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Commutators of fractional maximal operator on generalized Orlicz–Morrey spaces

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Abstract In the present paper, we shall give necessary and sufficient conditions for the Spanne and Adams type boundedness of the commutators of fractional maximal operator on generalized Orlicz–Morrey spaces, respectively. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators.

Keywords Generalized Orlicz–Morrey space · Fractional maximal operator · Commutator · BMO

Mathematics Subject Classification 42B25 · 42B35 · 46E30

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1 Introduction

For $1 \le p < \infty$ and a measurable function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the generalized Morrey space $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ is to be the set of all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the norm

$$\|f\|_{\mathcal{M}^{p,\varphi}} := \sup_{x\in\mathbb{R}^n,\ r>0} \varphi(x,r)^{-1}\, |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))}$$

is finite. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Note that, in the case $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, $0 \le \lambda \le n$, we get the classical Morrey space $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ from generalized Morrey space $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$. As is well known, Morrey spaces are widely used to investigate the local behavior of solutions to second order elliptic partial differential equations (PDE).

A natural step in the theory of functions spaces was to study Orlicz–Morrey spaces $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were first introduced and studied by Nakai [28]. Then another kind of Orlicz–Morrey spaces were introduced by Sawano et al. [37]. We point out that our definition of Orlicz–Morrey spaces introduced in [7,24] and used here is different from that of the papers [28] and [37]. In words of [19], our generalized Orlicz–Morrey space is the third kind and the ones in [28] and [37] are the first kind and the second kind, respectively. According to the examples in [12], one can say that the generalized Orlicz–Morrey space of the first kind and the second kind are different. Notice that the definition of the space of the third kind relies only on the fact that L^Φ is a normed linear space, which is independent of the condition that it is generated by modulars. On the other hand, the spaces of the first and the second kind are defined via the family of modulars.

Let $0 < \alpha < n$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \quad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

If $\alpha = 0$, then $M \equiv M_0$ is the well known Hardy-Littlewood maximal operator. Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x).$$

The commutators generated by a suitable function b and the operators M_α and I_α are formally defined by

$$[b, M_\alpha]f = M_\alpha(bf) - bM_\alpha(f), \quad [b, I_\alpha]f = I_\alpha(bf) - bI_\alpha(f),$$

respectively.

Given a measurable function b the operators $M_{b,\alpha}$ and $|b, I_\alpha|$ are defined by

$$M_{b,\alpha}(f)(x) := \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)|dy$$

and

$$|b, I_\alpha| f(x) := \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y)dy,$$

respectively. If $\alpha = 0$, then $M_{b,0} \equiv M_b$ is the sublinear commutator of the Hardy-Littlewood maximal operator. Recall that, for $0 < \alpha < n$,

$$M_{b,\alpha}(f)(x) \leq C|b, I_\alpha|(|f|)(x).$$

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid:

Let b be any non-negative locally integrable function. Then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}(f)(x), \quad x \in \mathbb{R}^n$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x), \quad x \in \mathbb{R}^n \tag{1.1}$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (see, for example, [39]).

The classical result by Hardy–Littlewood–Sobolev states that the operator I_α is of weak type $(p, np/(n - \alpha p))$, if $1 \leq p < n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p < n/\alpha$. Also the operator M_α is of weak type $(p, np/(n-\alpha p))$, if $1 \leq p \leq n/\alpha$ and of strong type $(p, np/(n - \alpha p))$, if $1 < p \leq n/\alpha$.

Around the 1970's, the Hardy–Littlewood–Sobolev theorem is extended from Lebesgue spaces to Morrey spaces by Spanne [32] and Adams [1], respectively. Although Adams type theorems provide a stronger estimate, theorems of Spanne type with a weaker estimate have a wider range of applicability. For more details we refer to survey paper [29].

Commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE, see for instance [2–5], where in particular in [3] it was shown that the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$.

Fractional maximal operator in Morrey spaces including their generalized versions were studied in various papers. The Spanne and Adams type results for the fractional maximal operator in generalized Morrey spaces were studied in [15,16,36]. In generalized Orlicz–Morrey spaces they were recently studied in [18,21], where references to the studies in the Morrey space setting may be found. Commutators in Morrey spaces were studied in a less generality. In the case of the classical Morrey spaces we refer for instance to [33] and [35], in the case of generalized Morrey spaces to [14–16,38], where other references may be also found. The boundedness of M_α and $M_{b,\alpha}$ in Orlicz spaces was studied in [6] and [11], respectively. The Spanne type results for $M_{b,\alpha}$ in the setting of generalized Orlicz–Morrey spaces was obtained in [18]. Note that, the conditions given in [18] for the boundedness of $M_{b,\alpha}$ was sufficient.

