{j=1}^k \lambda_j(s) \geq_H \prod_{j=1}^k \sigma_j(s). \tag{2.4}$$

The equations (2.3) and (2.4) imply that $\lambda_j =_H \sigma_j$ for all $j = 1, \ldots, r$. □

2.2.3   The Definition for Matrices

The structure at infinity of a rational matrix G is always $\{s^{-q_1}, s^{-q_2}, \ldots, s^{-q_r}\}$, where $r = \text{nrank}(G)$. The structure at infinity defined by Rosenbrock in [81] is the set of integers $\{q_1, q_2, \ldots, q_r\}$. Henceforth, the integers $q_1, q_2, \ldots, q_r$ are called *the structural indices*. They are defined for all matrix valued functions that are meromorphic at infinity [60].

In general, it is impossible to define the structure at infinity as a set of integers because of the complex nature of the structural functions, e.g., Example 2.2.11 and Example 2.2.18. However, one can use the diagonal form found in the previous section instead of the Smith-McMillan form to describe the behavior of a matrix at high frequencies. By Theorem 2.2.8, the structural indices of a rational matrix just denote the equivalence classes to which the structural functions belong. These observations lead to the following definition.

Definition 2.2.9. Let $\Lambda(s) = \text{diag}(q_1(s), \ldots, q_r(s))$ be the diagonal matrix in (2.1). The structure at infinity of $G(s)$ with respect to H is $\{[q_1]_H, \ldots, [q_r]_H\}$. The functions $q_1, q_2, \ldots, q_r$ are called structural functions with respect to H.

The structure at infinity with respect to some set of paths is a set of equivalence classes, where a equivalence class can occur multiple times. The structure at infinity is uniquely defined by Theorem 2.2.8. From now on, notation f is used instead of $[f]_H$ when referring to the structure at infinity if there is no risk of confusion.

It might be that H consists of all paths in some specified unbounded set $Q \subseteq \mathbb{C}$, i.e., $H = \{p \in \mathrm{P} \,|\, p(\alpha) \in Q\}$. In this case notations $\geq_Q$, $=_Q$ and $[\cdot]_Q$ can be used instead of $\geq_H$, $=_H$ and $[\cdot]_H$, respectively. If $f \geq_Q g$ ($f =_Q g$), it is said that f majorizes g along Q (f is equivalent to g along Q). If H is replaced

by a set Q in the above definition it is said that the structure at infinity is defined along Q.

Actually, a number of different structures at infinity have been defined. The different structures are related in the following way.

Theorem 2.2.10. *Let* $H_1, H_2 \subseteq \mathrm{P}$ *and* $G \in \mathbf{C}^{n \times m}$. *Let the structure at infinity of* G *with respect to* H_1 *be* $\{[q_1]_{H_1}, \ldots, [q_r]_{H_1}\}$. *If* $H_2 \subseteq H_1$, *then the structure at infinity of* G *with respect to* H_2 *is* $\{[q_1]_{H_2}, \ldots, [q_r]_{H_2}\}$.

Proof. Set $H = H_1$ in Theorem 2.2.7, and let U and V be the matrices in (2.1). If $f \geq_{H_1} g$, then $f \geq_{H_2} g$. Thus, the matrices U and V satisfy the uniform boundedness property respect to H_2 from which the claim follows. $\square$

Example 2.2.11. Consider the time-delay system $\Sigma_D (A_0, A_1, B, C)$ where

$$A_0 = \begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The transfer function of this system is

$$\mathcal{P}(s) = \begin{bmatrix} \frac{e^{-s}-1}{g(s)} & \frac{e^{-s}+1}{g(s)} & \frac{(e^{-s}+s)(e^{-s}-1)}{(s+1)g(s)} \\ \frac{s+1}{g(s)} & \frac{-(e^{-s}+1)}{(s+1)g(s)} & \frac{s(e^{-s}-1)}{(s+1)g(s)} \\ 0 & \frac{1}{s+1} & 0 \end{bmatrix}$$

where $g(s) = (s + 1)^2 + e^{-s} - 1$.

The behavior along the imaginary axis becomes crucial in Chapter 3 and Chapter 4, so the structure at infinity is defined along $i\mathbb{R}$. The structural functions are found by using the algorithm presented in the proof of Theorem 2.2.7.

It is easily verified that $\frac{1}{s+1}$ majorizes all the other elements of $\mathcal{P}(s)$ along $i\mathbb{R}$. The majorizing element is first brought to the (1,1)-position by interchanging the first and the last rows, and the first and the second columns. Now

$$\mathcal{P}_1(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathcal{P}(s) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ \frac{-(e^{-s}+1)}{(s+1)g(s)} & \frac{s+1}{g(s)} & \frac{s(e^{-s}-1)}{(s+1)g(s)} \\ \frac{e^{-s}+1}{g(s)} & \frac{e^{-s}-1}{g(s)} & \frac{(e^{-s}+s)(e^{-s}-1)}{(s+1)g(s)} \end{bmatrix}.$$

The non-diagonal elements in the first row are already zeros so one only needs to zero out the non-diagonal elements in the first column. Adding the first row multiplied by

$$-\left(\frac{1}{s+1}\right)^{-1}\frac{-(\mathrm{e}^{-s}+1)}{(s+1)g(s)}=\frac{(\mathrm{e}^{-s}+1)}{g(s)}$$

to the second one, and adding the first row multiplied by

$$-\left(\frac{1}{s+1}\right)^{-1}\frac{\mathrm{e}^{-s}+1}{g(s)}=-\frac{(\mathrm{e}^{-s}+1)(s+1)}{g(s)}$$

to the last one brings the transfer function to the form

$$\mathcal{P}_2(s)=\begin{bmatrix}1&0&0\\\frac{(\mathrm{e}^{-s}+1)}{g(s)}&1&0\\-\frac{(\mathrm{e}^{-s}+1)(s+1)}{g(s)}&0&1\end{bmatrix}\mathcal{P}_1(s)=\begin{bmatrix}\frac{1}{s+1}&0&0\\0&\frac{s+1}{g(s)}&\frac{s(\mathrm{e}^{-s}-1)}{(s+1)g(s)}\\0&\frac{\mathrm{e}^{-s}-1}{g(s)}&\frac{(\mathrm{e}^{-s}+s)(\mathrm{e}^{-s}-1)}{(s+1)g(s)}\end{bmatrix}.$$

The element $\frac{s+1}{g(s)}$ majorizes the elements of the submatrix received by deleting the first row and column. By adding the second column multiplied by

$$-\left(\frac{s+1}{g(s)}\right)^{-1}\frac{s(\mathrm{e}^{-s}-1)}{(s+1)g(s)}=-\frac{s(\mathrm{e}^{-s}-1)}{(s+1)^2}$$

to the last column, and the second row multiplied by

$$-\left(\frac{s+1}{g(s)}\right)^{-1}\frac{\mathrm{e}^{-s}-1}{g(s)}=-\frac{\mathrm{e}^{-s}-1}{s+1}$$

to the last row one, the diagonal form

$$\mathcal{P}_3(s)=\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{\mathrm{e}^{-s}-1}{s+1}&1\end{bmatrix}\mathcal{P}_2(s)\begin{bmatrix}1&0&0\\0&1&-\frac{s(\mathrm{e}^{-s}-1)}{(s+1)^2}\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{s+1}&0&0\\0&\frac{s+1}{g(s)}&0\\0&0&\frac{\mathrm{e}^{-s}-1}{(s+1)^2}\end{bmatrix}$$

is obtained. Multiply the diagonal form by $\mathrm{diag}\left(\frac{s+1}{s},\frac{g(s)}{s(s+1)},\frac{(s+1)^2}{s^2}\right)$ to get

$$\Lambda(s)=\mathrm{diag}\left(s^{-1},s^{-1},s^{-2}(\mathrm{e}^{-s}-1)\right). \tag{2.5}$$

One gets the matrices

$$V(s)=\left(\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{\mathrm{e}^{-s}-1}{s+1}&1\end{bmatrix}\begin{bmatrix}1&0&0\\\frac{(\mathrm{e}^{-s}+1)}{g(s)}&1&0\\-\frac{(\mathrm{e}^{-s}+1)(s+1)}{g(s)}&0&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}\frac{(\mathrm{e}^{-s}+1)(s+1)}{g(s)}&\frac{\mathrm{e}^{-s}-1}{s+1}&1\\-\frac{\mathrm{e}^{-s}+1}{g(s)}&1&0\\1&0&0\end{bmatrix}$$

and

$$
U(s) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{s(e^{-s}-1)}{(s+1)^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+1}{s} & 0 & 0 \\ 0 & \frac{g(s)}{s(s+1)} & 0 \\ 0 & 0 & \frac{(s+1)^2}{s^2} \end{bmatrix} \right)^{-1}
$$

$$
= \begin{bmatrix} 0 & \frac{s}{s+1} & 0 \\ \frac{s(s+1)}{g(s)} & 0 & \frac{s^2(e^{-s}-1)}{(s+1)g(s)} \\ 0 & 0 & \frac{s^2}{(s+1)^2} \end{bmatrix}
$$

for which $\mathcal{P}(s) = V(s)\Lambda(s)U(s)$ by multiplying the matrices corresponding to the matrix operations made during the algorithm. It is easy to verify that

$$
V(i\omega) \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

and

$$
U(i\omega) \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

as $\omega \to \pm\infty$, so the found matrices $U(s)$ and $V(s)$ satisfy the uniform boundedness property along $i\mathbb{R}$.

The diagonal form (2.5) reveals that a part of stucure at infinity of $\mathcal{P}(s)$ is purely rational, but another part is a combination of the periodically behaving function $e^{-s} - 1$ and the rational function s^{-2}. Note that the periodic behavior of the function $e^{-s}-1$ is essential when considering the behavior at infinity along the imaginary axis, because the zeros of $e^{-s} - 1$ on the imaginary axis have an accumulation point at infinity. Note that the found structural functions are not structural functions along $\mathbb{C}$ because the terms e^{-s} in $\mathcal{P}(s)$ grow exponentially when $\Re(s) \to -\infty$. ∎

2.2.4   The Definition for Linear Systems

Transfer functions of linear systems of the form (2.6) are discussed next, and the structure at infinity with respect to $H \subseteq \mathrm{P}$ for a linear system is defined. First, it is made clear what is meant by the transfer function of a given system.

Notation $\Sigma(A, B, C, D)$ is an abbreviation for a linear system of the form

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0 \in X, & \text{(2.6a)} \\
y(t) &= Cx(t) + Du(t), & t &\geq 0, & \text{(2.6b)}
\end{aligned}
$$

where $A : \mathcal{D}(A) \to X$ is the generator of a C_0-semigroup in a Hilbert space X, and $B \in \mathcal{B}(U, X)$, $C \in \mathcal{B}(X, Y)$, and $D \in \mathcal{B}(U, Y)$. The input and output spaces are $U = \mathbb{C}^m$ and $Y = \mathbb{C}^n$, respectively. The transfer function of a system $\Sigma(A, B, C, D)$ is defined to be

$$\mathcal{P}(s) = C R(s, A) B + D, \ s \in \rho(A) \tag{2.7}$$

where $R(s, A) = (sI - A)^{-1}$ denotes the resolvent of A, and $\rho(A)$ is the resolvent set of the operator A, i.e., the set where $R(s, A)$ is defined as a bounded linear operator in X. The spectrum of A is denoted by $\sigma(A) = \mathbb{C} \setminus \rho(A)$. It is said that a system $\Sigma(A, B, C, D)$ has *a transmission zero* at $s_0 \in \mathbb{C}$ if rank $(\mathcal{P}(s_0)) <$ nrank $(\mathcal{P}(s))$.

Time-delay systems form a special class of distributed systems and they appear in some examples of the thesis. A time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t), \qquad x(0) = x_0, \tag{2.8a}$$
$$y(t) = C x(t) \tag{2.8b}$$

where all the operators are matrices of suitable dimensions, is denoted by $\Sigma_D(A_0, A_1, B, C)$. A time-delay system can be presented in the form (2.6). Lemma 4.3.9 of [14] shows that its transfer function is

$$\mathcal{P}(s) = C(sI - A_0 - \mathrm{e}^{-s} A_1)^{-1} B. \tag{2.9}$$

The definition of transfer functions adopted here is the most convenient one to use in Chapter 3. However, there are other definitions available. One can always define the transfer function of a finite-dimensional system in many equivalent ways and it has a unique closed form expression as a rational matrix. It was shown in [102] that in the infinite-dimensional case the different definitions can lead to different transfer functions. Thus, the different definitions of transfer functions may lead to different structures at infinity.

A transfer function generally is an abstract operator valued function that has no closed form. The absence of the closed form is not a restriction from the theoretical point of view.

Definition 2.2.12. Let the transfer function of a linear system $\Sigma(A, B, C, D)$ be $\mathcal{P}(s)$, and let $H \subseteq \mathrm{P}$ be a set such that $p(\alpha) \subseteq \rho(A)$ for all $p \in H$ and $\alpha \geq 0$. The structure of $\Sigma(A, B, C, D)$ at infinity with respect to H is defined to be the structure of $\mathcal{P}(s)$ at infinity with respect to H.

Remark 2.2.13. *Define*

$$H_s = \{p : [0, \infty) \to \mathbb{C} \mid \mathrm{Re}(p(\alpha)) \to +\infty \ as \ \alpha \to \infty\}$$

Consider a delay system $\Sigma_D\left(A_0, A_1, B, C\right)$. The structure at infinity with respect to H_s (along $\mathbb{C}^+$) is the same as the strong (weak) structure in [62] in the sense that the integer set called the strong (weak) structure at infinity by Malabre and Rabah corresponds to a set of functions that is exactly the set of structural functions with respect to H_s (along $\mathbb{C}^+$).

Remark 2.2.14. *The definition of the structure at infinity given here is especially suitable for those systems that have a transfer function with a closed form expression that is a combination of fractional and exponential terms, because it is relatively easy to verify the majorization conditions for them respect to certain sets of paths. Obviously, the time-delay systems, but also several other distributed parameter systems [4], are of this type.*

Remark 2.2.15. *The algorithm that lead to the given definition of the structure at infinity involved only basic calculations with functions and their absolute values. Thus, the algorithm can be used in computer aided numerical calculations.*

By [13], a transfer function $\mathcal{P}(s)$ is said to be *proper*, if there exists $\alpha \in \mathbb{R}$, such that $\sup_{s \in \overline{\mathbb{C}^+} \backslash B_\alpha(0)} \|\mathcal{P}(s)\| < \infty$. A transfer function $\mathcal{P}(s)$ is *strictly proper* if $\sup_{s \in \overline{\mathbb{C}^+} \backslash B_\alpha(0)} \|\mathcal{P}(s)\| \to 0$, as $\alpha \to +\infty$. The transfer function is *well-posed* if there exists a constant $\alpha \in \mathbb{R}$ such that $\sup_{s \in \mathbb{C}^+_\alpha} \|\mathcal{P}(s)\| < \infty$. The following theorem gives a necessary and sufficient condition for properness, strictly properness, and well-posedness of a transfer function in terms of the structural functions.

Theorem 2.2.16. *Let $\mathcal{P}(s)$ be a transfer function and let $\{q_1, \ldots, q_r\}$ be its structure at infinity along $\mathbb{C}^+_\rho$ for some $\rho < 0$. The transfer function $\mathcal{P}(s)$ is proper/strictly proper/well-posed if and only if $q_1(s)$ is proper/strictly proper/ well-posed.*

Proof. Only the result concerning properness is proved. The results concerning strict properness and well-posedness can be proved similarly.

Sufficiency. Assume that $\mathcal{P}(s)$ is not proper. It follows, that for all $k \in \mathbb{N}$ there exists $x_k \in \overline{\mathbb{C}^+} \backslash B_k(0)$ such that $\|\mathcal{P}(x_k)\| > k$. Define $p \in \mathrm{P}$ to be such that $p(\beta) = x_k$ for $\beta \in [k-1, k)$. Now for all $\alpha > 0$

$$\sup_{s \in \overline{\mathbb{C}^+} \backslash B_\alpha(0)} |q_1(x_k)| \geq \sup_{x_k \in \overline{\mathbb{C}^+} \backslash B_\alpha(0)} |q_1(x_k)|$$

$$\geq \sup_{x_k \in \overline{\mathbb{C}^+} \backslash B_\alpha(0)} M_0 \left\| \begin{bmatrix} \Lambda(x_k) & 0 \\ 0 & 0 \end{bmatrix} \right\|$$

$$\geq \sup_{x_k \in \overline{\mathbb{C}^+} \backslash B_\alpha(0)} M_1 \|\mathcal{P}(x_k)\| = \infty,$$

where the existence of a suitable $M_0 > 0$ follows since $q_1(s)$ majorizes all the other diagonal elements, and the existence of a suitable $M_1 > 0$ follows by the uniform boundedness property of the diagonalizing matrices in (2.1).

Necessity. Assume that $q_1(s)$ is not proper. It follows that for all $k \in \mathbb{N}$ there exists $x_k \in \overline{\mathbb{C}^+} \setminus B_k(0)$ such that $|q_1(x_k)| > k$. It follows that for all $\alpha > 0$

$$
\sup_{\overline{\mathbb{C}^+} \setminus B_\alpha(0)} \|\mathcal{P}(s)\| \geq \sup_{x_k \in \overline{\mathbb{C}^+} \setminus B_\alpha(0)} \|\mathcal{P}(x_k)\|
$$

$$
\geq \sup_{x_k \in \overline{\mathbb{C}^+} \setminus B_\alpha(0)} m_0 \left\| \begin{bmatrix} \Lambda(x_k) & 0 \\ 0 & 0 \end{bmatrix} \right\|
$$

$$
\geq \sup_{x_k \in \overline{\mathbb{C}^+} \setminus B_\alpha(0)} m_1 |q_1(x_k)| = \infty,
$$

where the existence of a suitable $m_0 > 0$ follows by the uniform boundedness property of the diagonalizing matrices in (2.1), and the existence of a suitable $m_1 > 0$ follows by the properties of matrix norms. $\qquad\square$

Example 2.2.17. Recall the time-delay system $\Sigma_D(A_0, A_1, B, C)$ in Example 2.2.11. By the definition, the structure of $\Sigma_D(A_0, A_1, B, C)$ at infinity along $i\mathbb{R}$ is $\{s^{-1}, s^{-1}, s^{-2}(e^{-s} - 1)\}$. $\qquad\blacksquare$

Example 2.2.18. Consider the metal bar of Figure 2.1. It is heated with two heaters along the second and the fourth quarters of its length, and the temperature is measured along the first and the third quarters of its length. The resulting system can be written in the form (2.6) where $D = 0$ and the linear operators A, B, and C are defined below.

$$
Ax(z,t) = \frac{d^2 x}{dz^2}(z,t) - x(z,t), \text{ with}
$$

$$
\mathcal{D}(A) = \left\{ h \in \mathbf{L}_2(0,1) \,\middle|\, h, \frac{dh}{dz} \text{ are absolutely continuous,} \right.
$$

$$
\left. \frac{d^2 h}{dz^2} \in \mathbf{L}_2(0,1) \text{ and } \frac{dh}{dz}(0) = \frac{dh}{dz}(1) = 0 \right\},
$$

$$
(Bu)(z,t) = 4 \left[\mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}(z), \mathbf{1}_{[\frac{3}{4},1]}(z) \right] u(t),
$$

$$
y(t) = Cx(z,t) = \int_0^1 \begin{bmatrix} \mathbf{1}_{[0,\frac{1}{4}]}(z) \\ \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}(z) \end{bmatrix} x(z,t) dz
$$

where the state space is $X = \mathbf{L}_2(0,1)$, the input space and the output space is $U = Y = \mathbb{C}^2$, and the characteristic function $\mathbf{1}_{[a,b]}(z)$ is defined to be

$$
\mathbf{1}_{[a,b]}(z) = \begin{cases} 1, & z \in [a,b], \\ 0, & z \notin [a,b]. \end{cases}
$$
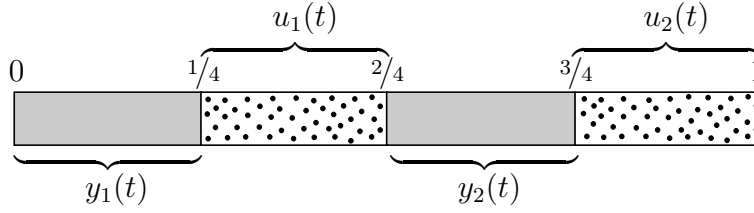
Figure 2.1: A heated metal bar with two heaters and sensors side by side.

The term $-x(z)$ in A represents the heat transfer to the environment. The operator A is a generator of a C_0-semigroup, and B and C are bounded operators. The spectrum of A consists of eigenvalues $\lambda_n = -1 - (n\pi)^2$ where $n \in \{, 1, \ldots\}$. The system is exponentially stable.

Calculations similar to those in [14, Example 4.3.11] show that the transfer function is

$$\mathcal{P}(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

where

$$g_{11}(s) = \frac{4 \sinh\left(\frac{1}{4}\sqrt{s+1}\right)\left(\sinh\left(\frac{3}{4}\sqrt{s+1}\right) - \sinh\left(\frac{1}{2}\sqrt{s+1}\right)\right)}{(s+1)^{\frac{3}{2}}\sinh\left(\sqrt{s+1}\right)},$$

$$g_{12}(s) = \frac{4\sinh^2\left(\frac{1}{4}\sqrt{s+1}\right)}{(s+1)^{\frac{3}{2}}\sinh\left(\sqrt{s+1}\right)},$$

$$g_{21}(s) = \frac{\left\{ \begin{array}{c} -4 + 4\cosh\left(\sqrt{\frac{1}{4}(s+1)}\right) + 2\cosh\left(\sqrt{\frac{1}{2}(s+1)}\right) \\ -4\cosh\left(\sqrt{\frac{3}{4}(s+1)}\right) + 2\cosh\left(\sqrt{s+1}\right) \end{array} \right\}}{(s+1)^{\frac{3}{2}}\sinh\left(\sqrt{s+1}\right)}, \text{ and}$$

$$g_{22}(s) = g_{11}(s).$$

Next the behavior of the transfer function is considered in the right half-plane $\mathbb{C}_{-1}^+$. If $s \in \mathbb{C}_{-1}^+$ and $|s| \to \infty$, then $\mathrm{Re}\left(\sqrt{s+1}\right) \to +\infty$. It is seen that $\left|\sinh\left(d\sqrt{s+1}\right)\right|$ and $\left|\cosh\left(d\sqrt{s+1}\right)\right|$ where $d > 0$ behave approximately like $\left|e^{d\sqrt{s+1}}\right|$ near infinity, if $s \in \mathbb{C}_{-1}^+$.

The above arguments show that

$$g_{11}(s) =_H g_{22}(s) =_H g_{21}(s) =_H s^{-\frac{3}{2}} \geq_H s^{-\frac{3}{2}}e^{-\frac{1}{2}\sqrt{s}} =_H g_{12}(s).$$

It follows that in 2.1 the diagonal matrix is $\Lambda(s) = \mathrm{diag}\left(s^{-\frac{3}{2}}, s^{-\frac{3}{2}}\right)$. Thus, the structure of $\mathcal{P}(s)$ at infinity along $\mathbb{C}_{-1}^+$ is $\{s^{-\frac{3}{2}}, s^{-\frac{3}{2}}\}$.

Note that the term $g_{12}(s)$ of the transfer function, which describes the effect of u_2 on y_1, decays much faster towards zero when approaching infinity in $\mathbb{C}_{-1}^+$ than the other terms. The exponential decay rate $\mathrm{e}^{-\frac{1}{2}\sqrt{s}}$ is a consequence of the gap of half a unit between the measurement y_1 and the input u_2. Thus, if there would be an interval of measurement that is not adjacent to any interval that is heated or vice versa, then there would be an exponentially decaying structural function of the form $\mathrm{e}^{-d\sqrt{s}}$ where $d > 0$.   ∎

2.3   Calculation of Structure at Infinity

In this section, different ways to find the structure at infinity of a given matrix are discussed. The main result of this chapter is a method for calculating the structure at infinity by using minors. It generalizes the method of [44, 90] to transfer functions of infinite-dimensional systems. Computer aided computation of the structure at infinity is also considered.

Before proceeding to the main result of this section, a technical lemma is given. The existence of the supremums in the following discussion follows by Lemma 2.2.3.

Lemma 2.3.1. *Let $A \in \mathbf{C}^{n \times m}$ and $B \in \mathbf{C}^{m \times k}$. If $1 \geq_H |A|^i_{\mathbf{r},\mathbf{c}}$ for all i, $\mathbf{r}$, and $\mathbf{c}$, then $\sup_{\mathbf{r},\mathbf{c}}\left\{|B|^i_{\mathbf{r},\mathbf{c}}\right\} \geq_H \sup_{\mathbf{r},\mathbf{c}}\left\{|AB|^i_{\mathbf{r},\mathbf{c}}\right\}$. If $1 \geq_H |B|^i_{\mathbf{r},\mathbf{c}}$ for all i, $\mathbf{r}$ and $\mathbf{c}$, then $\sup_{\mathbf{r},\mathbf{c}}\left\{|A|^i_{\mathbf{r},\mathbf{c}}\right\} \geq_H \sup_{\mathbf{r},\mathbf{c}}\left\{|AB|^i_{\mathbf{r},\mathbf{c}}\right\}$.*

Proof. By the Binet-Cauchy formula (A.2) and Lemma 2.2.2,

$$\sup_{\mathbf{r},\mathbf{c}}\left\{|B|^i_{\mathbf{r},\mathbf{c}}\right\} \geq_H \sup_{\mathbf{r},\mathbf{c},\mathbf{l}}\left\{|A|^i_{\mathbf{r},\mathbf{l}}|B|^i_{\mathbf{l},\mathbf{c}}\right\} \geq_H \sup_{\mathbf{r},\mathbf{c}}\left\{\sum_{\mathbf{l}}|A|^i_{\mathbf{r},\mathbf{l}}|B|^i_{\mathbf{l},\mathbf{c}}\right\} = \sup_{\mathbf{r},\mathbf{c}}\left\{|AB|^i_{\mathbf{r},\mathbf{c}}\right\}.$$

One can show that $\sup_{\mathbf{r},\mathbf{c}}\left\{|B|^i_{\mathbf{r},\mathbf{c}}\right\} \geq_H \sup_{\mathbf{r},\mathbf{c}}\left\{|BA|^i_{\mathbf{r},\mathbf{c}}\right\}$ by using similar arguments.   □

Theorem 2.3.2. *Let $H \subseteq \mathrm{P}$, and denote the supremum with respect to order $\geq_H$ of all the equivalence classes of $i \times i$-minors of $G(s)$ by $[\mu_i(s)]_H$. Let r be the size of the largest non-zero minor. A set of structural functions of G with respect to H is given by $q_1(s) = \mu_1(s)$, and $q_i(s) = \frac{\mu_i(s)}{\mu_{i-1}(s)}$ for $i = 2, \ldots, r$.*

Proof. Let $G(s)$ be of the form (2.1). The uniform boundedness property implies that all the minors of U and V are majorized by 1. By using Lemma 2.3.1

repeatedly, one gets

$$[q_1 \cdots q_i]_H = \sup_{\mathbf{r,c}} \left\{ |\Lambda(s)|^i_{\mathbf{r,c}} \right\}$$

$$\geq_H \sup_{\mathbf{r,c}} \left\{ \left| U(s) \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} V(s) \right|^i_{\mathbf{r,c}} \right\}$$

$$\geq_H \sup_{\mathbf{r,c}} \left\{ \left| U^{-1}(s)U(s) \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} V(s)V^{-1}(s) \right|^i_{\mathbf{r,c}} \right\} = [q_1 \cdots q_i]_H .$$

Thus, $q_1 \cdots q_i =_H \mu_i$. This implies that $q_i =_H \frac{\mu_i}{\mu_{i-1}}$. $\qquad\square$

Example 2.3.3. Consider the transfer function of Example 2.2.11. It is obvious that $\mu_1(s) = \frac{1}{s+1}$ is a 1×1-minor that majorizes all the other minors of the same size along $i\mathbb{R}$. The determinant of the plant is

$$\mu_3(s) = \det(\mathcal{P}(s)) = \frac{e^{-s} - 1}{(s+1)^2 g(s)} \tag{2.10}$$

and all the non-zero 2×2-minors are

$$\frac{-(e^{-s}+1)}{(s+1)g(s)}, \quad \pm\frac{e^{-s}-1}{(s+1)g(s)}, \quad \frac{1}{g(s)}, \quad \pm\frac{(e^{-s}+s)(e^{-s}-1)}{(s+1)^2 g(s)}, \quad \text{and} \quad \frac{s(e^{-s}-1)}{(s+1)^2 g(s)},$$

where $g(s) = (s+1)^2 + e^{-s} - 1$. The minor $\mu_2(s) = \frac{1}{g(s)}$ majorizes all the other 2×2-minors. The following structural functions along $i\mathbb{R}$ are found by using Theorem 2.3.2:

$$\mu_1(s) = \frac{1}{s+1}, \frac{\mu_2(s)}{\mu_1(s)} = \frac{s+1}{g(s)} \text{ and } \frac{\mu_3(s)}{\mu_2(s)} = \frac{e^{-s}-1}{(s+1)^2}.$$

These are the ones also found in Example 2.2.11. $\qquad\blacksquare$

Remark 2.2.15 shows that the computer aided calculation of the behavior of structural functions is possible – at least in theory. In practice, the calculations involved are heavy for large systems. Luckily, there is a more convenient way to do the calculations.

Let $G(s)$ be a matrix valued function. The matrix $G(s_0)$ has singular value decomposition $U(s_0)\Lambda(s_0)V(s_0)$ where $U(s_0)$ and $V(s_0)$ are unitary matrices at a given point s_0. The axiom of choice allows one to define $U(s)$ and $V(s)$ by fixing the unitary matrices at each point of the domain of $G(s)$. Since $U(s)$ and $V(s)$ are unitary for all s, they satisfy the uniform boundedness property. Thus, the structural functions are given by the singular values, and one has the following theorem.

Theorem 2.3.4. *Denote the singular values of a $n \times m$-matrix valued function $G(s)$ by $\sigma_1(s), \sigma_2(s), \ldots, \sigma_r(s)$ for all s, and assume that the singular values are in decreasing order. The structure at infinity of $G(s)$ with respect to $H \subseteq \mathrm{P}$ is $\{[\sigma_1(s)]_H, [\sigma_2(s)]_H, \ldots, [\sigma_r(s)]_H\}$.*

The above theorem enables numerical calculation of the structural functions, if one is able to calculate the value of the transfer function at a point. One gets the following interesting relation between the singular values and the structure at infinity for rational matrices.

Corollary 2.3.5. *Let $G(s)$ be a rational matrix with $\mathrm{nrank}\,(G(s)) = k$, and denote the non-zero singular values of $G(s)$ by $\sigma_1(s) > \sigma_2(s) > \ldots > \sigma_k(s)$. Let $q_1, q_2, \ldots, q_k$ be the structural indices of $G(s)$. There exist positive constants M, m, and ρ such that $\frac{m}{|s|^{q_i}} < \sigma_i(s) < \frac{M}{|s|^{q_i}}$ for $i = 1, 2, \ldots, k$ and for all $s \in \mathbb{C}$ such that $|s| > \rho$.*

Proof. Every structural index q_i corresponds to a structural function for form s^{-q_i}. Theorem 2.3.4 shows that $s^{-q_i} =_{\mathbb{C}} \sigma_i(s)$. The claim follows by Definition 2.2.1. $\qquad\square$

2.4   Invariance of Structure at Infinity Under Feedback

2.4.1   Invariance under State Feedback

The invariance under a constant state feedback of form

$$u = Kx + v$$

is studied in this section. The results generalize the well-known results on structural properties of finite-dimensional systems presented in [24] to infinite-dimensional systems. The closed loop system with the constant state feedback is

$$\dot{x} = (A + BK)x + Bv, \qquad\qquad x(0) = x_0 \qquad\quad (2.11\text{a})$$
$$y = (C + DK)x + Dv. \qquad\qquad\qquad\qquad\qquad (2.11\text{b})$$

Its transfer function is

$$\mathcal{P}_K(s) = (C + DK)(sI - (A + BK))^{-1}B + D.$$

Substitute $X = sI - A$, $Y = -BK$, and $Z = I = V$ to (A.1) to see that

$$\begin{aligned}
\mathcal{P}_K(s) &= (C + DK)(sI - (A + BK))^{-1}B + D \\
&= C(sI - (A + BK))^{-1}B + D + DK(sI - (A + BK))^{-1}B \\
&= C(sI - A)^{-1}B + D + (C(sI - A)^{-1}B + D)K(sI - (A + BK))^{-1}B \\
&= \mathcal{P}(s)(I + K(sI - (A + BK))^{-1}B).
\end{aligned}$$

Consequently, the state-feedback does not alter the structure at infinity with respect to $H \subset \mathrm{P}$ if $I + K(sI - (A + BK))^{-1}B$ satisfies the uniform boundedness property with respect to H. This leads to the following lemma.

Lemma 2.4.1. *Choose $\beta \in \mathbb{R}$. The structure at infinity along $\mathbb{C}_\beta^+$ is not affected by the state feedback $u = Kx + v$, if there exists $\rho \geq 0$ such that*

$$\sup_{s \in \mathbb{C}_\beta^+ \setminus B_\rho(0)} \|K(sI - (A + BK)^{-1}B\| < 1. \tag{2.12}$$

Proof. Assume that the condition (2.12) holds for some $\rho > 0$. In that case, $I + K(sI - (A + BK))^{-1}B$ is invertible in $\mathbb{C}_\beta^+ \setminus B_\rho(0)$, and both $I + K(sI - (A + BK))^{-1}B$ and its inverse are uniformly bounded there. Thus, $I + K(sI - (A + BK))^{-1}B$ satisfies the uniform boundedness condition along $\mathbb{C}_\beta^+$ and the claim follows. $\qquad\square$

Lemma 2.4.2. *If the system $\Sigma(A, B, C, 0)$ that has finite-dimensional input and output spaces is exponentially stabilizable or exponentially detectable, then there exists $\beta < 0$ such that $\sup_{\substack{|s| > \rho \\ s \in \mathbb{C}_\beta^+}} \|\mathcal{P}(s)\| \to 0$ as $\rho \to \infty$.*

Proof. Only the case where the system is exponentially stabilizable is proved. The proof for exponentially detectable systems follows by similar arguments.

By [14, Theorem 5.2.6] there exists a constant $\epsilon > 0$ such that the state-space can be decomposed into a direct sum $X = X_+ \oplus X_-$ where X_+ is a finite-dimensional space corresponding to the finite point spectrum of A in $\mathbb{C}_{-\epsilon}^+$. Denote the decompositions of the operators C, B, A, and $R(s, A)$ with respect to the decomposition of the state-space by $C = \begin{bmatrix} C_+ & C_- \end{bmatrix}$,

$$B = \begin{bmatrix} B_+ \\ B_- \end{bmatrix}, \quad A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}, \text{ and } R(s, A) = \begin{bmatrix} R(s, A_+) & 0 \\ 0 & R(s, A_-) \end{bmatrix},$$

respectively. The transfer function is of the plant is

$$\mathcal{P}(s) = C_+ R(s, A_+) B_+ + C_- R(s, A_-) B_- = \mathcal{P}_+(s) + \mathcal{P}_-(s).$$

Since $\Sigma(A_+, B_+, C_+, 0)$ is a finite-dimensional plant, $\mathcal{P}_+$ is a strictly proper rational matrix, and $\|\mathcal{P}_+(s)\| \to 0$, as $s \to \infty$. By the triangle inequality, it

remains to show that there exits a constant $\beta < 0$ such that $\|\mathcal{P}_-(s)\| \to 0$, as $s \to \infty$ in $\mathbb{C}_\beta^+$.

The operator A_- is an infinitesimal generator of an exponentially stable C_0-semigroup $T_-(t)$, so there exist constants $M > 0$ and $\epsilon_0 > 0$ such that $\|T_-(t)\| \leq Me^{-\epsilon_0}$ for all $t \geq 0$. Thus, $\int\limits_0^\infty \|e^{\epsilon t}C_-T_-(t)B_-\|^2 < \infty$ if $0 < \epsilon < \epsilon_0$. This shows that $e^{\epsilon t}C_-T_-(t)B_- \in \mathbf{L}_2([0,\infty), \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m))$. By the Paley-Wiener theorem [14, Theorem A.6.21], $\mathcal{P}_-(s - \epsilon) = \int_0^\infty e^{-st}e^{\epsilon t}C_-T_-(t)B_- dt \in H_0^2(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m))$. Choose $0 < \beta < \epsilon$. By [36, Theorem 6.4.2], $\|\mathcal{P}_-(s)\| \to 0$, as $s \to \infty$ in $\mathbb{C}_\beta^+$ $\square$

Theorem 2.4.3. *If $\Sigma(A, B, C, D)$ is exponentially stabilizable, there exists $\beta < 0$ such that a state-feedback of the form $u = Kx + v$ does not change the structure at infinity along $\mathbb{C}_\beta^+$.*

Proof. Since $\Sigma(A, B, C, D)$ is exponentially stabilizable $\Sigma(A + BK, B, K, 0)$ is too. Lemma 2.4.2 shows that $K(sI - (A + BK))^{-1}B$ satisfies the condition (2.12), so the claim follows by Lemma 2.4.1. $\square$

2.4.2   Invariance under Output Feedback

Consider output feedback of the form

$$u = Ly + v. \tag{2.13}$$

The resulting closed-loop transfer function is

$$\mathcal{P}_L(s) = (I - \mathcal{P}(s)L)^{-1}\mathcal{P}(s). \tag{2.14}$$

The structure at infinity is not invariant under output feedback – not even in the finite-dimensional case – as it is shown in the following example. In the example the feedback takes one finite pole to the infinity. However, the zeros at infinity are invariant under output feedback in the finite-dimensional case [24]. The next theorem generalizes the result to infinite-dimensional systems.

Example 2.4.4. Set $A = B = C = D = L = 1$. Now $\mathcal{P}(s) = \frac{1}{s-1} + 1$ and $\mathcal{P}_L(s) = -s$, so the structure at infinity was changed by the output feedback. $\blacksquare$

Theorem 2.4.5. *Let $\Sigma(A, B, C, D)$ be such that $I - DL$ is invertible and $CR(p(\alpha), A)B \to 0$, as $\alpha \to \infty$, for all $p \in H \subseteq \mathrm{P}$. The output feedback (2.13) does not change the structure at infinity with respect to H.*

Proof. Fix $p \in H$. Since $CR(p(\alpha), A)BL \to 0$, as $\alpha \to \infty$, it follows that $I - \mathcal{P}(p(\alpha))L \to I - DL$. Thus, $I - \mathcal{P}(p(\alpha))L$ has the uniform boundedness property, and the claim follows by (2.14). $\square$

A direct consequence of Theorem 2.4.5 and Lemma 2.4.2 is given by the following corollary:

Corollary 2.4.6. *If $\Sigma(A, B, C, D)$ is exponentially stabilizable or exponentially detectable and $I - DL$ is invertible, then there exists a $\beta > 0$ such that the output feedback (2.13) does not change the structure at infinity along $\mathbb{C}_\beta^+$.*

2.5 Robustness of Structure at Infinity

Robustness properties of the structure at infinity are studied in this section. It is seen that the robustness properties are generally very weak. Nonetheless, there might exist some important classes of perturbations that do not change the structure at infinity. The robustness properties are of importance since in Section 3.3 it is shown that the robustness properties of the structure at infinity are related to the robustness of regulation.

The next example illustrates that small perturbations may change the structure at infinity of a linear system. Actually, almost every perturbation changes the structure at infinity of certain systems. However, for systems with bounded operators and one dimensional input or output space there exists a neighborhood where the perturbed transfer functions cannot vanish faster than the original plant transfer function as shown by Theorem 2.5.3. Example 2.5.2 shows that this property fails even for finite-dimensional MIMO-systems.

Example 2.5.1. Let $\Sigma(A, B, C, D)$ be a finite-dimensional plant with one dimensional input and output spaces that satisfies $D = CA^{i-1}B = 0$ for $i = 1, \ldots, j-1$ and $CA^jB \neq 0$. Since $\mathcal{P}(s) = D + \sum_{i=1}^{\infty} 1/s^i CA^{i-1}B$ the structure at infinity is $1/s^j$. Any perturbation to D changes the structure at infinity. In addition, it is easy to see that certain arbitrarily small perturbations change $CA^{i-1}B$ to be non-zero. Such perturbations change the structure at infinity. ∎

Example 2.5.2. Consider the system $\Sigma(A, B, C, 0)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let the output operator be subject to the additive perturbation

$$\Delta C = \begin{bmatrix} 0 & 0 & 0 & -\epsilon \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The transfer functions of the original and the perturbed systems are

$$\mathcal{P}(s) = \begin{bmatrix} \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix} \text{ and } \mathcal{P}'(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{\epsilon}{s} \\ 0 & \frac{1}{s^2} \end{bmatrix},$$

and the structures at infinity are $\{s^{-2}, s^{-2}\}$ and $\{s^{-1}, s^{-3}\}$, respectively. Thus, even an arbitrarily small perturbation can increase the order of the zero at infinity.   ∎

In the above example, the number of zeros at infinity counting multiplicities remains the same. The perturbation lowers the order of the first zero and increases the order of the second one. This is not possible if there is only one input or output.

Theorem 2.5.3. *If the input space or the output space of $\Sigma(A, B, C, D)$ with bounded operators is one dimensional, then there exists $\epsilon > 0$ such that the structural functions of $\Sigma(A', B', C', D')$ majorize the structural functions of $\Sigma(A, B, C, D)$ for all systems $\Sigma(A', B', C', D')$ such that $\|A - A'\| < \epsilon$, $\|B - B'\| < \epsilon$, $\|C - C'\| < \epsilon$, and $\|D - D'\| < \epsilon$.*

Proof. The plant transfer function has the following Laurent expansion at infinity:

$$\mathcal{P}(s) = D + \sum_{i=1}^{\infty} \frac{1}{s^i} C A^{i-1} B.$$

Denote the lowest order non-zero coefficient in the Laurent expansion by Ψ. This term determines the structural index of $\mathcal{P}$. If $\Psi = D$ and $0 < \epsilon < \|D\|$, then $D' \neq 0$ whenever $\|D - D'\| < \epsilon$. Thus, the structure at infinity remains invariant. Assume, that $\Psi = CA^{i-1}B$ for some $i > 1$. Since Ψ is a continuous function of A, B, and C, it is clear that there exists $\epsilon > 0$ such that $\Psi' = C'(A')^{i-1}B' \neq 0$, if $\|A - A'\| < \epsilon$, $\|B - B'\| < \epsilon$, and $\|C - C'\| < \epsilon$. It follows, that the structural index cannot increase.   □

The next examples illustrate that arbitrarily small perturbations can change the structure at infinity of infinite-dimensional SISO-systems. In the first example, an infinite number of finite transmission zero locations are perturbed. The perturbation changes the structure at infinity, because the zeros have an accumulation point at infinity. The second example shows that in small perturbations can result in a big difference in the structural properties of a system. It also illustrates that structure at infinity can be robust with respect to some important class of perturbations.

Example 2.5.4. Consider the time-delay system $\Sigma_D\left(A_0, A_1, B, C, D\right)$ with

$$-A_0 = A_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } D = 0.$$

The transfer function is $\mathcal{P}(s) = 1/s^2 \left(1 - e^{-s}\right)$. There is a finite transmission zero at $2\pi k \mathrm{i}$ for all $k \in \mathbb{Z}$.

Consider an additive perturbation of form $\Delta A_0 = (e^{-i\epsilon} - 1)A_0$. Since $\Delta A_0 \to 0$ as $\epsilon \to 0$ the norm of the perturbation can be chosen to be arbitrarily small. The perturbed transfer function is $\mathcal{P}'(s) = 1/s^2 \left(e^{-i\epsilon} - e^{-s}\right)$, so the finite transmission zero locations have been perturbed to $(2\pi k + \epsilon)\mathrm{i}$ where $k \in \mathbb{Z}$. The two transfer functions are not comparable at infinity along $\mathrm{i}\mathbb{R}$ by the majorization relation, so the structure at infinity has changed. ∎

Example 2.5.5. The heated bars of Figure 2.2 have one heater u and one measurement y, and there is some uncertainty in the location of the heater. Let the heat transfer be modelled by using models similar to that in Example 2.2.18. Let the transfer functions of the systems be $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$, first of which is related to Figure 2.2(a) and the latter of which to Figure 2.2(b).

(a) Non-collocated
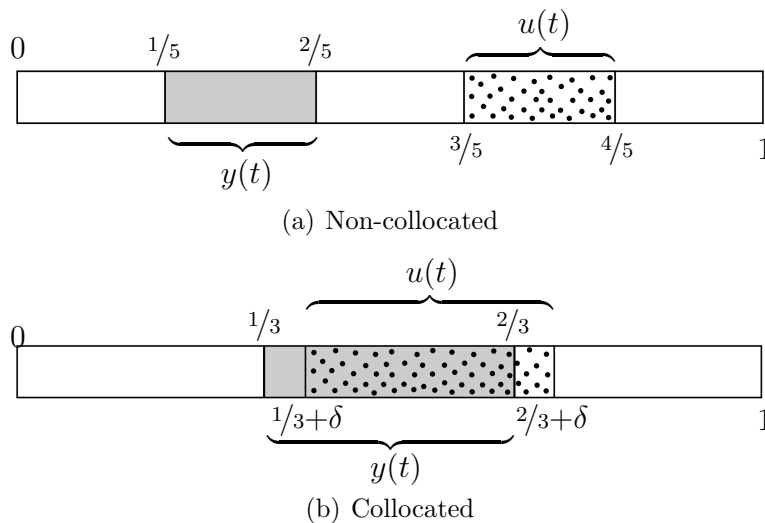
(b) Collocated

Figure 2.2: Two heated bars: one with a non-collocated heater and sensor pair [2.2(a)] and one with a collocated heater and sensor pair [2.2(b)].

The structure at infinity along $\mathrm{i}\mathbb{R}$ of $\mathcal{P}_1(s)$ is $\frac{1}{s\sqrt{s}}e^{-\left(\frac{1}{5}+\delta\right)\sqrt{s+1}}$. It is seen, that an arbitrarily small perturbation in the location changes the structure at infinity. Furthermore, the perturbations that move the heater farther away from the measurement make the transfer function to vanish faster as the infinity is approached along the imaginary axis. The perturbations that move the heater

closer to the measurement make the transfer function to vanish slower. The change in the decay rate between the perturbed and the original plant transfer functions is $e^{-\delta\sqrt{s+1}}$. Thus, even a small perturbation changes the decay rate by an exponential degree.

The structure at infinity along $i\mathbb{R}$ of $\mathcal{P}_2(s)$ is $\frac{1}{s}$ as long as $\delta < \frac{1}{3}$. Thus, the structure at infinity is robust with respect to small perturbations in the location of the heater. Similar robustness property is also evident with respect to the location of the measurement. ∎

2.6   Global Structure of Transfer Functions

The structure at infinity describes the rate of convergence or divergence as the frequency approaches infinity, However, it does not take into account the finite structure, i.e., the finite poles and zeros. In Chapters 3 and 4, the finite structure and the structure at infinity are of interest. That is why it is convenient to combine these two. This is done next by defining the global structure.

If one carefully examines the way the structure at infinity was defined, it is easy to see that one only needs to take the finite zeros and poles into account when defining the majorization relation. This leads to the following global majorization relation.

Definition 2.6.1. Let $f, g \in \mathbf{C}$ and $H \subseteq \mathrm{P}$. If

$$\forall p \in H : \sup_{\alpha \geq 0} \left| \frac{g(p(\alpha))}{f(p(\alpha))} \right| < \infty$$

it is said that f globally majorizes g with respect to H. This is denoted by $f \geqq_H g$. If $f \geqq_H g$ and $g \geqq_H f$, notation $f \equiv_H g$ is used.

Actually, all the properties of Lemma 2.2.2 hold for the relations $\geqq_H$ and $\equiv_H$. Furthermore, if $[[f]]_H$ denotes the equivalence class of f with respect to relation $\equiv_H$, then Lemma 2.2.3 and Lemma 2.2.4 hold if $[\cdot]_H$ is replaced by $[[\cdot]]_H$, and $=_H$ by $\equiv_H$. This means that the structure algorithm in the proof of Theorem 2.2.7 constructs diagonalizing matrices U and V having the global uniform boundedness property defined below. Thus, the same proof shows the stronger version of Theorem 2.2.7 that is stated in Theorem 2.6.3.

Definition 2.6.2. It is said that $G : \mathcal{D}(G) \subseteq \mathbb{C} \to \mathbb{C}^{n \times n}$ satisfies the global uniform boundedness property with respect to $H \subseteq \mathrm{P}$ if $M(p(\alpha))$ is invertible and $\|M(p(\alpha))\|$, $\|M^{-1}(p(\alpha))\|$ are uniformly bounded in $[0, \infty)$ for all $p \in H$.

Theorem 2.6.3. *Let $H \subseteq \mathrm{P}$. A matrix $G(s) \in \mathbf{C}^{n \times m}$ can be written in the form*

$$G(s) = V(s) \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} U(s) \tag{2.15}$$

where $\Lambda(s) = \mathrm{diag}\,(q_1(s), \ldots, q_r(s))$ is a diagonal matrix with non-zero elements, $q_1 \geqq_H q_2 \geqq_H \cdots \geqq_H q_r$, the zero blocks may be non-existent, and U and V have the global uniform boundedness property.

The result of Theorem 2.2.8 generalizes to the diagonal elements of Λ in (2.15), so they are unique up multiplication by elements $f \equiv_H 1$, and it makes sense to define the global structure in the following way.

Definition 2.6.4. Let $\Lambda(s) = \mathrm{diag}\,(q_1(s), \ldots, q_r(s))$ be the diagonal matrix in (2.15). The global structure at infinity of $G(s)$ with respect to H is the set $\{[[q_1]]_H, \ldots, [[q_r]]_H\}$. The functions $q_1, q_2, \ldots, q_r$ are called global structural functions with respect to H.

Consider the global structure along a set $Q \subseteq \mathbb{C}$. A matrix valued function G with full normal rank can have a transmission zero at $s_0 \in Q$ if and only if global structural function $q_r(s)$ has a zero at s_0. Similarly G can have a pole at $s_0 \in Q$ if and only if $q_1(s)$ has a pole at s_0. Since the global structural functions are also structural functions it is evident that the global structure captures the finite structure and the structure at infinity.

The reason for defining the global structure was explained in the beginning of the section. One might wonder where is the structure at infinity needed. The reason for defining the structure at infinity are the invariance properties found in Section 2.4. The global structure does not have the invariance properties, because feedback affects the finite structure even though the structure at infinity remains intact.

Chapter 3

Regulation in the Time Domain

3.1 Introduction to Robust Regulation in the Time Domain

In this section, the time domain output regulation problem is introduced. The purpose is to review the related results appearing in the literature so that the time domain and the frequency domain problems and their solvability conditions can be compared and that the smoothness properties of the reference and the disturbance signals required for solvability can be studied.

The output regulation problem is to find any controller that is capable to asymptotically track the reference signal. The two controller configurations considered in this thesis are the feedforward control configuration of Figure 1.2(a) and the error feedback control configuration depicted in Figure 1.2(b). It is also possible to combine these two controller designs [42].

The robust regulation problem studied here is the one from [34] where the exosystem is an infinite-dimensional diagonal system with an infinite number of unstable poles on the imaginary axis. The state spaces as well as the input and the output spaces are assumed to be Hilbert spaces. This is not the most general formulation appearing in the literature, see for example [39, 70], but is the closest one to the frequency domain regulation problem studied in the next chapter.

3.1.1 The Plant

In the time domain, the plants have the following state space formulation:

$$\dot{x} = Ax + Bu + d_1, \qquad x(0) = x_0 \in X, \qquad (3.1a)$$

$$y = Cx + Du + d_2. \qquad (3.1b)$$

The state disturbance d_1 and *the measurement disturbance* d_2 are defined below. The state operator A is assumed to be an infinitesimal generator of a

C_0-semigroup on a Hilbert space X. In addition, $B \in \mathcal{B}(U, X)$, $C \in \mathcal{B}(X, Y)$, and $D \in \mathcal{B}(U, Y)$ where the input space U and the output space Y are assumed to be Hilbert spaces. Choosing the spaces to be Hilbert spaces is not crucial. Most of the results appearing in the literature hold in some form even if X, U, and Y are Banach spaces.

3.1.2   The Reference and Disturbance Signals

It is clear that no controller can regulate every possible reference signal, nor can it get rid of every disturbance signal. However, usually only a certain class of reference and disturbance signals needs to be regulated at a time. The allowed reference and disturbance signals are often assumed to be generated by an exosystem.

The exosystem considered in this thesis is of form

$$\dot{v} = Sv, \qquad v(0) = v_0 \in W, \tag{3.2a}$$

$$y_r = F_1 v, \tag{3.2b}$$

$$d_1 = E_1 v, \tag{3.2c}$$

$$d_2 = E_2 v, \tag{3.2d}$$

where $F_1 \in \mathcal{B}(W, Y)$ and $E_1 \in \mathcal{B}(W, X)$, $E_2 \in \mathcal{B}(W, Y)$, and S is a linear diagonal operator

$$S = \sum_{k \in \mathbb{Z}} \mathrm{i}\omega_k \left\langle \cdot, \phi_k \right\rangle \phi_k, \tag{3.3}$$

with domain

$$\mathcal{D}(S) = \left\{ v \in W \; \middle| \; \sum_{k \in \mathbb{Z}} |\left\langle v, \phi_k \right\rangle|^2 < \infty \right\}$$

in a Hilbert-space W with an orthonormal basis $(\phi_k)_{k \in \mathbb{Z}}$. The operator has point spectrum $\mathrm{i}\omega_k$ on the imaginary axis. The following standing assumption states that there is a uniform gap between the poles of the exosystem.

Assumption 3.1.1. *The constants $\omega_k \in \mathbb{R}$ in (3.3) are in increasing order and there exists such a positive constant $\gamma > 0$ that $\omega_k - \omega_{k-1} > 4\gamma$ for all $k \in \mathbb{Z}$.*

In the frequency domain, one often considers only input disturbances, because the transfer functions give only an input-to-output description and the output disturbance can be seen as a part of the reference signal. *The input disturbance signal* can be written in the form $d_0 = E_0 v$. If there are no other types of disturbances, the disturbance to the state of the plant is $d_1 = B d_0$

and disturbance to the measurement $d_2 = Dd_0$; therefore, one has $E_1 = BE$ and $E_2 = DE$ in (3.2). In this thesis, there are no other types of disturbances affecting the system at the same time with an input disturbance.

With exosystems of the proposed form one can generate the class of almost periodic signals. Particularly, all periodic signals belong to this class of signals. More general signals can be generated if Jordan blocks are allowed in S. In this case, the signals can be polynomially increasing. The simple form of the exosystem makes controller design easier. A very general generator that can generate all the bounded periodic signals with one generator was considered in [40, 39], but actual controller design for such a controller is very difficult.

The smoothness of the reference and disturbance signals is crucial in regulation. There are two ways to control the smoothness of the signals, either by setting conditions on the allowed initial states of the exosystem [70, 68] or by setting conditions on the reference operator F_1 and disturbance operators E_1 and E_2 [34]. However, the latter approach allows different smoothness properties of the reference and disturbance signals to be set, while the former does not. This is why the latter approach is adopted here.

3.1.3 The Controller

There are two controller types considered in this thesis. The first one is feedforward controllers with a stabilizing feedback loop. *A feedforward controller* is a control law

$$u = Kx + Lv, \tag{3.4}$$

where $K \in \mathcal{B}(X, U)$ and $L \in \mathcal{B}(W, U)$.

The second controller type is error feedback controllers. It is a dynamic controller of form

$$\dot{z} = \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) = z_0, \tag{3.5a}$$
$$u = Kz, \tag{3.5b}$$

where $\mathcal{G}_1$ is an infinitesimal generator of a C_0-semigroup on a Hilbert space Z, $\mathcal{G}_2 \in \mathcal{B}(Y, Z)$ and $K \in \mathcal{B}(Z, U)$. The input of the controller is the error $e = y_r - y$ between the reference and output signals. The transfer function of a dynamic controller is $\mathcal{C}(s) = KR(s, \mathcal{G}_1)\mathcal{G}_2$.

Combining the plant and the error feedback controller gives the extended closed loop system

$$\dot{x}_e = A_e x_e + B_e v, \qquad x_e(0) = x_{e0} = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \tag{3.6a}$$

$$e = C_e x_e + D_e v, \tag{3.6b}$$

$$\tag{3.6c}$$

where $C_e = \begin{bmatrix} -C & -DK \end{bmatrix}$, $D_e = F_1 - E_2$,

$$A_e = \begin{bmatrix} A & BK \\ -\mathcal{G}_2 C & \mathcal{G}_1 - \mathcal{G}_2 DK \end{bmatrix} \text{ and } B_e = \begin{bmatrix} E_1 \\ \mathcal{G}_2(F_1 - E_2) \end{bmatrix}. \qquad (3.7)$$

From now on denote $E = E_1$ and $F = F_1 - E_2$.

3.1.4   The Feedforward Output Regulation Problem

The feedforward output regulation problem is to find a feedforward control law (3.4) such that

1. $A + BK$ is a generator of an exponentially stable C_0-semigroup, and

2. the tracking error $e(t) = y(t) - y_r(t) \to 0$, as $t \to \infty$.

Feedforward controllers are widely used in applications, if not by themselves, along with feedback controllers to improve performance. Thus, it is important to understand their possibilities and limitations. The most severe weakness of feedforward controllers is the lack of robustness properties. However, the output regulation problem without robustness is solvable by a feedforward controller if and only if it is solvable by a feedback controller under certain stabilizability conditions [6]. Consequently, if the stabilizability conditions are satisfied the solvability conditions of the output regulation problem with a feedforward controller are necessary for the robust output regulation problem too.

Necessary and sufficient conditions for the solvability of the output regulation problem with a feedforward controller in the finite-dimensional case have been known since the 1970s [16]. Since then, many authors have generalized these results to infinite-dimensional plants with finite-dimensional exosystems, see for example [6, 73]. A case study with a feedforward controller and an infinite-dimensional exosystem in [5] revealed a connection between the solvability of the problem and the behavior of the plant transfer function at high frequencies. In principle, the faster the transfer function approaches zero the smoother the reference and the disturbance signals should be for the problem to be solvable. This connection was later studied in [41]. A similar phenomenon was reported with dynamic error feedback controllers in [34, 70].

An instrumental tool in the study of the feedforward regulation problem are the so called regulator equations:

$$\Sigma S = A\Sigma + B\Gamma + E \qquad \text{in } \mathcal{D}(S) \qquad (3.8a)$$
$$F = C\Sigma + D\Gamma. \qquad (3.8b)$$

These equations are from [6] where they were used with a finite-dimensional exosystem and no feed through term, i.e., $D = 0$. Later, Immonen used the

equations with an infinite-dimensional exosystem [39, 41]. In this thesis, the plant is required to be exponentially stabilizable and the solvability of the equations is actually a necessary and sufficient condition for the feedforward output regulation problem to be solvable. If only strong stability of the plant is needed, then the solvability of the regulator equations is only a sufficient condition for the solvability of the feedforward regulation problem [39].

3.1.5 The Error Feedback Output Regulation Problem

The error feedback output regulation problem is to find an error feedback controller (3.5) such that

1. A_e is strongly stable, and

2. the tracking error $e(t) = y(t) - y_r(t) \to 0$ as $t \to \infty$ for all initial states $x_{e0} \in X_e$ and $z_0 \in W$.

Denote the set of all 7-tuples

$$(A', B', C', D', F_1', E_1', E_2'),$$

where $A' \in \mathcal{L}(X)$ is a infinitesimal generator of a C_0-semigroup, $B' \in \mathcal{B}(U, X)$, $C' \in \mathcal{B}(X, Y)$, $D' \in \mathcal{B}(U, Y)$, $F_1' \in \mathcal{B}(W, Y)$, $E_1' \in \mathcal{B}(W, X)$ and $E_2' \in \mathcal{B}(W, Y)$ by Ω. The nominal plant (3.1) and the unperturbed reference and disturbance operators in (3.2) corresponds to the 7-tuple

$$(A, B, C, D, F_1, E_1, E_2) \in \Omega.$$

In this thesis, perturbations are allowed in the operators of the plant and in the reference and the disturbance operators. When the nominal plant or the reference and the disturbance operators are subject to a perturbation the resulting plant and the operators are assumed to be an element of a subset $\mathcal{V}$ of Ω. Thus, $\mathcal{V}$ defines the class of allowed perturbations. It is assumed that $(A, B, C, D, F_1, E_1, E_2) \in \mathcal{V}$, but, at this point, no further assumptions on this set are made. *The robust error feedback output regulation problem* is to find an error feedback controller (3.5) such that

1. it solves the error feedback output regulation problem for the nominal plant (3.1), and

2. it solves the error feedback output regulation problem for all plants in $\mathcal{V}$ preserving strong stability of the closed loop.

For finite-dimensional linear systems and finite-dimensional signal generators, the error feedback output regulation problem and the robust error feedback output regulation problem were solved during the 1970s. Francis, Wonham, and Davison had a central role in this work [15, 18, 28, 29, 97]. The results for finite-dimensional plants have been generalized to infinite-dimensional plants by several authors [2, 33, 70, 74, 75, 80, 89]. In what follows, the focus is on the robust regulation problem with an infinite-dimensional exosystem. For a more comprehensive list of references and some discussion on their contributions an interested reader should see [32, 39, 68].

A fundamental result called the internal model principle first stated by Francis and Wonham in [28] gives a necessary and sufficient condition for a controller to be robustly regulating. Roughly stated, a controller can robustly regulate the dynamics generated by an exosystem if and only if it contains a suitably reduplicated copy of the exosystem.

This famous result was generalized to infinite-dimensional exosystems in the form of a repetitive control system by Hara et al. in [35, 99]. The idea of the repetitive control is that since all L-periodic function can be generated by using a free time-delay loop with delay length L, including such a loop in a controller would make it robustly regulating. Hara et al. formulated the internal model principle in frequency domain terms. However, the results in [35, 99] suffered from relatively narrow applicability.

In his thesis and articles, Immonen provided a state space formulation of the internal model principle that is suitable for infinite-dimensional systems and exosystems, see [39] and the references therein. His formulation of the internal model principle was called the internal model structure. The controller is said to possess the internal model structure of the exosystem if for all $\Delta \in \mathcal{B}(W, Y)$ and for all such $\Gamma \in \mathcal{B}(W, Z)$ that $\Gamma \mathcal{D}(S) \subset \mathcal{D}(\mathcal{G}_1)$

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \Rightarrow \Delta = 0.$$

It was shown that a controller possessing the internal model structure solves the robust regulation problem. Furthermore, the internal model structure is a necessary condition for a controller that exponentially stabilizes the extended system to be robustly regulating in the traditional sense, by which it is meant that small enough bounded linear additive perturbations do not destroy the output regulation. In other words, the controller solves the error feedback regulation problem for all systems in $\mathcal{V}$ containing all 7-tuples

$$(A', B', C', D', F_1', E_1', E_2') \in \Omega$$

satisfying $\|A - A'\| < \epsilon$, $\|B - B'\| < \epsilon$, $\|C - C'\| < \epsilon$, and $\|D - D'\| < \epsilon$ for some small enough $\epsilon > 0$. Since linear additive perturbations were considered

the operator $A - A'$ is a bounded operator, although A and A' can be unbounded operators, and its norm is well defined.

Despite the nice robustness results with exponential stability, Immonen preferred strong stability. This is because he used an infinite-dimensional exosystem. If one wants to find a robustly regulating controller, an internal model of the exosystem should be build into the controller. Consequently, with an error feedback controller this infinite-dimensional exosystem becomes a part of the closed loop, so it is often impossible to exponentially stabilize the extended system. This difficulty was already pointed out by Hara et al. in [35], who showed that it is impossible to obtain exponential stability with their repetitive control system if the transfer function is strictly proper.

The lack of exponential stability in the closed loop results in weaker robustness of stability. It is possible that an arbitrarily small perturbation destroys strong stability, e.g. [39, Example 6.5], and it is difficult to find classes of perturbations preserving strong stability. This is why Immonen merely considered the robustness of regulation under the assumption that the allowed perturbations preserve the strong stability of the closed loop. Immonen called this "conditional robustness", but later Paunonen used the term somewhat differently as seen below. In this thesis, conditional robustness is to be understood in the sense of Paunonen, and robustness of regulation is considered under the assumption that the perturbations preserve closed loop stability as was done in [34, 68, 70, 72].

Another characterization of the internal model principle suitable for infinite-dimensional setting was given by Hämäläinen and Pohjolainen in [34]. This characterization originates from a proof in [27] and was called the $\mathcal{G}$-conditions by Paunonen since the conditions are related to operators $\mathcal{G}_1$ and $\mathcal{G}_2$ of the controller [70]. For the diagonal exosystem in (3.3), the controller is said to satisfy the $\mathcal{G}$-conditions if for all $k \in \mathbb{Z}$

$$\mathcal{R}\left(\mathrm{i}\omega_k I - \mathcal{G}_1\right) \cap \mathcal{R}\left(\mathcal{G}_2\right) = \{0\}, \tag{3.9a}$$

$$\mathcal{N}\left(\mathcal{G}_2\right) = \{0\}. \tag{3.9b}$$

This characterization was extended for exosystems that were allowed to contain non-trivial Jordan blocks by Paunonen in [70] who also showed that the two characterizations above are equivalent. Related to the size of the non-trivial Jordan blocks, an additional $\mathcal{G}$-condition was needed. Under an additional assumption $\sigma\left(A_e\right) \cap \sigma\left(S\right) = \emptyset$, Paunonen was able to show in [70] that what he called the n-copy internal model is equivalent to the afore mentioned characterizations of the internal model principle. For the exosystem with diagonal operator (3.3), the controller is said to contain an n-copy of the exosystem if for all $k \in \mathbb{Z}$

$$\dim\left(\mathrm{i}\omega_k I - \mathcal{G}_1\right) \geq \dim\left(Y\right).$$

Perhaps the most important contribution of [70] was that it was shown that the internal model structure and the $\mathcal{G}$-conditions, which were known to be sufficient conditions for a controller to be robustly regulating, are also necessary provided the class of allowed perturbations is appropriately chosen. The class of perturbations considered in [34, 70] was chosen so that the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e \qquad \text{on } \mathcal{D}(S) \tag{3.10}$$

has such a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ that $\Sigma \mathcal{D}(S) \subseteq \mathcal{D}(A_e)$ for all perturbed plants. It was noted that the solvability of the above equation is related to the smoothness of the reference and the disturbance signals. In principle, the smoother the signals the easier it is to solve the equation (3.10). The required smoothness of the signals is related to the high frequency behavior of the plant transfer function [34, 39]. Conditions on the smoothness of the signals can be imposed by restricting the set of initial conditions of the exosystem [70] or by restricting the set of allowed reference and disturbance operators F_1, E_1, and E_2 [34]. It should be noted that the solution Σ does not need to be unique in order a controller with an internal model to solve the robust output regulation problem.

In his thesis [68], Paunonen presented yet another equivalent condition called conditional robustness for a controller to be robustly regulating. A controller is said to be conditionally robust if the implication

$$\Sigma S = A_e \Sigma + B_e \Rightarrow C_e \Sigma + D_e = 0 \tag{3.11}$$

is satisfied for all operators A_e, B_e, C_e, and D_e in the extended system (3.6) with fixed controller parameters $\mathcal{G}_1$, $\mathcal{G}_2$, and K and exosystem state operator S. Paunonen allowed the reference and the disturbance operators to be arbitrary, and restricted the set of the allowed initial values of the exosystem. If the set of initial states is restricted, then it is possible to consider the solvability of the Sylvester equation (3.10) in a related subspace of W. In this thesis, the set of initial values is not restricted but the reference and the disturbance operators are; consequently, one needs to consider the solvability of (3.10) in W. However, as seen later, this is not a restriction to theory and the two approaches to control the smoothness of signals can be used interchangeably.

In [49], it was shown that if $\sigma(A_e) \cap \sigma(S) = \emptyset$, then (3.11) is satisfied for S defined in (3.3) if and only if

$$P_e(\mathrm{i}\omega_k) = C_e R(\mathrm{i}\omega_k, A_e) B_e + D_e = 0 \tag{3.12}$$

for all $k \in \mathbb{Z}$. This is called the blocking zero condition and generalizes the finite-dimensional result in [26] to infinite-dimensional systems and exosystems. If there are non-trivial Jordan blocks in the exosystem, then the orders of the blocking zeros should correspond to the sizes of the blocks.

Actual time-domain controller design for infinite-dimensional exosystems can be found in [34, 40, 72]. The controllers Immonen proposed in [40] generalizes the finite-dimensional controllers of Davison [17] and Francis [29]. The design procedure of controllers proposed by Immonen requires equations similar to the regulator equations (3.8) to be solvable. An observer-based controller design procedure in [34] by Hämäläinen and Pohjolainen gives a definite method of how one can implement an infinite-dimensional internal model into a controller and simultaneously stabilize the closed loop system. For finite-dimensional input and output spaces, Hämäläinen and Pohjolainen were able to strongly stabilize the closed loop system, but they had to settle for weak stability in regard to infinite-dimensional input and output spaces. Later similar controller design was used by Paunonen and Pohjolainen for SISO-systems. They used pole placement to stabilize the system and under the assumption

$$|\mathcal{P}(\mathrm{i}\omega_k)| \geq M \left(1 + |\omega_k|\right)^\alpha$$

for some $\alpha > 0$ and $M > 0$, and they were able to achieve polynomial stability in addition to strong stability. Requiring polynomial stability in addition to strong stability is advantageous since recent results show that this type of stability has relatively good robustness properties, see [69] and the references therein. It should be noted that the controllers in [34, 72] are designed assuming conditions implying existence of a unique solution to (3.10).

This introduction section is concluded by emphasizing some important aspects of the robust regulation problem and its solvability. First, the characterizations of the internal model principle are just properties of the controller under which asymptotic tracking is guaranteed if the closed loop system is stable. Since the characterizations of the internal model principle provide the necessary and sufficient conditions for a controller to be robustly regulating, it follows that the robustness of regulation should be understood in the sense that the closed loop stability should imply asymptotic tracking. Secondly, it is often impossible to simultaneously obtain exponential stability and asymptotic tracking if the exosystem is infinite-dimensional. Consequently, some weaker stability types should be considered. A valid choice would be strong stability, but, if possible, additionally requiring polynomial stability would improve robustness of stability. The third and final point to be emphasized is that it is customary to assume that there exists a solution to the Sylvester equation (3.10) when designing controllers in order to guarantee sufficient smoothness of the reference and disturbance signals.

3.1.6 Organization and Contributions of the Chapter

The purpose of this chapter is not to give a detailed study of the robust regulation problem but merely to understand the trade-off between solvability and

generality of the classes of possible plants and signals when one has an infinite-dimensional exosystem. The introduction above gives a fairly complete picture of what kind of limitations an infinite-dimensional exosystem causes. It was pointed out that the solvability and the smoothness of the reference and the disturbance signals are related. The rest of this chapter is dedicated to study this relation and is organized as follows.

Section 3.2: In this section, the solvability of the feedforward regulation problem is studied in detail. Special attention is paid to the relation between the solvability of the problem and the high frequency behavior of the plant transfer function. The results of this section can be found in articles [52, 53] by the author of the thesis.

In Section 3.2.1, new sum conditions for the solvability are presented and discussed. These sum conditions generalize the similar result in [41] where the input space was assumed to be one dimensional. The sum conditions to be presented in this thesis allow even infinite-dimensional input and output spaces. In addition, the sum conditions exist in terms of the exosystem and the original plant, in contrast with the results in [41] where the sum conditions were given in terms of the exosystem and the stabilized plant.

Section 3.2.2 is dedicated to the study of the required smoothness properties of the reference signals. A relation between the solvability of the feedforward regulation problem and the high frequency behavior of the plant transfer function is made precise. An instrumental tool in this work is the global structure defined in Chapter 2. This leads to a characterization of all the reference signals generated by the proposed exosystem for which the output regulation problem is solvable by using a feedforward controller.

Section 3.3: As mentioned in the introduction, it is customary to assume that there exists a solution to the Sylvester equation (3.10) when designing controllers in order to guarantee sufficient smoothness of the reference and disturbance signals. The purpose of the section is to clarify the smoothness properties such an assumption implies. For simplicity, the plants are assumed to have one dimensional input and output spaces. A simple relation between the smoothness of the signals and the solvability of the Sylvester equation is found and the connection to the structure at infinity is established. It is also shown that the robustly regulating controller in [72] works under the found smoothness condition, which is more general than the one used in [72].

3.2 Solvability of the Feedforward Output Regulation Problem

The purpose of this section is to characterize the solvability of the feedforward output regulation problem. The plants are supposed to satisfy the assumption given below.

Assumption 3.2.1. *1. The pair (A, B) is exponentially stabilizable.*

2. The spectrum of S is contained in the resolvent set of A, i.e., $\mathrm{i}\omega_k \in \rho(A)$.

3. The transfer function $\mathcal{P}(\mathrm{i}\omega_k) = C(\mathrm{i}\omega_k I - A)^{-1}B + D$ is left or right invertible for all $k \in \mathbb{Z}$.

The following useful characterization of bounded operators is used extensively.

Lemma 3.2.2. *Let W and V be Hilbert spaces and let $(\phi_k)_{k\in\mathbb{Z}}$ be an orthonormal basis of W. A linear operator $T : W \to V$ is bounded if and only if $\sup_{\|h\|\leq 1} \sum_{k\in\mathbb{Z}} |\langle T\phi_k, h\rangle|^2 < \infty$.*

Proof. Necessity. Assume that T is bounded and let h_0 be an arbitrary fixed element of V satisfying $\|h_0\| \leq 1$. Set $w_N = \sum_{|k|\leq N} \overline{\langle T\phi_k, h_0\rangle}\phi_k$. By boundedness of T

$$
\begin{aligned}
\|T\|\|w_N\| &\geq \|Tw_N\| \\
&= \sup_{\|h\|\leq 1} |\langle Tw_N, h\rangle| \\
&\geq |\langle Tw_N, h_0\rangle| \\
&= \sum_{|k|\leq N} |\langle T\phi_k, h_0\rangle|^2 .
\end{aligned}
$$

Since $\|w_N\|^2 = \sum_{|k|\leq N} |\langle T\phi_k, h_0\rangle|^2$ one has $\sum_{|k|\leq N} |\langle T\phi_k, h_0\rangle|^2 \leq \|T\|^2$ for all $N \in \mathbb{N}$. Thus, $\sup_{\|h\|\leq 1} \sum_{k\in\mathbb{Z}} |\langle T\phi_k, h\rangle|^2 < \infty$.

Sufficiency. Assume that $M = \sup_{\|h\|\leq 1} \sum_{k\in\mathbb{Z}} |\langle T\phi_k, h\rangle| < \infty$. Let $w \in W$. Since $(\phi_k)_{k\in\mathbb{Z}}$ is an orthonormal basis, one can write $w = \sum_{k\in\mathbb{Z}} \langle w, \phi_k\rangle \phi_k$. By Cauchy-Schwartz inequality

$$
\begin{aligned}
\|Tw\| &= \sup_{\|h\|\leq 1} |\langle Tw, h\rangle| \\
&\leq \sup_{\|h\|\leq 1} \sum_{k\in\mathbb{Z}} |\langle w, \phi_k\rangle||\langle T\phi_k, h\rangle| \\
&\leq \sup_{\|h\|\leq 1} \left(\sum_{k\in\mathbb{Z}} |\langle w, \phi_k\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} |\langle T\phi_k, h\rangle|^2\right)^{\frac{1}{2}} \\
&= \sqrt{M}\|w\|
\end{aligned}
$$

$\square$

In what follows the following two sum conditions turn out to be crucial for the solvability of the feedforward output regulation problem:

$$\sup_{\|u\|\leq 1} \sum_{k\in\mathbb{Z}} |\langle \gamma_k, u \rangle|^2 < \infty \tag{3.13a}$$

and

$$\sup_{\|x\|\leq 1} \sum_{k\in\mathbb{Z}} |\langle \pi_k, x \rangle|^2 < \infty \tag{3.13b}$$

where $\gamma_k = \mathcal{P}^{rl}(i\omega_k)(F - C(i\omega_k I - A)^{-1}E)\phi_k$, $\pi_k = (i\omega_k I - A)^{-1}(B\gamma_k + E\phi_k)$, and $\mathcal{P}^{rl}(i\omega_k)$ is left or right inverse of $\mathcal{P}(i\omega_k)$, which ever exists.

By Lemma 3.2.2 operators

$$\Gamma = \sum_{k\in\mathbb{Z}} \langle \cdot, \phi_k \rangle \gamma_k \tag{3.14a}$$

and

$$\Pi = \sum_{k\in\mathbb{Z}} \langle \cdot, \phi_k \rangle \pi_k \tag{3.14b}$$

are bounded linear operators if and only if the sum conditions (3.13) hold. These operators in turn satisfy the so called regulator equations. The regulator equations in [6, 41] give a necessary and sufficient condition for the solvability. The following result was given only for the single output case in [41, Theorem 3.1]. However, the proof is valid also in the current case with minor modifications.

Theorem 3.2.3. *The feedforward output regulation problem is solvable by a state-feedback control law of form (3.4) if and only if $A + BK$ is exponentially stable and there exists a decomposition $L = \Gamma - K\Pi$, where $\Gamma : W \to U$ and $\Pi : W \to X$ are bounded linear operators such that for all $k \in \mathbb{Z}$*

$$E\phi_k = \Pi S\phi_k - A\Pi\phi_k - B\Gamma\phi_k, \tag{3.15a}$$
$$F\phi_k = C\Pi\phi_k + D\Gamma\phi_k. \tag{3.15b}$$

Proof. Denote

$$A_e = \begin{bmatrix} A + BK & BL + E \\ 0 & S \end{bmatrix} \in \mathcal{L}(X \times W).$$

The operator A_e generates the C_0-semigroup

$$T_{A_e}(t) = \begin{bmatrix} T_{A+BK}(t) & \int_0^t T_{A+BK}(s)(BL+E)T_S(t-s)ds \\ 0 & T_S(t) \end{bmatrix}. \tag{3.16}$$

If the regulator equation (3.15a) holds and $L = \Gamma - K\Pi$, then

$$(BL + E)\phi_k = (\Pi S - (A + BK)\Pi)\phi_k = (\mathrm{i}\omega_k I - (A + BK))\Pi\phi_k$$

and

$$
\int_0^t T_{A+BK}(s)(BL + E)T_S(t - s)v_0 ds
$$
$$
= \int_0^t T_{A+BK}(s)(BL + E)T_S(t - s) \sum_{k\in\mathbb{Z}} \langle v_0, \phi_k\rangle\phi_k ds
$$
$$
= \sum_{k\in\mathbb{Z}} \langle v_0, \phi_k\rangle \int_0^t T_{A+BK}(s)(BL + E)T_S(t - s)\phi_k ds
$$
$$
= \sum_{k\in\mathbb{Z}} \langle v_0, \phi_k\rangle \int_0^t T_{A+BK}(s)(BL + E)e^{\mathrm{i}\omega_k(t-s)}\phi_k ds
$$
$$
= \sum_{k\in\mathbb{Z}} \langle v_0, \phi_k\rangle \int_0^t e^{\mathrm{i}\omega_k(t-s)}T_{A+BK}(s)(\mathrm{i}\omega I - (A + BK))\Pi\phi_k ds
$$
$$
= \sum_{k\in\mathbb{Z}} \langle v_0, \phi_k\rangle \int_0^t \frac{d}{ds}\left(-e^{\mathrm{i}\omega_k(t-s)}T_{A+BK}(s)\right)\Pi\phi_k ds
$$
$$
= \sum_{k\in\mathbb{Z}} \langle v_0, \phi_k\rangle \left(e^{\mathrm{i}\omega_k t}I - T_{A+BK}(s)\right)\Pi\phi_k
$$
$$
= \Pi T_S(t)v_0 - T_{A+BK}(t)\Pi v_0. \tag{3.17}
$$

Changing the order of the integration and the summation above is justified by the Lebesgue dominated convergence theorem. Furthermore, by substituting (3.17) into (3.16), one finds out that

$$
e(t) = \begin{bmatrix} -(C + DK) & F - DL \end{bmatrix} T_{A_e}(t) \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}
$$
$$
= (C + DK)T_{A+BK}(t)\left(\Pi v_0 - x_0\right) + \left(F - D\Gamma - C\Pi\right)T_S(t)v_0. \tag{3.18}
$$

Necessity. Assume that the feedforward regulation problem is solvable by the feedforward control law (3.4). By the problem formulation $A + BK$ is exponentially stable. The growth bound of $T_{A+BK}(t)$ is strictly negative and the growth bound of $T_S(t)$ is zero so [96, Corollary 8] states that there exists a unique solution $\Pi \in \mathcal{B}(W, Z)$ to the Sylvester equation

$$\Pi S - (A + BK)\Pi = BL + E$$

on $\mathcal{D}(S)$. By choosing $\Gamma = L + K\Pi$ one sees that the regulator equation (3.15a) is satisfied. Thus, the error is given by (3.18). Since $T_{A+BK}(t)$ is exponentially stable one must have that $\left(F - D\Gamma - C\Pi\right)T_S(t)v_0 \to 0$ for all $v_0 \in W$, as $t \to \infty$. This is possible only if $F - D\Gamma - C\Pi = 0$, so the regulator equation (3.15b) also holds.

Sufficiency.   Assume that $A + BK$ is exponentially stable and that the regulator equations (3.15) hold. By (3.15b) $F - D\Gamma - C\Pi = 0$, so (3.18) and exponential stability of $T_{A+BK}(t)$ implies $e(t) \to 0$, as $t \to \infty$. $\square$

Before proceeding, the regulator equations are formulated so that the plant transfer function is present. This is done in the next lemma.

Lemma 3.2.4. *The operators Γ and Π satisfy the regulator equations* (3.15a) *and* (3.15b) *if and only if for all $k \in \mathbb{Z}$ they satisfy*

$$\mathcal{P}(\mathrm{i}\omega_k)\Gamma\phi_k = F\phi_k - C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k, \qquad (3.19a)$$

$$\Pi\phi_k = (\mathrm{i}\omega_k I - A)^{-1}(B\Gamma + E)\phi_k. \qquad (3.19b)$$

Proof. Necessity. Assume that the regulator equations (3.15) hold. By noting that ϕ_k is the eigenvector of S corresponding to the eigenvalue $\mathrm{i}\omega_k$, one can write (3.15a) in the form

$$\Pi\phi_k = (\mathrm{i}\omega_k I - A)^{-1}(B\Gamma + E)\phi_k.$$

Left multiplying the above equation by C, add $D\Gamma\phi_k$ to both sides and use (3.15b) to see that

$$C\Pi\phi_k + D\Gamma\phi_k = \mathcal{P}(\mathrm{i}\omega_k)\Gamma\phi_k + C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k = F\phi_k.$$

Sufficiency. Assume that the equations (3.19) hold. The equation (3.19b) implies

$$\Pi S\phi_k - A\Pi\phi_k - B\Gamma\phi_k = (\mathrm{i}\omega_k I - A)\Pi\phi_k - B\Gamma\phi_k = E\phi_k.$$

By (3.19), one sees that

$$C\Pi\phi_k + D\Gamma\phi_k = \mathcal{P}(\mathrm{i}\omega_k)\Gamma_k + C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k = F\phi_n.$$

$\square$

3.2.1   Solvability Conditions

Next it is shown that the sum conditions (3.13) are crucial for the solvability of the feedforward output regulation problem. The sum conditions are similar to the one given in [41, Corollary 4.7] for SISO plants. However, the sum conditions given here cover general Hilbert spaces as input and output spaces, and they give solvability conditions directly in terms of the original plant instead of the closed loop plant.