The purpose of this paper is twofold. First, we discuss the necessity of the conditions given in [18] for the Spanne type boundedness of $M_{b,\alpha}$ in the generalized Orlicz–Morrey spaces. Secondly, we give necessary and sufficient condition for the Adams type boundedness of $M_{b,\alpha}$ in these spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

2.1 On Young functions and Orlicz spaces

Orlicz spaces were first introduced by Orlicz in [30,31] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz spaces are also an appropriate substitute for L^1 when L^1 does not work.

First, we recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \quad \text{for } r \ge 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty &, \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \quad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \quad r \ge 0$$

for some $C > 1$.

A Young function Φ is said to satisfy the Δ'-condition, denoted also as $\Phi \in \Delta'$, if

$$\Phi(tr) \le C\Phi(t)\Phi(r), \quad t, r \ge 0$$

for some $C > 1$.

Note that, each element of Δ'-class is also an element of Δ_2-class.

We refer [34] for more details about those classes of Young functions.

Definition 2.2 (*Orlicz space*). For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^n) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. The space $L^{\Phi}_{\text{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1 \right\}.$$

We note that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi}}}\Big)dx \le 1. \tag{2.2}$$

Lemma 2.1 [7] *For a Young function Φ and $B = B(x, r)$, the following inequality is valid:*

$$\|f\|_{L^1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right) \|f\|_{L^{\Phi}(B)},$$

where $\|f\|_{L^\Phi(B)} := \|f\chi_B\|_{L^\Phi}$.

By elementary calculations we have the following.

Lemma 2.2 *Let* Φ *be a Young function and* B *be a set in* $\mathbb{R}^n$ *with finite Lebesgue measure. Then*

$$\|\chi_B\|_{L^\Phi} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

The following theorem is an analogue of Lebesgue differentiation theorem in Orlicz spaces.

Theorem 2.1 [22] *Suppose that* Φ *is a Young function and let* $f \in L^\Phi(\mathbb{R}^n)$ *be nonnegative. Then*

$$\liminf_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \ge f(x), \quad \text{for almost every } x \in \mathbb{R}^n. \tag{2.3}$$

If we moreover assume that $\Phi \in \Delta'$, *then*

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = f(x), \quad \text{for almost every } x \in \mathbb{R}^n. \tag{2.4}$$

2.2 Generalized Orlicz–Morrey spaces

We find it convenient to define generalized Orlicz–Morrey spaces in the form as follows.

Definition 2.3 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz–Morrey space, the space of all functions $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} := \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L^\Phi(B(x,r))} < \infty.$$

In the case $\varphi(x,r) = \frac{\Phi^{-1}\left(|B(x,r)|^{-1}\right)}{\Phi^{-1}\left(|B(x,r)|^{-\lambda/n}\right)}$, we get the Orlicz–Morrey space $\mathcal{M}^{\Phi,\lambda}(\mathbb{R}^n)$ from generalized Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$. We refer to [10, Lemmas 2.9 and 2.10] for more information about Orlicz–Morrey spaces. Also, according to this definition, we recover the generalized Morrey space $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ under the choice $\Phi(r) = r^p$, $1 \le p < \infty$ and if $\varphi(x,r) = \Phi^{-1}(|B(x,r)|^{-1})$, then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ coincides with the Orlicz space $L^\Phi(\mathbb{R}^n)$.

Lemma 2.3 *[10, Lemma 2.13] Let* Φ *be a Young function and* φ *be a positive measurable function on* $\mathbb{R}^n \times (0,\infty)$.

(i) If

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} = \infty \quad \textit{for some } t>0 \textit{ and for all } x \in \mathbb{R}^n,$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau>0 \textit{ and for all } x \in \mathbb{R}^n,$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.1 For the case Lemma 2.3 (ii), we imposed the condition $\Phi \in \Delta'$ in [10] since we used (2.4) in the proof. But this condition is superfluous. It is enough to use (2.3) to prove this fact, the details being omitted.