It is clear that if one has less control than measurements, then it is practically impossible to solve the feedforward output regulation problem. This is why the case of right invertible transfer functions is more interesting. However, for thorough analysis the case when $\mathcal{P}(s)$ is left invertible is also considered. This is done next.

Theorem 3.2.5. *Assume that the plant transfer function is left invertible at* $\mathrm{i}\omega_k$ *for all* $k \in \mathbb{Z}$. *The feedforward output regulation problem is solvable if and only if the equations* (3.13) *hold for some choice of* $\mathcal{P}^{rl}(\mathrm{i}\omega_k) = \mathcal{P}^l(\mathrm{i}\omega_k)$ *and*

$$(I - \mathcal{P}(\mathrm{i}\omega_k)\mathcal{P}^l(\mathrm{i}\omega_k))(F\phi_k - C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k) = 0. \qquad (3.20)$$

Proof. Necessity. Assume that the feedforward output regulation problem is solvable. It follows that (3.19a) and (3.19b) hold. Left invertibility and the equation (3.19a) show $\Gamma\phi_k = \gamma_k$. The equation (3.19b) implies that $\Pi\phi_k = \pi_k$. The sum conditions (3.13) hold by Lemma 3.2.2. Since (3.19a) implies $F\phi_k - C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k \in \mathcal{R}\left(\mathcal{P}(\mathrm{i}\omega_k)\right)$ and $\mathcal{P}(\mathrm{i}\omega_k)\mathcal{P}^l(\mathrm{i}\omega_k)$ is a projection operator on $\mathcal{R}\left(\mathcal{P}(\mathrm{i}\omega_k)\right)$, condition (3.20) holds.

Sufficiency. Under the sum conditions (3.13) Lemma 3.2.2 shows that the operators (3.14) are bounded. By Assumption (3.20), it is clear that (3.19a) and (3.19b) are satisfied by the defined operators. □

Condition (3.20) is natural. One has less control than measurements with non-square left invertible plants. As the reference signals live on the measurement space, it is clear that one cannot have control over all the reference signals. Roughly speaking, (3.20) means that the part of a reference signal that is not controllable must be compensated by the error signal.

Remark 3.2.6. *No matter how one chooses the left inverse* $\mathcal{P}^l(\mathrm{i}\omega_k)$, *one has the same value for* γ_k. *Thus, the operators* Γ *and* Π *solving the regulator equations are uniquely determined.*

In the remaining part of this section, the case of right invertible transfer functions is considered. The solvability condition in this case turns out to be similar to the one given in the case of left invertible transfer functions, but no additional assumptions need to be made.

Theorem 3.2.7. *Assume that the plant transfer function is right invertible at* $\mathrm{i}\omega_k$ *for all* $k \in \mathbb{Z}$. *The feedforward output regulation problem is solvable if and only if for some choice of* $\mathcal{P}^{rl}(\mathrm{i}\omega_k) = \mathcal{P}^r(\mathrm{i}\omega_k)$ *the sum conditions* (3.13) *hold.*

Proof. Necessity. Assume that the feedforward output regulation problem is solvable. It follows that (3.19a) and (3.19b) hold. By (3.19a) $\Gamma\phi_k = \gamma_k$ for some choice of right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ where k varies over $\mathbb{Z}$. From (3.19b) one gets $\Pi\phi_k = \pi_k$. The sum conditions (3.13) hold by Lemma 3.2.2.

Sufficiency. Under the sum conditions (3.13) Lemma 3.2.2 implies that the operators in (3.14) are bounded. It is a matter of trivial calculation to verify that (3.19a) and (3.19b) are satisfied by the defined operators. □

For right invertible transfer functions, the operators Γ and Π are not in general defined uniquely since different choices of right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ lead to different operators. However, the sum conditions (3.13) can be used to characterize all the regulating controllers of form (3.4). The next corollary follows by the proof of the above theorem and Lemma 3.2.4.

Corollary 3.2.8. *If the sum conditions* (3.13) *hold for a choice of* $\mathcal{P}^{rl}(\mathrm{i}\omega_k) = \mathcal{P}^r(\mathrm{i}\omega_k)$, $k \in \mathbb{Z}$, *then the feedback control law* (3.4), *where K exponentially stabilizes the pair (A, B) and $L = \Gamma - K\Pi$ with the operators Γ and Π in* (3.14), *solves the feedforward output regulation problem. All the feedback control laws solving the feedforward output regulation problem are of this form.*

It have been seen above that the solvability of the feedforward output regulation problem is characterized by two sum conditions. Both of them are needed, as shown by the following two examples.

Example 3.2.9. In this example, the plant and the exosystem are chosen so that the sum condition (3.13a) fails while the sum condition (3.13b) is satisfied. Set $\omega_k = k$, for $k \in \mathbb{Z}$, and $U = Y = X = \mathbb{C}$. Choose $E = 0$, $D = 0$, $B = C = 1$, $A = -1$, and $F = \langle \cdot, w_0 \rangle$ where $w_0 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k} \phi_k \in W$. Assumption 3.2.1 is now satisfied and

$$\gamma_k = (\mathrm{i}k + 1) \langle \phi_k, w_0 \rangle = \frac{\mathrm{i}k + 1}{k} \text{ and } \pi_k = \langle \phi_k, w_0 \rangle$$

for $k \in \mathbb{Z} \setminus \{0\}$. It is clear that

$$\sup_{\|u\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle \gamma_k, u \rangle|^2 = \infty$$

and

$$\sup_{\|x\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle \pi_k, x \rangle|^2 = \|w_0\|^2 < \infty.$$

∎

Example 3.2.10. In this example, the plant and the exosystem are chosen so that the sum condition (3.13b) fails while the sum condition (3.13a) is satisfied. Let $\omega_k = k$, for $k \in \mathbb{Z}$, and $U = Y = X = Z$. Choose X to be a Hilbert space with an orthonormal basis $\{\phi_k \in X \mid k \in \mathbb{Z}\}$. Choose $B = D = E = F = I$,

$$A = \sum_{k \in \mathbb{Z}} \left(\mathrm{i}k - \frac{1}{|k| + 1} \right) \langle \cdot, \phi_k \rangle \phi_k$$

with

$$\mathcal{D}(A) = \left\{ (x_k)_{k \in \mathbb{Z}} \in \ell^2 \left| \sum_{k \in \mathbb{Z}} |k|^2 |\langle x_k, \phi_k \rangle|^2 < \infty \right. \right\},$$

and

$$C = \sum_{k \in \mathbb{Z}} \frac{1}{|k| + 1} \langle \cdot, \phi_k \rangle \phi_k.$$

Elementary calculations show that

$$\mathcal{P}^{-1}(s) = \sum_{k \in \mathbb{Z}} \frac{(|k| + 1)(s - ik) + 1}{(|k| + 1)(s - ik) + 2} \langle \cdot, \phi_k \rangle \phi_k.$$

All the standing assumptions are now satisfied and

$$\gamma_k = -\frac{1}{2} \phi_k \text{ and } \pi_k = \frac{|k| + 1}{2} \phi_k.$$

It is now clear that $\sup_{\|u\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle \gamma_k, u \rangle|^2 = \frac{1}{2} < \infty$. Since $\sum_{k \in \mathbb{Z}} |\langle \pi_k, \phi_k \rangle|^2 = \frac{(|k|+1)^2}{4}$, the sum condition (3.13b) fails. ∎

Since the supremum over an infinite number of series is hard to find in practice, one seeks some additional conditions that simplify the situation. This is what is done in the remaining part of this section.

Lemma 3.2.11. *Let B be a finite rank operator and let $\sum_{k \in \mathbb{Z}} \|\gamma_k\|^2 < \infty$. The sum conditions* (3.13) *hold if and only if*

$$\sup_{\|x\| \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle (i\omega_k I - A)^{-1} E \phi_k, x \right\rangle \right|^2 < \infty. \tag{3.21}$$

Proof. By assumptions of the lemma, it follows that

$$\sup_{\|u\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle \gamma_k, u \rangle|^2 \leq \sum_{k \in \mathbb{Z}} \sup_{\|u\| \leq 1} |\langle \gamma_k, u \rangle|^2 = \sum_{k \in \mathbb{Z}} \|\gamma_k\|^2 < \infty.$$

The exponential stabilizability of (A, B) implies that X can be decomposed as $X = X_+ \oplus X_-$, where X_+ is finite-dimensional. The operator A has a corresponding decomposition $A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}$, where A_+ is a generator of C_0-semigroup in X_+ and A_- is a generator of an exponentially stable C_0-semigroup in X_- [14, Theorem 5.2.6]. By [14, Lemma 2.1.11], $\|(sI - A_-)^{-1}\|$ is uniformly bounded in some right half-plane including the imaginary axis. Since A_+ is an operator on a finite-dimensional space $\|(i\omega I - A_+)^{-1}\| \to 0$, as $|\omega| \to \infty$. It

follows that there exists $M > 0$ such that $\|(\mathrm{i}\omega_k I - A)^{-1}\| < M$ for all $k \in \mathbb{Z}$, because $\mathrm{i}\omega_k$ do not cluster at any finite point. Since B is bounded,

$$\sup_{\|u\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle (\mathrm{i}\omega_k I - A)^{-1} B\gamma_k, u \rangle|^2 \leq \sum_{k \in \mathbb{Z}} \|(\mathrm{i}\omega_k I - A)^{-1} B\gamma_k\|^2$$
$$\leq M\|B\| \sum_{k \in \mathbb{Z}} \|\gamma_k\|^2 < \infty.$$

It follows that $\sup_{\|u\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle \pi_k, u \rangle|^2 < \infty$ if and only if (3.21) holds.   □

Theorem 3.2.12. *If the output space Y is finite-dimensional and B is a finite-rank operator, then the feedforward output regulation problem is solvable if and only if*

$$\sum_{k \in \mathbb{Z}} \|\gamma_k\|^2 < \infty \tag{3.22a}$$

and

$$\sup_{\|x\| \leq 1} \sum_{k \in \mathbb{Z}} \left| \langle (\mathrm{i}\omega_k I - A)^{-1} E\phi_k, x \rangle \right|^2 < \infty \tag{3.22b}$$

for some choice of $\mathcal{P}^{rl}(\mathrm{i}\omega_k) = \mathcal{P}^r(\mathrm{i}\omega_k)$, $k \in \mathbb{Z}$.

Proof. By Lemma 3.2.11, it is sufficient to show that $\sup_{\|x\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle \gamma_k, x \rangle|^2 < \infty$ if and only if $\sum_{k \in \mathbb{Z}} \|\gamma_k\|^2 < \infty$, which is true by the Riesz representation theorem.   □

It is obvious that the latter sum condition is satisfied, if the disturbance operator is of finite-rank. This leads to the following corollary.

Corollary 3.2.13. *Let the output space Y be finite-dimensional and the operators B and E be of finite-rank. The feedforward output regulation problem is solvable if and only if*

$$\sum_{k \in \mathbb{Z}} \|\gamma_k\|^2 < \infty$$

for some choice of $\mathcal{P}^{rl}(\mathrm{i}\omega_k) = \mathcal{P}^r(\mathrm{i}\omega_k)$, $k \in \mathbb{Z}$.

3.2.2 On the Required Smoothness Properties of Reference Signals

In this section, the required smoothness properties of the reference signals for the feedforward output regulation problem to be solvable are characterized in terms of the global structure of the plant transfer function. The focus is the

smoothness of the reference signals, and the disturbance signals are assumed to be smooth enough. In what follows, all the reference operators for which the feedforward output regulation problem is solvable are characterized.

Since the global structure is only defined for such plants that have finite-dimensional input and output spaces, choice $U = \mathbb{C}^m$ and $Y = \mathbb{C}^n$ is made. In the case of right invertible transfer functions, the operators Γ and Π were shown to be non-unique. The question arises whether there is a systematic way to choose Γ and Π such that the feedforward output regulation problem is solvable if and only if the chosen operators satisfy the regulator equations. The next theorem gives a positive answer to this question. Using the global structure of plant transfer function is convenient here, but the result can also be formulated by using the structure of plant transfer function at infinity since the convergence of the sums defined in Section 3.2.1 depends only on the high frequency behavior of the plant transfer function if Assumption 3.2.1 holds. Since only the behavior of the plant transfer function along the imaginary axis is of interest, one defines

$$H = \{ p \in \mathrm{P} \,|\, p(\alpha) \in \mathrm{i}\mathbb{R} \text{ for all } \alpha \in [0, \infty) \} . \tag{3.23}$$

Theorem 3.2.14. *Write $\mathcal{P}(s)$ as in Theorem 2.6.3 with H from (3.23). The feedforward output regulation problem is solvable if and only if the sum conditions (3.22) hold with the choice $\mathcal{P}^{rl}(\mathrm{i}\omega_k) = V^{-1}(\mathrm{i}\omega_k) \begin{bmatrix} \Lambda^{-1}(\mathrm{i}\omega_k) \\ 0 \end{bmatrix} U^{-1}(\mathrm{i}\omega_k).$*

Proof. Sufficiency is obvious, so only the necessity remains to be proved. By Corollary 3.2.12 there exists a choice $\mathcal{P}^r(\mathrm{i}\omega_k)$ for all $k \in \mathbb{Z}$ such that (3.22) holds. Now one needs to show that (3.22) holds if $\mathcal{P}^r(\mathrm{i}\omega_k)$ is replaced by

$$\mathcal{P}_+(\mathrm{i}\omega_k) = U^{-1}(\mathrm{i}\omega_k) \begin{bmatrix} \Lambda^{-1}(\mathrm{i}\omega_k) \\ 0 \end{bmatrix} V^{-1}(\mathrm{i}\omega_k).$$

Every right inverse of $\mathcal{P}(s)$ can be written in the form

$$\mathcal{P}^r(s) = U^{-1}(s) \begin{bmatrix} \Lambda^{-1}(s) \\ J(s) \end{bmatrix} V^{-1}(s).$$

It is sufficient to show that there exists an $M > 0$ such that $\|\mathcal{P}^r(s)y\| \geq M\|\mathcal{P}_+(s)y\|$ for any $y \in Y$.

Let y be an arbitrary element of Y and set $\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \Lambda^{-1}(s) \\ J(s) \end{bmatrix} V^{-1}(s)y.$ Now $\begin{bmatrix} \Lambda^{-1}(s) \\ 0 \end{bmatrix} V^{-1}(s)y = \begin{bmatrix} y_1(s) \\ 0 \end{bmatrix}$ so the problem is reduced to showing that

there exists $M > 0$ such that $\left\| U(i\omega_k) \begin{bmatrix} y_1(i\omega_k) \\ y_2(i\omega_k) \end{bmatrix} \right\| \geq M \left\| U(i\omega_k) \begin{bmatrix} y_1(i\omega_k) \\ 0 \end{bmatrix} \right\|$. For any invertible matrix U one has

$$\frac{1}{\|U\|} \left\| U \begin{bmatrix} y_1 \\ 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} y_1 \\ 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|$$

$$= \left\| U^{-1} U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|$$

$$\leq \left\| U^{-1} \right\| \left\| U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|.$$

Since $U(s)$ satisfies the global uniform boundedness property respect to H, there exists $M > 0$ such that $\frac{1}{\|U(i\omega_k)\|\|U^{-1}(i\omega_k)\|} > M$ and the claim follows by the above inequality. □

The previous theorem shows that the solvability of the feedforward output regulation problem is closely related to the global structure of the plant transfer function. The next lemma gives a simple solvability condition in terms of the plant transfer function and the smoothness of the reference signal, provided that the disturbance signal is smooth enough. To guarantee the required smoothness for the disturbance signal, it is assumed that

$$\sum_{k\in\mathbb{Z}} \left\| \mathcal{P}_+(i\omega_k) C(i\omega_k I - A)^{-1} E\phi_k \right\|^2 < \infty, \tag{3.24}$$

where $\mathcal{P}_+(i\omega_k) = U^{-1}(i\omega_k) \begin{bmatrix} \Lambda^{-1}(i\omega_k) \\ 0 \end{bmatrix} V^{-1}(i\omega_k)$ and U, Λ and V are from Theorem 2.6.3 with H from (3.23).

Lemma 3.2.15. *Write $\mathcal{P}(s)$ as in Theorem 2.6.3 with H from* (3.23)*. Set*

$$\mathcal{P}^{rl}(i\omega_k) = \mathcal{P}_+(i\omega_k) = U^{-1}(i\omega_k) \begin{bmatrix} \Lambda^{-1}(i\omega_k) \\ 0 \end{bmatrix} V^{-1}(i\omega_k).$$

Assume, that the sum conditions (3.21) *and* (3.24) *hold. The feedforward output regulation problem is solvable if and only if $\sum_{k\in\mathbb{Z}} \left\| \mathcal{P}_+(i\omega_k) F\phi_k \right\|^2 < \infty$.*

Proof. Sufficiency. Assume that $\sum_{k\in\mathbb{Z}} \left\| \mathcal{P}_+(i\omega_k) F\phi_k \right\|^2 < \infty$. By the triangle inequality,

$$\|\gamma_k\|^2 \leq 2 \left(\left\| \mathcal{P}^r(i\omega_k) F\phi_k \right\|^2 + \left\| \mathcal{P}_+(i\omega_k) C(i\omega_k I - A)^{-1} E\phi_k \right\|^2 \right).$$

By Theorem 3.2.12, sufficiency follows.

Necessity. Assume that the feedforward output regulation problem is solvable. By the reverse triangle inequality,

$$\sum_{k\in\mathbb{Z}}\left|\|\mathcal{P}_+(\mathrm{i}\omega_k)F\phi_k\| - \left\|\mathcal{P}_+(\mathrm{i}\omega_k)C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k\right\|\right|^2 \le \sum_{k\in\mathbb{Z}}\|\gamma_k\|^2 < \infty.$$

Since $\sum_{k\in\mathbb{Z}}\left\|\mathcal{P}_+(\mathrm{i}\omega_k)C(\mathrm{i}\omega_k I - A)^{-1}E\phi_k\right\|^2 < \infty$, the claim follows. $\square$

It is now possible to characterize all the reference signals that can be regulated by a feedforward controller (3.4) under the assumption that the disturbance signals are smooth enough or nonexistent. To this end, write the reference operator $F : W \to Y$ in the form

$$F = \sum_{k\in\mathbb{Z}} \langle \cdot, \phi_k \rangle V(\mathrm{i}\omega_k) \begin{bmatrix} \langle f_1, \phi_k \rangle \\ \vdots \\ \langle f_n, \phi_k \rangle \end{bmatrix}, \tag{3.25}$$

where $V(s)$ is from Theorem 2.6.3. That every bounded operator from W to Y can be presented in this form follows from the Riesz representation theorem, boundedness properties of $V(\mathrm{i}\omega_k)$ and Lemma 3.2.2.

Theorem 3.2.16. *Let the global structure of $\mathcal{P}(s)$ respect to H from (3.23) be $\{q_1(s),\dots,q_n(s)\}$ and write F as in (3.25). Assume that the sum conditions (3.21) and (3.24) hold. The feedforward output regulation problem is solvable if and only if $\left(\frac{\langle f_j, \phi_k \rangle}{q_j(\mathrm{i}\omega_k)}\right)_{k\in\mathbb{Z}} \in \ell^2$ for all $j = 1,\dots,n$.*

Proof. By Lemma 3.2.15, the feedforward output regulation problem is solvable if and only if $\sum_{k\in\mathbb{Z}}\|\mathcal{P}_+(\mathrm{i}\omega_k)F\phi_k\|^2 < \infty$, where

$$\mathcal{P}_+(\mathrm{i}\omega_k) = U^{-1}(\mathrm{i}\omega_k) \begin{bmatrix} \Lambda^{-1}(\mathrm{i}\omega_k) \\ 0 \end{bmatrix} V^{-1}(\mathrm{i}\omega_k)$$

with U, V and Λ from Theorem 2.6.3. The result follows immediately by writing F as in (3.25) and noting that $\|U^{-1}(\mathrm{i}\omega_k)\| < M < \infty$ and $\|U(\mathrm{i}\omega_k)\| < M < \infty$ for some $M > 0$. $\square$

The following example illustrates the importance of the above theorem. It shows that for certain signals the required smoothness properties are far stricter than for others.

Example 3.2.17. Consider the metal bar of Figure 3.1 that is heated with two heaters and its temperature is measured along two intervals. The resulting
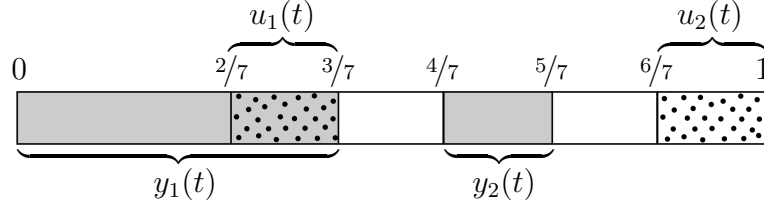
Figure 3.1: A heated metal bar with two heaters and sensors.

system is

$$\frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) - x(z,t) + 7 \cdot \mathbf{1}_{[\frac{2}{7},\frac{3}{7}]}(z)u_1(t) + 7 \cdot \mathbf{1}_{[\frac{6}{7},1]}(z)u_2(t),$$

$$y_1(t) = \int_0^{\frac{3}{7}} x(z,t)\mathrm{d}z,$$

$$y_2(t) = \int_{\frac{4}{7}}^{\frac{5}{7}} x(z,t)\mathrm{d}z,$$

$$x(z,0) = x_0(z),$$

$$\frac{\partial x}{\partial z}(0,t) = 0 = \frac{\partial x}{\partial z}(1,t),$$

where $\mathbf{1}_{[a,b]}(z)$ is the characteristic function of $[a,b]$.

Calculations similar to those in [14, Example 4.3.11] show that the transfer function is

$$\mathcal{P}(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix},$$

where

$$
p_{11}(s) = \frac{1}{s+1} + 7 \frac{\left\{ \begin{array}{c} -\cosh\left(\sqrt{s+1}\right) + \cosh\left(\frac{6}{7}\sqrt{s+1}\right) \\ -\cosh\left(\frac{2}{7}\sqrt{s+1}\right) \\ +\cosh\left(\frac{1}{7}\sqrt{s+1}\right) \end{array} \right\}}{2(s+1)^{3/2}\sinh\left(\sqrt{s+1}\right)},
$$

$$
p_{12}(s) = \frac{7}{(s+1)^{3/2}} \frac{\sinh\left(\frac{1}{7}\sqrt{s+1}\right)\sinh\left(\frac{3}{7}\sqrt{s+1}\right)}{\sinh\left(\sqrt{s+1}\right)},
$$

$$
p_{21}(s) = 7 \frac{\left\{ \begin{array}{c} -\cosh\left(\frac{6}{7}\sqrt{s+1}\right) - 2\cosh\left(\frac{5}{7}\sqrt{s+1}\right) \\ +\cosh\left(\frac{4}{7}\sqrt{s+1}\right) \\ +2\cosh\left(\frac{1}{7}\sqrt{s+1}\right) - 4 \end{array} \right\}}{2(s+1)^{3/2}\sinh\left(\sqrt{s+1}\right)}, \text{ and}
$$

$$
p_{22}(s) = \frac{7\sinh\left(\frac{1}{7}\sqrt{s+1}\right)\left(\sinh\left(\frac{5}{7}\sqrt{s+1}\right) - \sinh\left(\frac{4}{7}\sqrt{s+1}\right)\right)}{(s+1)^{3/2}\sinh\left(\sqrt{s+1}\right)}.
$$

The global structure $\mathcal{P}(s)$ along the imaginary axis is

$$
\left\{ \frac{1}{s+1}, \frac{1}{(s+1)^{3/2}} e^{-\frac{1}{7}\sqrt{s+1}} \right\}.
$$

Consider the feedforward output regulation problem with $\omega_k = k$ and $E = 0$, and define the following two reference operators:

$$
F_1 = \sum_{k\in\mathbb{Z}} \langle \cdot, \phi_k \rangle V(\mathrm{i}k) \begin{bmatrix} \langle f_1, \phi_k \rangle \\ 0 \end{bmatrix},
$$

and

$$
F_2 = \sum_{k\in\mathbb{Z}} \langle \cdot, \phi_k \rangle V(\mathrm{i}k) \begin{bmatrix} 0 \\ \langle f_2, \phi_k \rangle \end{bmatrix},
$$

where $V(\cdot)$ is the matrix valued function in (2.15). The boundedness properties of $V(\mathrm{i}k)$ imply that the smoothness of the reference signals generated by using the reference operators F_1 and F_2 are essentially the same, if $f_1 = f_2$. By Theorem 3.2.16, the feedforward output regulation problem with the reference operator F_1 is solvable if and only if $(\langle f_1, \phi_k \rangle (\mathrm{i}k+1))_{k\in\mathbb{Z}} \in \ell^2$. Similarly, the feedforward output regulation problem with the reference operator F_2 is solvable if and only if $\left(\langle f_2, \phi_k \rangle (\mathrm{i}k+1)^{3/2} e^{\sqrt{\mathrm{i}k+1}/7} \right)_{k\in\mathbb{Z}} \in \ell^2$. The required smoothness of the reference signals generated by using F_2 is of exponential type, while using F_1 only polynomial smoothness is required. ∎

In [6], results answering to the question when it is possible to solve the feed-forward output regulation problem for all reference and disturbance operators were presented. By this it is not meant that a single controller is able to do the task, which would mean the controller to be robustly regulating. The exosystem considered in [6] was finite-dimensional. Even if Assumption 3.2.1 is satisfied, Theorem 3.2.16 shows that there exists reference operators for which the problem with infinite-dimensional exosystems is not solvable. However, there is a controller solving the problem for all the reference signals if the set of reference operators is restricted appropriately as shown by the following corollary.

Corollary 3.2.18. *Write $\mathcal{P}(s)$ as in Theorem 2.6.3 with H from (3.23) and F in (3.25). Assume that the sum conditions (3.21) and (3.24) hold. Choose a sequence of strictly positive real numbers $(z_k)_{k\in\mathbb{Z}} \in \ell^\infty$ and denote*

$$\mathcal{Y}(z_k) = \left\{ F \in \mathcal{B}(W,Y) \,\middle|\, \sum_{k\in\mathbb{Z}} |z_k|^{-2}\|F\phi_k\|^2 < \infty \right\}.$$

The feedforward output regulation problem is solvable for all $F \in \mathcal{Y}(z_k)$ if and only if $\left(\alpha_k \frac{z_k}{q_n(\mathrm{i}\omega_k)}\right)_{k\in\mathbb{Z}} \in \ell^2$ for all sequences $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2$ of positive real numbers.

Proof. By Theorem 3.2.16, the feedforward output regulation problem is solvable for all $F \in \mathcal{Y}(z_k)$ if and only if

$$\sum_{k\in\mathbb{Z}} \frac{\|F\phi_k\|^2}{|q_n(\mathrm{i}\omega_k)|^2} = \sum_{k\in\mathbb{Z}} \frac{\|F\phi_k\|^2}{z_k^2} \frac{z_k^2}{|q_n(\mathrm{i}\omega_k)|^2} < \infty.$$

The claim follows by noting that $\left(\frac{\|F\phi_k\|}{z_k}\right)_{k\in\mathbb{Z}} \in \ell^2$ and for each $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2$ there exists such an $F \in \mathcal{Y}(z_k)$ that $\alpha_k = \frac{\|F\phi_k\|}{z_k}$. $\qquad\square$

Example 3.2.19. Let I be an infinite subset of $\mathbb{Z}$, and set $\omega_k = \frac{2\pi k}{p}$ for $k \in I$. It is assumed that ω_k are in strictly increasing order. Consider the solvability of the disturbance free feedforward output regulation problem with the time-delay system from Example 2.2.11 and the reference operators from $\mathcal{Y}(z_k)$ defined in Corollary 3.2.18 with index set I. The system is already exponentially stable as $\det(sI - A_0 - e^{-s}A_1) \neq 0$ when $s \in \{x \in \mathbb{C} \,|\, \Re(x) \geq 0\}$ [14, Theorem 5.1.5]. Thus, Assumption 3.2.1 holds if there are no transmission zeros at frequencies $\mathrm{i}\omega_k = \mathrm{i}\frac{2\pi k}{p}$, $k \in I$. The determinant of the transfer function, readily calculated in (2.10), shows that the time-delay system has finite transmission zeros on the imaginary axis at $s = \mathrm{i}2\pi l$, $l \in \mathbb{Z}$.

Assume that $\omega_k \neq 2\pi l$ for all $k \in I$ and $l \in \mathbb{Z}$. Since the structure of the system at infinity is $\{(s+1)^{-1}, (s+1)^{-1}, (s+1)^{-2}(e^{-s}-1)\}$ and the structural

functions do not have zeros at $i\omega_k$, $k \in I$, Corollary 3.2.18 gives a necessary and sufficient condition

$$\forall (\alpha_k)_{k \in I} \in \ell^2 : \sum_{k \in I} \alpha_k^2 \left| (i\omega_k)^{-2} (e^{-i\omega_k} - 1) \right|^{-2} z_k^2 < \infty$$

which is equivalent to

$$\forall (\alpha_k)_{k \in I} \in \ell^2 : \sum_{k \in I} \frac{k^4 z_k^2 \alpha_k^2}{\left| e^{\frac{-2\pi k i}{p}} - 1 \right|^2} < \infty \tag{3.27}$$

for the problem to be solvable for all $F \in \mathcal{Y}(z_k)$. Note that changing the index set from $\mathbb{Z}$ to I or using the structure at infinity instead of global structure is not a problem here, because the transmission zeros are assumed to be separate from $i\omega_k$. In particular, if one chooses $p = 2$ and $I = \{2k + 1 \,|\, k \in \mathbb{Z}\}$ the points $i\omega_k$ and transmission zeros both lie on the imaginary axis, but they do not cluster at infinity. Thus, the term $|e^{i\omega_k} - 1|$ remains bounded and bounded away zero for $k \in I$ and a necessary and sufficient solvability condition is

$$\sum_{k \in I} \alpha_k^2 k^4 z_k^2 < \infty \text{ for all } (\alpha_k)_{k \in \mathbb{Z}} \in \ell^2.$$

Consequently, $\sum_{k \in I} k^3 z_k^2 < \infty$ is a necessary condition for the solvability.

To show that finite transmission zeros are a crucial part of the infinite structure, set $p = 2\pi$ and consider an infinite set $I \subset \mathbb{Z}$ of elements $k \in \mathbb{Z}$ that satisfy $|2\pi l - k| < \frac{\rho}{|k|}$ for some $l \in \mathbb{Z}$, $k \neq 0$ and fixed $\rho > 0$. Such a set I exists, because there are infinitely many pairs of integers (k, l) such that $\left| \frac{1}{2\pi} - \frac{l}{k} \right| < \frac{1}{k^2}$ [84, Corollary 5.4.].

Now the transmission zeros and terms $i\omega_k$ cluster at infinity, i.e., the minimum distance of $i\omega_k$ to transmission zeros approach zero as $k \to \infty$. It follows that $\lim_{k \to \pm\infty} e^{i\omega_k} - 1 = 0$. The faster the rate of convergence the more unlikely it is that the condition (3.27) holds.

One cannot say for sure at which rate $i\omega_k$ approaches the set of transmission zeros. Irrationality measure gives insight into the rate of convergence. It is believed that the irrationality measure of π is 2 in which case $e^{i\omega_k} - 1 \to 0$ by rate proportional to $\frac{1}{|k|}$ and (3.27) is equivalent to

$$\sum_{k \in I} \alpha_k^2 k^6 z_k^2 < \infty \text{ for all } (\alpha_k)_{k \in \mathbb{Z}} \in \ell^2.$$

So far the best approximation for the irrationality measure is $7.6063 \ldots$ due to Salikhov [83]. Thus, by approximating the irrationality measure of π by 8 it is seen, that $s^{-2}(e^{-s} - 1) \geq_H s^{-10}$. Corollary 3.2.18 gives a sufficient condition

$$\sum_{k \in I} k^{22} z_k^2 < \infty.$$

∎

In the above discussion, it was assumed that the disturbance signal is smooth enough or even nonexistent. One may wonder what happens if the roles of the disturbance signals and reference signals in the discussion are changed. Assume for a moment that the reference operator is zero and the plant transfer function is right invertible at $i\omega_k$ for all $k \in \mathbb{Z}$. Now $\gamma_k = \mathcal{P}^r(i\omega_k)C(i\omega_k I - A)^{-1}E\phi_k$ and $\pi_k = (i\omega_k I - A)^{-1}(B\gamma_k + E\phi_k)$. It is seen that the smoothness of the disturbance signals is not related to the boundedness of the operator Γ in such a direct manner as the smoothness of the reference signals is due to the term $C(i\omega_k I - A)^{-1}$. This is why there is no such a simple characterization for the required smoothness properties of the disturbance signals than there is for the smoothness of the reference signals. However, generally one would expect the required smoothness properties to be less strict for the disturbance signals since the resolvent operator $(i\omega_k I - A)^{-1}$ of an exponentially stabilizable operator A generally smoothens the term γ_k.

3.3 Solvability of the Error Feedback Robust Regulation Problem

The robust regulation was studied in [34, 70] under the assumption that the solvability of the Sylvester equation (3.10) is preserved by the perturbations. However, no research on the robustness of this assumption was made. Here the connection between solvability of the regulation equation (3.10) and the smoothness of the reference and disturbance signals is considered with a diagonal exosystem and SISO plants. The main aim is to study the smoothness of the signals. However, the found results are simply conditions for solvability of the Sylvester equation (3.10) and are important as such. They also provide insight into what perturbations are allowed to the parameters.

It is quite straightforward to include an internal model into a controller, but it is hard to stabilize the controller [34, 72]. For stabilization of the extended system to be as simple as possible it is advantageous if the plant is exponentially stabilizable and detectable and if the internal model is not complicated. The assumed properties of the nominal plant are given in the next assumption. Then some assumptions on the controller are made.

Assumption 3.3.1. *The nominal plant* (3.1) *is assumed to satisfy the following conditions:*

1. *The plant has one dimensional input and output spaces,*

2. *the pair (A, B) is exponentially stabilizable,*

3. *the pair (A, C) is exponentially detectable, and*

4. the plant transfer function $\mathcal{P}(s)$ is right invertible at $\mathrm{i}\omega_k$ for all $k \in \mathbb{Z}$.

Assumption 3.3.2. *The controller* (3.5) *is assumed to satisfy the following conditions:*

1. *$\mathrm{i}\omega_k$ is an isolated spectrum point of $\mathcal{G}_1$ for all $k \in \mathbb{Z}$,*

2. *if γ_k is a simple closed curve that separates $\mathrm{i}\omega_k$ from the rest of the spectrum and $P_k = -\frac{1}{2\pi\mathrm{i}} \int_{\gamma_k} R\left(s, \mathcal{G}_1\right) ds$, then $Z_k = P_k Z$ is finite-dimensional, and*

3. *the controller satisfies the $\mathcal{G}$-conditions* (3.9).

Remark 3.3.3. *If item 1 of Assumption 3.3.2 holds P_k in item 2 is a projection and Z is a direct sum of Z_k and $Z'_k = (I - P_k)Z$ by [45, III-§4].*

When the robust error feedback regulation problem was formulated in Section 3.1.5 a class of allowed perturbations was defined by a set $\mathcal{V}$ of 7-tuples. However, the $\mathcal{V}$ was not defined explicitly. This is due to the fact that for a given controller it is not a trivial task to describe all the perturbed plants for which the controller is stabilizing or regulating. In the recent literature, it have been customary to assume that the Sylvester equation (3.10) is solvable for all the perturbed plants. With the exosystem (3.2) a solution to (3.10) is always unique provided that the controller strongly stabilizes the extended system [71, Theorem 8]. To see why such an assumption was made note that if the Sylvester equation has a solution, then the error can be written as

$$e(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e)v(t),$$

where $T_e(t)$ is the C_0-semigroup generated by A_e. A robustly regulating controller is conditionally robust, so $(C_e \Sigma + D_e)v(t) = 0$ by (3.11). Since $T_e(t)$ is strongly stable $e(t) \to 0$. The solvability was assumed in [34, 72] for all perturbations, but the robustness of solvability was not explicitly studied.

In [34, 72], a simplifying assumption that $\mathrm{i}\omega_k \in \rho\left(A_e\right)$ was made. This assumption allows one to give an explicit formula for the unique solution of the Sylvester equation (3.10). By [34, Lemma 6], the unique solution to (3.10) is

$$\Sigma = \sum_{k \in \mathbb{Z}} \langle \cdot, \phi_k \rangle R\left(\mathrm{i}\omega_k, A_e\right) B_e \phi_k,$$

and it exists as a bounded operator in W if and only if

$$\sup_{\|x_e\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle R\left(\mathrm{i}\omega_k, A_e\right) B_e \phi_k, x_e \rangle|^2 < \infty \tag{3.28}$$

by Lemma 3.2.2. This sum condition relates the operators E and F to the closed loop system consisting of the plant and the controller. Thus, it gives a

relation between the solvability and the smoothness of the reference and the disturbance signals. This connection is to be clarified later in this section.

It is assumed in this section that the spectrum separation condition $i\omega_k \in \rho(A_e)$ holds for all perturbed plants. The sum condition (3.28) relates the smoothness of the signals and the behavior of the plant transfer function at high frequencies. In [70, 72] the smoothness of the signals was controlled by restricting the set of initial states of the exosystem. A class of initial states of the exosystem is defined by a set

$$\Im(f_k) \left\{ v \in W \,\middle|\, \sum_{k \in \mathbb{Z}} f_k^{-2} |\langle v, \phi_k \rangle|^2 < \infty \right\}, \tag{3.29}$$

where $(f_k)_{k \in \mathbb{Z}}$ is a fixed bounded sequence of strictly positive real numbers. In this thesis, the smoothness is controlled by restricting the set of allowed reference and disturbance operators. To this end, one defines

$$\mathcal{Y}(f_k) = \left\{ F \in \mathcal{B}(W, Y) \,\middle|\, \sum_{k \in \mathbb{Z}} |f_k|^{-2} \|F\phi_k\|^2 < \infty \right\} \tag{3.30}$$

and

$$\mathcal{D}(f_k) = \left\{ E \in \mathcal{B}(W, X) \,\middle|\, \sum_{k \in \mathbb{Z}} |f_k|^{-2} \|E\phi_k\|^2 < \infty \right\}, \tag{3.31}$$

where $(f_k)_{k \in \mathbb{Z}}$ is a fixed bounded sequence of strictly positive real numbers. In order to clarify the relation between the two approaches to control the smoothness of the reference signals note that the reference signal generated by the exosystem (3.2) are of form

$$y_r(t) = \sum_{k \in \mathbb{Z}} \langle v_0, \phi_k \rangle e^{i\omega_k t} F_1 \phi_k. \tag{3.32}$$

Assume that a reference signal $y_r(t)$ can be written in form (3.32), where $v_0 \in \Im(f_k)$ and $F_1 \in \mathcal{B}(W, Y)$. If one defines

$$v_0' = \sum_{k \in \mathbb{Z}} \frac{\langle v_0, \phi_k \rangle}{f_k} \phi_k \text{ and } F_1' = \sum_{k \in \mathbb{Z}} f_k \langle \cdot, \phi_k \rangle F \phi_k,$$

then it is easy to see that $v_0' \in W$, $F_1' \in \mathcal{Y}(f_k)$ and

$$y_r(t) = \sum_{k \in \mathbb{Z}} \langle v_0', \phi_k \rangle e^{i\omega_k t} F_1' \phi_k. \tag{3.33}$$

Analogously, for every pair $v_0' \in W$ and $F_1' \in \mathcal{Y}(f_k)$ there exists such a pair $v_0 \in \Im(f_k)$ and $F_1 \in \mathcal{B}(W, Y)$ that (3.32) and (3.33) define the same reference

signal. Similar observation holds also for the disturbance signals. It is not required that one chooses the same sequence $(f_k)_{k\in\mathbb{Z}}$ in (3.30) and (3.31). Thus, the only difference between the two approaches to control the smoothness of the reference and the disturbance signals is that the control through the reference and the disturbance operators allow one to set different smoothness properties for the reference and the disturbance signals.

In what follows, the robustness of regulation is considered under the assumption that $i\omega_k \in \rho(A_e)$ and that certain smoothness of the reference and the disturbance signals is guaranteed. In addition, it is assumed that the spectra of the exosystem is contained in the resolvent set of the perturbed state operator of the plant and that the perturbations preserve the exponential stabilizability of the plant. This extra assumption is made in order to simplify the results and does not restrict the generality much. The assumptions made define the set of allowed perturbations, or in other words, one is now able to define the set $\mathcal{V}$.

Definition 3.3.4. Let an error feedback controller (3.5) be given and fix S to be the operator in (3.3). Fix two strictly positive sequences of real numbers $(f_k)_{k\in\mathbb{Z}} \in \ell^\infty$ and $(g_k)_{k\in\mathbb{Z}} \in \ell^\infty$. The class of perturbed plants and the perturbed reference and disturbance signals $\mathcal{V} \subseteq \Omega$ is defined to be the set of all 7-tuples $(A', B', C', D', E'_1, E'_2, F'_1)$ for which

1. $F'_1, E'_2 \in \mathcal{Y}(f_k)$ and $E'_1 \in \mathcal{D}(g_k)$,

2. $i\omega_k \in \rho(A')$,

3. $i\omega_k \in \rho(A'_e)$,

4. A'_e generates a strongly stable C_0-semigroup, and

5. the Sylvester equation (3.10) is solvable.

Here A'_e is the perturbed state operator of the extended plant in (3.7).   ▲

In the above definition of $\mathcal{V}$, the perturbations were defined with respect to a controller. It would be interesting to know what kind of perturbations $\mathcal{V}$ allows in the plant. In general, the second and the third property in the above definition tolerate perturbations relatively well, especially if the perturbations preserve strong stability. The smoothness of the reference signals can be seen as a matter of choice and the smoothness of the disturbance signals may be justified by some other arguments, e.g., by physical restrictions in a modelling problem. The robustness of strong stability is out of scope of this thesis. Thus, what is done in this thesis is that one tries to identify a conditions for perturbations that preserve the solvability of the Sylvester equation.

Some preliminary results are given before proceeding. First, the resolvent $R(s, A_e) B_e$ is written in terms of the plant and the controller transfer functions, which simplifies the analysis. In $\rho(A)$,

$$sI - A_e = \begin{bmatrix} I & 0 \\ -\mathcal{G}_2 CR(s, A) & I \end{bmatrix} \begin{bmatrix} sI - A & 0 \\ 0 & sI - \mathcal{G}_1 + \mathcal{G}_2 \mathcal{P}(s)K \end{bmatrix} \begin{bmatrix} I & -R(s, A) BK \\ 0 & I \end{bmatrix}.$$

Denote $W(s) = (sI - \mathcal{G}_1 + \mathcal{G}_2 \mathcal{P}(s)K)^{-1}$. Now,

$$\begin{aligned} R(s, A_e) &= \begin{bmatrix} I & R(s, A) BK \\ 0 & I \end{bmatrix} \begin{bmatrix} R(s, A) & 0 \\ 0 & W(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ -\mathcal{G}_2 CR(s, A) & I \end{bmatrix} \\ &= \begin{bmatrix} R(s, A)(I - BKW(s)\mathcal{G}_2 CR(s, A)) & R(s, A) BKW(s) \\ -W(s)\mathcal{G}_2 CR(s, A) & W(s) \end{bmatrix}. \end{aligned} \quad (3.34)$$

Set $X = sI - \mathcal{G}_1$, $Y = \mathcal{G}_2$, $Z = I$, and $V = \mathcal{P}(s)K$ in the inversion formula (A.1) to get

$$\begin{aligned} W(s) &= R(s, \mathcal{G}_1) - R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)KR(s, \mathcal{G}_1)\mathcal{G}_2)^{-1} \mathcal{P}(s)KR(s, \mathcal{G}_1) \\ &= R(s, \mathcal{G}_1) - R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \mathcal{P}(s)KR(s, \mathcal{G}_1) \end{aligned}$$

and

$$\begin{aligned} W(s)\mathcal{G}_2 &= R(s, \mathcal{G}_1) \mathcal{G}_2 \left(I - (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \mathcal{P}(s)\mathcal{C}(s) \right) \\ &= R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1}. \end{aligned}$$

Substitute this into (3.34) to get

$$\begin{aligned} R(s, A_e) B_e &= \begin{bmatrix} R(s, A)(I - B\mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} CR(s, A)) \\ -R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} CR(s, A) \end{bmatrix} E \\ &\quad + \begin{bmatrix} R(s, A) B\mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \\ R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \end{bmatrix} F \end{aligned} \quad (3.35)$$

For the input disturbance case, one substitutes $E = BE_0$ and $F = F_1 - DE_0$ into (3.35) and gets

$$\begin{aligned} R(s, A_e) B_e &= \begin{bmatrix} R(s, A) B(I - \mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \mathcal{P}(s)) \\ -R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \mathcal{P}(s) \end{bmatrix} E_0 \\ &\quad + \begin{bmatrix} R(s, A) B\mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \\ R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \end{bmatrix} F_1. \end{aligned} \quad (3.36)$$

The last preliminary result shows that one can divide the sum condition (3.28) into two parts and analyze both of them separately.

Lemma 3.3.5. *Let X and Z be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Z$, respectively. Define the inner product of the product space $X_e = X \times Z$ by setting $\langle (x_1, z_1), (x_2, z_2) \rangle = \langle x_1, x_2 \rangle_X + \langle z_1, z_2 \rangle_Z$. Let the norms $\| \cdot \|_X$, $\| \cdot \|_Z$, and $\| \cdot \|$ be induced by $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_Z$, and $\langle \cdot, \cdot \rangle$, respectively. For a fixed sequence $((x_k, z_k))_{k \in \mathbb{Z}} \subset X_e$,*

$$\sup_{\|(x,z)\| \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \begin{bmatrix} x_k \\ z_k \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \right\rangle \right|^2 < \infty$$

if and only if

$$\begin{cases} \sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} |\langle x_k, x \rangle_X|^2 < \infty, \\ \sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} |\langle z_k, z \rangle_Z|^2 < \infty. \end{cases}$$

Proof. Necessity. By the definition of the inner product,

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} |\langle x_k, x \rangle_X|^2 = \sup_{\|(x,0)\| \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \begin{bmatrix} x_k \\ z_k \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle \right|^2$$

$$\leq \sup_{\|(x,z)\| \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \begin{bmatrix} x_k \\ z_k \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \right\rangle \right|^2 < \infty.$$

Similarly one can show, that $\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} |\langle z_k, z \rangle_Z|^2 < \infty$.

Sufficiency. By the assumptions,

$$\sup_{\|(x,z)\| \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \begin{bmatrix} x_k \\ z_k \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \right\rangle \right|^2 = \sup_{\|(x,z)\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle x_k, x \rangle_X + \langle z_k, z \rangle_Z|^2$$

$$\leq 2 \sup_{\|(x,z)\| \leq 1} \sum_{k \in \mathbb{Z}} |\langle x_k, x \rangle_X|^2 + |\langle z_k, z \rangle_Z|^2$$

$$\leq 2 \sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} |\langle x_k, x \rangle_X|^2 + 2 \sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} |\langle z_k, z \rangle_Z|^2$$

$$< \infty$$

where the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ was used. $\qquad\square$

3.3.1 Solvability of $\Sigma S + A_e \Sigma = B_e$ and Smoothness of Reference Signals

Here the relation between the solvability of the Sylvester equation (3.10) and the smoothness properties of the reference signals is considered for SISO-systems. The next theorem makes the connection between the smoothness of the reference signals, the controller and the plant apparent. Since the focus is the reference

signals, one sets $E = 0$. It is justified to consider the smoothness properties of the reference and disturbance signals separately since perturbations in the operators F and E are allowed. It is clear that the measurement disturbance and the reference signal have similar effect on the error. This is why the case with no measurement disturbance needs not to be considered separately.

Theorem 3.3.6. *Assume that* $\mathrm{i}\omega_k \in \rho(A)$ *for all* $k \in \mathbb{Z}$, $\sigma(S) \cap \sigma(A_e) = \emptyset$, *and* $U = Y = \mathbb{C}$. *Let the controller satisfy Assumption 3.3.2. Write* $K = \langle \cdot, k_0 \rangle$ *and* $\mathcal{G}_2 y = y g_2$ *where* $k_0, g_2 \in Z$. *If* $E = 0$, *then the sum condition* (3.28) *holds if and only if*

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 \left| \frac{F\phi_k}{\mathcal{P}(\mathrm{i}\omega_k)} \right|^2 < \infty, \qquad (3.37)$$

where ν_k *is the order of the pole of* $R(s, \mathcal{G}_1)$ *at* $\mathrm{i}\omega_k$.

Proof. It follows from Assumption 3.3.2 that the resolvent $R(s, \mathcal{G}_1)$ has a pole at $\mathrm{i}\omega_k$. Substituting (3.35) into (3.28) and using Lemma 3.3.5 gives the following necessary and sufficient pair of sum conditions.

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \lim_{s \to \mathrm{i}\omega_k} R(s, A) \, B\mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} F\phi_k, x \right\rangle_X \right|^2 < \infty, \quad (3.38a)$$

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \lim_{s \to \mathrm{i}\omega_k} R(s, \mathcal{G}_1) \, \mathcal{G}_2 \left(I + \mathcal{P}(s)\mathcal{C}(s)\right)^{-1} F\phi_n, z \right\rangle_Z \right|^2 < \infty. \quad (3.38b)$$

The limits above exist because $i\omega_k \in \rho(A_e)$.

The next step is to calculate the limits. Fix $k \in \mathbb{Z}$. By [70, Lemma 6.4.] there exists exactly one eigenvector z_1 at $\mathrm{i}\omega_k$. Since Z_k in the second item of Assumption 3.3.2 is finite-dimensional, it follows by [45, III-§5] that $Z_k = \mathrm{span}\{z_1, \ldots, z_{\nu_k}\}$ where z_i, $i = 1, \ldots, \nu_k$, are the generalized eigenvectors. Furthermore, in a neighborhood of $\mathrm{i}\omega_k$ the resolvent has the Laurent expansion $R(s, \mathcal{G}_1) = \sum_{i=1}^{\nu_k} \frac{1}{(s - \mathrm{i}\omega_k)^i} (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{i-1} P_k + \sum_{i=0}^{\infty} (s - \mathrm{i}\omega_k)^i R_i$ where matrices R_i are of no interest.

It is easy to see that $Z = \mathrm{span}\{z_{\nu_k}\} \oplus \mathcal{R}(\mathrm{i}\omega_k I - \mathcal{G}_1)$. For (3.9a) to hold it is necessary that $g_2 = \alpha z_{\nu_k} + z$, where $z \in \mathcal{R}(\mathrm{i}\omega_k I - \mathcal{G}_1)$ and $\alpha \neq 0$. It follows, that $(\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2 = \alpha z_1$. By [70, Lemma 6.4.], $K(\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2 = \alpha K z_1 \neq 0$.

If $\mathcal{P}(\mathrm{i}\omega_k) \neq 0$ the above arguments show that

$$\lim_{s \to \mathrm{i}\omega_k} \mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} = \frac{1}{\mathcal{P}(\mathrm{i}\omega_k)} \qquad (3.39)$$

and

$$\lim_{s \to \mathrm{i}\omega_k} R(s, \mathcal{G}_1) \, \mathcal{G}_2 \left(I + \mathcal{P}(s)\mathcal{C}(s)\right)^{-1} = \frac{(\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2}{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z \, \mathcal{P}(\mathrm{i}\omega_k)}. \qquad (3.40)$$

Note that the sum condition (3.38) holds if and only if $\mathcal{P}(i\omega_k) \neq 0$ or $F\phi_k = 0$. Since $R(i\omega_k, A)$ exists for all $k \in \mathbb{Z}$ it is seen by (3.39) and (3.40) that (3.38) can be written as

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \frac{R(i\omega_k, A) B}{\mathcal{P}(i\omega_k)} F\phi_k, x \right\rangle_X \right|^2 < \infty, \tag{3.41a}$$

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \frac{(\mathcal{G}_1 - i\omega_k I)^{\nu_k - 1} P_k g_2}{\langle (\mathcal{G}_1 - i\omega_k I)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z \mathcal{P}(i\omega_k)} F\phi_k, z \right\rangle_Z \right|^2 < \infty. \tag{3.41b}$$

The condition (3.41b) is (3.37) in a slightly different form. Thus, the claim follows if (3.41a) follows from (3.41b). Substituting $z = k_0$ gives a lower bound $\sum_{k \in \mathbb{Z}} \left| \frac{F\phi_k}{\mathcal{P}(i\omega_k)} \right|^2$ for the supremum in (3.41b). By the Cauchy-Schwarz inequality $\sum_{k \in \mathbb{Z}} \left| \frac{F\phi_k}{\mathcal{P}(i\omega_k)} \right|^2 < \infty$ is a sufficient condition for (3.41a) to hold. $\square$

The above theorem shows how the solvability of the Sylvester equation is dependent on the choice of the controller, the plant and the smoothness of the reference signals. In particular, it shows that the degree of robustness can be increased by setting stricter smoothness requirements on the reference signals.

Theorem 3.3.6 establishes a direct link between the smoothness of the signal class and the robustness of the structure of the plant transfer function at infinity. Unfortunately, as was seen in Section 2.5, an arbitrarily small perturbations may change the structure at infinity drastically, so the solvability of the Sylvester equation (3.10) has weak robustness properties.

3.3.2 Solvability of $\Sigma S + A_e \Sigma = B_e$ and Smoothness of Disturbance Signals

The required smoothness of the disturbance signals is studied next. The smoothness properties of the output reference signals are already discussed above, so only the smoothness of the state disturbance and the input disturbance are of interest. The required smoothness properties of the state disturbance signals are considered first.

Theorem 3.3.7. *Assume that $i\omega_k \in \rho(A)$ for all $k \in \mathbb{Z}$, $\sigma(S) \cap \sigma(A_e) = \emptyset$, and $U = Y = \mathbb{C}$. Let the controller satisfy Assumption 3.3.2. Write $K = \langle \cdot, k_0 \rangle$ and $\mathcal{G}_2 y = y g_2$ where $k_0, g_2 \in Z$. If $F = 0$, then the sum condition (3.28) holds if and only if*

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} |\langle R(i\omega_k, A) E\phi_k, x \rangle_X|^2 < \infty, \tag{3.42a}$$

and

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 \left| \frac{CR(i\omega_k, A) E\phi_k}{\mathcal{P}(i\omega_k)} \right|^2 < \infty \qquad (3.42b)$$

where ν_k is the order of the pole of $R(s, \mathcal{G}_1)$ at $i\omega_k$.

Proof. Substituting (3.35) into (3.28) and using Lemma 3.3.5 results in the following necessary and sufficient pair of sum conditions:

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \lim_{s \to i\omega_k} R(s, A) \Big(I - B\mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} CR(s, A) \Big) E\phi_k, x \right\rangle_X \right|^2 < \infty, \tag{3.43a}$$

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \lim_{s \to i\omega_k} R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} CR(s, A) E\phi_k, z \right\rangle_Z \right|^2 < \infty. \tag{3.43b}$$

The limits above exist because $i\omega_k \in \rho(A_e)$. Use (3.39) and (3.40) to show that (3.43) is equivalent to

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} \left| \langle R(i\omega_k, A) E\phi_k, x \rangle_X - \frac{CR(i\omega_k, A) E\phi_k}{\mathcal{P}(i\omega_k)} \langle R(i\omega_k, A) B, x \rangle_X \right|^2 < \infty, \tag{3.44a}$$

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \frac{CR(i\omega_k, A) E\phi_k}{\mathcal{P}(i\omega_k)} \right|^2 < \infty. \tag{3.44b}$$

Conditions (3.44b) and (3.42b) are the same, so it remains to show that (3.42a) and (3.44a) are equivalent if (3.42b) holds. Exponential stabilizability implies uniform boundedness of terms $\langle R(i\omega_k, A) B, x \rangle_X$. Substituting $z = k_0$ into (3.42a) shows that

$$\sup_{\|x\|_X \leq 1} \sum_{k \in \mathbb{Z}} |\langle R(i\omega_k, A) B, x \rangle_X|^2 \left| \frac{CR(s, A) E\phi_k}{\mathcal{P}(i\omega_k)} \right|^2 < \infty,$$

which in turn shows that (3.42a) and (3.44a) are equivalent provided (3.42b) holds. $\square$

Generally, the resolvent operator $R(i\omega_k, A)$ smoothens the reference signal so it is natural that the smoothness requirements for the reference signals tend to be stricter compared to those of disturbance signals. Since the disturbance operator E is not a finite-rank operator in general, the condition (3.42a) is required, and the stricter smoothness requirements for the reference signals may occur. The condition (3.42a) always holds with finite-rank disturbance operators, so for input disturbances the smoothness requirement are very mild as shown by the following theorem.

Theorem 3.3.8. *Assume that* $i\omega_k \in \rho(A)$ *for all* $k \in \mathbb{Z}$, $\sigma(S) \cap \sigma(A_e) = \emptyset$, *and* $U = Y = \mathbb{C}$. *Let the controller satisfy Assumption 3.3.2. Write* $K = \langle \cdot, k_0 \rangle$ *and* $\mathcal{G}_2 y = y g_2$, *where* $k_0, g_2 \in Z$. *If* $F_1 = 0$ *and there is only input disturbance, then the sum condition* (3.28) *holds if and only if*

$$\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 |E_0 \phi_k|^2 < \infty, \qquad (3.45)$$

where ν_k *is the order of the pole of* $R(s, \mathcal{G}_1)$ *at* $i\omega_k$.

Proof. By using (3.36), (3.39), and (3.40), one can show that

$$\lim_{s \to i\omega_k} R(s, A_e) B_e = \begin{bmatrix} 0 \\ \frac{(\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2}{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} E_0 \end{bmatrix}.$$

It follows that (3.28) holds if and only if (3.45) holds. $\qquad \square$

3.3.3   Solvability of the Robust Regulation Problem

The aim of Sections 3.3.1 and 3.3.2 was to study the type of robustness defined by the perturbation class $\mathcal{V}$ from Definition 3.3.4. In this section, the solvability of the robust regulation problem is studied.

It is sensible to assume that a controller solves the output regulation problem for all the reference and the disturbance signals that can be generated by the exosystem (3.2) with any operators $F_1, E_2 \in \mathcal{Y}(f_k)$ and $E_1 \in \mathcal{D}(g_k)$. A sufficient condition can be found by using the results from the preceding two sections.

Theorem 3.3.9. *Assume, that the pair* (A, B) *is exponentially stabilizable,* $\sigma(S) \subset \rho(A_e) \cap \rho(A)$ *and* $U = Y = \mathbb{C}$. *In addition, assume that* $(f_k)_{k \in \mathbb{Z}} \in \ell^\infty$ *and* $(g_k)_{k \in \mathbb{Z}} \in \ell^2$ *are sequences of strictly positive real numbers. Let the controller satisfy Assumption 3.3.2. Write* $K = \langle \cdot, k_0 \rangle$ *and* $\mathcal{G}_2 y = y g_2$ *where* $k_0, g_2 \in Z$. *The Sylvester equation* (3.10) *is solvable for all* $F_1, E_2 \in \mathcal{Y}(f_k)$ *and* $E_1 \in \mathcal{D}(g_k)$ *if only if for all sequences* $(\alpha_k)_{k \in \mathbb{Z}} \in \ell^2$ *of positive real numbers and all* $E' \in \mathcal{B}(W, X)$

$$\sum_{k \in \mathbb{Z}} \alpha_k^2 \left| \frac{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 \frac{f_k^2}{|\mathcal{P}(i\omega_k)|^2} < \infty \qquad (3.46a)$$

and

$$\sum_{k \in \mathbb{Z}} \alpha_k^2 \left| \frac{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - i\omega_k)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 |CR(i\omega_k, A) E' \phi_k|^2 \frac{g_k^2}{|\mathcal{P}(i\omega_k)|^2} < \infty. \quad (3.46b)$$

Proof. It is easy to see that the Sylvester equation is solvable if the conditions (3.37) and (3.42) hold simultaneously. By the exponential stabilizability of A and the assumption $\sigma(S) \subset \rho(A)$, the sum condition (3.42a) holds for all $E_1 \in \mathcal{D}(g_k)$. Since the zero operators belong to $\mathcal{Y}(f_k)$ and $\mathcal{D}(g_k)$ it is clear that it is sufficient to show that (3.46a) is a necessary and sufficient condition for (3.37) to hold for any $F = F_1 - E_2 \in \mathcal{Y}(f_k)$, and that (3.46b) is a necessary and sufficient condition for (3.42b) to hold for any $E = E_1 \in \mathcal{D}(f_k)$.

Note that $F' = \sum_{k \in \mathbb{Z}} f_k^{-1} \langle \cdot, \phi_k \rangle F \phi_k \in \mathcal{B}(W, Y)$ if $F \in \mathcal{Y}(f_k)$ and that $(H\phi_k)_{k \in \mathbb{Z}} \in \ell^2$ if and only if $H \in \mathcal{B}(W, \mathbb{C})$. By writing

$$\frac{|F\phi_k|^2}{|\mathcal{P}(\mathrm{i}\omega_k)|^2} = \frac{|F\phi_k|^2}{f_k^2} \frac{f_k^2}{|\mathcal{P}(\mathrm{i}\omega_k)|^2},$$

one sees that (3.46a) holds if and only if (3.37) holds for all $F = F_1 - E_2 \in \mathcal{Y}(f_k)$. By similar arguments, one can show that (3.46b) holds if and only if (3.42b) holds for all $E = E_1 \in \mathcal{D}(g_k)$. $\qquad \square$

The above theorem gives a condition for a controller to be regulating for all the signals with certain smoothness. The theorem shows that if one is able to construct a stabilizing controller satisfying Assumption 3.3.2 and conditions (3.46) hold, then the controller is robustly regulating in $\mathcal{V}$ from Definition 3.3.4.

Theorem 3.3.10. *Let $(f_k)_{k \in \mathbb{Z}} \in \ell^\infty$ and $(g_k)_{k \in \mathbb{Z}} \in \ell^2$ be bounded sequences of strictly positive real numbers and let $U = Y = \mathbb{C}$. If the plant* (3.1) *satisfies Assumption 3.3.1 and*

$$\sum_{k \in \mathbb{Z}} \frac{|f_k|^2 + |g_k|^2}{\sqrt{1 + |k|}|\mathcal{P}(\mathrm{i}\omega_k)|^2} < \infty, \tag{3.47}$$

the robust regulation problem is solvable for the perturbation class $\mathcal{V}$ from Definition 3.3.4.

To prove the above theorem, a robustly regulating controller needs to be constructed. This section is concluded by proposing a robustly regulating controller for a plant satisfying the assumptions of the theorem. To that end, recall the observer based robustly regulating controller from [72]. Only a minor modification is needed to make the controller design work under the current assumptions.

Consider the controller (3.5) where $Z = X \times W$, $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$,

$$\mathcal{G}_1 = \begin{bmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & S \end{bmatrix}, \text{ and } \mathcal{G}_2 = \begin{bmatrix} -L \\ G_2 \end{bmatrix}.$$

The operator L is chosen so that $A + LC$ is exponentially stable. In addition, $G_2 u = u g_2$, where g_2 is chosen so, that $\langle g_2, \phi_k \rangle \neq 0$ for all $k \in \mathbb{Z}$. Let $A + BK_{11}$

be exponentially stable. By [72, Lemma 14], the unique solution Σ_0 to the Sylvester equation

$$S\Sigma_0 = \Sigma_0(A + BK_{11}) + G_2(C + DK_{11})$$

that satisfies $\Sigma_0\left(\mathcal{D}\left(A\right)\right) \subseteq \mathcal{D}\left(S\right)$ is given by

$$\Sigma_0 v = \sum_{k \in \mathbb{Z}} \langle G_2(C + DK_{11})R\left(\mathrm{i}\omega_k, A + BK_{11}\right)v, \phi_k\rangle\phi_k. \qquad (3.48)$$

Setting $K_1 = K_{11} + K_2\Sigma_0$, where K_2 is chosen so that $S + (\Sigma_0 B + G_2 D)K_2$ is strongly stable and $\sigma\left(S\right) \cap \sigma\left((\Sigma_0 B + G_2 D)K_2\right) = \emptyset$, yields a strongly stabilizing controller that satisfies the $\mathcal{G}$-conditions [72, Theorems 12 and 13].

The problem is now reduced to choosing K_2 such that it strongly stabilizes (S, B_0), where $B_0 = \Sigma_0 B + G_2 D$. In [72], it was desired that the spectrum approaches the imaginary axis polynomially in order to get a nice polynomial bound for the resolvent of the closed loop system. This assumption is not necessary for the strong stabilizability. It is enough to find an operator K_2 such that $S + B_0 K_2$ is a Riesz spectral operator and the spectrum of $S + B_0 K_2$ is in the open left half plane $\{s \in \mathbb{C} \mid \Re(s) < 0\}$. If that is the case, then the strong stability follows by Arendt-Batty-Lyubich-Phong theorem [1, 57]. An appropriate operator K_2 can be found by using Theorem A.2.1. It is easy to verify that the hypotheses $H1 - H3$ of the theorem hold. Set

$$\mu_k = \mathrm{i}\omega_k - \frac{|f_k|^2 + |g_k|^2}{\sqrt{1 + |k|}}|\langle g_2, \phi_k\rangle|. \qquad (3.49)$$

Denote the transfer function of the plant $\Sigma\left(A + BK_{11}, B, C + DK_{11}, D\right)$ by $\mathcal{P}_K(s)$. Since $\mathrm{i}\omega_k \in \rho\left(A\right) \cap \rho\left(A + BK_{11}\right)$, and $\mathcal{P}(\mathrm{i}\omega_k) \neq 0$ by (3.47), the transfer function of the stabilized plant cannot have a transmission zero at $\mathrm{i}\omega_k$ [42, Lemma 4.3]. By using (3.48), one easily shows that $\langle B_0, \phi_k\rangle = \langle g_2, \phi_k\rangle\mathcal{P}_K(\mathrm{i}\omega_k)$. Now,

$$\sum_{k \in \mathbb{Z}} \left|\frac{\mu_k - \mathrm{i}\omega_k}{\langle B_0, \phi_k\rangle}\right|^2 = \sum_{k \in \mathbb{Z}} \frac{|f_k|^2 + |g_k|^2}{\sqrt{1 + |k|}|P_K(\mathrm{i}\omega_k)|^2} < \infty$$

by Theorem 2.4.3 and (3.47). Theorem A.2.1 is now applicable and if $K_2 = \langle \cdot, h\rangle$, where

$$h = \sum_{k \in \mathbb{Z}} h_k\phi_k, \text{ where } \overline{h}_k = \frac{\mu_k - \mathrm{i}\omega_k}{\langle B_0, \phi_k\rangle}\prod_{\substack{l \in \mathbb{Z} \\ l \neq k}}\frac{\mathrm{i}\omega_k - \mu_l}{\mathrm{i}\omega_k - \mathrm{i}\omega_l}, \qquad (3.50)$$

then $S + B_0 K_2$ is a Riesz spectral operator, has spectrum $\sigma\left(S + B_0 K_2\right) = \{\mu_k \mid k \in \mathbb{Z}\}$ and is strongly stable.

The closed loop system is now strongly stable and the controller satisfies the
$\mathcal{G}$-conditions. Thus, the controller solves the robust regulation problem if one
can show that the Sylvester equation (3.10) has a solution for all $F_1, E_2 \in \mathcal{Y}(f_k)$
and $E_1 \in \mathcal{D}(g_k)$. To this end, note that (3.37) is equivalent to (3.38b). Denote
$R_1 = A + BK_1 + L(C + DK_1)$, $R_0 = B + LD$, and $\mathcal{C}_0(s) = K_2 R(s, S) G_2$. One
has

$$
R(s, \mathcal{G}) \mathcal{G}_2 = \begin{bmatrix} -R(s, R_1) L + R(s, R_1) R_0 \mathcal{C}_0(s) \\ R(s, S) G_2 \end{bmatrix}.
$$

Furthermore, $\mathcal{C}(s) = -K_1 R(s, R_1) L + (1 + K_1 R(s, R_1) R_0) \mathcal{C}_0(s)$. Since
(R_1, R_0) is exponentially stabilizable, $K_1 R(i\omega, R_1) R_0 \to 0$ as $\omega \to \pm\infty$ by
Lemma 2.4.2. For large enough $|k|$ one has $1 + K_1 R(i\omega_k, R_1) R_0 \neq 0$ and

$$
\lim_{s \to i\omega_k} R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} = \begin{bmatrix} R(i\omega_k, R_1) R_0 \frac{(1 + K_1 R(i\omega_k, R_1) R_0)}{\mathcal{P}(i\omega_k)} \\ \frac{1}{\langle \phi_k, h \rangle \mathcal{P}(i\omega_k)} \phi_k \end{bmatrix}
$$

where h is from (3.50). Substitute this into (3.38b) and note that the discussion
above shows that $1 + K_1 R(s, R_1) R_0 =_{i\mathbb{R}} 1$. By Lemma 3.3.5, the condition
(3.38b) holds if

$$
\sup_{\|z\|_Z \leq 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle \phi_k, z \rangle}{\langle \phi_k, h \rangle \mathcal{P}(i\omega_k)} F \phi_k \right|^2 < \infty. \tag{3.51}
$$

Note that $\langle \phi_k, h \rangle = \overline{h}_k$ defined in (3.50). By using (3.49) one can write

$$
\frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} = 1 + i \frac{(|f_l|^2 + |g_l|^2)|\langle g_2, \phi_l \rangle|}{\sqrt{1 + |l|}(\omega_l - \omega_k)} = 1 + u_{kl}.
$$

Clearly $\left| \frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} \right| > 1$, so $\left| \prod_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} \right| > 1$. Apply the estimate for infinite
products from [82, Lemma 15.3] to see that

$$
\begin{aligned}
\left| \prod_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} \right| &\leq \exp\left(\sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} |u_{kl}| \right) \\
&= \exp\left(\sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{(|f_l|^2 + |g_l|^2)|\langle g_2, \phi_l \rangle|}{\sqrt{1 + |l|}|\omega_l - \omega_k|} \right) \\
&\leq \exp\left(M_0 \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{|\langle g_2, \phi_l \rangle|}{\gamma |l - k|} \right) < M < \infty
\end{aligned}
$$

where one has used estimate $0 < \frac{|f_l|^2 + |g_l|^2}{\sqrt{1+|l|}} < M_0$. The existence of a bound $M > 0$ that is independent of $k \in \mathbb{Z}$ follows by Lemma 4.3.2. This shows, that whether (3.51) holds does not depend on the infinite products in terms $\overline{h}_k$.

Substituting $\langle z, \phi_k \rangle = z_k$ and $\langle \phi_k, h \rangle = \overline{h}_k$ into (3.51) and removing the infinite sums appearing in $\overline{h}_k$, which is allowed by the above discussion, shows that (3.51) is equivalent to

$$\sup_{\sum_{k \in \mathbb{Z}} |z_k|^2 \leq 1} \sum_{k \in \mathbb{Z}} \frac{F\phi_k}{\sqrt{1+|k|}(|f_k|^2 + |g_k|^2)} \left| \frac{\mathcal{P}_K(\mathrm{i}\omega_k)}{\mathcal{P}(\mathrm{i}\omega_k)} \right|^2 |z_k|^2 < \infty.$$

Note that $\left(\frac{F\phi_k}{\sqrt{1+|k|}(|f_k|^2 + |g_k|^2)} \right)_{k \in \mathbb{Z}}$ is a bounded sequence, since $F \in \mathcal{Y}(f_k)$. The supremum above is finite since $\mathcal{P}(s) =_{\mathrm{i}\mathbb{R}} \mathcal{P}_K(s)$ by Theorem 2.4.3. Thus, (3.46a) holds. By similar arguments, (3.46a) holds. The controller is robustly regulating by Theorem 3.3.9.

Chapter 4

Regulation in the Frequency Domain

4.1 Introduction to Robust Regulation in the Frequency Domain

The robust regulation problem considered in the previous chapter has an alternative formulation in frequency domain terms. Powerful yet simple results in the frequency domain that allow simple parametrization of all robustly regulating controllers and quantitative measurement of robustness of stability in terms of a metric, for example graph metric [92], make the frequency domain approach appealing. A survey on stabilization and regulation in the frequency domain was given in [56] where several related topics including especially robustness of regulation were addressed.

The purpose of this section is introduce the robust regulation in the frequency domain and review related literature. In addition, some necessary preliminary results concerning stability are stated.

4.1.1 Stability

As was seen earlier, the stability type chosen in the robust regulation problem is of importance in the time domain. It is even more important in the frequency domain, since the plants and the controllers considered are determined by the stability type. First, one chooses a ring of stable functions $\mathbf{R}$ that is assumed to be commutative, to have no zero divisors, and to have a multiplicative identity element. The field of fractions over $\mathbf{R}$ is denoted by $\mathbf{F_R}$. The plant $\mathcal{P}$ and the controller $\mathcal{C}$ considered are the matrices in $\mathcal{M}(\mathbf{F_R})$. A transfer function is said to be $\mathbf{R}$-*stable* if it is in $\mathcal{M}(\mathbf{R})$. If the ring of stable transfer functions is clear from the context, then $\mathbf{R}$-stability is meant when speaking of stability.

There are several rings of stable transfer functions that all have special features; some are more suitable for certain problems than others [56]. The transfer functions of finite-dimensional plants are proper rational matrices. In this case, one can choose $\mathbf{R}$ to be the set of all proper rational functions with poles in some prescribed set $H \subseteq \mathbb{C}^- = \{s \in \mathbb{C} \,|\, \Re(s) < 0\}$. Then $\mathbf{F_R}$ is the set of all rational functions. For infinite-dimensional plants, commonly used rings of stable transfer functions are the rings of H^∞ and H^2 functions, i.e., the set of bounded and analytic functions in $\mathbb{C}^+$ and the Hardy space of square integrable functions. In the Callier-Desoer class of transfer functions, the stable functions $\hat{\mathcal{A}}_-$ are Laplace-transforms of a certain impulse responses, see Section A.3. The transfer functions in the Callier-Desoer algebra are of form $\frac{\hat{n}}{\hat{d}}$, where $\hat{n} \in \hat{\mathcal{A}}_-$ and $\hat{d} \in \hat{\mathcal{A}}_\infty \subsetneq \hat{\mathcal{A}}_-$, so it is a proper subset of $\mathbf{F}_{\hat{\mathcal{A}}_-}$.

Stability in the frequency domain is an input-to-output property, but stability in the time domain involves internal dynamics of the plant. Roughly speaking, the frequency domain stability is a weaker property. For finite-dimensional plants, the frequency domain stability implies stability in time domain provided that the plant is stabilizable and detectable. The result generalizes for plants with transfer functions in Callier-Desoer -algebra [14]. However, for more general transfer function classes or weaker stability types than exponential stability the connection between time domain and frequency domain stability is not so clear.

Note that if one considers only a proper subset of $\mathbf{F_R}$, one might be able to prove stronger theoretical results. For example, in the Callier-Desoer class $\mathcal{M}\left(\hat{\mathcal{B}}(\beta)\right)$ transfer functions have coprime factorizations while this is not the case for all elements in $\mathcal{M}\left(\mathbf{F}_{\hat{\mathcal{A}}_-}\right)$.

4.1.2   Coprime Factorization Approach to Stabilization

An algebraic approach to stabilization of a frequency domain plant was given in [94] where coprime factorizations were used. The assumptions made on the ring were exactly the same made in the previous section for $\mathbf{R}$ except that $\mathbf{R}$ was assumed to be a topological ring when robustness was considered. The use of coprime factorizations is a restriction, e.g., in $\mathbf{F}_{H^\infty}$ there are functions with no coprime factorization [54]. In addition, it might be hard to find a coprime factorization for a given plant even if it exists. Therefore, it would be more sophisticated to use theory that is not based on the coprime factorizations. For stability results using non-coprime factorizations approach see [66, 79] and the references therein. In this thesis, the coprime factorization approach is used in stabilization because the focus is regulation and the coprime factorization approach provides simple and strong stability results. In addition, many plants of interest have coprime factorizations, e.g. all stable plants. Results for plants

with either a right or a left coprime factorization are available [65]. A review of results from [94] is given next.

A pair of transfer functions $(N_{pr}, D_{pr}) \in \mathbf{R}^{n \times m} \times \mathbf{R}^{m \times m}$ is called *a right coprime factorization* of $\mathcal{P} \in \mathbf{F}_{\mathbf{R}}^{n \times m}$ if $\mathcal{P} = N_{pr} D_{pr}^{-1}$, $\det{(D_{pr})} \neq 0$ and (N_{pr}, D_{pr}) is a right coprime pair of matrices, i.e., there exist such matrices $X \in \mathbf{R}^{m \times n}$ and $Y \in \mathbf{R}^{m \times m}$ that

$$X N_{pr} + Y D_{pr} = I.$$

Similarly, a pair of transfer functions $(N_{pl}, D_{pl}) \in \mathbf{R}^{n \times m} \times \mathbf{R}^{n \times n}$ is *a left coprime factorization* of $\mathcal{P} \in \mathbf{F}_{\mathbf{R}}^{n \times m}$ if $\mathcal{P} = D_{pl}^{-1} N_{pl}$, $\det{(D_{pl})} \neq 0$ and (N_{pl}, D_{pl}) is a left coprime pair of matrices, i.e., there exist such matrices $X \in \mathbf{R}^{m \times n}$ and $Y \in \mathbf{R}^{n \times n}$ that

$$N_{pr} X + D_{pr} Y = I.$$

It is easy to characterize all coprime factorizations once one of them is known [92, Lemma 8.2]. A square matrix G in $\mathcal{M}(\mathbf{R})$ is said to be $\mathbf{R}$-*unimodular* if there exists a matrix $G^{-1} \in \mathcal{M}(\mathbf{R})$ such that $GG^{-1} = G^{-1}G = I$.

Lemma 4.1.1. *Let* $G \in \mathbf{F}_{\mathbf{R}}^{n \times m}$ *be given.*

1. *If* G *has a right coprime factorization* (N, D)*, then the set of all right coprime factorizations of* G *is the set*

$$\left\{ (NU, DU) \,\middle|\, U \in \mathbf{R}^{m \times m} \text{ is } \mathbf{R} - unimodular \right\}.$$

2. *If* G *has a left coprime factorization* (N, D)*, then the set of all left coprime factorizations of* G *is the set*

$$\left\{ (UN, UD) \,\middle|\, U \in \mathbf{R}^{n \times n} \text{ is } \mathbf{R} - unimodular \right\}.$$

Consider the feedback loop of Figure 1.3. The transfer function from combined vector $(\hat{d}, \hat{y}_r)$ of the disturbance and the reference signals to the combined vector $(\hat{u}, \hat{e})$ of the input and error vectors is

$$\mathcal{H}(\mathcal{P}, \mathcal{C}) = \begin{bmatrix} I - \mathcal{C}(I + \mathcal{PC})^{-1}\mathcal{P} & \mathcal{C}(I + \mathcal{PC})^{-1} \\ -(I + \mathcal{PC})^{-1}\mathcal{P} & (I + \mathcal{PC})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} (I + \mathcal{CP})^{-1} & (I + \mathcal{CP})^{-1}\mathcal{C} \\ -\mathcal{P}(I + \mathcal{CP})^{-1} & I - \mathcal{P}(I + \mathcal{CP})^{-1}\mathcal{C} \end{bmatrix}. \tag{4.1}$$

A controller $\mathcal{C}$ is said to be $\mathbf{R}$-*stabilizing*, or shortly stabilizing if the ring $\mathbf{R}$ is clear from the context, for $\mathcal{P}$ if $\mathcal{H}(\mathcal{P}, \mathcal{C})$ is $\mathbf{R}$-stable and $\det{(I + \mathcal{PC})} \neq 0$.

The following lemma is a version of [94, Lemma 3.1]. The roles of the $\mathcal{P}$ and the $\mathcal{C}$ can be changed, so the lemma also gives a necessary and sufficient condition for a plant to be stabilizable with a controller having a coprime factorization. The proof is included since it provides valuable insight into how to find a coprime factorization of a given stabilizing controller.

Lemma 4.1.2.   *1. If (N_{pr}, D_{pr}) is a right coprime factorization of a plant $\mathcal{P}$, then the following are equivalent:*

*i. The controller $\mathcal{C}$ **R**-stabilizes $\mathcal{P}$.*

ii. There exists such a left coprime factorizations (N_{cl}, D_{cl}) of $\mathcal{C}$ that $N_{cl}N_{pr} + D_{cl}D_{pr} = I$.

*iii. $\mathcal{C}$ has a left coprime factorization and $N_{cl}N_{pr} + D_{cl}D_{pr}$ is **R**-unimodular for all left coprime factorizations (N_{cl}, D_{cl}) of $\mathcal{C}$.*

2. If (N_{pl}, D_{pl}) is a left coprime factorization of a plant $\mathcal{P}$, then the following are equivalent:

*i. The controller $\mathcal{C}$ **R**-stabilizes $\mathcal{P}$.*

ii. There exists such a right coprime factorizations (N_{cr}, D_{cr}) of $\mathcal{C}$ that $N_{pl}N_{cr} + D_{pl}D_{cr} = I$.

*iii. $\mathcal{C}$ has a right coprime factorization and $N_{pl}N_{cr} + D_{pl}D_{cr}$ is **R**-unimodular for all right coprime factorizations (N_{cr}, D_{cr}) of $\mathcal{C}$.*

Proof. The proof is for a plant having a left coprime factorization. The case with a right coprime factorization is similar. By using Lemma 4.1.1, it is easy to show that the second item and the third item are equivalent. Next, it is shown that the first and the second one are equivalent.

Assume, that the second item holds. Then

$$(I + \mathcal{P}\mathcal{C})^{-1} = (D_{pl}^{-1}D_{pl}D_{cr}D_{cr}^{-1} + D_{pl}^{-1}N_{pl}N_{cr}D_{cr}^{-1}) = D_{cr}D_{pl}.$$

It follows that

$$\mathcal{H}(\mathcal{P}, \mathcal{C}) = \begin{bmatrix} I - N_{cr}N_{pl} & N_{cr}D_{pl} \\ D_{cr}N_{pl} & D_{cr}D_{cl} \end{bmatrix} \in \mathcal{M}(\mathbf{R}).$$

Thus, the second item implies the first one.

It remains to show, that the first item implies the second one. To this end, assume that $\mathcal{C}$ is stabilizing. Denote $D_{cr} = (D_{pl} + N_{pl}\mathcal{C})^{-1}$ and $N_{cr} = \mathcal{C}D_{cr}$. It is shown that (N_{cr}, D_{cr}) is the desired right coprime factorization of $\mathcal{C}$ guaranteed by the second item. Since

$$\det(D_{cr}) = \det\left((I + \mathcal{P}\mathcal{C})^{-1} D_{pl}^{-1}\right) = \det\left((I + \mathcal{P}\mathcal{C})^{-1}\right)\det(D_{pl}) \neq 0$$

the matrices D_{cr} and N_{cr} are well defined. Since

$$
\begin{aligned}
N_{pl}N_{cr} + D_{pl}D_{cr} &= D_{pl}\left(D_{pl}^{-1}N_{pl}N_{cr} + D_{cr}\right) \\
&= D_{pl}\left(\mathcal{P}\mathcal{C} + I\right)D_{cr} \\
&= D_{pl}\left(I + \mathcal{P}\mathcal{C}\right)\left(I + \mathcal{P}\mathcal{C}\right)^{-1}D_{pl}^{-1} = I,
\end{aligned}
$$

the proof is completed if the defined matrices are in $\mathcal{M}\left(\mathbf{R}\right)$.