Remark 2.2 Let Φ be a Young function. We denote by Ω_Φ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} \right\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.3, we always assume that $\varphi \in \Omega_\Phi$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C>0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{1}{\Phi^{-1}(t^{-n})}\varphi(t)$ is almost increasing. In the case $\Phi(t) = t^p$, $1 \le p < \infty$, we denote the class $\mathcal{G}_\Phi$ by $\mathcal{G}_p$. Note that the class $\mathcal{G}_p$ was first defined by Nakai in [27].

The following lemma plays a key role in our main results.

Lemma 2.4 [9] *Let* $B_0 := B(x_0, r_0)$. *If* $\varphi \in \mathcal{G}_\Phi$, *then there exist* $C>0$ *such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \le \frac{C}{\varphi(r_0)}.$$

We need the following theorem about the boundedness of the maximal commutator operator M_b on generalized Orlicz–Morrey spaces for proving our main results.

Theorem 2.2 [8] *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $\varphi \in \Omega_\Phi$ and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-n}\big) \mathop{ess\ inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi(x,r), \tag{2.5}$$

where C does not depend on x and r. Then the operator M_b is bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

The following theorems are Spanne and Adams type results for M_α in generalized Orlicz–Morrey spaces, respectively.

Theorem 2.3 [18] *Let $0 \le \alpha < n$, Φ, Ψ be Young functions, $\Psi^{-1}(t) = \Phi^{-1}(t)t^{-\alpha/n}$, $\varphi_1 \in \Omega_\Phi$ and $\varphi_2 \in \Omega_\Psi$. Let also $\Phi, \Psi \in \nabla_2$, and the functions (φ_1, φ_2) and (Φ, Ψ) satisfy the condition*

$$\sup_{r<t<\infty} \Psi^{-1}\big(t^{-n}\big) \mathop{ess\ inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi_2(x,r),$$

where C does not depend on x and r. Then the operator M_α is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Theorem 2.4 [10] *Let $0 < \alpha < n$, $\Phi \in \nabla_2$, $\varphi \in \Omega_\Phi$, $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$ for some $\beta \in (0,1)$. Let φ and Φ satisfy the conditions*

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-n}\big) \mathop{ess\ inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi(x,r)$$

and

$$r^\alpha \varphi(x,r) + \sup_{r<t<\infty} t^\alpha\,\varphi(x,t) \le C\varphi(x,r)^\beta \tag{2.6}$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Then the operator M_α is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Remark 2.3 Note that, for $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$, $\varphi \in \Omega_\Phi$ implies that $\eta \in \Omega_\Psi$.

Remark 2.4 The condition (2.6) in Theorem 2.4 attributes to Hendra Gunawan [20], see also [13].

3 Spanne type results for $M_{b,\alpha}$ in $\mathcal{M}^{\Phi,\varphi}$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 3.1 Suppose that $f \in L^1_{\rm loc}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f_{B(x,r)}|dy,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{f \in L^1_{\rm loc}(\mathbb{R}^n) \ : \ \|f\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Before proving our theorems, we need the following lemmas.

Lemma 3.1 [26] *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad for \quad 0 < 2r < t, \tag{3.1}$$

where C is independent of b, x, r, and t.

Lemma 3.2 [17] *Let $f \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$, then*

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(|B(x,r)|^{-1}\big)\, \left\|f(\cdot) - f_{B(x,r)}\right\|_{L^\Phi(B(x,r))}. \tag{3.2}$$

Remark 3.1 For (3.2), see for instance [23] and [25], where more general statements of rearrangement invariant spaces and also for variable exponent Lebesgue spaces may be found, respectively.

For proving our main results, we need the following estimate.

Lemma 3.3 *If $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{B_0}| \le C M_{b,\alpha}\chi_{B_0}(x) \ \ for\ every \ \ x \in B_0.$$

Proof It is well-known that

$$\mathrm{M}_{b,\alpha} f(x) \le 2^{n-\alpha} M_{b,\alpha} f(x), \tag{3.3}$$

where $\mathrm{M}_{b,\alpha}(f)(x) := \sup\limits_{B\ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x) - b(y)||f(y)|dy$.