It is shown that D_{cr} and N_{cr} are in $\mathcal{M}\left(\mathbf{R}\right)$. By stability $\left(I + \mathcal{P}\mathcal{C}\right)^{-1} = D_{cr}D_{pl} \in \mathcal{M}\left(\mathbf{R}\right)$ and $\left(I + \mathcal{P}\mathcal{C}\right)^{-1}\mathcal{P} = D_{cr}N_{pl} \in \mathcal{M}\left(\mathbf{R}\right)$. Since $\left(N_{pl}, D_{pl}\right)$ is a left coprime pair of matrices, there exist $X, Y \in \mathcal{M}\left(\mathbf{R}\right)$ such that $N_{pl}X + D_{pl}Y = I$. Thus, $D_{cr} = D_{cr}(N_{pl}X + D_{pl}Y) = D_{cr}N_{pl}X + D_{cr}D_{pl}Y \in \mathcal{M}\left(\mathbf{R}\right)$. Since $\mathcal{C}(I + \mathcal{P}\mathcal{C})^{-1} = N_{cr}D_{pl} \in \mathcal{M}\left(\mathbf{R}\right)$ and $\mathcal{C}(I + \mathcal{P}\mathcal{C})^{-1}\mathcal{P} = N_{cr}N_{pl} \in \mathcal{M}\left(\mathbf{R}\right)$, similar arguments show that $N_{cr} \in \mathcal{M}\left(\mathbf{R}\right)$. $\qquad\square$

It was seen in the proof of the previous lemma that the closed loop transfer function $\mathcal{H}\left(\mathcal{P}, \mathcal{C}\right)$ has a simple representation provided that there exists a left coprime factorization of $\mathcal{P}$. Corresponding representation can also be found if there exists a right coprime factorization. This observation is used later and is stated as a separate lemma.

Lemma 4.1.3.   *1. If $\left(N_{pr}, D_{pr}\right)$ is a right coprime factorization of a plant $\mathcal{P}$ and $\left(N_{cl}, D_{cl}\right)$ is such a left coprime factorizations of $\mathcal{C}$ that $N_{cl}N_{pr} + D_{cl}D_{pr} = I$, then*

$$
\begin{aligned}
\mathcal{H}\left(\mathcal{P}, \mathcal{C}\right) &= \begin{bmatrix} \left(I + \mathcal{C}\mathcal{P}\right)^{-1} & \left(I + \mathcal{C}\mathcal{P}\right)^{-1}\mathcal{C} \\ -\mathcal{P}\left(I + \mathcal{C}\mathcal{P}\right)^{-1} & I - \mathcal{P}\left(I + \mathcal{C}\mathcal{P}\right)^{-1}\mathcal{C} \end{bmatrix} \\
&= \begin{bmatrix} D_{pr}D_{cl} & D_{pr}N_{cl} \\ -N_{pr}D_{cl} & I - N_{pr}N_{cl} \end{bmatrix}.
\end{aligned}
$$

2. If $\left(N_{pl}, D_{pl}\right)$ is a left coprime factorization of a plant $\mathcal{P}$ and $\left(N_{cr}, D_{cr}\right)$ is such a right coprime factorizations of $\mathcal{C}$ that $N_{pl}N_{cr} + D_{pl}D_{cr} = I$, then

$$
\begin{aligned}
\mathcal{H}\left(\mathcal{P}, \mathcal{C}\right) &= \begin{bmatrix} I - \mathcal{C}\left(I + \mathcal{P}\mathcal{C}\right)^{-1}\mathcal{P} & \mathcal{C}\left(I + \mathcal{P}\mathcal{C}\right)^{-1} \\ \left(I + \mathcal{P}\mathcal{C}\right)^{-1}\mathcal{P} & \left(I + \mathcal{P}\mathcal{C}\right)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I - N_{cr}N_{pl} & N_{cr}D_{pl} \\ D_{cr}N_{pl} & D_{cr}D_{pl} \end{bmatrix}.
\end{aligned}
$$

The set of all stabilizing controllers can be parametrized in a simple manner provided that the plant possesses both right and left coprime factorizations. This lemma was presented in [94, Lemma 3.2], where [21] was cited as proof. For ease of reference, proof is provided here.

Lemma 4.1.4. *Let* (N_{pr}, D_{pr}) *and* (N_{pl}, D_{pl}) *be a right and a left coprime factorization of* $\mathcal{P}$. *Let* $X, Y, \tilde{X}, \tilde{Y} \in \mathbf{R}$ *be such, that* $XN_{pr} + YD_{pr} = I$ *and* $N_{pl}\tilde{X} + D_{pl}\tilde{Y} = I$.

1. *A controller* $\mathcal{C}$ *stabilizes* $\mathcal{P}$ *if and only if it possesses a left coprime factorization* (N_{cl}, D_{cl}) *where*

$$N_{cl} = X + RD_{pl}$$

and

$$D_{cl} = Y - RN_{pl},$$

for some $R \in \mathcal{M}(\mathbf{R})$ *for which* $\det(Y - RN_{pl}) \neq 0$.

2. *A controller* $\mathcal{C}$ *stabilizes* $\mathcal{P}$ *if and only if it possesses a right coprime factorization* (N_{cr}, D_{cr}) *where*

$$N_{cr} = \tilde{X} + D_{pr}R$$

and

$$D_{cl} = \tilde{Y} - N_{pr}R,$$

for some $R \in \mathcal{M}(\mathbf{R})$ *for which* $\det(Y - RN_{pl}) \neq 0$.

Proof. The proof is provided only for the first item since the proof of the second item is analogous. Assume first that D_{cl} possesses a left coprime factorization (N_{cl}, D_{cl}), where $N_{cl} = X + RD_{pl}$ and $D_{cl} = Y - RN_{pl}$. The controller is stabilizing by Lemma 4.1.2 since

$$(X + RD_{pl})N_{pr} + (Y - RN_{pl})D_{pr} = I + R(D_{pl}N_{pr} - N_{pl}D_{pr})$$
$$= I + RD_{pl}(\mathcal{P} - \mathcal{P})D_{pr} = I.$$

Assume that $\mathcal{C}$ is stabilizing. By Lemma 4.1.2 it has such a left coprime factorization (N_{cl}, D_{cl}) that $N_{cl}N_{pr} + D_{cl}D_{pr} = I$. Denote

$$U_1 = \begin{bmatrix} Y & X \\ -N_{pl} & D_{pl} \end{bmatrix} \text{ and } U_2 = \begin{bmatrix} D_{pr} & -\tilde{X} \\ N_{pr} & \tilde{Y} \end{bmatrix}.$$

By direct calculation

$$U_3 = U_1U_2 = \begin{bmatrix} I & R_1 \\ 0 & I \end{bmatrix},$$

where $R_1 = X\tilde{Y} - Y\tilde{X}$. It follows that $U_2^{-1} = U_3^{-1}U_1$. Since $\begin{bmatrix} D_{cl} & N_{cl} \end{bmatrix} U_2 = \begin{bmatrix} I & R_2 \end{bmatrix}$, where $R_2 = N_{cl}\tilde{Y} - D_{cl}\tilde{X}$, one has

$$\begin{bmatrix} D_{cl} & N_{cl} \end{bmatrix} = \begin{bmatrix} I & R_2 \end{bmatrix} U_2^{-1} = \begin{bmatrix} I & R \end{bmatrix} \begin{bmatrix} Y & X \\ -N_{pl} & D_{pl} \end{bmatrix},$$

where $R = R_2 - R_1$. This is exactly what was claimed. $\qquad\square$

4.1.3 Reference and Disturbance Signals

In the time domain, all reference and disturbance signals were generated by an exogenous system. In the frequency domain, it is customary that the reference and disturbance signals are generated by using transfer functions Θ_1 and Θ_2. This is to say, that the reference and the disturbance signals are

$$\hat{y}_r = \Theta_1 \hat{y}_0 \qquad \text{and} \qquad \hat{d} = \Theta_2 \hat{d}_0,$$

where $\hat{y}_0$ and $\hat{d}_0$ varies over all stable vectors of appropriate size. The simplest generator would be of form $\theta^{-1}I$ where $\theta \in \mathbf{R}$. This kind of generator was used for example in [101]. In [30, 92] robust regulation was considered for rational matrices and therefore the generator had a left coprime factorization $\Theta = D^{-1}N$.

Another approach is to simply define the sets of reference and disturbance signals. The choice of the signal class is usually motivated by certain time domain signals that can be generated by a finite-dimensional exosystem, see [33, 80]. Thus, the corresponding generator in the frequency domain is a rational matrix. An appropriate choice of generator is not evident for a more general class of signals.

4.1.4 Robust Regulation

Let $\hat{\mathcal{Y}}$ and $\hat{\mathcal{D}}$ be sets of reference and disturbance signals, respectively. Let $\mathcal{P} \in \mathcal{M}(\mathbf{F_R})$ be given. *The output regulation problem* is to find such a controller $\mathcal{C} \in \mathcal{M}(\mathbf{F_R})$, that

1. $\mathcal{C}$ stabilizes $\mathcal{P}$.

2. For all $\hat{y}_r \in \hat{\mathcal{Y}}$ and $\hat{d} \in \hat{\mathcal{D}}$

$$\hat{e} = \begin{bmatrix} (I + \mathcal{PC})^{-1}\mathcal{P} & (I + \mathcal{PC})^{-1} \end{bmatrix} \begin{bmatrix} \hat{d} \\ \hat{y}_r \end{bmatrix} \in \mathcal{M}(\mathbf{R}).$$

If a plant-controller pair $(\mathcal{P}, \mathcal{C})$ is such that the second item holds, then it is said that $\mathcal{C}$ *regulates* $\mathcal{P}$.

It is desired that a regulating controller of a given plant also solves the regulation problem for all plants near the given nominal plant. To define what being close means one must define a topology. Let a topology τ on the set of plants be given. A controller $\mathcal{C}$ is said to be robustly regulating if it solves the output regulation problem for all the plants in a neighborhood of $\mathcal{P}$.

It makes sense to set some restrictions for the topology. At least closed loop stability should be a robust property. For general topological rings, the weakest topology having this property was introduced in [94], and, for rational transfer functions, this topology is called the graph topology [92]. It is also possible to measure the degree of robustness. This is done by introducing a metric that induces the topology. There are several known metrics inducing the graph topology, e.g., Graph metric [92, 93], gap metric [22], pointwise gap metrics [78], and ν-metric [31, 95].

4.1.5   Solvability of the Robust Regulation Problem

Solvability of the robust regulation problem is well known for rational matrices with graph topology [30, 92]. Stability is a robust property in graph topology, so a stabilizing controller solves the robust regulation problem provided that it regulates every plant it stabilizes. Consequently, one can see the problem is divided into stabilization and regulation parts. The regulation part finds conditions under which regulation follows from stability, which is exactly the same characterization of robustly regulating controllers that was adopted in the time domain.

Every rational generator Θ has a left coprime factorization $D^{-1}N$. It was shown in [92] that if θ is the largest invariant factor of D, then a stabilizing $\mathcal{C}$ is robustly regulating if and only if θ divides D_{cr}, where D_{cr} is the denominator of a right coprime factorization (N_{cr}, D_{cr}) of $\mathcal{C}$. This is the frequency domain version of the internal model principle. The robust regulation problem was then shown to be solvable if and only if the pair $(\theta I, N_{pr})$ is left coprime, where N_{pr} is the numerator of a right coprime factorization (N_{pr}, D_{pr}) of $\mathcal{P}$. The set of all stabilizing controller for a plant $\mathcal{P}$ was shown to be the set of all stabilizing controllers for the plant $\theta^{-1}\mathcal{P}$.

The solvability conditions mentioned above are based on the coprime factorizations, which is an obstacle when generalizing them for more general transfer function classes. However, the theory generalize to some transfer function classes very straightforwardly, e.g., the Callier-Desoer class of transfer functions [7]. In addition, for stable transfer functions both coprime factorizations exist, and the stability results of Section 4.1.1 hold. Then it is possible to parametrize a simple robustly regulating controller for certain sets reference and disturbance

signals. This was done in [33] for stable plants in the Callier-Desoer class of transfer functions with a rational generator of form $\Theta = \theta^{-1}I$ with unstable poles on the imaginary axis. The results of [33] were later generalized for H_-^∞-stable plants with Hilbert spaces as input and output spaces and a rational generator in [80] and for H_-^∞-stable plants with a generator having an infinite number of poles on the imaginary axis in [101].

4.1.6 Robust Regulation with a Generator Having an Infinite Number of Poles

Generalizing the theory of rational transfer matrices to the H^∞-setting was discussed in Chapter 8 of [92] on a general level, but no detailed study was made. While many of the results can be generalized in a straightforward manner, there are some unavoidable problems. As mentioned, the theory relying on coprime fractions is restrictive. Another problem is that the Smith form is guaranteed to exist only in a principal ideal domain. Consequently, the invariant factors cannot be defined for all stable matrices, and the frequency domain version of internal model principle becomes superfluous with some generators.

The repetitive control system considered by Hara et al. in [35, 99] is the first attempt to tackle the robust regulation problem in the frequency domain terms with an infinite-dimensional exosystem. The main obstacle turned out to be that a controller designed for a strictly proper time domain plant cannot have an exponentially stabilizing realization. A similar restriction was encountered in [101] where the robust regulation problem with H_-^∞-stable plants and generators having an infinite number of poles on the imaginary axis was discussed. The pole locations $i\omega_k$ were assumed to have a uniform gap; therefore, they approach infinity as k approaches infinity. It was shown that there should be a uniformly bounded sequence of right inverses of $\mathcal{P}(i\omega_k)$, where $k \in \mathbb{Z}$, in order the robust regulation problem to be solvable. This means that the plant should not vanish at infinity at the zero locations. For example, if $\mathcal{P}(s) = C(sI - A)^{-1}B + D$ is the transfer function of an exponentially stabilizable time domain plant, then D should be invertible. The problem is caused by a conflict between stability type and requirement that the controller possess an internal model, which is evident for SISO-systems. Indeed, if a controller $\mathcal{C}$ is to be stabilizing for a H^∞-stable plant $\mathcal{P}$, then $\mathcal{P}N_{cr} + D_{cr} = I$. But if D_{cr} is to posses an internal model, then $D_{cr}(i\omega_k) = 0$ for all $k \in \mathbb{Z}$, so $\mathcal{P}(i\omega_k)N_{cr}(i\omega_k) = I$. Since N_{cr} is bounded, $\mathcal{P}$ should be bounded away from zero in a neighborhood of each $i\omega_k$.

4.1.7   Relation of the Time Domain and Frequency Domain Robust Regulation Problems

Some remarks on the connection between the time domain and frequency domain robust regulation problems are made before proceeding. The two problems are very similar, although the mathematical tools to tackle the problems are quite different. The theories are based on the same basic ideas, such as the internal model principle, the need for sufficiently weak concept of stability when the signals contain infinite number of unstable dynamics, and the proper characterization of robustness.

The connection is well-understood for finite-dimensional systems. In the finite-dimensional framework, the time domain robust regulation requires the closed loop transfer function have a blocking zero of suitably high order, and this should be a robust property [26]. This implies that the controller transfer function contains a frequency domain internal model of the unstable dynamics produced by the exosystem. Since the time domain stability implies stability in the frequency domain, the robustness of regulation is inherited from the time domain to the frequency domain. Consider the converse problem where a proper rational plant transfer function and a robustly regulating controller for it are given. For any realizations the regulation property is conserved since $\hat{e} \in \mathbf{R}$ implies $e \to 0$, as $t \to \infty$. Here $\mathbf{R}$ is the set of all proper rational functions not having poles in $\overline{\mathbb{C}^+}$ and e the inverse Laplace-transform of $\hat{e}$. It is well-known that the $\mathbf{R}$-stability of the transfer function guarantees exponential stability of the realization only for minimal realizations, so one has to pay attention to stability part. If one assumes minimality one also fixes the dimension of the state space, and the robustness of exponential stability and conservation of the regulation property imply robust regulation in the time domain.

For infinite-dimensional framework the situation is in general more complicated. For the Callier-Desoer class of transfer functions the connection between the time domain and frequency domain problems is much like in the finite-dimensional case [7], but in general stability issues may rise. The conditional robustness implies the blocking zero condition [49], so the controller transfer function would contain an internal model in some sense. This would mean that in certain cases the time domain robustness would be inherited to the frequency domain. When infinite-dimensional realizations are considered, there are at least two problems to deal with. First, $\hat{e} \in \mathbf{R}$ does not generally imply $e \to 0$, as $t \to \infty$, e.g., if $\mathbf{R} = H^\infty$. This may not be such a big problem from the physical point of view, e.g., if $\mathbf{R} = H^\infty_\beta$ with $\beta < 0$ this would imply fast convergence of the energy [80]. In addition, the implication can hold under certain smoothness assumptions on the signal generators [25]. A similar situation occurs in the time domain theory where certain smoothness properties were needed. Secondly, a minimal realization of $\mathbf{R}$-stable functions is not always

exponentially stable, e.g. if $\mathbf{R} = H^\infty$ [55]. However, one can restrict the class of transfer functions so that the exponential stability is implied by minimality, see [100] and the references therein, or it might be that some weaker stability type is inherited. Here one should point out that the stability properties transmitted from the frequency domain to the time domain are dependent on the time domain topology chosen [103].

4.1.8 Organization and Contributions of the Chapter

An introduction to the robust regulation in frequency domain terms and a review of related results are given in this section. The author of the thesis has written two research articles based on the theory to be presented in this chapter [48, 50]. The rest of this chapter is organized as follows:

Section 4.2 A new ring $\mathbf{P}$ of stable transfer functions is introduced to be used in the robust regulation problem. The new stability type is motivated by the polynomial stability in the time domain. The rationale behind the ring is to relax the boundedness requirement of H^∞-functions on the imaginary axis in order to allow polynomial growth there.

The motivation to use such a ring of stable transfer functions dwells from observation that in the time domain the exponential stability is too strong for the robust regulation problem to be solvable. It is seen in Section 4.6.1 and Section 4.6.2 that choosing $\mathbf{P}$-stability instead of H^∞-stability makes the robust regulation problem solvable for a wider class of transfer functions. The resulting field of fractions contains most of the fractions over H^∞. In particular, $\mathbf{P}$ contains all transfer functions from Callier-Desoer algebra $\hat{\mathcal{B}}(0)$. The ring $\mathbf{P}$ appeared for the first time in the article [49] by the author of the thesis.

Section 4.3 In this section, signals to be regulated are defined. They are Laplace transformations of the signals generated by the exosystem (3.2). Thus, a reference signal is a superposition of infinitely many sinusoidal signals. The smoothness properties of the signals are discussed in Section 4.3.1. Two different generators for the signals are introduced and compared in Section 4.3.2. The introduced generators are of simple form and have infinite number of poles on the imaginary axis.

The found generators are natural frequency domain counterparts of the exosystem (3.2) as is seen in Theorem 4.3.3 and Theorem 4.3.4. They give a concrete way of generating the desired signals. In addition, they are minimal in the sense that any other generator generating the desired signals generates a larger class of signals than the ones in question, see

Theorem 4.3.6. These results are new and unlike [101], where the generators were defined through a set of assumptions, they provide a link between the time domain and the frequency domain signals and address the minimality of the generators. The importance of the generators is revealed by the internal model principle, see Section 4.5.1.

Section 4.4 The output regulation problem and the robust output regulation problem are introduced in this section. The problem formulations are motivated by the time domain problems studied in Section 3.1.5.

Section 4.5 In this section, the solvability of the robust regulation problem is considered. It is seen in Section 4.5.1 that a robustly regulating controller should contain a generator generating the reference signals. This generalizes the frequency domain internal model principle [92] for the reference signals that are superposition of infinitely many sinusoidal signals. It is also shown that partial internal model can be sufficient for robust disturbance decoupling. In Section 4.5.2, the solvability condition given by [92, Theorem 7.5.2] for rational transfer matrices using the largest invariant factor of the generator and coprime factorization of the plant transfer function is generalized to plants in $\mathcal{M}\left(\mathbf{F_P}\right)$ and the class of signals defined in Section 4.3 with an infinite-number of poles.

Only a left or a right coprime factorization of a given plant is assumed to be known. This is important since it is not entirely sure if a $\mathbf{P}$-stablizable plant has both coprime factorizations, as is the case with H^∞-stabilizable transfer functions. Furthermore, choice $\mathbf{R} = \mathbf{P}$ plays a role only when proving the internal model principle in Section 4.5.1, so the results of Section 4.5.2 generalize to a any algebraic setting where the internal model principle holds.

Section 4.6 In this section, design of robustly regulating controllers is considered. First, a robustly regulating controller is designed for H_-^∞-stable plants in Section 4.6.1. Then a general design procedure is given in Section 4.6.2 for unstable plants with a right or a left coprime factorization. Finally, in Section 4.6.3, it is shown that the proposed controller design is well-suited the transfer functions in the Callier-Desoer class, and an example is provided.

The controller design in Section 4.6.1 is based on the work by Davison [15, 17, 18]. The Davison's controller structure was generalized to infinite-dimensional systems in [33, 80] where the signals only had a finite number of poles on the imaginary axis. A similar controller was designed in [101] for signals with infinite number of poles on the imaginary axis. In

[101], H^∞-stability was considered, and the solvability of the robust regulation problem set severe restrictions on the allowed plants. Use of the **P**-stability solves such a problem, and Theorem 4.6.5 shows that polynomial decay is allowed on the imaginary axis.

The controller design for unstable plants presented in Section 4.6.2 shows that a robustly regulating controller can be found in a very straightforward way provided that one has left or right coprime factorization and one can design a robustly regulating controller for stable plants. The design procedure combines nicely with the simple controller design of Section 4.6.1. In addition, it shows how to generalize the controller design of stable plants in [33, 80] to unstable plants.

Section 4.7 A cursory study on the meaning of the smoothness of the reference and disturbance signals is made in this section. Using examples, it illustrates that while the smoothness of the signals does not play a role in the solvability of the robust regulation problem, it might be of use if some smoothness of the error term is required.

Section 4.8 A new ring **S** of stable transfer functions is introduced and the robust regulation is considered with **S**-stability. The idea behind **S**-stability is to remove the polynomial boundedness assumption of **P**-stability on the imaginary axis.

The results of Section 4.5, including the internal model principle and the controller design of Section 4.6, are also applicable with **S**-stability. The use of **S** instead of **P** allows the design of a robustly regulating controller for a more general class of transfer functions.

The blocking zero condition of the robust regulation problem first introduced by Francis and Wonham in [26] for finite-dimensional systems is generalized here for the generator with an infinite number of poles. The blocking zero condition was later generalized for infinite-dimensional systems in time domain terms and the frequency domain counterpart of the condition was also considered with **P**-stability in [49] by the author of this thesis and his coauthors. However, the blocking zero condition was not shown to be sufficient for the robust regulation with **P**-stability. With **S**-stability it can be shown that the blocking zero condition is a sufficient condition for a controller to be robustly regulating.

4.2   P-stability

This section introduces a ring of stable transfer functions that allows polynomial growth on the imaginary axis. Choosing such a ring of stable transfer functions

instead of the more commonly used H^∞ is justified by the discussion in Section 4.1.5.

To overcome the aforementioned difficulties with solvability of the robust output regulation problem, one should relax the boundedness conditions near the poles of the reference signals. In this thesis, the reference and disturbance signals have unstable poles only on the imaginary axis, so it is sufficient to relax the boundedness conditions there. Unfortunately, there are no suitable rings of stable transfer functions readily available.

In [3, Theorem 2.4], the following two conditions were shown to be equivalent:

$$\left\| T(t)A^{-1} \right\| = \mathcal{O}\left(t^{-\frac{1}{\alpha}} \right), \ t \to \infty, \tag{4.2}$$

and

$$\left\| R\left(\mathrm{i}\omega, A \right) \right\| = \mathcal{O}\left(|\omega|^\alpha \right), \ \omega \to \infty, \tag{4.3}$$

where $\alpha > 0$ is fixed, $T(t)$ is a bounded C_0-semigroup on a Hilbert space, and A satisfying $\mathrm{i}\mathbb{R} \subset \rho\left(A \right)$ is the generator of $T(t)$. The growth condition (4.2) shows that sufficiently smooth initial states are guaranteed to vanish at a polynomial rate. Thus, certain polynomially stable systems have polynomially bounded transfer functions by (4.3). Thus, a complex function f is said to be **P**-*stable* if it satisfies the following conditions:

1. f is analytic in an open set containing $\overline{\mathbb{C}^+}$,

2. $f \in H_\beta^\infty$ for all $\beta > 0$, and

3. there exists constants $M > 0$ and $\alpha > 0$ such that $|f(\mathrm{i}\omega)| \le M \left(1 + |\omega| \right)^\alpha$.

The set of all **P**-stable functions is denoted **P**. It is a matter of an elementary exercise to show that **P** together with the pointwise addition and multiplication forms an integral domain with an identity. The following lemma establishes relations between $\mathbf{F_P}$ and some classes of transfer functions that frequently appear in the control theory literature.

Lemma 4.2.1. *One has the relations*

1. $\mathbb{C}_{pr}(s) \subset \mathbf{F_P}$,

2. $\hat{\mathcal{B}}(0) \subset \mathbf{F_P}$,

3. $\mathbf{F}_{H^\infty} \not\subset \mathbf{F_P}$,

4. $\{ f \in \mathbf{F}_{H^\infty} \,|\, f \text{ is analytic on } \mathrm{i}\mathbb{R} \} \subset \mathbf{F_P}$, *and*

5. $\mathbf{F}_{H_-^\infty} \subset \mathbf{F_P}$

Proof. The set of proper rational functions can be presented as a fraction of two proper rational functions with no poles in $\overline{\mathbb{C}^+}$ from which the first item follows. The second item follows by noting that $\hat{f} \in \hat{\mathcal{A}}_-(0)$ is bounded and holomorphic in $\overline{\mathbb{C}^+}$ so $\hat{\mathcal{A}}_\infty(0) \subset \hat{\mathcal{A}}_-(0) \subset \mathbf{P}$. The third item holds since a H^∞ function need not be analytic on the imaginary axis. A function f in $\{f \in \mathbf{F}_{H^\infty} \mid f \text{ is analytic on } i\mathbb{R}\}$ is a fraction of two H^∞ functions that are analytic on the imaginary axis, which shows the fourth item. The fifth item follows from the fourth one. $\square$

4.3 The Reference and Disturbance Signals

The reference and disturbance signals considered in this chapter are generated by a diagonal exosystem defined in (3.2). To make the frequency domain analysis possible, the Laplace transforms of the time-domain reference and disturbance signals are considered. Note that in the frequency domain the plant is subject to input disturbance.

Laplace transforming the reference and disturbance signals in (3.2) gives

$$\hat{y}_r(s) = F_1(sI - S)^{-1}v_0 = \sum_{k \in \mathbb{Z}} \langle v_0, \phi_k \rangle \frac{1}{s - i\omega_k} F_1 \phi_k \tag{4.4a}$$

and

$$\hat{d}(s) = \hat{d}_0(s) = E_0(sI - S)^{-1}v_0 = \sum_{k \in \mathbb{Z}} \langle v_0, \phi_k \rangle \frac{1}{s - i\omega_k} E_0 \phi_k \tag{4.4b}$$

The following standing assumption is made:

Assumption 4.3.1. *For all $k \in \mathbb{Z}$ terms $F_1 \phi_k \neq 0$.*

The assumption is made to guarantee that all the terms corresponding to any mode $i\omega_k$ are present in a reference signal for some initial state v_0. This is important when formulating the frequency domain version of the internal model principle.

Rewriting the signals (4.4) yields

$$\hat{y}_r(s) = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - i\omega_k} a_k \tag{4.5a}$$

and

$$\hat{d}(s) = \sum_{k \in \mathbb{Z}} \frac{\beta_k}{s - i\omega_k} b_k \tag{4.5b}$$

where $\alpha_k = \langle v_0, \phi_k \rangle \, \| F_1 \phi_k \|$, $a_k = \frac{F_1 \phi_k}{\|F_1 \phi_k\|}$ and $\beta_k = \langle v_0, \phi_k \rangle \, \| E_0 \phi_k \|$. If $E_0 \phi_k \neq 0$, then $b_k = \frac{E_0 \phi_k}{\|E_0 \phi_k\|}$. Otherwise $b_k = 0$.

In this thesis, robust regulation is studied in two separate cases. In the first case, F_1 and E_0 are allowed to vary arbitrarily. The set of reference (disturbance) signals in this class of signals is denoted by $\hat{\mathcal{Y}}_0$ ($\hat{\mathcal{D}}_0$). It is clear that there exists such F_1 that Assumption 4.3.1 is satisfied. This is why the above assumption does not reduce the generality. In the second case, F_1 and E_0 are assumed to be fixed. Assumption 4.3.1 is crucial in that case.

4.3.1   Smoothness of the Reference and Disturbance Signals

In the time domain regulation the smoothness of the reference and disturbance signals becomes crucial. There are two ways to control the smoothness of the reference and the disturbance signals, either by conditions on the allowed initial states of the exosystem [68, 70] or by conditions on the operators F and E [34].

By setting conditions on the allowed set of initial sates v_0, one can affect the rate at which $|\langle v_0, \phi_k \rangle|$ approaches zero as $|k|$ approaches zero. Conditions on F and E would affect the rate at which $\|F\phi_k\|$ and $\|E\phi_k\|$ approach zero as $|k|$ approaches infinity. Both approaches forces the terms α_k and β_k to approach zero at some rate. However, the latter approach allows different smoothness properties of reference and disturbance signals to be set while the first one does not. This is why the latter approach is adopted here.

In the frequency domain, *the smoothness* of reference and disturbance signals in (4.5) means the rate at which α_k and β_k vanish as $|k|$ approaches infinity. Consider the size of the term $|\alpha_k|$ in (4.5a). The sequence $(|\langle v_0, \phi_k \rangle|)_{k \in \mathbb{Z}}$ is square summable since $\{\phi_k \mid k \in \mathbb{Z}\}$ is an orthonormal basis. The sequence $(\|F\phi_k\|)_{k \in \mathbb{Z}}$ is square summable since F is a linear operator from an infinite-dimensional space to a finite-dimensional space. It follows that $\sum_{k \in \mathbb{Z}} |\alpha_k| < \infty$. Similar justifications show that $\sum_{k \in \mathbb{Z}} |\beta_k| < \infty$. Thus, $(\alpha_k)_{k \in \mathbb{Z}}$ and $(\beta_k)_{k \in \mathbb{Z}}$ are at least absolutely summable. However, the signals in (4.5) are well defined as frequency domain signals if $(\alpha_k)_{k \in \mathbb{Z}}$ and $(\beta_k)_{k \in \mathbb{Z}}$ are square summable, because there is a uniform gap between the poles $\mathrm{i}\omega_k$ which implies that $\left(\frac{1}{s - \mathrm{i}\omega_k} \right)_{k \in \mathbb{Z}}$ is square summable for all $s \notin \{\mathrm{i}\omega_k \mid k \in \mathbb{Z}\}$.

In this thesis, the smoothness of the reference and disturbance signals are controlled by a sequence $(f_k)_{k \in \mathbb{Z}} \in \ell^2$ of strictly positive real numbers. The following sets of reference and disturbance signals with respect to $(f_k)_{k \in \mathbb{Z}}$ are defined:

$$\hat{\mathcal{Y}}(f_k) = \{\hat{y}_r \text{ in } (4.5a) \mid a_k \in \mathbb{C}^n, \|a_k\| = 1, \exists M > 0 : \alpha_k \leq M f_k\} \qquad (4.6a)$$

and

$$\hat{\mathcal{D}}\left(f_k\right) = \left\{\hat{d} \text{ in (4.5b)} \,\middle|\, b_k \in \mathbb{C}^m, \|b_k\| = 1, \exists M > 0 : \beta_k \leq M f_k\right\}. \qquad (4.6b)$$

The above sets of signals corresponds to the case where F_1 and E_0 subject to arbitrary perturbations. This means that a_k and b_k can have arbitrary directions. If F_1 and E_0 are fixed, then a_k and b_k are fixed. In that case the following two sets are defined

$$\hat{\mathcal{Y}}\left(f_k, a_k\right) = \left\{\hat{y}_r \text{ in (4.5a)} \,\middle|\, \exists M > 0 : \alpha_k \leq M f_k\right\} \qquad (4.7a)$$

and

$$\hat{\mathcal{D}}\left(f_k, b_k\right) = \left\{\hat{d} \text{ in (4.5b)} \,\middle|\, \exists M > 0 : \beta_k \leq M f_k\right\}. \qquad (4.7b)$$

Note that the sequence $(f_k)_{k \in \mathbb{Z}}$ does not need to be the same in (4.6a) and (4.6b) (or in (4.7a) and (4.7b)), because one can choose different smoothness properties for the reference and disturbance signals independently of each other.

To justify the requirement that $(f_k)_{k \in \mathbb{Z}}$ is square summable, note that if one chooses $f_k = \frac{1}{1+|k|}$ say, then the sets (4.6) contains all the Laplace transforms of the signals generated by the diagonal exosystem (3.2). Furthermore, if $(f_k)_{k \in \mathbb{Z}}$ is some absolutely summable sequence, then there exist exosystems of the form (3.2) that generate reference signals not in $\hat{\mathcal{Y}}\left(f_k\right)$.

4.3.2 Generators for the Reference and Disturbance Signals

Consider a rational signal generator Θ, and let (N, D) be its left coprime factorization. The internal model principle can be formulated for rational transfer functions by using the largest invariant factor θ of D. The element θ has the property that the generator $\theta^{-1} I$ can generate all the signals the generator Θ can. In addition, θ is minimal in the sense that, if θ_0 is an element with the same property, then θ divides θ_0.

The purpose of this section is to find an generating element θ resembling the largest invariant factor, i.e., the generator of form $\theta^{-1} I$ needs to generate the signals in the signal classes from the previous section, and it needs to be minimal in the afore mentioned sense. In what follows it is seen that the internal model principle can be formulated by using such an element even if there is no concrete signal generator given. A repeatedly used technical lemma is given before proceeding.

Lemma 4.3.2. *Consider a series $\sum_{k \in Z} \frac{A_k}{s - i\omega_k}$, where $A_k \in \mathcal{M}\left(\mathbb{C}\right)$. Let $\omega_k \in \mathbb{R}$ be such that $\omega_{k+1} - \omega_k > 4\gamma$ for a constant $\gamma > 0$. If $\|A_k\| \in \ell^2$, then*

1.

$$\left\| \sum_{k \in \mathbb{Z}} \frac{A_k}{s - \mathrm{i}\omega_k} \right\| \leq \sqrt{\sum_{k \in Z} \|A_k\|^2} \sqrt{\sum_{n=1}^{\infty} \frac{2}{\gamma^2 n^2}} < \infty,$$

2.

$$\left\| \sum_{k \in \mathbb{Z}} \frac{1}{(s - \mathrm{i}\omega_k)(s - \mathrm{i}\omega_k + \epsilon)} \right\| \leq \sum_{n=1}^{\infty} \frac{2}{\gamma^2 n^2} < \infty$$

for all $s \in \overline{\mathbb{C}^+} \setminus \bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$ and $\epsilon > 0$.

Proof. Fix $s \in \mathbb{C}$, and denote the integer for which $\min_{k \in \mathbb{Z}}\{|s - \omega_k|\}$ is attained by l. By assumptions, $|s - \mathrm{i}\omega_k| > (|k - l| + 1)\gamma$ for all $k \neq l$. If $|s - \mathrm{i}\omega_k| > \gamma$, then by the Cauchy-Schwarz inequality

$$\left\| \sum_{k \in Z} \frac{A_k}{s - \mathrm{i}\omega_k} \right\| \leq \sum_{k \in \mathbb{Z}} \frac{\|A_k\|}{|s - \mathrm{i}\omega_k|}$$

$$\leq \sum_{k \in Z} \frac{\|A_k\|}{\gamma(|k - l| + 1)}$$

$$\sqrt{\leq \sum_{k \in Z} \|A_k\|^2} \sqrt{\sum_{n=1}^{\infty} \frac{2}{\gamma^2 n^2}}.$$

The second item follows from the first one by noting that $|s - \mathrm{i}\omega_k| > |s - \mathrm{i}\omega_k + \epsilon|$ in $\overline{\mathbb{C}}_+ \setminus \bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$. $\square$

In the next theorem, an element θ such that the signal generator $\theta^{-1}I$ generates all the signals of interest is presented. It is called *a generating element*. It becomes handy when parameterizing the robustly regulating controllers and giving solvability conditions for regulation problems.

Theorem 4.3.3. *Let $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ be a sequence of strictly positive real numbers in ℓ^2. Define*

$$\theta(s) = \left(1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - \mathrm{i}\omega_k} \right)^{-1}, \tag{4.8}$$

where the real numbers ω_k satisfy $\omega_{k+1} - \omega_k > 4\gamma$ for some fixed $\gamma > 0$ and for all $k \in Z$. Denote $\Phi_{1k} = (1 + \epsilon_1 \frac{h_k}{s - \mathrm{i}\omega_k})^{-1}$ and $\Phi_{2k}(s) = \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{h_l}{s - \mathrm{i}\omega_l}$. The constant $\epsilon_1 > 0$ is chosen so that

$$\epsilon_1 \|\Phi_{1k}(s)\Phi_{2k}(s)\| < \frac{1}{2} \tag{4.9}$$

for all $k \in \mathbb{Z}$ and $s \in B_\gamma(\mathrm{i}\omega_k) \cap \overline{\mathbb{C}^+}$.

1. *There exists a choice of ϵ_1 such that (4.9) is satisfied.*

2. *θ is in H^∞ and is analytic in an open set containing $\overline{\mathbb{C}^+}$.*

3. *If (f_k) is a sequence of strictly positive real numbers such that $f_k \leq M_0 h_k$ for some $M_0 > 0$ and for all $k \in Z$, then the signals in $\hat{\mathcal{Y}}(f_k)$ and $\hat{\mathcal{D}}(f_k)$ are of the form $\theta^{-1} w_0$, where w_0 is a H^∞-vector that is analytic in an open set containing $\overline{\mathbb{C}^+}$.*

Proof. If $\Re(s) > 0$, then $\Re\left(\frac{h_k}{s - \mathrm{i}\omega_k}\right) > 0$. Thus, $\left|1 + \epsilon_1 \frac{h_k}{s - \mathrm{i}\omega_k}\right| > 1$, and $\|\Phi_{1k}(s)\| < 1$ in $B_\gamma(\mathrm{i}\omega) \cap \overline{\mathbb{C}^+}$. There exists $M > 0$ such that $|\Phi_{2k}(s)| < M$ for all $s \in B_\gamma(\mathrm{i}\omega) \cap \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$ by Lemma 4.3.2. Thus, choosing $\epsilon_1 = \frac{1}{2M}$ completes the proof of the first item.

To prove the second item note that $1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - \mathrm{i}\omega_k}$ is analytic everywhere except at $\mathrm{i}\omega_k$ where $k \in \mathbb{Z}$. It is clear that $\theta(s)$ is analytic in $\overline{\mathbb{C}^+}$ provided that it is bounded there. Since $\Re\left(\frac{h_k}{s - \mathrm{i}\omega_k}\right) > 0$ whenever $\Re(s) > 0$, it follows that $\left|1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - \mathrm{i}\omega_k}\right| > 1$. This in turn implies the boundedness of θ.

The third item is proved by showing that the claim holds for an arbitrary $\hat{y}_r \in \hat{\mathcal{Y}}(f_k)$. The proof for disturbance signals is identical. Write $\hat{y}_r(s) = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - \mathrm{i}\omega_k} a_k$. One needs to show that $w_0 = \theta \hat{y}_r$ is in H^∞, and that w_0 is analytic in an open set containing $\overline{\mathbb{C}^+}$.

Again, w_0 is analytic, if it is bounded. Denote $U = \bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$. There exists $M_1 > 0$ such that $\|\hat{y}_r\| < M_1$ for all $s \in \overline{\mathbb{C}^+} \setminus U$ by Lemma 4.3.2. Since $\|\theta(s)\| < 1$ in $\overline{\mathbb{C}^+}$ it follows that $\|w_0(s)\| < M_1$ for all $s \in \overline{\mathbb{C}^+} \setminus U$.

The claim follows if one can show that w_0 is bounded in U. To this end, fix $k \in \mathbb{Z}$. Using the decomposition $\theta = \Phi_{1k}(1 + \epsilon_1 \Phi_{1k}(s)\Phi_{2k}(s))^{-1}$ one can write

$$w_0(s) = \left(\Phi_{1k}(s)\frac{\alpha_k}{s - \mathrm{i}\omega_k}a_k + \Phi_{1k}(s)\sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\alpha_l}{s - \mathrm{i}\omega_l}a_l\right)(1 + \epsilon_1 \Phi_{1k}(s)\Phi_{2k}(s))^{-1}.$$

(4.10)

The inequality (4.9) shows that $|(1 + \epsilon_1 \Phi_{1k}(s)\Phi_{2k}(s))^{-1}| < 2$ for all $s \in B_\gamma(\mathrm{i}\omega_k)$. It is also easy to verify that $\left\|\sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\alpha_l}{s - \mathrm{i}\omega_l}a_l\right\| < M_1$ in $B_\gamma(\mathrm{i}\omega_k)$.