Now let $x \in B_0$. By using (3.3), we get

$$M_{b,\alpha}\chi_{B_0}(x) \geq C\mathrm{M}_{b,\alpha}\chi_{B_0}(x) = C \sup_{B\ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x)-b(y)|\chi_{B_0}(y)dy$$
$$= C \sup_{B\ni x} |B|^{-1+\frac{\alpha}{n}} \int_{B\cap B_0} |b(x)-b(y)|dy \geq C|B_0|^{-1+\frac{\alpha}{n}} \int_{B_0\cap B_0} |b(x)-b(y)|dy$$
$$\geq \Big|C|B_0|^{-1+\frac{\alpha}{n}} \int_{B_0} (b(x)-b(y))dy\Big| = Cr_0^{\alpha}|b(x)-b_{B_0}|.$$

The following theorem is one of our main results.

Theorem 3.1 *Let* $0 \leq \alpha < n$, $\varphi_1 \in \Omega_\Phi$, $\varphi_2 \in \Omega_\Psi$ *and* $b \in BMO(\mathbb{R}^n)$.

1. *Let* $\Psi^{-1}(t) = \Phi^{-1}(t)t^{-\alpha/n}$ *and* $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, *then the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-n}\big)} \leq C\,\varphi_2(r) \tag{3.4}$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If* $\varphi_1 \in \mathcal{G}_\Phi$ *and* $\Psi \in \Delta_2$, *then the condition*

$$t^{\alpha}\varphi_1(t) \leq C\varphi_2(t) \tag{3.5}$$

for all $t > 0$, *where* $C > 0$ *does not depend on* t, *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let* $\Psi^{-1}(t) = \Phi^{-1}(t)t^{-\alpha/n}$ *and* $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. *If* $\varphi_1 \in \mathcal{G}_\Phi$ *satisfies the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)t^{\alpha}\,\varphi_1(t) \leq Cr^{\alpha}\varphi_1(r) \tag{3.6}$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *then the condition* (3.5) *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof The first part of the theorem is a corollary of [18, Theorem 5.13].

We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.3 we have $r_0^{\alpha}|b(x)-b_{B_0}| \leq CM_{b,\alpha}\chi_{B_0}(x)$. Therefore, by Lemmas 2.4 and 3.2

$$r_0^{\alpha} \leq C\frac{\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\cdot)-b_{B_0}\|_{L^\Psi(B_0)}} \leq \frac{C}{\|b\|_*}\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}\Psi^{-1}(|B_0|^{-1})$$
$$\leq \frac{C}{\|b\|_*}\varphi_2(r_0)\|M_{b,\alpha}\chi_{B_0}\|_{\mathcal{M}^{\Psi,\varphi_2}} \leq C\varphi_2(r_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi_1}} \leq C\frac{\varphi_2(r_0)}{\varphi_1(r_0)}.$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem.

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $p, q \in [1, \infty)$ at Theorem 3.1 we get the following new result for generalized Morrey spaces.

Corollary 3.1 *Let $p, q \in [1, \infty)$, $0 \le \alpha < n$, $\varphi_1 \in \Omega_p \equiv \Omega_{t^p}$, $\varphi_2 \in \Omega_q$ and $b \in BMO(\mathbb{R}^n)$.*

1. *Let $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then the condition*

$$\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\underset{t<s<\infty}{\text{ess inf}}\,\varphi_1(s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \le C\varphi_2(r)$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$.

2. *If $\varphi_1 \in \mathcal{G}_p$, then the condition (3.5) is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$.*

3. *Let $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\varphi_1 \in \mathcal{G}_p$ satisfies the condition (3.6), then the condition (3.5) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$.*

By (1.1) and Theorems 2.3 and 3.1 we get the following corollary.

Corollary 3.2 *Let $0 \le \alpha < n$, $b \in BMO(\mathbb{R}^n)$, $b^- \in L^\infty(\mathbb{R}^n)$, $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, $\Psi^{-1}(t) = \Phi^{-1}(t)t^{-\alpha/n}$, $\varphi_1 \in \Omega_\Phi$ and $\varphi_2 \in \Omega_\Psi$. If the functions (φ_1, φ_2) and (Φ, Ψ) satisfy the condition (3.4), then $[b, M_\alpha]$ is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

4 Adams type results for $M_{b,\alpha}$ in $\mathcal{M}^{\Phi,\varphi}$

In this section we shall give a characterization for the Adams type boundedness of the operator $M_{b,\alpha}$ on generalized Orlicz–Morrey spaces.