Since $|\alpha_k| \leq M f_k \leq M M_0 h_k$ for some $M, M_0 > 0$ independent of $k \in \mathbb{Z}$, and $\|a_k\| = 1$,

$$\left\|\Phi_{1k}(s)\frac{\alpha_k}{s - \mathrm{i}\omega_k}a_k\right\| = \frac{|\alpha_k|}{|s - \mathrm{i}\omega_k + \epsilon_1 h_k|} \leq M M_0 \epsilon_1^{-1}$$

for all $s \in \overline{\mathbb{C}^+} \cap B_\gamma(\mathrm{i}\omega_k)$. By (4.10),

$$\|w_0\| < \left(M M_0 \epsilon_1^{-1} + M_1\right)2$$

in $s \in \overline{\mathbb{C}^+} \cap B_\gamma(\mathrm{i}\omega_k)$. Since ϵ_1, M, M_0 and M_1 are independent of $k \in \mathbb{Z}$, w_0 is bounded in U, which completes the proof. $\square$

In the above generating element a sequence $(h_k)_{k\in\mathbb{Z}}$ must be fixed. A generating element with no such restriction is proposed below.

Theorem 4.3.4. *Define*

$$\theta(s) = \left(1 + \sum_{k\in\mathbb{Z}} \frac{\epsilon_2^2}{(s - \mathrm{i}\omega_k)(s - \mathrm{i}\omega_k + \epsilon_2)}\right)^{-1}, \qquad (4.11)$$

where the real numbers ω_k satisfy $\omega_{k+1} - \omega_k > 4\gamma$ for some fixed $\gamma > 0$ and for all $k \in \mathbb{Z}$. Denote $\Phi_{1k}(s) = \left(1 + \frac{\epsilon_2^2}{(s-\mathrm{i}\omega_k)(s-\mathrm{i}\omega_k+\epsilon_2)}\right)^{-1}$ and $\Phi_{2k}(s) = \sum_{\substack{l\in\mathbb{Z}\\l\neq k}} \frac{\epsilon_2}{(s-\mathrm{i}\omega_l)(s-\mathrm{i}\omega_l+\epsilon_2)}$. The constant $\epsilon_2 > 0$ is chosen so that

$$\epsilon_2^2 \left\| \sum_{k\in\mathbb{Z}} \frac{1}{(s - \mathrm{i}\omega_k)(s - \mathrm{i}\omega_k + \epsilon_2)} \right\| < \frac{1}{2} \qquad (4.12)$$

for all $s \in \overline{\mathbb{C}^+} \setminus \bigcup_{k\in\mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$, and

$$\epsilon_2 \|\Phi_{1k}(s)\Phi_{2k}(s)\| < \frac{1}{2} \qquad (4.13)$$

for all $s \in B_\gamma(\mathrm{i}\omega_k) \cap \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$.

1. *There exists a choice of ϵ_2 such that (4.12) and (4.13) hold.*

2. *θ is in H^∞ and is analytic in an open set containing $\overline{\mathbb{C}^+}$.*

3. *If $(f_k) \in \ell^2$, then all the signals in $\hat{\mathcal{Y}}(f_k)$ and $\hat{\mathcal{D}}(f_k)$ are of the form $\theta^{-1}w_0$, where w_0 is a H^∞-vector that is analytic in an open set containing $\overline{\mathbb{C}^+}$.*

Proof. Denote $U = \bigcup_{k\in\mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$. By Lemma 4.3.2, there exists a constant $M > 0$ such that $\left|\sum_{k\in\mathbb{Z}} \frac{1}{(s-\mathrm{i}\omega_k)(s-\mathrm{i}\omega_k+\epsilon_2)}\right| < M$ for all $s \in \overline{\mathbb{C}^+} \setminus U$. Thus, the equation (4.12) holds if $\epsilon_2 < \frac{1}{\sqrt{2M}}$ and the first item is shown if for small enough $\epsilon_2 > 0$ equation (4.13) holds. To this end, set $z = \frac{s - \mathrm{i}\omega_k}{\epsilon_2}$. Now

$$\Phi_{1k}(s) = \left(1 + \frac{1}{z(z+1)}\right)^{-1} = \frac{z(z+1)}{z^2 + z + 1}.$$

Since the poles of $z^2 + z + 1$ have real part $-\frac{1}{2}$ and the rational function $\frac{z(z+1)}{z^2+z+1}$ is strictly proper, $M_1 = \sup_{z\in\overline{\mathbb{C}^+}} \left\{\left|\frac{z(z+1)}{z^2+z+1}\right|\right\} < \infty$. The function Φ_{1k} is bounded in $B_\gamma(\mathrm{i}\omega_k)$ by a constant $M_1 > 0$ independent of k since $\Re(s) \geq 0$ implies $\Re(z) \geq 0$.

The existence of $M_2 > 0$ such that $|\Phi_{2k}(s)| < M_2$ for all $s \in B_\gamma(i\omega) \cap \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$ can be proved by using Lemma 4.3.2. The choice $\epsilon_2 < \min\left\{\frac{1}{\sqrt{2M}}, \frac{1}{2M_1 M_2}\right\}$ completes the proof of the first item.

Boundedness of θ in $\overline{\mathbb{C}^+} \setminus U$ follows by (4.12). Since the boundedness of θ in $\overline{C^+}$ implies that θ is analytic in $\overline{\mathbb{C}^+}$, the second item follows if one is able to show the boundedness of θ in U. Observe that $\theta = \Phi_{1k} \left(1 + \epsilon_2 \Phi_{1k} \Phi_{2k}\right)^{-1}$. Since Φ_{1k} is bounded in $B_\gamma(i\omega_k)$ by a constant independent of $k \in \mathbb{Z}$, the boundedness of θ in U follows by (4.13).

The proof of the third item is analogous to the proof of the third item in Theorem 4.3.3. $\qquad\square$

The two generators above are very similar, but in the first one a choice of a sequence must be made. Such a choice naturally affects the class of signals the generator can produce. A natural question arises whether the sets of signals the above signal generators can produce are different. The next theorem answers this question.

If θ is an element in $\mathbf{R}$ denote the set of signals θ^{-1} generates by $\theta^{-1}\mathbf{R} = \{\theta^{-1}w_0 \,|\, w_0 \in \mathbf{R}\}$. Before proceeding to the theorem, a simple lemma is given that illustrates the relation between the generators and the sets of the reference signals they generate.

Lemma 4.3.5. *For $\theta_1, \theta_2 \in \mathbf{R}$, $\theta_1^{-1}\mathbf{R} \subseteq \theta_2^{-1}\mathbf{R}$ if and only if $\theta_2\theta_1^{-1} \in \mathbf{R}$.*

Proof. Necessity. If $\theta_1^{-1}\mathbf{R} \subseteq \theta_2^{-1}\mathbf{R}$, then there exists $w_0 \in \mathbf{R}$ such that

$$\theta_2\theta_1^{-1} = \theta_2(\theta_1^{-1} \cdot 1) = \theta_2(\theta_2^{-1}w_0) = w_0 \in \mathbf{R}.$$

Sufficiency. Write $\theta_1^{-1}w_0 = \theta_2^{-1}\theta_2\theta_1^{-1}w_0 = \theta_2^{-1}v_0$ for a $w_0 \in \mathbf{R}$, where $v_0 = \theta_2\theta_1^{-1}w_0$. If $\theta_2\theta_1^{-1} \in \mathbf{R}$, then $v_0 \in \mathbf{R}$. Thus, $\theta_1^{-1}\mathbf{R} \subseteq \theta_2^{-1}\mathbf{R}$. $\qquad\square$

Theorem 4.3.6. *Fix $(h_k)_{k\in\mathbb{Z}} \in \ell^2$, and denote the functions (4.8) and (4.11) by θ_1 and θ_2, respectively. Assume that $\omega_k \in \mathbb{R}$ satisfy $\omega_{k+1} - \omega_k > 4\gamma > 0$ for some fixed γ and all $k \in \mathbb{Z}$. With these assumptions the following are true.*

1. *$\theta_1^{-1}H^\infty \subsetneq \theta_2^{-1}H^\infty$,*

2. *If $h_k > M(|\omega_k| + 1)^{-\alpha}$ for some fixed $\alpha, M > 0$ and for all $k \in \mathbb{Z}$, then $\theta_1^{-1}\mathbf{P} = \theta_2^{-1}\mathbf{P}$.*

3. *If there exist no $\alpha > 0$ and $M > 0$ such that $h_k > M(|\omega_k| + 1)^{-\alpha}$ for all $k \in \mathbb{Z}$, then $\theta_1^{-1}\mathbf{P} \subsetneq \theta_2^{-1}\mathbf{P}$.*

4. *If $\theta \in \mathbf{P}$ generates all the signals in (4.6a), where $f_k > M(|\omega_k| + 1)^{-\alpha}$ for some fixed constants $M, \alpha > 0$ and for all $k \in \mathbb{Z}$, then $\theta = \theta_1 w_0 = \theta_2 v_0$ where $w_0, v_0 \in \mathbf{P}$.*

Proof. Theorem 4.3.4 shows that there exists such $w_0 \in H^\infty$ that $\theta_2^{-1} w_0 = \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - \mathrm{i}\omega_k}$. Thus, $\theta_2^{-1}(\theta_2 + w_0) = 1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - \mathrm{i}\omega_k} = \theta_1^{-1}$. By Theorem 4.3.4, $\theta_2 \in H^\infty$. Thus, $\theta_2 \theta_1^{-1} \in H^\infty$. Lemma 4.3.5 shows that $\theta_1^{-1} H^\infty \subseteq \theta_2^{-1} H^\infty$. Since w_0 and θ_2 are analytic on the imaginary axis $\theta_1^{-1} \mathbf{P} \subseteq \theta_2^{-1} \mathbf{P}$.

Next it is shown that the inclusions in the first and third items are proper, and that the sets are equal in the second item. Consider $\theta_1 \theta_2^{-1}$. Note that it is uniformly bounded in $\overline{\mathbb{C}^+} \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$. Decompose $\theta_1 = \Phi_{1k} (1 + \epsilon_1 \Phi_{1k} \Phi_{2k})^{-1}$ where Φ_{1k} and Φ_{2k} are the functions defined in Theorem 4.3.3. By the choice of ϵ_1, one has $\left|(1 + \epsilon_1 \Phi_{1k} \Phi_{2k})^{-1}\right| > \frac{1}{2}$ for all $s \in \overline{\mathbb{C}^+} \cap B_\gamma(\mathrm{i}\omega_k)$. Thus,

$$\frac{1}{2} \left\| \Phi_{1k}(s) \left(1 + \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\epsilon_2^2}{(s - \mathrm{i}\omega_l)(s - \mathrm{i}\omega_l + \epsilon_2)} \right) \right\| - \left| \Phi_{1k}(s) \frac{\epsilon_2^2}{(s - \mathrm{i}\omega_k)(s - \mathrm{i}\omega_k + \epsilon_2)} \right| \right\|$$

$$\leq \left| \theta_1(s) \theta_2^{-1}(s) \right| \leq \frac{1}{2} \left| \Phi_{1k}(s) \theta_2^{-1}(s) \right|. \tag{4.14}$$

The supremum of $\left| \Phi_{1k}(s) \frac{\epsilon_2^2}{(s - \mathrm{i}\omega_k)(s - \mathrm{i}\omega_k + \epsilon_2)} \right|$ over $\overline{\mathbb{C}^+} \cap B_\gamma(\mathrm{i}\omega_k)$ is $\frac{\epsilon_2}{\epsilon_1 h_k}$. By the arguments presented in the proofs of Theorem 4.3.3 and Theorem 4.3.4, $\left| \Phi_{1k}(s) \left(1 + \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\epsilon_2^2}{(s - \mathrm{i}\omega_l)(s - \mathrm{i}\omega_l + \epsilon_2)} \right) \right|$ is bounded by a constant independent of k. Thus, $\theta_1(s) \theta_2^{-1}(s) \notin H^\infty$. The first item holds by Lemma 4.3.5.

If there exists no $\alpha > 0, M$ such that $h_k > M(|\omega_k| + 1)^{-\alpha}$, then there exists no $\alpha > 0, M$ such that $\frac{1}{h_k} < M^{-1}(|\omega_k| + 1)^\alpha$. The supremum of $\left| \Phi_{1k} \frac{\epsilon_2^2}{(s - \mathrm{i}\omega_k)(s - \mathrm{i}\omega_k + \epsilon_2)} \right|$ is attained at $\mathrm{i}\omega_k$ and is $\frac{\epsilon_2}{\epsilon_1 h_k}$. It follows that $\theta_1(s) \theta_2^{-1}(s) \notin \mathbf{P}$. The third item follows by Lemma 4.3.5.

To prove the second item, assume that $h_k > M(|\omega_k| + 1)^{-\alpha}$ for some $\alpha, M > 0$. The uniform boundedness of θ_2^{-1} in $\overline{\mathbb{C}}_+ \setminus \bigcup_{k \in \mathbb{Z}} B_a(\mathrm{i}\omega_k)$ for all $a > 0$ follows by Lemma 4.3.2. Furthermore, $\theta_2^{-1} \theta_1 \in H_\beta^\infty$ for all $\beta > 0$ since $\theta_1 \in H^\infty$. It remains to show that $\theta_2^{-1} \theta$ is polynomially bounded in $\bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$. The first absolute value on the first line of (4.14) is uniformly bounded in $\bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$. The second absolute value on the first line of (4.14) is bounded by $M(|\omega_k| + 1)^{-\alpha}$ in $B_\gamma(\mathrm{i}\omega_k)$ since its supremum in $B_\gamma(\mathrm{i}\omega_k)$ is $\frac{\epsilon_2}{\epsilon_1 h_k}$. The polynomial boundedness on the imaginary axis follows since there is a uniform gap between the real numbers ω_k where $k \in \mathbb{Z}$. Thus, $\theta_1 \theta_2^{-1} \in \mathbf{P}$. The second item holds by Lemma 4.3.5

The fourth item is shown next. Assume, that $\theta \in \mathbf{P}$ generates all the signals in (4.6a). In particular, this means that there exists $w_0 \in \mathbf{P}$ satisfying $\theta^{-1} w_0 = \sum_{k \in \mathbb{Z}} \frac{\epsilon f_k}{s - \mathrm{i}\omega_k}$. This implies that $\theta^{-1}(\theta + w_0) = 1 + \sum_{k \in \mathbb{Z}} \frac{\epsilon f_k}{s - \mathrm{i}\omega_k} = \theta_f^{-1}$. It is clear, that $\theta_f \in \mathbf{P}$ is another generator of form (4.8) for an appropriate choice of ϵ. By the second item, $\theta_f^{-1} v_0 = \sum_{k \in \mathbb{Z}} \frac{\epsilon_1 h_k}{s - \mathrm{i}\omega_k}$ for some $v_0 \in \mathbf{P}$. Thus, $\theta^{-1}(\theta + w_0)(\theta_f + v_0) = \theta_1^{-1}$, from which the assertion follows. $\qquad \square$

Remark 4.3.7. *Although the Theorem 4.3.6 states the result only for the scalar case, it can be trivially generalized to the case where w_0 is a vector.*

The second item in Theorem 4.3.6 reveals when the the function θ defined in (4.8), is a generating element for all the signals that are of form (4.5a) with an absolutely summable sequence $(\alpha_k)_{k\in\mathbb{Z}}$. The fourth item shows that this element is actually minimal in the sense that if $\theta_0^{-1}I$ generates all the signals of form (4.5a), then θ divides θ_0. Thus, it comes as no surprise that in Section 4.5 the generator serves as an internal model.

4.4 Problem Formulation in the Frequency Domain

4.4.1 The Regulation Problem

In this section, regulation of the signals defined in the previous section is considered in frequency domain terms. The basic setup is as in Section 4.1, but the ring of stable transfer functions is fixed to be $\mathbf{P}$. To avoid confusion, the regulation problem to be considered is specified below:

Let $\hat{\mathcal{Y}}$ and $\hat{\mathcal{D}}$ be sets of reference and disturbance signals, respectively. Let $\mathcal{P} \in \mathcal{M}(\mathbf{F_P})$ be given. The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-regulation problem is defined in the following way:

Find a controller $\mathcal{C} \in \mathcal{M}(\mathbf{F_P})$ such that

1. $\mathcal{C}$ is $\mathbf{P}$-stabilizing, and

2. for all $\hat{y}_r \in \hat{\mathcal{Y}}$ and $\hat{d} \in \hat{\mathcal{D}}$

$$\hat{e} = \begin{bmatrix} -(I + \mathcal{PC})^{-1}\mathcal{P} & (I + \mathcal{PC})^{-1} \end{bmatrix} \begin{bmatrix} \hat{d} \\ \hat{y}_r \end{bmatrix} \in \mathcal{M}(\mathbf{P}).$$

The above regulation problem is called the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-regulation problem to distinguish it from the general case. The regulation problems considered in this thesis are with respect to four different sets of reference and disturbance signals; those that are the Laplace transforms of the signals generated by the exosystem (3.2) with fixed F_1 and E_0, those that are the Laplace transforms of the signals generated by the exosystem (3.2) with any F_1 and E_0 satisfying certain smoothness properties, those that are the Laplace transforms of the signals generated by the exosystem (3.2) for completely arbitrary F_1 and E_0, and those generated by an generator of form $\theta^{-1}I$ where θ is the one in (4.8). The corresponding sets of the reference and disturbance signals defined in the previous section are denoted by $\hat{\mathcal{Y}}(f_k, \alpha_k)$ and $\hat{\mathcal{D}}(h_k, \beta_k)$, $\hat{\mathcal{Y}}(f_k)$ and $\hat{\mathcal{D}}(h_k)$, $\hat{\mathcal{Y}}_0$ and $\hat{\mathcal{D}}_0$, and $\theta^{-1}I_{n\times n}\mathbf{P}$ and $\theta^{-1}I_{m\times m}\mathbf{P}$, respectively.

Lemma 4.4.1. *Let $(f_k)_{k\in\mathbb{Z}} \in \ell^2$, and let θ be the generating element defined in (4.8) with $h_k = f_k$. A controller $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k), \{0\})$-regulation problem if and only if it solves the $(\theta^{-1}\mathbf{P}, \{0\})$-regulation problem.*

Proof. By Theorem 4.3.3, $\hat{\mathcal{Y}}(f_k) \subset \theta^{-1}\mathbf{P}$, so the sufficiency follows. Assume that $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k), \{0\})$-regulation problem. Then, $(I + \mathcal{P}\mathcal{C})^{-1}\hat{y}_r$ is stable for all $\hat{y}_r \in \hat{\mathcal{Y}}(f_k)$, in particular, for $\hat{y}_r = \sum_{k\in\mathbb{Z}} \frac{f_k}{s - \mathrm{i}\omega_k} e_i$ where e_i the ith natural basis vector of $\mathbb{C}^n$. It follows that $\sum_{k\in\mathbb{Z}} \frac{f_k}{s - \mathrm{i}\omega_k}(I + \mathcal{P}\mathcal{C})^{-1} \in \mathcal{M}(\mathbf{P})$ so $\left(1 + \epsilon_1 \sum_{k\in\mathbb{Z}} \frac{f_k}{s - \mathrm{i}\omega_k}\right)(I + \mathcal{P}\mathcal{C})^{-1} = \theta^{-1}(I + \mathcal{P}\mathcal{C})^{-1} \in \mathcal{M}(\mathbf{P})$, which shows the claim. $\qquad\square$

Theorem 4.4.2. *Let $\theta \in \mathbf{P}$, and let $\mathcal{P} \in \mathbf{F_P}$ have a left coprime factorization (N_{pl}, D_{pl}). If $\mathcal{C}$ is a stabilizing controller with a right coprime factorization (N_{cr}, D_{cr}), then it solves the $(\theta^{-1}\mathbf{P}, \theta^{-1}\mathbf{P})$-regulation problem if and only if $\theta^{-1}D_{cr} \in \mathbf{P}$.*

Proof. One can assume that $N_{pl}N_{cr} + D_{pl}D_{cr} = I$ by Lemma 4.1.1 and Lemma 4.1.2. By Lemma 4.1.3,

$$\hat{e} = \left[(I + \mathcal{P}\mathcal{C})^{-1}\mathcal{P} \quad (I + \mathcal{P}\mathcal{C})^{-1}\right] \begin{bmatrix} \hat{d} \\ \hat{y}_r \end{bmatrix} = \theta^{-1}\left[D_{cr}N_{pl} \quad D_{cr}N_{pl}\right] \begin{bmatrix} \hat{d}_0 \\ \hat{y}_0 \end{bmatrix}, \quad (4.15)$$

so clearly $\hat{e} \in \mathcal{M}(\mathbf{P})$ for all stable $\hat{d}_0$ and $\hat{y}_0$ if $\theta^{-1}D_{cr} \in \mathbf{P}$.

If $\mathcal{C}$ solves the $(\theta^{-1}I\mathbf{P}, \theta^{-1}I\mathbf{P})$-regulation problem, then $\hat{e}$ is $\mathbf{P}$-stable if one chooses $\hat{d}_0 = N_{cr}v$ and $\hat{y}_0 = D_{cr}v$ where v is an arbitrary constant vector. Thus, $\theta^{-1}D_{cr}v \in \mathcal{M}(\mathbf{P})$ for all constant vectors v by (4.15). This implies that $\theta^{-1}D_{cr} \in \mathcal{M}(\mathbf{P})$. $\qquad\square$

Remark 4.4.3. *It is easy to verify that the proof of the above theorem is valid if $\mathbf{P}$ is replaced by any ring of stable transfer functions as long as the results concerning stabilizability presented in Section 4.1.1 hold.*

In the time domain, an internal model is not needed in regulation, but the above theorem shows that this is not the case for $(\theta^{-1}\mathbf{P}, \theta^{-1}\mathbf{P})$-regulation problem. The reason for this is that the time domain counterpart of $(\theta^{-1}\mathbf{P}, \theta^{-1}\mathbf{P})$-regulation problem would be a regulation problem where the output regulation problem should be solved for all F_1 and E_0 simultaneously. Any controller that solves this time domain problem should contain an internal model.

4.4.2   The Robust Regulation Problem

Let $\hat{\mathcal{Y}}$ and $\hat{\mathcal{D}}$ be sets of reference and disturbance signals, respectively. Let $\mathcal{P} \in \mathcal{M}(\mathbf{F_P})$ be given. The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-robust regulation problem is defined in the following way:

Find a controller $\mathcal{C} \in \mathcal{M}(\mathbf{F_P})$, such that

1. $\mathcal{C}$ **P**-stabilizes $\mathcal{P}$, and

2. $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-regulation problem for all the plants it stabilizes.

Here the robustness is understood in the sense that the closed loop stability should imply regulation. This type of robustness have been already considered in the frequency domain in [7] where a design of robustly regulating controllers was proposed for the transfer functions in the Callier-Desoer algebra. However, the main motivation for choosing such a definition is due to the observation that, in the time domain, infinite-dimensional exosystems force one to settle for stability types that have bad robustness properties.

4.5   Solvability of the Robust Regulation Problem

4.5.1   The Internal Model Principle

Some results from [65] are needed in order to formulate the internal model principle for plants having only a right coprime factorization. These results generalize the stabilization results of Section 4.1.2 to plants with left or right coprime factorization but not necessarily both. They are stated here without proofs.

Lemma 4.5.1. *Let* $\mathcal{P} \in \mathbf{F}_\mathbf{P}^{n \times m}$*. Denote* $\mathcal{P}_1 = \begin{bmatrix} \mathcal{P}^T & 0 \end{bmatrix}^T \in \mathbf{F}_\mathbf{P}^{(n+m) \times m}$ *and* $\mathcal{P}_2 = \begin{bmatrix} \mathcal{P} & 0 \end{bmatrix} \in \mathbf{F}_\mathbf{P}^{n \times (n+m)}$*.*

1. *If* $\mathcal{P}$ *has a right coprime factorization, then* $\mathcal{P}_1$ *has both right and left coprime factorizations.*

2. *If* $\mathcal{P}$ *has a left coprime factorization, then* $\mathcal{P}_2$ *has both right and left coprime factorizations.*

Lemma 4.5.2. *Let* $\mathcal{P} \in \mathbf{F}_\mathbf{P}^{n \times m}$*. Denote* $\mathcal{P}_1 = \begin{bmatrix} \mathcal{P}^T & 0 \end{bmatrix}^T \in \mathbf{F}_\mathbf{P}^{(n+m) \times m}$ *and* $\mathcal{P}_2 = \begin{bmatrix} \mathcal{P} & 0 \end{bmatrix} \in \mathbf{F}_\mathbf{P}^{n \times (n+m)}$*.*

1. *If* $\mathcal{P}$ *has a right coprime factorization* (N_{pr}, D_{pr}) *that satisfies* $Y N_{pr} + X D_{pr} = I$ *for some* $X, Y \in \mathcal{M}(\mathbf{R})$*, and* (N_1, D_1) *is a left coprime factorization of* $\mathcal{P}_1$ *that satisfies* $N_1 Y_1 + D_1 X_1 = I$ *for some* $X_1, Y_1 \in \mathcal{M}(\mathbf{R})$*, then* $\mathcal{C}$ *stabilizes* $\mathcal{P}$ *if and only if it has a left coprime factorization* $\left(Y + R D_1 \begin{bmatrix} I & 0 \end{bmatrix}^T, X - R N_1 \right)$ *for some* $R \in \mathcal{M}(\mathbf{R})$*.*

2. *If $\mathcal{P}$ has a left coprime factorization (N_{pl}, D_{pl}) that satisfies $N_{pl}Y + D_{pl}X = I$ for some $X, Y \in \mathcal{M}(\mathbf{R})$, and (N_2, D_2) is a right coprime factorization of $\mathcal{P}_2$ that satisfies $Y_1 N_1 + X_1 D_1 = I$ for some $X_2, Y_2 \in \mathcal{M}(\mathbf{R})$, then $\mathcal{C}$ stabilizes $\mathcal{P}$ if and only if it has a right coprime factorization $\left(Y + \begin{bmatrix} I & 0 \end{bmatrix} D_1 R, X - N_1 R\right)$ for some $R \in \mathcal{M}(\mathbf{R})$.*

Remark 4.5.3. *One can use Lemma 4.1.4 and Lemma 4.5.2 to parametrize all the plants a controller stabilizes by changing the roles of the plant and the controller.*

Definition 4.5.4. Consider a controller $\mathcal{C} \in \mathbf{F}_{\mathbf{P}}^{n \times m}$.

1. If $\mathcal{C}$ has a right coprime factorization (N_{cr}, D_{cr}), then it is said that it contains $\theta \in \mathbf{P}$ as its internal model if $\theta^{-1} D_{cr} \in \mathcal{M}(\mathbf{P})$.

2. Let $\mathcal{C}$ have a left coprime factorization, and let $(\widetilde{N}_e, \widetilde{D}_e)$ be a right coprime factorization of $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$ which exists by Lemma 4.5.1. It is said that $\mathcal{C}$ contains $\theta \in \mathbf{P}$ as its internal model if $\theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e \in \mathcal{M}(\mathbf{P})$.

▲

To be precise, one needs to give a concrete definition of the internal model principle. In this thesis, the internal model principle for the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-robust regulation problem is formulated as follows:

The internal model principle. It is said that the internal model principle holds if there exists such an element $\theta \in \mathbf{P}$ that $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-robust regulation problem for a given plant with left or right coprime factorization if and only if it contains θ as its internal model.

In the next two theorems, the internal model principle is shown to hold for the $(\hat{\mathcal{Y}}(f_k), \{0\})$-robust regulation problem. The first theorem states the necessity and sufficiency of an internal model for plants that have a left coprime factorization, and the latter for those with a right coprime factorization. It should be noted that the existence of both right and left coprime factorizations is not assumed. A technical lemma is given before proceeding to the theorems.

Lemma 4.5.5. *Let $G \in \mathbf{P}^{p \times m}$. Fix a sequence $(f_k)_{k \in \mathbb{Z}} \in \ell^2$, and let θ be the function (4.8) with $h_k = f_k$. If $GQ\hat{y}_r \in \mathcal{M}(\mathbf{P})$ for all $\hat{y}_r \in \hat{\mathcal{Y}}(f_k, a_k)$ and $Q \in \mathcal{M}(\mathbf{P})$ of appropriate size, then $\theta^{-1}G \in \mathcal{M}(\mathbf{P})$.*

Proof. Recall that $\hat{y}_r$ is a n-vector. Choose $Q = e_i e_j^T$, where e_i is the ith natural basis vector of $\mathbb{C}^m$ and e_j^T denotes the transpose of the jth natural basis vector e_j of $\mathbb{C}^n$. In $\hat{y}_r = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - i\omega_k} a_k$ choose $\alpha_k = f_k \frac{|e_j^T a_k|}{e_j^T a_k}$ and denote $x_{jk} = f_k |e_j^T a_k|$. By

varying i and j, one can show that every element of G is divisible by $\sum_{k\in\mathbb{Z}}\frac{x_{jk}}{s-i\omega_k}$. Since $f_k = f_k\|a_k\| \leq \sum_{j=1}^{n} x_{jk}$, it is shown that $\sum_{k\in\mathbb{Z}}\frac{f_k}{s-i\omega_k}$ divides every element of G. Thus, $\theta^{-1}G = \left(1 + \epsilon_1\sum_{k\in\mathbb{Z}}\frac{f_k}{s-i\omega_k}\right)G \in \mathcal{M}\left(\mathbf{P}\right)$. $\square$

Theorem 4.5.6. *Let $\mathcal{P}$ have a left coprime factorization (N_{pl}, D_{pl}) and fix a sequence $(f_k)_{k\in\mathbb{Z}} \in \ell^2$. If θ is the function (4.8) with $h_k = f_k$, then a stabilizing controller $\mathcal{C}$ with a right coprime factorization (N_{cr}, D_{cr}) solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if $\theta^{-1}D_{cr} \in \mathcal{M}\left(\mathbf{P}\right)$.*

Proof. Since θ generates all the signals in $\hat{\mathcal{Y}}(f_k, a_k)$, the sufficiency follows by Theorem 4.4.2 and Lemma 4.4.1. It remains to show the necessity.

Assume that $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem. One can assume by Lemma 4.1.1 and Lemma 4.1.2 that $N_{pl}N_{cr}+D_{pl}D_{cr} = I$. Choose $Q = \frac{\delta}{(s+1)^\alpha}Q_0$ where $Q_0 \in \mathcal{M}\left(\mathbf{P}\right)$ and $\alpha, \delta > 0$ are such that $\det(D_{pl} + Q) \neq 0$ and $\|Q(s)\|\|D_{cr}(s)\| < \epsilon$ for all $s \in \overline{\mathbb{C}^+}$ and for some fixed $\epsilon < 1$. Such constants exist because $Q_0, D_{cr} \in \mathcal{M}\left(\mathbf{P}\right)$.

Consider the plant $\mathcal{P}' = (D_{pl} - Q)^{-1}N_{pl}$. Since $\|Q(s)D_{cr}(s)\| < \epsilon < 1$ $N_{pl}N_{cr}+(D_{pl}-Q)D_{cr} = I-QD_{cr}$ is $\mathbf{P}$-unimodular so the controller stabilizes $\mathcal{P}'$ by Lemma 4.1.2. By using the Neumann series of $(I-QD_{cr})^{-1}$ and $(I-D_{cr}Q)^{-1}$ [37], one shows that

$$
\begin{aligned}
(I + \mathcal{P}'\mathcal{C})^{-1} &= D_{cr}(I - QD_{cr})^{-1}(D_{pl} + Q) \\
&= D_{cr}\left(I + \sum_{i=1}^{\infty}(QD_{cr})^i\right)(D_{pl} + Q) \\
&= \left(I + \sum_{i=1}^{\infty}(D_{cr}Q)^i\right)(D_{cr}D_{pl} + D_{cr}Q) \\
&= (I - D_{cr}Q)^{-1}(D_{cr}D_{pl} + D_{cr}Q).
\end{aligned}
$$

The controller is regulating for $\mathcal{P}'$ by the assumption, so $(I+\mathcal{P}'\mathcal{C})^{-1}\hat{y}_r \in \mathcal{M}\left(\mathbf{R}\right)$ for every $\hat{y}_r \in \hat{\mathcal{Y}}(f_k, a_k)$. Let $\hat{y}_r \in \hat{\mathcal{Y}}(f_k, a_k)$ be arbitrary. Since $(I - D_{cr}Q)^{-1}$ is $\mathbf{P}$-unimodular, $(D_{cr}D_{pl} + D_{cr}Q)\hat{y}_r \in \mathcal{M}\left(\mathbf{P}\right)$. One has $D_{cr}D_{pl}\hat{y}_r \in \mathcal{M}\left(\mathbf{P}\right)$ since $\mathcal{C}$ is regulating for $\mathcal{P}$. Thus, $D_{cr}Q\hat{y}_r \in \mathcal{M}\left(\mathbf{P}\right)$.

Recall that $\hat{y}_r = \sum_{k\in\mathbb{Z}}\frac{\alpha_k}{s-i\omega_k}a_k$ for some sequence $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2$ satisfying $\alpha_k \leq Mf_k$ for some $M > 0$. The reference signal $\hat{y}_r$ is bounded in every right half plane $\mathbb{C}_\beta^+$ with $\beta > 0$ by Lemma 4.3.2, so only the boundary behavior on the imaginary axis can make $D_{cr}Q\hat{y}_r$ unstable. Consequently, $D_{cr}Q\hat{y}_r = \frac{\delta}{(s+1)^{-\alpha}}D_{cr}Q_0\hat{y}_r \in \mathcal{M}\left(\mathbf{P}\right)$ if only if $D_{cr}Q_0\hat{y}_r \in \mathcal{M}\left(\mathbf{P}\right)$. The result follows by Lemma 4.5.5. $\square$

Theorem 4.5.7. *Assume that $\mathcal{P} \in \mathbf{F}_{\mathbf{P}}^{n\times m}$ has a right coprime factorization (N_{pr}, D_{pr}) and that $\mathcal{C}$ is a stabilizing controller for $\mathcal{P}$. Fix a sequence $(f_k)_{k\in\mathbb{Z}} \in$*

ℓ^2. *Let θ be the function (4.8) with $h_k = f_k$. If $(\widetilde{N}_e, \widetilde{D}_e)$ is a right coprime factorization of $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} \in \mathbf{F}_{\mathbf{P}}^{m \times (n+m)}$, then $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if $\theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e \in \mathcal{M}(\mathbf{P})$.*

Proof. Sufficiency. Assume that $\theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D} \in \mathcal{M}(\mathbf{P})$, and denote $\mathcal{P}_e = \begin{bmatrix} P^T & 0 \end{bmatrix}^T$. It is a matter of an elementary calculation to show that $\mathcal{C}_e$ stabilizes $\mathcal{P}_e$. By Lemma 4.1.2, $\mathcal{P}_e$ has a left coprime factorization (N_e, D_e) that satisfies $N_e \widetilde{N}_e + D_e \widetilde{D}_e = I$. By the assumption,

$$\theta^{-1}(I + \mathcal{P}\mathcal{C})^{-1} = \theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} (I + \mathcal{P}_e \mathcal{C}_e)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$= \theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e D_e \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathcal{M}(\mathbf{P}).$$

The sufficiency follows by Lemma 4.4.1 and Theorem 4.4.2.

Necessity. Assume that $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem. Let (N_{cl}, D_{cl}) be the left coprime factorization of $\mathcal{C}$ that satisfies $N_{cl} N_{pr} + D_{cl} D_{pr} = I$. Remark 4.5.3 implies that every plant $\mathcal{P}'$ the controller $\mathcal{C}$ stabilizes can be written in form

$$\mathcal{P}' = \begin{bmatrix} I & 0 \end{bmatrix} \left(\begin{bmatrix} N_{pr} \\ 0 \end{bmatrix} + \widetilde{D}_e R \right) \left(D_{pr} + \widetilde{N}_e R \right)^{-1}$$

where $R \in \mathcal{M}(\mathbf{P})$ is such that $\det \left(D_{pr} + \widetilde{N}_e R \right) \neq 0$. It is a matter of simple calculation to show that

$$I - \mathcal{P}' (I + \mathcal{C}\mathcal{P}')^{-1} \mathcal{C} = I - N_{pr} N_{cl} - \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e R N_{cl}.$$

Fix an arbitrary element $\hat{y}_r \in \hat{\mathcal{Y}}(f_k, a_k)$. Since $\mathcal{C}$ is robustly regulating $(I - N_{pr} N_{cl})\hat{y}_r \in \mathcal{M}(\mathbf{P})$. It follows that $\begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e R N_{pr} \hat{y}_r \in \mathcal{M}(\mathbf{P})$. In particular, one can choose $R = -Q N_{pr}$ where Q is an arbitrary matrix of appropriate size. Now

$$\begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e R N_{cl} \hat{y}_r = \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e Q(-I + (I - N_{pr} N_{cl}))\hat{y}_r$$

$$= \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e Q(I - N_{pr} N_{cl})\hat{y}_r - \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e Q \hat{y}_r.$$

Since $(I - N_{pr} N_{cl})\hat{y}_r$ is $\mathbf{P}$-stable, $\begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e Q \hat{y}_r \in \mathcal{M}(\mathbf{P})$. The claim follows by Lemma 4.5.5. $\qquad \square$

Remark 4.5.8. *The sufficiency parts of Theorem 4.5.6 and Theorem 4.5.7 hold for any ring of stable transfer functions, because they do not use any properties of $\mathbf{P}$.*

Corollary 4.5.9. *If $\mathcal{P} \in \mathcal{M}(\mathbf{F_P})$ has left or right coprime factorization, then a controller $\mathcal{C}$ solves the $\left(\hat{\mathcal{Y}}(f_k, a_k), \emptyset\right)$-robust regulation problem if and only if it solves the $(\theta\mathbf{P}, \theta\mathbf{P})$-regulation problem.*

Proof. If $\mathcal{P}$ has a left coprime factorization, the claim follows directly from Theorem 4.4.2 and Theorem 4.5.6. Assume that $\mathcal{P}$ has a right coprime factorization. The sufficiency is obvious, so it remains to show the necessity. By Lemma 4.1.2 and Lemma 4.5.1, $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$ has a right coprime factorization $(\tilde{N}_e, \tilde{D}_e)$.

If $\mathcal{C}$ stabilizes a plant $\mathcal{P}'$, then $\mathcal{P}'_e = \begin{bmatrix} (\mathcal{P}')^T & 0 \end{bmatrix}^T$ has a left coprime factorization (N_e, D_e) that satisfies $N_e \tilde{N}_e + D_e \tilde{D}_e = I$. Now

$$\begin{bmatrix} -(I + \mathcal{P}'\mathcal{C})^{-1}\mathcal{P}' & (I + \mathcal{P}'\mathcal{C})^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \tilde{D}_e \begin{bmatrix} -N_e & D_e \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix},$$

so the claim follows Theorem 4.5.7. $\qquad\square$

It was seen above that a robustly regulating controller always contains the generating element (4.8) as its internal model even if the sequence $(a_k)_{k \in \mathbb{Z}}$ is fixed. The next example shows that a controller can be robustly regulating even if it does not contain an internal model if only the disturbance signals contain unstable dynamics. Thus, the internal model principle does not hold for the $(\{0\}, \hat{\mathcal{D}}(f_k, b_k))$-robust regulation problem. However, a controller containing the generatring element (4.8) as its internal model is $(\{0\}, \hat{\mathcal{D}}(f_k, b_k))$-robustly regulating by Theorem 4.4.2.

Example 4.5.10. Consider the $\left(\{0\}, \left\{\begin{bmatrix} \frac{1}{s} & 0 \end{bmatrix}^T\right\}\right)$-robust regulation problem. Let the plant and the controller be

$$\mathcal{P} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} \frac{s+1}{s} & 0 \\ 0 & 1 \end{bmatrix}.$$

The plant admits a right coprime factorization $(\mathcal{P}, I)$ which is also a left coprime factorization. A factorization of $\mathcal{C}$, that is both right and left coprime, is given by $(I, \mathcal{C}^{-1})$ because $\mathcal{C}^{-1}$ is $\mathbf{P}$-stable and $\mathcal{P} + \mathcal{C}^{-1} = I$. This also shows that $\mathcal{C}$ stabilizes $\mathcal{P}$. All plants $\mathcal{C}$ stabilizes are of the form

$$\mathcal{P}' = (\mathcal{P} + \mathcal{C}^{-1}R)(I - IR)^{-1}$$

where $\det(I - IR) \neq 0$. Since $\mathcal{P}'(I + \mathcal{C}\mathcal{P}') = (\mathcal{P} + \mathcal{C}^{-1}R)\mathcal{C}^{-1}$ and $\mathcal{C}^{-1} \begin{bmatrix} \frac{1}{s} & 0 \end{bmatrix}^T$ is stable the controller is robustly regulating. However, $\frac{1}{s}\mathcal{C}^{-1}$ is not stable, so the controller contains only a partial internal model.

4.5.2　A Necessary and Sufficient Condition for the Solvability

The next two theorems generalize the solvability condition of rational transfer functions and generators given in [92, Theorem 7.5.2] to the class of signals defined in Section 4.3 and to the plants in $\mathbf{F_P}$.

Theorem 4.5.11. *Let $\mathcal{P}$ have a left coprime factorization (N_{pl}, D_{pl}). Fix a sequence $(f_k)_{k \in \mathbb{Z}} \in \ell^2$ and let θ be the function (4.8) with $h_k = f_k$. If (N_e, D_e) is a right coprime factorization of $\mathcal{P}_e = \begin{bmatrix} \mathcal{P} & 0 \end{bmatrix}$, then the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem is solvable if and only if $(\theta I, N_e)$ is left coprime.*

Proof. Sufficiency. Assume that $(\theta I, N_e)$ is left coprime. There exist $\mathbf{P}$-stable matrices V and W such that $\theta V + N_e W = I$. Since (N_{pl}, D_{pl}) is left coprime and (N_e, D_e) is right coprime, there exist $\mathbf{P}$-stable matrices Y, X, Y_e, and X_e such that $N_{pl}Y + D_{pl}X = I$, and $Y_e N_e + X_e D_e = I$. Note that X can be chosen so that $\det(X) \neq 0$. All stabilizing controllers of $\mathcal{P}$ have a right coprime factorization of the form $\left(\begin{bmatrix} I & 0 \end{bmatrix} \left(\begin{bmatrix} Y^T & 0 \end{bmatrix}^T + D_e R \right), X - N_e R \right)$ by Lemma 4.5.2. Choose $R = WX$. Now

$$\theta^{-1} (X - N_e R) = \theta^{-1} (I - N_e W) X = \theta^{-1} (\theta V) X \in \mathcal{M}(\mathbf{P}).$$

By Theorem 4.5.6, the chosen controller is robustly regulating provided that $\det(X - N_e W X) = \det(I - N_e W) \det(X) \neq 0$. If this is not the case, then choose $R = (W + \theta R_0) X$ where R_0 is chosen so that $\det(I - N_e(W + \theta R_0)) \neq 0$. Similar arguments to those above show that the controller is robustly regulating, which completes the proof of the sufficiency part.