Lemma 4.1 [9] *If $0 < \alpha < n$ and $f, b \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ and $r > 0$ we get*

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}|b(x)-b(y)|dy \lesssim r^\alpha M_b f(x).$$

Theorem 4.1 *Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$. Let $\varphi \in \Omega_\Phi$ satisfy the conditions (2.5) and*

$$r^\alpha\varphi(x,r) + \sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)t^\alpha\,\varphi(x,t) \le C\varphi(x,r)^\beta \tag{4.1}$$

for some $\beta \in (0,1)$ and for every $x \in \mathbb{R}^n$ and $r > 0$. Define $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Then, the operator $M_{b,\alpha}$ is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B := B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 := f\chi_{2B}$ and $f_2 := f\chi_{\complement_{(2B)}}$.

Let x be an arbitrary point in B. If $B(x,t) \cap \{^{\complement}(2B)\} \neq \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap \{^{\complement}(2B)\}$, then $t > |x-y| \ge |x_0-y| - |x_0-x| > 2r - r = r$.

On the other hand, $B(x,t)\cap\{{}^{\complement}(2B)\} \subset B(x_0,2t)$. Indeed, if $y \in B(x,t)\cap\{{}^{\complement}(2B)\}$, then we get $|x_0 - y| \le |x - y| + |x_0 - x| < t + r < 2t$.

Hence

$$
\begin{aligned}
M_{b,\alpha}(f_2)(x) &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)\cap\{{}^{\complement}(2B)\}} |b(y)-b(x)||f(y)|dy \\
&\le 2^{n-\alpha} \sup_{t>r} \frac{1}{|B(x_0,2t)|^{1-\frac{\alpha}{n}}} \int_{B(x_0,2t)} |b(y)-b(x)||f(y)|dy \\
&= 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}} \int_{B(x_0,t)} |b(y)-b(x)||f(y)|dy.
\end{aligned}
$$

Therefore, for all $x \in B$ we have

$$
\begin{aligned}
M_{b,\alpha}(f_2)(x) &\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y)-b(x)||f(y)|dy \\
&\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}||f(y)|dy \\
&\quad + \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b_{B(x_0,t)}-b_B||f(y)|dy \\
&\quad + \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b_B-b(x)||f(y)|dy \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

Applying Hölder's inequality, by (2.1), (3.1), (3.2) and Lemma 2.1 we get

$$
\begin{aligned}
J_1 + J_2 &\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}||f(y)|dy \\
&\quad + \sup_{t>2r} t^{\alpha-n} |b_{B(x_0,r)}-b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|dy \\
&\lesssim \sup_{t>2r} t^{\alpha-n} \left\| b(\cdot)-b_{B(x_0,t)} \right\|_{L^{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L^{\Phi}(B(x_0,t))} \\
&\quad + \sup_{t>2r} t^{\alpha-n} |b_{B(x_0,r)}-b_{B(x_0,t)}| t^n \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \|f\|_{L^{\Phi}(B(x_0,t))} \\
&\lesssim \|b\|_* \sup_{t>2r} \Phi^{-1}(|B(x_0,t)|^{-1}) t^{\alpha} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^{\Phi}(B(x_0,t))} \\
&\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{t>2r} \Big(1+\ln\frac{t}{r}\Big) t^{\alpha}\varphi(x_0,t).
\end{aligned}
$$

A geometric observation shows $2B \subset B(x,3r)$ for all $x \in B$. Using Lemma 4.1, we get

$$J_0(x) := M_{b,\alpha}(f_1)(x) \lesssim |b, I_\alpha|(|f_1|)(x) = \int_{2B} \frac{|b(y)-b(x)|}{|x-y|^{n-\alpha}} |f(y)| dy$$
$$\lesssim \int_{B(x,3r)} \frac{|b(y)-b(x)|}{|x-y|^{n-\alpha}} |f(y)| dy \lesssim r^\alpha M_b f(x)$$

Consequently for all $x \in B$ we get

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* r^\alpha M_b f(x) + \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{t>2r} \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \varphi(x_0, t).$$

Thus, by (4.1) we obtain

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* \min\{\varphi(x_0, r)^{\beta-1} M_b f(x), \varphi(x_0, r)^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}\}$$
$$\lesssim \|b\|_* \sup_{s>0} \min\{s^{\beta-1} M_b f(x), s^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}\}$$
$$= \|b\|_* (M_b f(x))^\beta \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}.$$

Hence for every $x \in B$ we have

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* (M_b f(x))^\beta \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}. \tag{4.2}$$

By using the inequality (4.2) we have

$$\|J_0(\cdot) + J_1 + J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|(M_b f)^\beta\|_{L^\Psi(B)} \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}.$$

Note that from (2.2) we get

$$\int_B \Psi\left(\frac{(M_b f(x))^\beta}{\|M_b f\|^\beta_{L^\Phi(B)}}\right) dx = \int_B \Phi\left(\frac{M_b f(x)}{\|M_b f\|_{L^\Phi(B)}}\right) dx \le 1.$$

Thus $\|(M_b f)^\beta\|_{L^\Psi(B)} = \|M_b f\|^\beta_{L^\Phi(B)}$. Therefore, we have

$$\|J_0(\cdot) + J_1 + J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|M_b f\|^\beta_{L^\Phi(B)} \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}.$$

By (3.2), Lemma 2.1 and condition (4.1), we also get

$$\begin{aligned}\|J_3\|_{L^\Psi(B)} &= \Big\| \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}} \int_{B(x_0,t)} |b(\cdot)-b_B||f(y)|dy \Big\|_{L^\Psi(B)} \\ &\approx \|b(\cdot)-b_B\|_{L^\Psi(B)} \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |f(y)|dy \\ &\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(|B|^{-1}\big)} \sup_{t>2r} \Phi^{-1}(|B(x_0,t)|^{-1}) t^\alpha \|f\|_{L^\Phi(B(x_0,t))} \\ &\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(|B|^{-1}\big)} \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{t>2r} t^\alpha \varphi(x_0,t) \\ &\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(|B|^{-1}\big)} \|f\|_{\mathcal{M}^{\Phi,\varphi}} \varphi(x_0,r)^\beta.\end{aligned}$$

Consequently by using Theorem 2.2, we get

$$\begin{aligned}\|M_{b,\alpha} f\|_{\mathcal{M}^{\Psi,\eta}} &= \sup_{x_0\in\mathbb{R}^n, r>0} \eta(x_0,r)^{-1} \Psi^{-1}(|B|^{-1}) \|M_{b,\alpha} f\|_{L^\Psi(B)} \\ &\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \left(\sup_{x_0\in\mathbb{R}^n, r>0} \varphi(x_0,r)^{-1} \Phi^{-1}(|B|^{-1}) \|M_b f\|_{L^\Phi(B)} \right)^\beta \\ &\quad + \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}} \\ &\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}}.\end{aligned}$$

The following theorem is one of our main results.

Theorem 4.2 *Let* $0 < \alpha < n$, $\Phi \in \Delta_2$, $\varphi \in \Omega_\Phi$, $b \in BMO(\mathbb{R}^n)$, $\beta \in (0,1)$, $\eta(r) \equiv \varphi(r)^\beta$ *and* $\Psi(r) \equiv \Phi(r^{1/\beta})$.

1. *If* $\Phi \in \nabla_2$ *and* $\varphi(t)$ *satisfies* (2.5), *then the condition*

$$r^\alpha \varphi(r) + \sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \varphi(t) t^\alpha \le C \varphi(r)^\beta$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If* $\varphi \in \mathcal{G}_\Phi$, *then the condition*

$$r^\alpha \varphi(r) \le C \varphi(r)^\beta \tag{4.3}$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$. *If* $\varphi \in \mathcal{G}_\Phi$ *satisfies the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \varphi(t) t^\alpha \le C r^\alpha \varphi(r) \tag{4.4}$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition (4.3) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Proof The first part of the theorem is a corollary of Theorem 4.1.

We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.3 we have $r_0^\alpha |b(x) - b_{B_0}| \leq C M_{b,\alpha} \chi_{B_0}(x)$. Therefore, by Lemmas 2.4 and Lemma 3.2

$$\begin{aligned} r_0^\alpha &\leq C \frac{\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L^\Psi(B_0)}} \leq \frac{C}{\|b\|_*} \|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)} \Psi^{-1}(|B_0|^{-1}) \\ &\leq \frac{C}{\|b\|_*} \eta(r_0) \|M_{b,\alpha}\chi_{B_0}\|_{\mathcal{M}^{\Psi,\eta}} \leq C\eta(r_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \leq C\frac{\eta(r_0)}{\varphi(r_0)} \leq C\varphi(r_0)^{\beta-1}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem.