Necessity. Assume that $\mathcal{C}$ is a robustly regulating controller. Since $\mathcal{C}$ is stabilizing, Lemma 4.1.2 implies that it has a right coprime factorization (N_{cr}, D_{cr}) that satisfies $N_{pl}N_{cr} + D_{pl}D_{cr} = I$. Denote $\mathcal{C}_e = \begin{bmatrix} \mathcal{C}^T & 0 \end{bmatrix}^T$. Since

$$\begin{bmatrix} N_{pl} & 0 \end{bmatrix} \begin{bmatrix} N_{cr}^T & 0 \end{bmatrix}^T + D_{pl}D_{cr} = I$$

$\mathcal{C}_e$ stabilizes $\mathcal{P}_e$. By Lemma 4.1.2, $\mathcal{C}_e$ has a left coprime factorization $(\widetilde{N}_e, \widetilde{D}_e)$ satisfying $N_e \widetilde{N}_e + D_e \widetilde{D}_e = I$. By Lemma 4.1.3,

$$I - N_e \widetilde{N}_e = (I + \mathcal{P}_e \mathcal{C}_e)^{-1} = D_{cr} D_{pl}.$$

Theorem 4.5.6 shows that $\theta^{-1} \left(I - N_e \widetilde{N}_e \right) = V \in \mathcal{M}(\mathbf{P})$. The claim follows since $N_e \widetilde{N}_e + \theta V = I$. $\qquad \square$

Theorem 4.5.12. *Let $\mathcal{P}$ have a right coprime factorization (N_{pr}, D_{pr}). Fix a sequence $(f_k)_{k \in \mathbb{Z}} \in \ell^2$ and let θ be the function (4.8) with $h_k = f_k$. The $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem is solvable if and only if $(\theta I, N_{pr})$ is left coprime.*

Proof. Sufficiency. Assume that $(\theta I, N_{pr})$ is left coprime. There exist **P**-stable matrices V and W such that $\theta V + N_{pr} W = I$. Since (N_{pr}, D_{pr}) is right coprime there exists stable matrices X and Y such that $Y N_{pr} + X D_{pr} = I$.

By Lemma 4.5.1, $\mathcal{P}_e = \begin{bmatrix} \mathcal{P}^T & 0 \end{bmatrix}$ has a left coprime factorization (N_e, D_e). It also has the right coprime factorization $\left(\begin{bmatrix} N_{pr}^T & 0 \end{bmatrix}, D_{pr} \right)$. It is easy to verify that $\mathcal{C}_0 = \begin{bmatrix} X^{-1} Y & 0 \end{bmatrix}$ stabilizes $\mathcal{P}_e$, so $\mathcal{C}_0$ has a right coprime factorization (N_0, D_0) such that $N_e N_0 + D_e D_0 = I$ and $\det(D_0) \neq 0$.

Choose $R = \begin{bmatrix} W + \theta R_0 & 0 \end{bmatrix} D_0$ where R_0 is chosen so that

$$\det \left(D_0 - \begin{bmatrix} N_{pr}^T & 0 \end{bmatrix}^T R \right) \neq 0.$$

The controller

$$
\begin{aligned}
\mathcal{C}_e &= (N_0 + D_{pr} R) \left(D_0 - \begin{bmatrix} N_{pr}^T & 0 \end{bmatrix}^T R \right)^{-1} \\
&= (N_0 + D_{pr} R) D_0^{-1} \begin{bmatrix} I - N_{pr} W + \theta N_{pr} R_0 & 0 \\ 0 & I \end{bmatrix}^{-1} \\
&= \left(\mathcal{C}_0 + \begin{bmatrix} D_{pr}(W + \theta R_0) & 0 \end{bmatrix} \right) \begin{bmatrix} \theta(V + N_{pr} R_0)^{-1} & 0 \\ 0 & I \end{bmatrix}
\end{aligned}
$$

stabilizes $\mathcal{P}_e$ by Lemma 4.1.4. The defined controller is of the form $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$. It is a matter of an easy calculation to show that $\mathcal{C}$ stabilizes $\mathcal{P}$. Theorem 4.5.7 shows that $\mathcal{C}$ is robustly regulating since

$$\begin{bmatrix} I & 0 \end{bmatrix} \left(D_0 - \begin{bmatrix} N_{pr}^T & 0 \end{bmatrix}^T R \right) = \theta \begin{bmatrix} V + N_{pr} R_0 & 0 \end{bmatrix} D_0.$$

Thus, the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem is solvable.

Necessity. Let $\mathcal{C}$ be a robustly regulating controller. Since it stabilizes $\mathcal{P}$ it possesses a left coprime factorization (N_{cl}, D_{cl}) such that $N_{cl} N_{pr} + D_{cl} D_{pr} = I$. Let $(\tilde{N}_e, \tilde{D}_e)$ be a right coprime factorization of $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$. Since $\mathcal{P}_e = \begin{bmatrix} \mathcal{P}^T & 0 \end{bmatrix}$ is stabilized by $\mathcal{C}_e$, it has a left coprime factorization (N_e, D_e) such

that $N_e \widetilde{N}_e + D_e \widetilde{D}_e = I$. By Theorem 4.5.7,

$$\theta^{-1}(I - N_{pr}N_{cr}) = \theta^{-1}(I + \mathcal{P}\mathcal{C})^{-1}$$

$$= \theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} (I + \mathcal{P}_e\mathcal{C}_e)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$= \theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e D_e \begin{bmatrix} I \\ 0 \end{bmatrix} = V \in \mathcal{M}(\mathbf{P}).$$

Thus $\theta V + N_{pr}N_{cr} = I$, which completes the proof. $\qquad\square$

A stable plant $\mathcal{P}$ has the right coprime factorization $(\mathcal{P}, I)$. The above theorem implies the following corollary.

Corollary 4.5.13. *Let $\mathcal{P}$ be a $\mathbf{P}$-matrix. Fix a sequence $(f_k)_{k\in\mathbb{Z}} \in \ell^2$ and let θ be the function (4.8) with $h_k = f_k$. The $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem is solvable if and only if $(\theta I, \mathcal{P})$ is left coprime.*

Remark 4.5.14. *The results of this section do not use any properties of the ring $\mathbf{P}$. Only the internal model principle is needed. Thus, the results can be generalized as long as the internal model is required for a controller to be robustly regulating. This is the case for the $(\theta^{-1}\mathbf{P}, \theta^{-1}\mathbf{P})$-regulation problem if the plant has a right coprime factorization.*

4.6   Design of Robustly Regulating Controllers

A robustly regulating controller design for a plant with left or right coprime factorization is proposed in this section. The controller is a combination of an arbitrary stabilizing controller and a robustly regulating controller designed for the numerator matrix of the plant. It is shown first how to design a robustly regulating controller for a H_-^∞-stable plant. Then a general design procedure is proposed for unstable plants with a coprime factorization.

4.6.1   Design of Robustly Regulating Controllers for H_-^∞-stable Plants

The main result of this section is given by the following theorem. The theorem gives a necessary and sufficient condition for the solvability of the robust regulation problem with an H_-^∞-stable plant.

Theorem 4.6.1. *If $\mathcal{P}(s)$ is H_-^∞-stable, then there exists a controller solving the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if there exist right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ of $\mathcal{P}$ at $\mathrm{i}\omega_k$ where $k \in \mathbb{Z}$ and constants $M, \alpha > 0$, such that $\|\mathcal{P}^r(\mathrm{i}\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for all $k \in \mathbb{Z}$.*

The proof of the above theorem is divided into two theorems; Theorem 4.6.3 shows the necessity, and Theorem 4.6.5 shows the sufficiency. The necessity holds for all **P**-stable plants with a right coprime factorization, whereas the sufficiency is proved only for H_-^∞-stable functions.

Remark 4.6.2. *If $\mathcal{P} \in \mathcal{M}\left(H_-^\infty\right)$, the solvability of the $(\hat{\mathcal{Y}}\left(f_k, a_k\right), \{0\})$-robust regulation problem is independent of the sequence $(f_k)_{k\in\mathbb{Z}}$ by Theorem 4.6.1. Thus, the smoothness of the reference signals does not affect the solvability of the robust regulation problem.*

Theorem 4.6.3. *Let $\mathcal{P} \in \mathbf{P}^{n\times m}$. If the $(\hat{\mathcal{Y}}\left(a_k, f_k\right), \{0\})$-robust regulation problem is solvable, then $\mathcal{P}$ is right invertible at $\mathrm{i}\omega_k$ and the right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ can be chosen so that $\|\mathcal{P}^r(\mathrm{i}\omega)\| < M(1+|\omega_k|)^\alpha$ for some fixed constants $\alpha, M > 0$ and for all $k \in \mathbb{Z}$.*

Proof. Let $\mathcal{C}$ solve the $(\hat{\mathcal{Y}}\left(a_k, f_k\right), \{0\})$-robust regulation problem. There exists a right coprime factorization of (N_{cr}, D_{cr}) of $\mathcal{C}$ satisfying $\mathcal{P}N_{cr} + D_{cr} = I$ since $(\mathcal{P}, I)$ is a left coprime factorization of $\mathcal{P}$ and $\mathcal{C}$ is a stabilizing controller.

It follows easily from Theorem 4.5.6 that $D_{cr}(\mathrm{i}\omega_k) = 0$ for all $k \in \mathbb{Z}$, so $\mathcal{P}(\mathrm{i}\omega_k)N_{cr}(\mathrm{i}\omega_k) = I$ for all $k \in \mathbb{Z}$. Thus, $N_{cr}(\mathrm{i}\omega_k)$ is a right inverse of $\mathcal{P}(\mathrm{i}\omega_k)$. Since $N_{cr} \in \mathcal{M}(\mathbf{P})$ it follows that $\|N_{cr}(\mathrm{i}\omega)\| < M(1 + |\omega|)^\alpha$ for some $\alpha, M > 0$ and for all $\omega \in \mathbb{R}$. In particular, this holds if $\omega = \omega_k$. $\qquad\square$

It is shown next that there exists a robustly regulating controller for an H_-^∞-stable plant if the plant has at most polynomial decay rate at $\mathrm{i}\omega_k$, as $|k| \to \infty$. To this end, a controller solving the robust regulation problem is presented. The controller considered here is based on the one in [80]. The controller is of the form

$$\mathcal{C}_\epsilon(s) = \epsilon\left(\mathcal{C}_0(s) + \sum_{k\in\mathbb{Z}} \frac{K_k}{s - \mathrm{i}\omega_k}\right), \tag{4.16}$$

where $\mathcal{C}_0$ is H_-^∞-stable and $(\|K_k\|)_{k\in\mathbb{Z}} \in \ell^2$. The only difference to the controller considered in [80] is that the sum here is infinite. Some assumptions over the design parameters K_k are needed.

Assumption 4.6.4. *Denote $G_k = \mathcal{P}(\mathrm{i}\omega_k)K_k$. It is assumed that*

1. *G_k is invertible for all $k \in \mathbb{Z}$,*

2. *there exist $\alpha, M > 0$ such that $\|G_k^{-1}\| \leq M(1 + |\omega_k|)^\alpha$ for all $k \in \mathbb{Z}$,*

3. *there exists $M > 0$ such that $\left\|\left(I + \frac{G_k}{z}\right)^{-1}\right\| < M$ for all $z \in \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$, and*

4. there exists $M > 0$ such that $\left\|\left(I + zG_k^{-1}\right)^{-1}\right\| < M$ for all $z \in \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$.

The first assumption is crucial for the controller to contain an internal model. The second assumption is needed for closed loop stability. Since $K_k G_k^{-1}$ is a right inverse of $P(\mathrm{i}\omega_k)$ it restricts the convergence speed of the plant. The last two are technical assumptions that are used to guarantee polynomial bound of the closed loop transfer function. In the last condition polynomial bound would be sufficient.

Theorem 4.6.5. *Let $\mathcal{P}(s) \in \mathcal{M}\left(H_-^\infty\right)$. If there exist right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ of $\mathcal{P}(\mathrm{i}\omega_k)$ and constants $M, \alpha > 0$ such that $\|\mathcal{P}^r(\mathrm{i}\omega_k)\| \le M(1 + |\omega_k|)^\alpha$ for all $k \in \mathbb{Z}$, then there exist parameters K_k and a constant $\epsilon^* > 0$ such that the controller (4.16) solves the robust output regulation problem for all $\epsilon \in (0, \epsilon^*]$.*

The proof of the above theorem is divided into a series of lemmas. First, a technical lemma is given. Lemma 4.6.7 and Lemma 4.6.8 show that, if Assumption 4.6.4 holds, the proposed controller is stabilizing. Lemma 4.6.9 shows that the controller is regulating. Finally, Lemma 4.6.10 shows that a choice of parameters that satisfies Assumption 4.6.4 actually exists.

Lemma 4.6.6. *Let $\mathcal{P}(s) \in \mathcal{M}\left(H_\beta^\infty\right)$ for some $\beta \in \mathbb{R}$. For any fixed $\epsilon > 0$, there exists a positive real number $M > 0$ such that $\left\|\frac{\mathcal{P}(s) - \mathcal{P}(s_0)}{s - s_0}\right\| < M$ for all $s, s_0 \in \mathbb{C}_{\beta+\epsilon}^+$.*

Proof. Fix $s_0 \in \mathbb{C}_{\beta+\epsilon}^+$ and choose $\delta < \min\left\{\frac{\epsilon}{3}, 1\right\}$. By assumption, there exists $M_0 > 0$ such that $\|\mathcal{P}(s)\| < M_0$ for all $s \in \mathbb{C}_{\beta+}^+$. Let γ be the simple positively oriented circular path around s_0 with radius 2δ. By the Cauchy's differentiation formula,

$$\left\|\mathcal{P}^{(n)}(s_0)\right\| = \left\|\frac{n!}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathcal{P}(s)}{(s - s_0)^{n+1}} ds\right\|$$

$$\le \frac{n!}{2\pi} \oint_\gamma \frac{\|\mathcal{P}(s)\|}{|s - s_0|^{n+1}} ds$$

$$\le \frac{n!}{2\pi} \oint_\gamma \frac{M_0}{(2\delta)^{n+1}} ds = \frac{M_0 n!}{(2\delta)^n}.$$

Let $s \in B_\delta(s_0)$ be arbitrary. By using the above equation and the Taylor series

of $\mathcal{P}(s)$ at s_0,

$$
\left\| \frac{\mathcal{P}(s) - \mathcal{P}(s_0)}{s - s_0} \right\| = \left\| \frac{\sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{P}^{(n)}(s_0)(s - s_0)^n - \mathcal{P}(s_0)}{s - s_0} \right\|
$$

$$
= \left\| \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{P}^{(n)}(s_0)(s - s_0)^{n-1} \right\|
$$

$$
\leq \sum_{n=1}^{\infty} \frac{1}{n!} \| \mathcal{P}^{(n)}(s_0) \| |s - s_0|^{n-1}
$$

$$
< \sum_{n=1}^{\infty} \frac{1}{n!} \frac{M_0 n!}{(2\delta)^n} \delta^{n-1}
$$

$$
= \frac{M_0}{\delta} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{2M_0}{\delta}.
$$

Thus, $\left\| \frac{\mathcal{P}(s) - \mathcal{P}(s_0)}{s - s_0} \right\| < \frac{2M_0}{\delta}$ for all $s \in B_\delta(s_0)$. It is easy to see, that the same upper bound also holds true for all $s \in \mathbb{C}^+_{\beta+\epsilon} \setminus B_\delta(s_0)$. The claim follows since the limit does not depend on s_0. $\qquad\square$

Lemma 4.6.7. *Let* $\mathcal{P} \in \mathcal{M}\left(H^\infty_-\right)$. *If the design parameters* K_k *in* (4.16) *satisfy Assumption 4.6.4, then there exists a constant* $\epsilon^* > 0$ *such that* $(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1} \in \mathcal{M}\left(\mathbf{P} \cap H^\infty\right)$ *for all* $\epsilon \in (0, \epsilon^*]$.

Proof. Denote $U = \overline{\mathbb{C}^+} \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. Lemma 4.3.2 and the uniform boundedness of $\mathcal{P}(s)$ show that $\mathcal{P}(s) \sum_{k \in \mathbb{Z}} \frac{K_k}{s - i\omega_k}$ is uniformly bounded in $\overline{\mathbb{C}^+} \setminus U$. Thus, $\mathcal{P}(s)\mathcal{C}_\epsilon(s)$ is uniformly bounded in $\overline{\mathbb{C}^+} \setminus U$. Choose $M_1 > 0$ such that $\left\| \mathcal{P}(s) \sum_{k \in \mathbb{Z}} \frac{K_k}{s - i\omega_k} \right\| \leq M_1$ for all $s \in \overline{\mathbb{C}^+} \setminus U$ and set $\epsilon_1 = \frac{1}{2M_1}$. By using the Neumann series [37, Corollary 5.6.16],

$$
\left\| (I + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1} \right\| = \left\| \sum_{i=0}^{\infty} (-\mathcal{P}(s)\mathcal{C}_\epsilon(s))^i \right\| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2
$$

for all $\epsilon \in (0, \epsilon_1]$.

Consider the decomposition $(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1} = \Phi_{1k} \left(I + \epsilon \Phi_{2k} \Phi_{1k}\right)^{-1}$, where

$$
\Phi_{1k}(s) = \left(I + \epsilon \frac{\mathcal{P}(i\omega_k)K_k}{s - i\omega_k}\right)^{-1} \tag{4.17}
$$

and

$$
\Phi_{2k}(s) = \frac{\mathcal{P}(s) - \mathcal{P}(i\omega_k)}{s - i\omega_k} K_k + \mathcal{P}(s)\mathcal{C}_0(s) + \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\mathcal{P}(s)K_l}{s - i\omega_k}. \tag{4.18}
$$

Assumption 4.6.4 implies that there exists a positive constant M_2 such that $\|\Phi_{1k}(s)\| < M_2$ since the real part of $z = \frac{s - \mathrm{i}\omega_k}{\epsilon}$ is positive whenever $s \in \overline{\mathbb{C}}_+$. A constant $M_3 > 0$ such that $\|\Phi_{2k}(s)\| < M_3$ for all $s \in B_\gamma(\mathrm{i}\omega_k)$ exists by Lemma 4.6.6 and Lemma 4.3.2. Note that M_2 and M_3 can be chosen so that they are independent of k. Choose $\epsilon_2 = \frac{1}{2M_2 M_3}$. Now, $\left\|(I + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1}\right\| \leq 2M_2$ for all $\epsilon \in (0, \epsilon_2]$ and $s \in \overline{\mathbb{C}^+} \cap B_\gamma(\mathrm{i}\omega_k)$. Since M_2 does not depend on $k \in \mathbb{Z}$, the claim follows by choosing $\epsilon^* = \min\{\epsilon_1, \epsilon_2\}$. $\square$

Lemma 4.6.8. *If the design parameters K_k in (4.16) satisfy Assumption 4.6.4, then there exists such a constant $\epsilon^* > 0$ that $\mathcal{C}_\epsilon(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1} \in \mathcal{M}(\mathbf{P})$ for all $\epsilon \in (0, \epsilon^*]$.*

Proof. Lemma 4.6.7 shows that there exists a choice of $\epsilon_1 > 0$ such that $(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1}$ is $\mathbf{P}$-stable for all $\epsilon \in (0, \epsilon_1]$. The controller $\mathcal{C}_\epsilon$ is uniformly bounded in $\overline{\mathbb{C}^+} \setminus \bigcup_{k \in \mathbb{Z}} B_\delta(\mathrm{i}\omega_k)$ for all $\delta < \gamma$ by Lemma 4.3.2. It follows that $\mathcal{C}_\epsilon(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1} \in H_\beta^\infty$ for all $\beta > 0$. Thus, the polynomial boundedness of $\mathcal{C}_\epsilon(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1}$ needs to be shown only in $\overline{\mathbb{C}^+} \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(\mathrm{i}\omega_k)$.

Observe that $\mathcal{C}_\epsilon(s)(I + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1}$ can be written in the form

$$\frac{\epsilon K_k}{s - \mathrm{i}\omega_k}(I + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1} + \left(\mathcal{C}_0(s) + \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\epsilon K_l}{s - \mathrm{i}\omega_l}\right)(I + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1}.$$

Lemma 4.3.2 and Lemma 4.6.7 show that the last term is bounded in $B_\gamma(\mathrm{i}\omega_k)$ by a bound independent of $k \in \mathbb{Z}$.

It remains to analyze the behavior of $\frac{\epsilon K_k}{s - \mathrm{i}\omega_k}(I + \mathcal{P}(s)\mathcal{C}(s))^{-1}$ in $B_\gamma(\mathrm{i}\omega_k)$. Decompose $(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1}$ into $\Phi_{1k}(I + \epsilon\Phi_{2k}\Phi_{1k})^{-1}$, where Φ_{1k} and Φ_{2k} are the functions defined in (4.17) and (4.18). By Assumption 4.6.4, there exist $M > 0$ and $\epsilon_0 > 0$ such that $\left\|(I + \epsilon\Phi_{2k}\Phi_{1k})^{-1}\right\| < M$ in $B_\gamma(\mathrm{i}\omega_k)$ for all $k \in \mathbb{Z}$ and $\epsilon \in (0, \epsilon_0]$. Thus, it is sufficient to analyze the behavior of $\frac{\epsilon K_k}{s - \mathrm{i}\omega_k}\Phi_{1k}(s)$. To this end, write $z = \frac{s - \mathrm{i}\omega_k}{\epsilon}$. By Assumption 4.6.4,

$$\frac{\epsilon K_k}{s - \mathrm{i}\omega_k}\Phi_{1k}(s) = \frac{K_k}{z}\left(I + \frac{G_k}{z}\right)^{-1} = K_k G_k^{-1}\left(I + zG_k^{-1}\right)^{-1}$$

is bounded by $M(1 + |\omega_k|)^\alpha$ where $M > 0$ and $\alpha > 0$ can be chosen to be independent of $k \in \mathbb{Z}$. This completes the proof. $\square$

Lemma 4.6.9. *Let θ be the function (4.8). If the controller $\mathcal{C}_\epsilon$ in (4.16) $\mathbf{P}$-stabilizes $\mathcal{P} \in \mathcal{M}\left(H_-^\infty\right)$, then $\theta^{-1}(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1} \in \mathcal{M}(\mathbf{P})$.*

Proof. The proof is analogous to the proof of Lemma 4.6.8 and is therefore skipped. $\square$

Lemma 4.6.10. *Let $\mathcal{P} \in \mathcal{M}\left(H^\infty_-\right)$ be right invertible at $\mathrm{i}\omega_k$ for all $k \in \mathbb{Z}$, and assume that the right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ can be chosen so that $\|\mathcal{P}^r(\mathrm{i}\omega_k)\| \leq M(1 + |\omega_k|)^\alpha$ for some fixed $\alpha, M > 0$ and for all $k \in \mathbb{Z}$. In (4.16) choose $K_k = g_k \mathcal{P}^r(\mathrm{i}\omega_k)$. The constants $g_k > 0$ can be chosen so that $(\|K_k\|)_{k\in\mathbb{Z}} \in \ell^2$ and Assumption 4.6.4 holds.*

Proof. Since $\|\mathcal{P}^r(\mathrm{i}\omega_k)\| < M(1 + |\omega_k|)^\alpha$ there exists a sequence $(g_k)_{k\in\mathbb{Z}} \in \ell^2$ of positive real numbers such that $g_k > \frac{1}{M_0}(1 + |\omega_k|)^{-\alpha_0}$ and $(\|K_k\|)_{k\in\mathbb{Z}} \in \ell^2$. Since $G_k = \mathcal{P}(\mathrm{i}\omega_k)K_k = g_k I$, the first and the second item of Assumption 4.6.4 hold. Since $(I + z^{-1}G_k)^{-1} = z(z + g_k)^{-1}I$ and $\left(I + zG_k^{-1}\right)^{-1} = (1 + zg_k^{-1})^{-1}I$, it is easy to see that for all $z \in \overline{\mathbb{C}^+}$ $\left\|(I + z^{-1}G_k)^{-1}\right\| < 1$ and $\left\|\left(I + zG_k^{-1}\right)^{-1}\right\| < 1$. It follows that the third and the fourth items of Assumption 4.6.4 hold. $\qquad\square$

Proof of Theorem 4.6.5. Lemma 4.6.7, Lemma 4.6.8, and Lemma 4.6.10 show that there exists a choice of $\epsilon^* > 0$ and the design parameters K_k in (4.16) such that $(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1}$ and $\mathcal{C}_\epsilon(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1}$ are **P**-stable for all $\epsilon \in (0, \epsilon^*]$. It follows that $\mathcal{C}_\epsilon$ is a **P**-stabilizing controller by **P**-stability of $\mathcal{P}$ for all $\epsilon \in (0, \epsilon^*]$.

Since $(\mathcal{P}, I)$ is a left coprime factorization of $\mathcal{P}$, there exists a right coprime factorization (N_{cr}, D_{cr}) of $\mathcal{C}_\epsilon$ that satisfies $\mathcal{P}N_{cr} + D_{cr} = I$. By Lemma 4.6.9,

$$\theta^{-1}(I + \mathcal{P}\mathcal{C}_\epsilon)^{-1} = \theta^{-1}(D_{cr}D_{cr}^{-1} + \mathcal{P}N_{cr}D_{cr}^{-1})^{-1} = \theta^{-1}D_{cr} \in \mathcal{M}\left(\mathbf{P}\right).$$

The controller $\mathcal{C}_\epsilon$ is robustly regulating by Theorem 4.5.6. The claim follows since this holds true for any $\epsilon \in (0, \epsilon^*]$. $\qquad\square$

Corollary 4.6.11. *Let θ be the function (4.11). If $\mathcal{P} \in H^\infty_-$, then $(\theta I, \mathcal{P})$ is left coprime if and only if $\mathcal{P}$ has such right inverses $\mathcal{P}^r(\mathrm{i}\omega_k)$ at $\mathrm{i}\omega_k$ that $\|\mathcal{P}^r(\mathrm{i}\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for some fixed positive constants M and α and for all $k \in \mathbb{Z}$.*

Proof. Theorem 4.5.12, Theorem 4.6.3, and Theorem 4.6.5 show that the claim holds for the generators defined in (4.8) such that the sequence $(h_k)_{k\in\mathbb{Z}}$ has at most polynomial decay rate. The claim follows by Theorem 4.3.6. $\qquad\square$

Corollary 4.6.12. *Let θ be the function (4.11). Let the structure at infinity of $\mathcal{P} \in H^\infty_-$ along $\mathrm{i}\mathbb{R}$ be $\{q_1, \ldots, q_n\}$. The pair $(\theta I, \mathcal{P})$ is left coprime if and only if $\mathcal{P}(\mathrm{i}\omega)$ has full rank for all $k \in \mathbb{Z}$, and there exist positive constants α, M and M_0 such that $|q_n(\mathrm{i}\omega_k)| > M(1 + |\omega_k|)^\alpha$ whenever $|k| > M_0$.*

Example 4.6.13. Recall the transfer function of Example 2.2.18. The zeros of $\sinh(\sqrt{s+1})$ are $-1 - k^2\pi^2$, $k \in \mathbb{Z}$, so the transfer function does not have poles in $\mathbb{C}^+_{-1}$. Since the structure at infinity along $\mathbb{C}^+_{-1}$ is

$$\left\{(s+1)^{-\frac{3}{2}}, (s+1)^{-\frac{3}{2}}\right\}$$

it is seen that $\mathcal{P} \in H^\infty_-$. Corollary 4.5.13 and Corollary 4.6.12 show that the controller (4.16) is robustly regulating for $\mathcal{P}$.

Similar arguments show that the transfer function of Example 3.2.17 is in $\mathcal{P} \in H^\infty_-$. The structure at infinity along $\mathrm{i}\mathbb{R}$ is

$$\left\{ \frac{1}{s+1}, \frac{1}{(s+1)^{3/2}} e^{-\frac{1}{7}\sqrt{s+1}} \right\}.$$

Theorem 4.6.3 implies that there are no robustly regulating controllers for the transfer function since the structural function $\frac{1}{(s+1)^{3/2}} e^{-\frac{1}{7}\sqrt{s+1}}$ has an exponential decay rate on the imaginary axis. ∎

A controller of the form (4.16) was discussed in [101] by Ylinen et al., but they were not able to show whether the controller works in their setting. The next example illustrates that the answer is negative. In the example only SISO-systems are considered and the signal generator is chosen to be $\theta(s) = 1 - \mathrm{e}^{-s}$. However, one can use similar justification to show the results for MIMO-systems and every signal generator satisfying the assumptions in [101].

Example 4.6.14. Consider the reference signals of form $\hat{y}_r(s) = \frac{\gamma(s)}{\theta(s)}$, where $\gamma(s) \in H^\infty$ and $\theta(s) = 1 - \mathrm{e}^{-s}$. The generator has zeros at $2\pi k\mathrm{i}$ where $k \in \mathbb{Z}$, so $\omega_k = 2\pi k$ in this case. Let $\mathcal{P}(s)$ be a SISO-plant such that $|\mathcal{P}(\mathrm{i}\omega_k)| > M$ for some fixed $M > 0$ and for all $k \in \mathbb{Z}$.

Consider the controller (4.16) where $\mathcal{C}_0(s) = 0$. Write K_k in the form $K_k = \frac{g_k}{\mathcal{P}(\mathrm{i}\omega_k)}$. Since the controller should be defined at $s = 1$, say, necessarily $K_k \to 0$ as $|k| \to \infty$. It follows, that $g_k \to 0$ as $|k| \to \infty$.

The decomposition of $(I + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1}$ from the proof of Lemma 4.6.7 gives

$$\frac{1}{\theta(s)} (1 + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1} = \frac{1}{\theta(s)} \Phi_{1k}(s) \left(I + \epsilon \Phi_{2k}(s)\Phi_{1k}^{-1}(s) \right)^{-1}.$$

Here one can choose sufficiently small $\epsilon > 0$ so that $\epsilon \Phi_{2k}(s)\Phi_{1k}^{-1}(s) < \frac{1}{2}$ near $\mathrm{i}\omega_k$. It follows, that

$$\lim_{s \to \mathrm{i}\omega_k} \left| (1 + \mathcal{P}(s)\mathcal{C}(s))^{-1} \times \frac{1}{\theta(s)} \right| \geq \frac{1}{2} \lim_{s \to \mathrm{i}\omega_k} \left| \frac{s - \mathrm{i}\omega_k}{(s - \mathrm{i}\omega_k + \epsilon g_k)\theta(s)} \right| = \frac{1}{2\epsilon|g_k|}.$$

The last term approaches infinity as $|k| \to \infty$, so there exists no $\epsilon^* > 0$ such that $\frac{1}{\theta(s)}(1 + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1} \in H^\infty$ for all $\epsilon \in (0, \epsilon^*]$. ∎

4.6.2   Design of Robustly Regulating Controllers for Unstable Plants

The results of Section 4.4 give conditions for the solvability of the robust regulation problem, but they do not give any actual controllers for unstable plants.
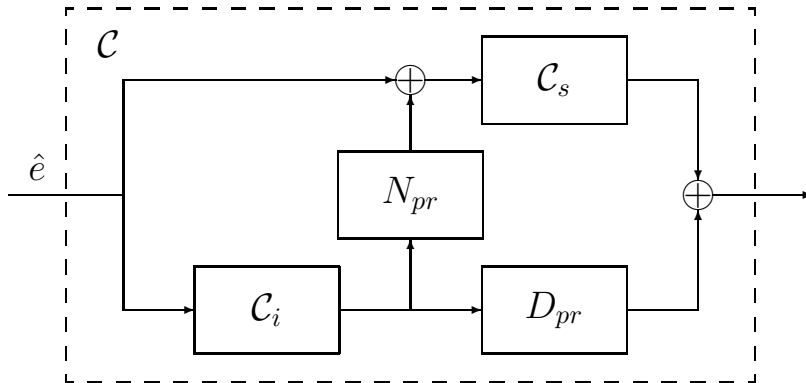
Figure 4.1: Structure of the designed robustly regulating controller.

An idea how to find a robustly regulating controller can be found in the proofs of Theorem 4.5.12 and Theorem 4.5.11. In what follows, this idea is made precise and a procedure for finding a robustly regulating controller is proposed. For a plant with a right coprime factorization (N_{pr}, D_{pr}) the proposed controller consists of a stabilizing controller $\mathcal{C}_s$ and a controller $\mathcal{C}_i$ that robustly regulates N_{pr} and contains an internal model. The controller has the structure depicted in Figure 4.1.

In [25] it was shown that a regulating controller should contain an internal model cascaded with a loop stabilizer. Roughly speaking, a regulating controller can be constructed by adding an internal model to the plant and stabilizing the resulting system. Such a design procedure of robustly regulating controllers was also suggested for rational transfer functions in [92]. Although the controller designed here necessarily has the suggested structure, the design procedure is somewhat different. Here a stabilizing controller is designed for the plant alone including no internal model and the robustly regulating controller is designed for a stable part of the plant.

Controller Design for Plants with a Right Coprime Factorization

Let $\mathcal{P}$ have a right coprime factorization (N_{pr}, D_{pr}). A robustly regulating controller can be found by using the following procedure:

1. Find a stabilizing controller $\mathcal{C}_s$ for $\mathcal{P}$.

2. Find a robustly regulating controller $\mathcal{C}_i$ for N_{pr}. Denote

$$\widetilde{D}_i = (I + N_{pr}\mathcal{C}_i)^{-1} \text{ and } \widetilde{N}_i = \mathcal{C}_i\widetilde{D}_i.$$

3. A robustly regulating controller is given by

$$\mathcal{C} = \mathcal{C}_s\widetilde{D}_i^{-1} + D_{pr}\mathcal{C}_i = \begin{bmatrix} \mathcal{C}_s & I \end{bmatrix} \begin{bmatrix} I & N_{pr} \\ 0 & D_{pr} \end{bmatrix} \begin{bmatrix} I \\ \mathcal{C}_i \end{bmatrix}.$$

Note that the second step is possible if and only if the robust regulation problem is solvable by Theorem 4.5.12. Similar justifications to those made in the proof of Lemma 4.1.2 show that the factorization of $\mathcal{C}_i$ in the second step is right coprime and satisfies $N_{pr}\widetilde{N}_i + \widetilde{D}_i = I$. To show that $\mathcal{C}$ is robustly regulating, let $(\widetilde{N}_e, \widetilde{D}_e)$ be a right coprime factorization of $\mathcal{C}_e = \begin{bmatrix} \mathcal{C}_s & 0 \end{bmatrix}$. Now

$$\mathcal{C} = \left(\widetilde{N}_e + D_{pr}R\right) \left(\widetilde{D}_e - \begin{bmatrix} N_{pr} \\ 0 \end{bmatrix} R\right)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

where $R = \begin{bmatrix} \widetilde{N}_i & 0 \end{bmatrix}\widetilde{D}_e$. By Lemma 4.1.4, $\begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$ stabilizes $\begin{bmatrix} \mathcal{P}^T & 0 \end{bmatrix}^T$ so it is easy to show that $\mathcal{C}$ stabilizes $\mathcal{P}$. In addition, $\left(\widetilde{N}_e + D_{pr}R, \widetilde{D}_e - \begin{bmatrix} N_{pr}^T & 0 \end{bmatrix}^T R\right)$ is a right coprime factorization of $\begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$. Since $\theta^{-1}\widetilde{D}_i \in \mathcal{M}(\mathbf{P})$ by Theorem 4.5.6 and $\begin{bmatrix} I & 0 \end{bmatrix}\left(\widetilde{D}_e - \begin{bmatrix} N_{pr}^T & 0 \end{bmatrix}^T R\right) = \begin{bmatrix} \widetilde{D}_i & 0 \end{bmatrix}\widetilde{D}_e$, Theorem 4.5.7 implies that $\mathcal{C}$ is robustly regulating.

Controller Design for Plants with a Left Coprime Factorization

Let $\mathcal{P}$ have a left coprime factorization (N_{pl}, D_{pl}). A robustly regulating controller can be found by using the following procedure:

1. Find a stabilizing controller $\mathcal{C}_s$ for $\mathcal{P}$. Denote

$$D_{cr} = (D_{pl} + N_{pl}\mathcal{C}_s)^{-1} \text{ and } N_{cr} = \mathcal{C}_s D_{cr}.$$

2. Find a right coprime factorization (N_e, D_e) of $\mathcal{P}_e = \begin{bmatrix} \mathcal{P} & 0 \end{bmatrix}$. A way to do this can be found in [65].

3. Find a robustly regulating controller $\mathcal{C}_i$ for N_e. Denote

$$\widetilde{D}_i = (I + N_e\mathcal{C}_i)^{-1} \text{ and } \widetilde{N}_i = \mathcal{C}_i\widetilde{D}_i.$$

4. A robustly regulating controller is given by

$$\mathcal{C} = \mathcal{C}_s\widetilde{D}_i^{-1} + \begin{bmatrix} I & 0 \end{bmatrix} D_e\mathcal{C}_i = \begin{bmatrix} \mathcal{C}_s & I \end{bmatrix} \begin{bmatrix} I & N_e \\ 0 & \begin{bmatrix} I & 0 \end{bmatrix} D_e \end{bmatrix} \begin{bmatrix} I \\ \mathcal{C}_i \end{bmatrix}.$$

The proof of Lemma 4.1.2 shows that, in the first step, the selected factorization of the controller is right coprime and satisfies $N_{pl}N_{cr} + D_{pl}D_{cr} = I$. Similarly, $N_e\widetilde{N}_i + \widetilde{D}_i = I$. By Theorem 4.5.11 and Corollary 4.5.13, the third step is possible if and only if the robust regulation problem is solvable. To see that $\mathcal{C}$ actually is robustly regulating, note that

$$\mathcal{C} = \left(N_{cr} + \begin{bmatrix} I & 0 \end{bmatrix} D_e R\right)(D_{cr} - N_e R)^{-1},$$

where $R = \widetilde{N}_i D_{cr}$. Since $D_{cr} - N_e R = \widetilde{D}_i D_{cr}$, Lemma 4.5.2 shows that the controller is well defined and stabilizing. Furthermore, $\theta^{-1}\widetilde{D}_i \in \mathcal{M}(\mathbf{P})$ by Theorem 4.5.6, so the same theorem implies that the controller is robustly regulating.

4.6.3 Design of Robustly Regulating Controllers in the Callier-Desoer Algebra

The power of the controller design procedure proposed in Section 4.6.2 becomes apparent when one considers plants that have a coprime factorization with the numerator matrix in H_-^∞. The controller (4.16) is robustly regulating for the stable numerator matrix of such plants. In particular, all the matrices in the Callier-Desoer class have a right coprime factorization (N, D) where $N \in H_-^\infty$, as was stated in Section A.3. In addition, standard techniques can be used to stabilize a plant in $\mathcal{M}\left(\hat{\mathcal{B}}(0)\right)$. Thus, the proposed design procedure is especially suitable for the plants in the Callier-Desoer class of transfer functions.

The output regulation problem in the Callier-Desoer class of transfer functions has been studied earlier in [7, 25]. However, no robustness properties were studied in [25], and the signal generator had only a finite number of unstable dynamics in both articles. In [34], robustly regulating controllers were designed, but the plants were assumed to be stable, and the reference and disturbance signals consisted of finite sums of sinusoidal functions. Thus, the results presented here extends the existing theory to infinite-dimensional reference signals.

Corollary 4.6.11 gives a necessary and sufficient condition for the left coprimeness of H_-^∞-stable matrix and the function (4.11). The next corollary gives similar conditions for all the plants in $\mathcal{M}\left(\hat{\mathcal{B}}(0)\right)$.

Corollary 4.6.15. *1. Let $\mathcal{P} \in \mathbf{F_P}$ have a left coprime factorization, and let (N_e, D_e) be a right coprime factorization of $\mathcal{P}_e = \begin{bmatrix} \mathcal{P} & 0 \end{bmatrix}$. If $N_e \in \mathcal{M}\left(H_-^\infty\right)$, then the $(\hat{\mathcal{Y}}(f_k, a_k), \emptyset)$-regulation problem is solvable if and only if N_e is right invertible at points $i\omega_k$ and the right inverses $N_e^r(i\omega_k)$ can be chosen so that there exist $\alpha > 0$ and $M > 0$ such that $\|N_e^r(i\omega_k)\| < M(|\omega_k| + 1)^\alpha$ for all $k \in \mathbb{Z}$.*

2. *Let $\mathcal{P} \in \mathbf{F_P}$ have a right coprime factorization (N_{pr}, D_{pr}). If $N_{pr} \in \mathcal{M}\left(H_-^\infty\right)$ then the $(\hat{\mathcal{Y}}(f_k, a_k), \emptyset)$-regulation problem is solvable if and only if N_{pr} is right invertible at points $\mathrm{i}\omega_k$, and the right inverses $N_{pr}^r(\mathrm{i}\omega_k)$ can be chosen so that there exist $\alpha > 0$ and $M > 0$ such that $\left\|N_{pr}^r(\mathrm{i}\omega_k)\right\| < M(|\omega_k| + 1)^\alpha$ for all $k \in \mathbb{Z}$.*

Proof. The claim follows by Corollary 4.6.11, Theorem 4.5.11, and Theorem 4.5.12. $\qquad\square$

Note that the above result and the proposed design procedure are valid for a more general class of transfer functions than $\mathcal{M}\left(\hat{\mathcal{B}}(0)\right)$. An example of such a class of transfer functions is given by the stabilizable transfer functions in $\mathbf{F}_{H^\infty}$ since they have both coprime factorizations over H_-^∞ by [87]. The following example illustrates the design procedure of Section 4.6.2 for a plant in $\mathcal{M}\left(\hat{\mathcal{B}}(0)\right)$.