If we take $\Phi(t) = t^p,\ p \in [1, \infty)$ and $\beta = \frac{p}{q}$ with $p < q < \infty$ at Theorem 4.2 we get the following new result for generalized Morrey spaces.

Corollary 4.1 *Let $0 < \alpha < n$, $1 \leq p < q < \infty$, $\varphi \in \Omega_p$ and $b \in BMO(\mathbb{R}^n)$.*

1. *If $1 < p < \infty$ and $\varphi(t)$ satisfies*

$$\sup_{r<t<\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi(s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \leq C\varphi(r),$$

then the condition

$$r^\alpha \varphi(r) + \sup_{r<t<\infty} \left(1 + \ln \frac{t}{r}\right) \varphi(t) t^\alpha \leq C\varphi(r)^{\frac{p}{q}}$$

for all $r > 0$ and $C > 0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_p$, then the condition*

$$r^\alpha \varphi(r) \leq C\varphi(r)^{\frac{p}{q}} \tag{4.5}$$

for all $r > 0$ and $C > 0$ does not depend on r, is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. *Let $1 < p < \infty$. If $\varphi \in \mathcal{G}_p$ satisfies the condition*

$$\sup_{r<t<\infty} \left(1 + \ln \frac{t}{r}\right) \varphi(t) t^\alpha \leq C\varphi(r)^{\frac{p}{q}}$$

for all $r > 0$ and $C > 0$ does not depend on r, then the condition (4.5) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

By (1.1) and Theorems 2.4 and 4.1 we get the following corollary.

Corollary 4.2 *Let $0 < \alpha < n$, $\beta \in (0,1)$, $b \in BMO(\mathbb{R}^n)$, $b^- \in L^\infty(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$. Let $\varphi \in \Omega_\Phi$ satisfy the conditions (2.5) and (4.1). Define $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Then the operator $[b, M_\alpha]$is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

To compare, we formulate the following theorem proved in [9] and remark below.

Theorem 4.3 *Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, $\Phi \in \Delta_2$, $\varphi \in \Omega_\Phi$, $\beta \in (0,1)$, $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.*

1. *If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (2.5), then the condition*

$$r^\alpha \varphi(r) + \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(t) t^\alpha \frac{dt}{t} \le C\varphi(r)^\beta$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition (4.3) is necessary for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

3. *Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(t) t^\alpha \frac{dt}{t} \le C r^\alpha \varphi(r) \tag{4.6}$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition (4.3) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Remark 4.1 Although $M_{b,\alpha}$ is pointwise dominated by $|b, I_\alpha|$, and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the boundedness of $M_{b,\alpha}$ under weaker assumptions than it derived from the results for the operator $|b, I_\alpha|$. More precisely, for $\varphi \in \mathcal{G}_\Phi$, integral condition (4.6) imply the supremal condition (4.4). Indeed, by (2.1) we have

$$\Phi^{-1}\big(s^{-n}\big) \approx \Phi^{-1}\big(s^{-n}\big) s^n \int_s^\infty \frac{dt}{t^{n+1}} \lesssim \int_s^\infty \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

It follows from this inequality

$$\begin{aligned} r^\alpha \varphi(r) &\gtrsim \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \varphi(t) \frac{dt}{t} \gtrsim \int_s^\infty \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \varphi(t) \frac{dt}{t} \\ &\gtrsim \frac{\varphi(s)}{\Phi^{-1}\big(s^{-n}\big)} s^\alpha \Big(1 + \ln \frac{s}{r}\Big) \int_s^\infty \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \gtrsim \Big(1 + \ln \frac{s}{r}\Big)\varphi(s) s^\alpha, \end{aligned}$$

where we took $s \in (r, \infty)$ arbitrarily, so that

$$\sup_{s>r} \Big(1 + \ln \frac{s}{r}\Big)\varphi(s) s^\alpha \lesssim r^\alpha \varphi(r).$$

Remark 4.2 For $\varphi \in \mathcal{G}_\Phi$, the condition (2.5) becomes

$$\sup_{r<t<\infty} \left(1 + \ln \frac{t}{r}\right)\varphi(t) \leq C\,\varphi(r). \tag{4.7}$$

By Remark 4.1, we have (4.6) implies (4.4). Also, note that (4.4) implies (4.7).

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