Example 4.6.16. In what follows, a robustly regulating controller is constructed for the transfer function

$$\mathcal{P}(s) = \begin{bmatrix} 0 & \frac{1}{s} & \frac{1}{s-2} \\ \frac{1}{s+1} & \frac{1}{s} + \frac{1-e^{-s}}{s(s+1)} & 0 \end{bmatrix}$$

stemming from a delay system. First a right coprime factorization is to be searched. To this end, the transfer function is presented as a sum of a rational transfer function that contains the unstable poles of the plant, and a stable transfer function.

$$\mathcal{P}(s) = \begin{bmatrix} 0 & \frac{1}{s} & \frac{1}{s-2} \\ 0 & \frac{1}{s} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1-e^{-s}}{s(s+1)} & 0 \end{bmatrix} = \mathcal{P}_r(s) + \mathcal{P}_s(s).$$

A right coprime factorization of the rational matrix $\mathcal{P}_r$ is (N_r, D_r) where

$$N_r(s) = \begin{bmatrix} 0 & \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & \frac{1}{s+1} & 0 \end{bmatrix} \text{ and } D_r(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{s}{s+1} & 0 \\ 0 & 0 & \frac{s-2}{s+1} \end{bmatrix}.$$

Set $X = I$ and

$$Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}.$$

Now, $Y N_r + X D_r = I$. Choose $D_{pr} = D_r$ and

$$N_{pr} = N_r + \mathcal{P}_s D_r = \begin{bmatrix} 0 & \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1-e^{-s}}{(s+1)^2} & 0 \end{bmatrix}.$$

The pair (N_{pr}, D_{pr}) is a right coprime factorization since

$$N_{pr}D_{pr}^{-1} = N_r D_r^{-1} + \mathcal{P}_s = \mathcal{P}_r + \mathcal{P}_s = \mathcal{P}$$

and

$$Y N_{pr} + (X - Y\mathcal{P}_s)D_{pr} = YN_r + Y\mathcal{P}_s D_r + XD_r - Y\mathcal{P}_s D_r = I. \qquad (4.19)$$

Set $N_0 = Y$ and

$$D_0 = X - Y\mathcal{P}_s = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{s+1} & 1 - \frac{1-e^{-s}}{s(s+1)} & 0 \\ \frac{3}{s+1} & \frac{3(1-e^{-s})}{s(s+1)} & 1 \end{bmatrix}.$$

The equation (4.19) shows that $\mathcal{C}_0 = D_0^{-1}N_0$ is a stabilizing controller of $\mathcal{P}$.

A robustly regulating controller for N_{pr} is given next. It is easy to see that the structure at infinity of N_{pr} is $\left\{\frac{1}{s+1}, \frac{1}{s+1}\right\}$, so a robustly regulating controller for N_{pr} exists by Corollary 4.6.12 and Corollary 4.5.13. Since

$$N_{pr}(s)\begin{bmatrix} 0 & s+1 \\ 0 & 0 \\ s+1 & 0 \end{bmatrix} = N_{pr}(s)N_{pr}^r(s) = I$$

a right inverse of $N_{pr}(\mathrm{i}\omega_k)$ is $N_{pr}^r(\mathrm{i}\omega_k)$ for all $k \in \mathbb{Z}$. Choose $\epsilon^* = \frac{1}{60\gamma}$. A robustly regulating controller for N_{pr} is

$$\mathcal{C}_{i\epsilon}(s) = \sum_{k\in\mathbb{Z}} \frac{\epsilon N_{pr}^r(\mathrm{i}\omega_k)}{(s - \mathrm{i}\omega_k)(1 + |\omega_k|)^2} = \begin{bmatrix} 0 & c_\epsilon(s) \\ 0 & 0 \\ c_\epsilon(s) & 0 \end{bmatrix},$$

where $c_\epsilon(s) = \sum_{k\in\mathbb{Z}} \frac{\epsilon(1+\mathrm{i}\omega_k)}{(s-\mathrm{i}\omega_k)(1+|\omega_k|)^2}$ and ϵ is any positive real number in $(0, \epsilon^*]$.

One now has a right coprime factorization (N_{pr}, D_{pr}) of $\mathcal{P}$, a stabilizing controller $\mathcal{C}_0$ of $\mathcal{P}$, and a robustly regulating controller $\mathcal{C}_i$ of N_{pr}. Thus, a robustly regulating controller for $\mathcal{P}$ is

$$\mathcal{C}_\epsilon(s) = \begin{bmatrix} \mathcal{C}_0(s) & I \end{bmatrix}\begin{bmatrix} I & N_{pr}(s) \\ 0 & D_{pr}(s) \end{bmatrix}\begin{bmatrix} I \\ \mathcal{C}_{i\epsilon}(s) \end{bmatrix} = \begin{bmatrix} 0 & c_\epsilon(s) \\ 0 & \frac{s(1+s+c_\epsilon(s))}{s(s+1)+e^{-s}-1} \\ 3 + c_\epsilon(s) & \frac{-3s(1+s+c_\epsilon(s))}{s(s+1)+e^{-s}-1} \end{bmatrix},$$

where $c_\epsilon(s) = \sum_{k\in\mathbb{Z}} \frac{\epsilon(1+\mathrm{i}\omega_k)}{(s-\mathrm{i}\omega_k)(1+|\omega_k|)^2}$ and ϵ is a real number in $\left(0, \frac{1}{60\gamma}\right]$. Recall that $\gamma > 0$ is any number less than one quarter of the minimum distance between the poles $\mathrm{i}\omega_k$. Note that the tuning parameter $\epsilon > 0$ enables online tuning of the controller if necessary.   ∎

4.7   On Smoothness of the Reference and Disturbance Signals

A cursory survey on the role of the signal smoothness in frequency domain regulation is made in this section. The purpose is not to do a comprehensive study on the subject, but to illustrate some related ideas by examples. The plants are assumed to be in $\mathcal{M}\left(\hat{\mathcal{B}}(0)\right)$.

It was shown in Section 3.3 that, if a robustly regulating controller was given in the time domain, the transfer functions of all perturbed plants should satisfy certain conditions related to the smoothness of the signals. Corollary 4.6.15 shows that for plants in Callier-Desoer class of transfer functions the solvability of the robust regulation problem depends solely on the asymptotic behavior of the numerator matrix $N_{pr}(\mathrm{i}\omega_k)$, as $k \to \infty$. Thus, unlike in the time domain, the solvability of the frequency domain robust regulation problem does not depend on the smoothness of the reference or disturbance signals.

In the frequency domain, one might want the error signal to be in H^∞ for all reference and disturbance signals. The smoothness properties are of use in that case. The required smoothness is related to the asymptotic behavior of the plant as shown by the following example.

Example 4.7.1. Let $\mathcal{P}(s) \in H^\infty_-$ be such that the $(\hat{\mathcal{Y}}\left(f_k\right), \{0\})$-robust regulation problem is solvable. It is shown that, if the reference signals are smooth enough, it is possible to find such a robustly regulating controller that $\hat{e} \in H^\infty$ for any reference signals.

Consider a controller of form

$$\mathcal{C}_\epsilon(s) = \sum_{k \in \mathbb{Z}} \frac{\epsilon g_k}{\mathcal{P}(\mathrm{i}\omega_k)(s - \mathrm{i}\omega_k)}.$$

It is desired that $\hat{e} = (1 + \mathcal{P}\mathcal{C})^{-1}\hat{y}_r \in H^\infty$ for all $\hat{y}_r \in \hat{\mathcal{Y}}\left(f_k\right)$. This implies that $(1 + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1} \sum_{k \in \mathbb{Z}} \frac{f_k}{s - \mathrm{i}\omega_k}$ should be bounded on the imaginary axis. The value of $(1 + \mathcal{P}(s)\mathcal{C}_\epsilon(s))^{-1} \sum_{k \in \mathbb{Z}} \frac{f_k}{s - \mathrm{i}\omega_k}$ at $\mathrm{i}\omega_k$ is $\frac{f_k}{\epsilon g_k}$. It is seen that, if $\sup_{k \in \mathbb{Z}} \frac{f_k}{g_k} < \infty$, then one can choose ϵ to be small enough to guarantee that $\hat{e} \in H^\infty$.

The terms g_k are restricted by that $\mathcal{C}_\epsilon(s)$ should be well defined for almost all $s \in \overline{\mathbb{C}^+}$. This happens if $\left(\frac{g_k}{\mathcal{P}(\mathrm{i}\omega_k)}\right)_{k \in \mathbb{Z}} \in \ell^2$. If one chooses $f_k < \frac{|\mathcal{P}(\mathrm{i}\omega_k)|^{\frac{1}{2} + \epsilon_0}}{(1 + |\omega_k|)}$ and $g_k = \frac{|\mathcal{P}(\mathrm{i}\omega_k)|^{\frac{1}{2} + \epsilon_0}}{(1 + |\omega_k|)}$, then $\mathcal{C}_\epsilon$ is well defined and $\sup_{k \in \mathbb{Z}} \frac{f_k}{g_k} < \infty$. Thus, choosing ϵ small enough makes the given controller to satisfy the required properties. ∎

In the next example, the growth rate of the closed loop is considered. There a robustly regulating controller is given for a plant, and it is shown that there exists perturbations such that the closed loop remains **P**-stable while the polynomial growth rate $(1 + |\mathrm{i}\omega|)^\alpha$ of $\hat{e}$ on the imaginary axis can grow arbitrarily large.

Example 4.7.2. Choose $\mathcal{P}(s) = 1$. Set $\omega_k = k$ and $f_k = (1 + |k|)^{-1}$. The controller $\mathcal{C} = \sum_{k \in \mathbb{Z}} \frac{\epsilon}{(1 + |k|)(s - i\omega_k)}$ solves the $(\hat{\mathcal{Y}}(f_k), \{0\})$-robust regulation problem for $\epsilon > 0$. Here ϵ is chosen so that for all $k \in \mathbb{Z}$

$$\epsilon \left| \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{1}{(1 + |k|)(s - i\omega_k)} \right| < \frac{1}{4}.$$

Consider the perturbed systems

$$\mathcal{P}'(s) = \left(1 + \sum_{k \in \mathbb{Z}} \frac{\epsilon h_k}{s - i\omega_k + z_k} \right)^{-1} \in \mathbf{P}$$

where $h_k = (1 + |k|)^{-\alpha}$ and $z_k = (1 + |k|)^{-\beta}$ for some $\alpha > 1$ and $\beta > 1$. It is easy to verify that $\frac{1}{\mathcal{P}'(s)} \in \mathbf{P}$ and

$$\left(\frac{1}{\mathcal{P}'(s)} + \mathcal{C}(s) \right)^{-1} = \left(1 + \sum_{k \in \mathbb{Z}} \left(\frac{\epsilon h_k}{s - i\omega_k + z_k} + \frac{1}{(1 + |k|)(s - i\omega_k)} \right) \right)^{-1} \in \mathbf{P}.$$

Thus, $(1 + \mathcal{P}'\mathcal{C})^{-1} = (\mathcal{P}')^{-1} \left((\mathcal{P}')^{-1} + \mathcal{C} \right)^{-1} \in \mathbf{P}$. An analysis similar to that in the proof of Theorem 4.3.3 shows that $\mathcal{C} \left((\mathcal{P}')^{-1} + \mathcal{C} \right)^{-1} \in \mathbf{P}$, so $\mathcal{C}(1 + \mathcal{P}'\mathcal{C})^{-1} \in \mathbf{P}$. It follows that the closed loop system remains $\mathbf{P}$-stable in the perturbation. Taking the limit at $i\omega_k$ shows that there exists a constant $M > 0$ such that

$$\left| (1 + \mathcal{P}'(i\omega)\mathcal{C}(i\omega))^{-1} \sum_{k \in \mathbb{Z}} \frac{f_k}{s - i\omega_k} \right| > M \frac{h_k}{z_k} = M(1 + |k|)^{\beta - \alpha}.$$

Since the constants $\alpha > 1$ and $\beta > 1$ can be chosen arbitrarily, it follows that $\hat{e}(s)$ is polynomially bounded on the imaginary axis while the polynomial growth rate $\beta - \alpha$ can be arbitrarily large. ∎

Example 4.7.1 shows that the error signals are bounded on the imaginary axis, if the smoothness of the reference signals is restricted appropriately. On the other hand, Example 4.7.2 shows that some restrictions over the set of perturbations is needed, if the perturbations should preserve the boundedness of the error signal. The situation is somewhat similar to that in Chapter 3 with time domain plants. In the time domain, sufficient smoothness of the signals is guaranteed by the solvability of the regulator equation, but it is hard to give simple conditions that imply sufficient smoothness properties of the signals in the frequency domain.

4.8   S-stability and Robust Regulation in $\mathbf{F_S}$

When the robust regulation problem with **P**-stability was considered it was seen that the solvability of the problem was related to the decay rate of the plant. The purpose of this section is to get rid of this restriction. The first step to that direction is to weaken the stability type further, which leads to the definition of **S**-stability. The results of Section 4.4 and Section 4.6 hold if **P**-stability is replaced by the new stability type. The only exception is that the condition on the decay rate of $\mathcal{P}$ can be removed.

Unlike polynomial stability, strong stability does not limit the growth rate of the resolvent operator's norm on the imaginary axis. It was seen in Section 3.3 that, if strong stability is considered, one can always restrict the smoothness of the reference and disturbance signals appropriately so that the robust regulation problem is solvable. With polynomial stability the norm of the original resolvent operator should be polynomially bounded. It is seen below that the situation is very similar also in the frequency domain; the removal of the polynomial boundedness assumption of **P**-stability allows the robust regulation problem to be solved in a more general class of transfer functions.

A complex function f is said to be **S**-*stable* if it satisfies the following conditions:

1. f is analytic in an open set containing $\overline{\mathbb{C}^+}$, and

2. $f \in H_\beta^\infty$ for all $\beta > 0$.

The set of all **S**-stable functions is denoted by **S**. Together with pointwise addition and multiplication **S** forms an integral domain with an identity.

To avoid confusion, the regulation problem to be considered is formulated in terms of **S**-stability. Let $\hat{\mathcal{Y}}$ and $\hat{\mathcal{D}}$ be sets of reference and disturbance signals, respectively. Let $\mathcal{P} \in \mathcal{M}(\mathbf{F_S})$ be given. The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-regulation problem is defined in the following way: Find a controller $\mathcal{C} \in \mathcal{M}(\mathbf{F_S})$ such that

1. $\mathcal{C}$ **S**-stabilizes $\mathcal{P}$, and

2. for all $\hat{y}_r \in \hat{\mathcal{Y}}$ and $\hat{d} \in \hat{\mathcal{D}}$

$$\hat{e} = \begin{bmatrix} (I + \mathcal{P}\mathcal{C})^{-1}\mathcal{P} & (I + \mathcal{P}\mathcal{C})^{-1} \end{bmatrix} \begin{bmatrix} \hat{d} \\ \hat{y}_r \end{bmatrix} \in \mathcal{M}(\mathbf{S}).$$

The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-robust regulation problem is defined in the following way: Find a controller $\mathcal{C} \in \mathcal{M}(\mathbf{F_S})$ such that

1. $\mathcal{C}$ **S**-stabilizes $\mathcal{P}$, and

2. $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$-regulation problem for all the plants it stabilizes.

Since $\mathbf{P} \subseteq \mathbf{S}$ it is clear that the signal generators from Section 4.3.2 generate all the desired signals. The proofs of Theorem 4.5.6 and Theorem 4.5.6 apply to the current problem so one has the following theorem.

Theorem 4.8.1. *Consider a plant $\mathcal{P} \in \mathbf{F_S}^{n \times m}$. Let θ be the function (4.8) with any sequence of positive real numbers $(h_k)_{k \in \mathbb{Z}} \in \ell^2$. Fix a sequence $(f_k)_{k \in \mathbb{Z}} \in \ell^2$.*

1. *If $\mathcal{P}$ has a left coprime factorization and $\mathcal{C}$ is a stabilizing controller with a right coprime factorization (N_{cr}, D_{cr}), then $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if $\theta^{-1} D_{cr} \in \mathcal{M}(\mathbf{S})$.*

2. *Let $\mathcal{P}$ have a right coprime factorization, and let $\mathcal{C}$ be a stabilizing controller. If $(\widetilde{N}_e, \widetilde{D}_e)$ is a right coprime factorization of $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} \in \mathbf{F_S}^{m \times (n+m)}$, then $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if $\theta^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e \in \mathcal{M}(\mathbf{S})$.*

Note that the assumption $h_k = f_k$ was not needed in the above theorem. This is due to the fact that the set $\theta^{-1}\mathbf{S}$ where θ is the function (4.8) does not depend on the sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ chosen.

If θ is the function (4.8), then $\theta^{-1} \in H_\beta^\infty$ for all $\beta > 0$. Consequently, if $D \in \mathcal{M}(\mathbf{S})$ then $\theta^{-1} D \in \mathcal{M}(\mathbf{S})$ if and only if $D(\mathrm{i}\omega_k) = 0$ for all $k \in \mathbb{Z}$. This shows the following corollary.

Corollary 4.8.2. *Consider a plant $\mathcal{P} \in \mathbf{F_S}^{n \times m}$. Let θ be the function (4.8) with any sequence of positive real numbers $(h_k)_{k \in \mathbb{Z}} \in \ell^2$. Fix a sequence $(f_k)_{k \in \mathbb{Z}} \in \ell^2$.*

1. *If $\mathcal{P}$ has a left coprime factorization (N_{pl}, D_{pl}) and $\mathcal{C}$ is a stabilizing controller with a right coprime factorization (N_{cr}, D_{cr}), then $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if $D_{cr}(\mathrm{i}\omega_k) = 0$ for all $k \in \mathbb{Z}$.*

2. *Let $\mathcal{P}$ have a right coprime factorization (N_{pr}, D_{pr}) and let $\mathcal{C}$ be a stabilizing controller. If $(\widetilde{N}_e, \widetilde{D}_e)$ is a right coprime factorization of $\mathcal{C}_e = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} \in \mathbf{F_S}^{m \times (n+m)}$, then $\mathcal{C}$ solves the $(\hat{\mathcal{Y}}(f_k, a_k), \{0\})$-robust regulation problem if and only if $\begin{bmatrix} I & 0 \end{bmatrix} \widetilde{D}_e(\mathrm{i}\omega_k) = 0$ for all $k \in \mathbb{Z}$.*

It is now possible to state a result that corresponds to the time domain blocking zero condition for robustly regulating controllers in [26].

Corollary 4.8.3. *Consider a plant $\mathcal{P} \in \mathcal{M}(\mathbf{F_S})$. Let θ be the function (4.8) with any sequence of positive real numbers $(h_k)_{k \in \mathbb{Z}} \in \ell^2$, and assume that $\mathcal{P}$ has right or left coprime factorization. A controller $\mathcal{C}$ solves the $(\theta\mathbf{S}, \theta\mathbf{S})$-robust regulation problem if and only if it $\mathbf{S}$-stabilizes $\mathcal{P}$ and*

$$\begin{bmatrix} -(I + \mathcal{P}'(\mathrm{i}\omega_k)\mathcal{C}(\mathrm{i}\omega_k))^{-1}\mathcal{P}'(\mathrm{i}\omega_k) & (I + \mathcal{P}'(\mathrm{i}\omega_k)\mathcal{C}(\mathrm{i}\omega_k))^{-1} \end{bmatrix} = 0$$

for all $\mathcal{P}'$ which $\mathcal{C}$ $\mathbf{S}$-stabilizes.

Since the internal model principle holds by Theorem 4.8.1, the results of Section 4.5.2 are applicable by Remark 4.5.14. Thus, the left coprimeness of the numerator matrix of a right coprime factorization and the generating element characterizes the solvability of the robust regulation problem.

The controller design of Section 4.6 can be carried out with **S**-stability. In fact the proofs are easier, because no attention to the behavior on the imaginary axis needs to be payed as long as the closed loop system remains analytic on the imaginary axis. This is why the polynomial decay rate of a plant $\mathcal{P}(s) \in \mathcal{M}\left(H_-^\infty\right)$ is not needed; only right invertibility of $\mathcal{P}(i\omega_k)$ is required for all $k \in \mathbb{Z}$.

Theorem 4.8.4. *Let θ be the function* (4.8). *If $\mathcal{P} \in \mathcal{M}\left(H_-^\infty\right)$, then $(\theta I, \mathcal{P})$ is left coprime if and only if $\mathcal{P}(i\omega_k)$ is right invertible for all $k \in \mathbb{Z}$.*

Since the controller design procedure of Section 4.6.2 for unstable plants is applicable, the above theorem gives an easy characterization for the solvability of the robust regulation problem with **S**-stability for all plants in the Callier-Desoer class of transfer functions.

Example 4.8.5. Recall the transfer functions in Example 2.2.18 and Example 3.2.17. They both belong to H^∞ and do not have transmission zeros on the imaginary axis. Thus, the $(\hat{\mathcal{Y}}\left(f_k, a_k\right), \{0\})$-robust regulation problem with **S**-stability is solvable with both of the transfer functions. A comparison to Example 4.6.13 illustrates the difference between the robust regulation problems with **S**-stability and **P**-stability. ∎

Chapter 5

Concluding Remarks and Directions for Further Research

5.1 Concluding Remarks

In this thesis, the robust regulation problem with an infinite-dimensional exosystem was studied. The main emphasis was in the frequency domain formulation of the problem. The thesis was divided into three somewhat independent topics – the structure at infinity, the time domain robust regulation problem and the frequency domain robust regulation problem – each of which was studied in its own chapter. Relations between the topics were discussed.

In Chapter 2, a new definition for the structure at infinity was given. The strength of the definition, when compared to the definitions based on the Smith-McMillan form, is that it is available for a more general class of transfer functions, including the transfer functions of plants with an infinite-dimensional state operator. However, the structure at infinity has weak algebraic properties compared to the conventional definition, which is an unavoidable trade-off caused by the complicated high frequency behavior of transfer functions that are not meromorphic at infinity. The structure at infinity describes the convergence or divergence rate of the transfer function at high frequencies. This property was exploited in Chapter 3 and Chapter 4. The structure at infinity of exponentially stabilizable plants was shown to be invariant with respect to state feedback and output feedback, but it was also shown that in general it has weak robustness properties.

Chapter 3 was dedicated to the study of the robust regulation problem in the time domain. In the introduction of the chapter, special attention was paid to how the robustness should be defined and what is an appropriate type of stability. The research results of the chapter focused on the relation between the structure at infinity of the plant transfer function and the solvability of the output regulation problem. The solvability of the feedforward output regula-

tion problem was characterized and the required smoothness of the reference signals was revealed in Section 3.2 by using the structure at infinity of the plant defined in Chapter 2. Roughly speaking, the faster the transfer function approaches zero at high frequencies the smoother the signals should be for the robust regulation to be solvable. A similar connection between the robust error feedback output regulation problem and the high frequency behavior of the plant transfer function was found in Section 3.3. Since a small perturbation can change the structure at infinity drastically, see Example 2.5.5, the found connection shows that the solvability of the Sylvester equation (3.10) has weak robustness properties. Consequently, the robustness properties of the controller are generally weak even if the perturbations are assumed to retain the closed loop stability. It was shown that the degree of robustness can be increased by restricting the set of reference and disturbance signals to a signal class with better smoothness properties.

The formulation of the robust regulation problem in the frequency domain in Chapter 4 used the ideas of Chapter 3, i.e., the definition of robustness, the stability types, and the classes of the reference and disturbance signals were all inspired by their time domain counterparts. The results found in this thesis show that this approach is reasonable.

The reference and disturbance signals were chosen to be the Laplace transforms of those generated by the exosystem (3.2). The signals had an infinite number of poles on the imaginary axis. Consequently, the robust regulation problem could not be solved if H^∞-stability of the closed loop system was required. This problem was solved by introducing two new rings of stable transfer functions, $\mathbf{P}$ and $\mathbf{S}$, resembling polynomial stability and strong stability in the time domain. The reason why these stability types made the robust regulation problem solvable even if the plant transfer function was strictly proper was that they allowed the closed loop transfer function to grow on the imaginary axis.

The use of the new rings of stable transfer functions lead to another problem, namely, in what sense should the robustness be understood. Again, the answer was found by using a time domain idea. The robustness type considered was defined by saying that a controller is robustly regulating if it solves the regulation problem for all the plants it stabilizes. It should be noted that the same type of robustness was considered already in [7], where a design procedure of a robustly regulating controller was proposed in the Callier-Desoer class of transfer functions. Defining robustness this way made it possible to consider the robustness of regulation without bothering the robustness of stability. This is why there was no need to find topologies for the proposed rings of stable transfer functions. Although omitting the topological aspect might seem to restrict the usability of the theory, one should note that the classical concept of robustness necessitates a robustly regulating controller to posses an internal

model, both in the time domain and in the frequency domain. This implies that for situations where the classical notion of robustness is feasible, the two concepts of robustness are equivalent.

The theory of robust regulation for rational transfer functions in [92] made use of coprime factorizations and the invariant factors. In particular, the frequency domain version of the internal model principle has the following form for rational transfer functions:

The internal model principle for rational transfer functions: Let
the pair (N, D) be a left coprime factorization of the generator Θ and (N_{cr}, D_{cr}) be a right coprime factorization of a stabilizing controller $\mathcal{C}$. If θ is the largest invariant factor of D, then the controller is robustly regulating if and only if θ divides D_{cr}.

In this thesis, coprime factorizations of plants and controllers were used, but invariant factors could not be defined in the algebraic structures considered. However, a generating function θ for the signals considered in this thesis that corresponds to the largest invariant factor in the above formulation of the internal model principle was found in Section 4.3.2. This function served as an internal model and a generalization of the above formulation of the internal model principle to infinite-dimensional systems was found in Section 4.5.1. In addition, solvability of the robust regulation problem was characterized by using the function θ and a coprime factorization of the plant.

In Section 4.6.1, it was shown that the Davison's robust controller design for finite-dimensional stable plants can be generalized in a straightforward manner for H_-^∞-stable plants if the closed loop was required to be **P**-stable. This controller design and the solvability condition from Section 4.5.2 revealed a simple solvability condition for H_-^∞-stable plants, i.e., the robust regulation problem with **P**-stability was shown to be solvable if and only if the decay rate of the plant transfer function at the zeros of the generator θ was at most polynomial. Later in Section 4.8, it was shown that this solvability condition further simplifies if only the **S**-stability of the closed loop is required; in this case, the plant transfer function was required to be only invertible at the zeros of the generator function θ.

A simple way to design a controller for a plant transfer function with a right coprime factorization (N_{pr}, D_{pr}) was presented in 4.6.2, provided one is able to robustly regulate the numerator N_{pr}. To that end, one needed to find any stabilizing controller $\mathcal{C}_s$ for the plant and a robustly regulating controller $\mathcal{C}_i$ for N_{pr}. The controller $\mathcal{C}_i$ contained the internal model and appropriately combining the two controllers resulted into a stabilizing controller with an internal model. The power of the controller design lies in that one can separately design a stabilizing controller – several techniques readily available – and a robustly regulating controller for a stable plant, which is generally easier task than to

design a robustly regulating controller for an unstable plant directly. In Section 4.6.3, it was seen that the proposed controller design is especially suitable for plants in the Callier-Desoer class of transfer functions since the plants in the Callier-Desoer class possess a right coprime factorization with the numerator matrix in $\mathcal{M}\left(H^{\infty}_{-}\right)$.

5.2   Directions for Further Research

A fairly comprehensive study on the robust regulation problem was made in this thesis, but there are still many related topics that need to be studied. These topics are discussed in this section. The topics are mainly related in one way or another to the connection between the time domain and the frequency domain description of the robust regulation problem, which was not studied in this thesis. This connection is important for applicability of the theory developed in this thesis due to dominating role of state space representations in the modern control engineering.

The structure of transfer functions at infinity defined in Chapter 2 described the rate of the convergence or divergence of the transfer function at high frequencies and was sufficient to make the relation between the smoothness of the signals and the solvability of the time domain output regulation problem apparent. In many problems, for example model matching, the structure needs to be described in a certain algebraic structure and the algorithm used when defining the structure should work in that algebraic structure. In such a case, one would need the algebraic structure to be a principal ideal domain so that the Smith form exists. Consequently, the definition of the structure at infinity given here is too weak. However, many of the algebraic structures related to infinite-dimensional plants are not principal ideal domains. Two interesting research topics arise. First, would it be possible to find a way to find easily verifiable conditions under which a diagonal form similar to Smith form exists for plant transfer functions of infinite-dimensional systems? Secondly, would it be possible to define the structure at infinity without using any algorithm at all, particularly in the way that one could define the structure of a linear system with infinite-dimensional input and output spaces?

In the time domain theory for robust regulation developed in [34, 39, 70, 72] the perturbations were assumed to preserve the solvability of the Sylvester equation $\Sigma S + A_e \Sigma = B_e$. In Chapter 3, the solvability of the equation with SISO plants was shown to be related to the structure of the plant transfer function at infinity. The results in the same chapter for feedforward controllers indicates that the solvability results should extend to MIMO plants. Furthermore, the relation between the solvability and the structure at infinity shows the importance of understanding the robustness properties of the structure at infinity. Some

results to this direction were presented in Chapter 2, but a more detailed study on this subject should be made in order to get a better idea on the perturbations that a controller possessing an internal model tolerates.

It was mentioned that the relation between the time domain and the frequency domain formulations of the robust regulation problem should be studied. Maybe the most important part of this work is providing realization theory for the frequency domain controllers. A specific research question would be: under which conditions does a realization of a frequency domain controller solving the robust regulation problem solve the time domain robust output regulation problem? A part of the research topic would be finding out in what sense realizations of $\mathbf{P}$-stabilizing or $\mathbf{S}$-stabilizing controllers stabilize the extended system in the time domain. It was mentioned that $\mathbf{P}$-stability and $\mathbf{S}$-stability resemble polynomial stability and the strong stability in the time domain, respectively. Under the assumption that the state operator of a plant is polynomially and strongly stable and generates a bounded semigroup, one knows that its transfer function is $\mathbf{P}$ [3], but it would be hard to say when $\mathbf{P}$-stability of the plant transfer function implies polynomial stability in the time domain. The connection between the strong and $\mathbf{S}$-stability is even more fuzzy. Strong stability allows accumulation points of point spectra on the imaginary axis, but $\mathbf{S}$-stability does not allow accumulation points of poles there. Even the current definition of $\mathbf{S}$-stability is sufficient to tackle the problem considered in this thesis, one might want to revise the definition in order to facilitate more complicated signal generators or plants and to have stronger connection between the strong stability in the time domain and the $\mathbf{S}$-stability in the frequency domain.

With the type of robustness considered in this thesis, the perturbations a robustly regulating controller tolerates are just the perturbations preserving closed loop stability. Thus, the robustness of stability is of importance. To be able to use stability theory for topological rings studied by Vidyasagar et al. in [94], one needs to define a topology to the rings $\mathbf{P}$ and $\mathbf{S}$. It would be even better if one is able to find a metric inducing a suitable topology, since it would enable quantitative measurement of robustness and deeper understanding of robustness of stability. The author believes that the robustness properties of $\mathbf{P}$-stability would be better and the task of finding a suitable topology would be easier for $\mathbf{P}$ compared to finding it for $\mathbf{S}$-stability, because the transfer function behavior on the imaginary axis is generally better in $\mathbf{P}$ than in $\mathbf{S}$. At least in the time domain, Paunonen has been able to find classes of perturbations preserving polynomial stability [69].

The time domain theory of robust regulation in [39, 68, 72] is suitable for more general signal classes than those considered in this thesis. The theory allows the state operator of the exosystem to be a block diagonal operator with an infinite number of non trivial Jordan blocks on its diagonal. The signals gener-

ated by such operators correspond to signals with higher order poles. One possible direction for future research is to generalize the results in this thesis for such signals. The generalization for signals with a finite number of unstable higher order poles and an infinite number of simple poles is believed to be straightforward; one could possibly combine the finite-dimensional controller design for signals with higher order poles from [33] and the controller for infinite-number of simple poles presented in this thesis. The theoretical results concerning the solvability of the problem might also generalize in a straightforward manner for generators with infinite number of higher order poles. However, since the controller design in [33] relied on the finite-dimensionality of the exosystem, the actual controller design for signals with an infinite number of unstable higher order poles might be hard or even impossible.

The class of linear systems with bounded input and output operators is quite restricted. One possible research direction would be to extend the time domain theory to systems with unbounded input and output operators to allow boundary control and measurement. It would be also interesting to allow the plants to have some nonlinearities. One possible class of systems are the semilinear ones. Although the robust regulation of nonlinear systems has been studied before, see [88] and the references therein, it would be interesting to see whether the new insights presented in this thesis hold for a more general class of systems.

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Appendix A

Selected Results from Literature

A.1 Some Results from Linear Algebra

The following inversion formula called the Woodbury matrix identity is used extensively in this thesis:

$$(X + YZV)^{-1} = X^{-1} - X^{-1}Y(Z^{-1} + VX^{-1}Y)^{-1}VX^{-1}, \qquad \text{(A.1)}$$

where X and Z are invertible.

The Binet-Cauchy formula [81, Theorem 1.3] states, that if A and B are $n \times m$ and $m \times l$ matrices, respectively, and $i \leq \min\{n, l\}$, then

$$|AB|_{\mathbf{r},\mathbf{c}}^{i} = \sum_{\mathbf{h}} |A|_{\mathbf{r},\mathbf{h}}^{i} |B|_{\mathbf{h},\mathbf{c}}^{i}. \qquad \text{(A.2)}$$

A.2 Riesz Basis Assignment

The following result is a part of [98, Theorem 1]. Let G be the infinitesimal generator of a C_0-semigroup on a separable Hilbert space H. One makes the following hypotheses.

H1 The operator G has a compact resolvent, and its spectrum is simple and is denoted by $\sigma(G) = \{\lambda_k \,|\, k \in \mathbb{N}\}$.

H2 The domain $\mathcal{D}(G^*)$ of the adjoint operator G^* is a Hilbert space with the graph norm. $\mathcal{D}'(G^*)$ is the topological dual of $\mathcal{D}(G)$. It is supposed that $b \in \mathcal{D}'(G^*)$.

H3 The eigenvectors $\{\phi_k \in H \,|\, k \in \mathbb{N}\}$ of G form a Riesz basis in H and the biorthogonal sequence corresponding to the eigenvectors of G^* is denoted by $\{\psi_k \in H \,|\, k \in \mathbb{N}\}$. Set $b_k = (\psi_k, b)$ where $(\cdot, \cdot)$ is the classical duality product on $\mathcal{D}(G^*) \times \mathcal{D}'(G^*)$. Suppose that $b_k \neq 0$ for all $k \in \mathbb{N}$. Denote

$d_k = \inf_{k \neq l \in \mathbb{N}} \{|\lambda_k - \lambda_l|\}$. Suppose that there exists a constant $M > 0$ such that for all $\lambda \notin \bigcup_{k \in \mathbb{N}} B_{\frac{d_k}{3}}(\lambda_k)$ and $l \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \left| \frac{b_k}{\lambda - \lambda_k} \right|^2 \leq M \tag{A.3}$$

and

$$\sum_{\substack{k \in \mathbb{N} \\ k \neq l}} \left| \frac{b_k}{\lambda_l - \lambda_k} \right|^2 \leq M. \tag{A.4}$$

Denote the unique extension of $G : \mathcal{D}(G) \to H \subseteq \mathcal{D}'(G^*)$ in $\mathcal{L}(H, \mathcal{D}'(G^*))$ by $\hat{G}$. The linear operator $G_h : \mathcal{D}(G) \to \mathcal{D}'(G^*)$ admits a unique extension from H to $\mathcal{D}'(G^*)$ still denoted by G_h, and

$$G_h x = \hat{G} x + b\langle x, b \rangle.$$

The domain of G_h is defined to be

$$\mathcal{D}(G_h) = \left\{ x \in H \,\middle|\, \hat{A} x + b\langle x, h \rangle \in H \right\}.$$

Theorem A.2.1. *Assume that the hypotheses H1-H3 above hold.*

1. *For every $h \in H$ the operator G_h is regular spectral, i.e., it has a compact resolvent and its eigenvectors form a Riesz basis of H, and the spectrum $\sigma(G_h) = \{\nu_k \,|\, k \in \mathbb{N}\}$ satisfies condition*

$$\sum_{k \in \mathbb{N}} \left| \frac{\nu_k - \lambda_k}{b_k} \right|^2. \tag{A.5}$$

2. *If $\Lambda = \{\nu_k \,|\, k \in \mathbb{N}\}$ is a given set satisfying $\nu_k \neq \nu_l$ for $k \neq l$, then there exists an element $h \in H$ such that the operator G_h has spectrum $\sigma(G_h) = \Lambda$ if and only if the set Λ satisfies the condition (A.5). Moreover, the feedback is given by*

$$h = \sum_{k \in \mathbb{N}} h_k \psi_k \quad where \quad \overline{h}_k = \frac{\nu_k - \lambda_k}{\overline{b}_k} \prod_{\substack{l \in \mathbb{N} \\ l \neq k}}.$$

The infinite product above is understood as a limit in ℓ^2.

A.3 The Callier-Desoer Class of Transfer Functions

Here the Callier-Desoer class of transfer functions and some of its basic properties are presented. The Callier-Desoer algebra was first defined by Callier and Desoer in [9]. The results presented here are from [8, 14].

Let β be a real number, and denote the class of functions that Lebesgue measurable and absolutely integrable on the interval $[0, \infty)$ by $\mathbf{L}_1(0, \infty)$. Notation $\mathcal{A}(\beta)$ is used for the class of complex functions that can be presented in the form

$$f(t) = \begin{cases} f_0(t) + \sum_{i \in \mathbb{N}} f_i \delta(t - t_i), & \text{for } t \geq 0, \\ 0, & \text{for } t < 0, \end{cases}$$

where $t_i \in [0, \infty)$, $t_1 = 0$, $t_i > 0$ for $i \geq 2$, $\delta(t - t_i)$ is the t_i-centered delta distribution, $\mathrm{e}^{-\beta \cdot} f_0(\cdot) \in \mathbf{L}_1(0, \infty)$, and $(f_i)_{i \in \mathbb{N}}$ is a sequence of complex numbers such that $\sum_{i \in \mathbb{N}} = |f_i| \mathrm{e}^{-\beta t_i} < \infty$.

The set $\mathcal{A}(\beta)$ with norm

$$\|f\| = \int_0^\infty \mathrm{e}^{-\beta t} |f_0(t)| dt + \sum_{i \in \mathbb{N}} |f_i| \mathrm{e}^{-\beta t_i},$$

and the convolution product is a commutative Banach algebra with identity. The algebraic properties of $\mathcal{A}(\beta)$ are inherited by $\hat{\mathcal{A}}(\beta)$, the set of Laplace transforms of $\mathcal{A}(\beta)$. Subalgebras $\hat{\mathcal{A}}_-(\beta)$ and $\hat{\mathcal{A}}_\infty(\beta)$ of $\hat{\mathcal{A}}(\beta)$ are defined by

$$\hat{\mathcal{A}}_-(\beta) = \left\{ \hat{f} \in \hat{\mathcal{A}}(\beta) \,\middle|\, \hat{f} \in \hat{\mathcal{A}}(\beta_0) \text{ for some } \beta_0 < \beta \right\}$$

and

$$\hat{\mathcal{A}}_\infty(\beta) = \left\{ \hat{f} \in \hat{\mathcal{A}}_-(\beta) \,\middle|\, \inf_{\substack{s \in \overline{\mathbb{C}}_\beta^+ \\ |s| > \rho}} |\hat{f}(s)| > 0 \text{ for some } \rho > 0 \right\}.$$

Note that $\hat{\mathcal{A}}_\infty(\beta)$ is a subalgebra of $\hat{\mathcal{A}}_-(\beta)$.

The Callier-Desoer class of scalar transfer functions is defined to be

$$\hat{\mathcal{B}}(\beta) = \left\{ \hat{f} = \frac{\hat{n}}{\hat{d}} \,\middle|\, \hat{n} \in \hat{\mathcal{A}}_-(\beta) \text{ and } \hat{d} \in \hat{\mathcal{A}}_\infty(\beta) \right\}.$$

The Callier-Desoer class of transfer functions is defined to be $\mathcal{M}\left(\hat{\mathcal{B}}(\beta)\right)$.

Every matrix $\mathcal{P} \in \mathcal{M}\left(\hat{\mathcal{B}}(\beta)\right)$ can be presented in the form

$$\mathcal{P} = N_{pr} D_{pr}^{-1} = D_{pl}^{-1} N_{pl},$$

where D_{pr} and D_{pl} are square matrices satisfying $\det(D_{pr}), \det(D_{pl}) \in \hat{\mathcal{A}}_-(\beta) \setminus \{0\}$, and there exist matrices $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{M}\left(\hat{\mathcal{A}}_-(\beta)\right)$ such that

$$XN_{pr} + YD_{pr} = I$$

and

$$N_{pl}\tilde{X} + D_{pl}\tilde{Y} = I$$

on $\overline{\mathbb{C}_\beta^+}$. The pairs (N_{pr}, D_{pr}) and (N_{pl}, D_{pl}) are said to be a right coprime factorization and a left coprime factorization of $\mathcal{P}$, respectively.

Lemma A.7.47 in [14] states that the Laplace transform $\hat{\tilde{f}}$ of an $f \in \mathcal{A}(\beta)$ is holomorphic and bounded on $\overline{\mathbb{C}_\beta^+}$. Consequently, if (N, D) is a right or a left coprime factorization of a transfer function in $\mathcal{M}\left(\hat{\mathcal{B}}(0)\right)$, then $N \in H_-^\infty$.