This is the post print version of the article, which has been published in Linear Algebra and Its Applications. 2018, 558, 1-24.
https://doi.org/10.1016/j.laa.2018.08.022

Inertia, positive definiteness and ℓ_p norm of GCD and LCM matrices and their unitary analogs

Pentti Haukkanen
Faculty of Natural Sciences
FI-33014 University of Tampere, Finland
pentti.haukkanen@uta.fi

László Tóth
Department of Mathematics
University of Pécs, Ifjúság útja 6
H-7624 Pécs, Hungary
ltoth@gamma.ttk.pte.hu

August 14, 2018

Abstract

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. The GCD matrix $(S)_f$ on S associated with f is defined as the $n \times n$ matrix having f evaluated at the greatest common divisor of x_i and x_j as its ij entry. The LCM matrix $[S]_f$ is defined similarly. We consider inertia, positive definiteness and ℓ_p norm of GCD and LCM matrices and their unitary analogs. Proofs are based on matrix factorizations and convolutions of arithmetical functions.

2010 AMS Mathematics Subject Classification: Primary 11C20; secondary 11A25, 11N37, 15B36.

Keywords and phrases: GCD type matrix, inertia, positive definiteness, matrix norm, arithmetical convolution, asymptotic formula.

1 Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function (function from the positive integers into the reals). The GCD matrix $(S)_f$ on S associated with f is defined as the $n \times n$ matrix having f evaluated at the greatest common divisor of x_i and x_j as its ij entry. The LCM matrix $[S]_f$ is defined similarly. For $f(x) = x$ we obtain the usual GCD and LCM matrices (S) and $[S]$ on S, and for $f(x) = x^m$ we obtain the power GCD and LCM matrices on S.

H.J.S. Smith calculated $\det(S)_f$ on factor-closed sets [37, (5.)] and $\det[S]_f$ in a more special case [37, (3.)]. There is a large number of generalizations and analogues of these determinants in the literature. For general accounts, see [21, 34]. We assume that the reader is familiar with the modern terminology of GCD and LCM matrices.

Various properties of GCD and LCM matrices and their analogues and generalizations are presented in the literature. Since Smith, determinant, inverse and factorizations have been studied very extensively. Lately, more attention is paid to eigenvalues, positive definiteness and norms. Computational aspects are also brought forth to the agenda of the study of these type matrices [27].

In this paper we present some further results on inertia, positive definiteness and ℓ_p norm of GCD and LCM matrices and their unitary analogs. The background material on unitary analogs is presented in Section 2. Inertia means the numbers of positive, negative and zero eigenvalues and is explained in more detail in Section 4. The study of inertia and positive definiteness utilizes the factorizations presented in Section 3 and Sylvester's law of inertia. The study of ℓ_p norm bases on convolutional methods, see Section 6.

2 Preliminaries

2.1 The Dirichlet convolution

The Dirichlet convolution of arithmetical functions f and g is defined as

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

The function δ (defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise) serves as the identity under the Dirichlet convolution. The Möbius function μ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function μ is the inverse of the constant function 1 under the Dirichlet convolution.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and define

$$A_f(x_i) = \sum_{\substack{d|x_i \\ d\nmid x_t \\ t<i}} \phi_f(d) \tag{2.1}$$

for all $i = 1, 2, \ldots, n$, where

$$\phi_f(x) = \sum_{d|x} f(d)\mu(x/d) = (f \star \mu)(x). \tag{2.2}$$

If S is factor-closed, then $A_f(x_i) = \phi_f(x_i)$. If $f(x) = x$ for all x, then $\phi_f = \phi$, Euler's totient function. If $f(x) = x^m$ for all x, then $\phi_f = J_m$, Jordan's totient function J_m. General accounts on the Dirichlet convolution can be found in [31, 36].

2.2 Unitary divisors and convolution

A divisor $d \in \mathbb{N}_+$ of $n \in \mathbb{N}_+$ is said to be a unitary divisor of n and is denoted by $d \parallel n$ if $(d, n/d) = 1$. For example, the unitary divisors of 72 $(= 2^3 3^2)$ are $1, 8, 9, 72$. If $d \parallel n$, we also say that n is a unitary multiple of d. The greatest common unitary divisor (GCUD) of m and n exists for all $m, n \in \mathbb{N}_+$ but, unfortunately, the least common unitary multiple (LCUM) of m and n does not always exist. For example, the LCUM of 2 and 4 does not exist. The GCUD of m and n is denoted by $(m, n)^{**}$ and the LCUM is denoted by $[m, n]^{**}$ when it exists.

Hansen and Swanson [14] overcame the difficulty of the nonexistence of the LCUM by defining

$$[m, n]^{**} = \frac{mn}{(m, n)^{**}}. \tag{2.3}$$

It is easy to see that $mn/(m, n)^{**}$ exists for all $m, n \in \mathbb{N}_+$ and is equal to the usual LCUM of m and n when the usual LCUM exists. Therefore $[m, n]^{**}$ in (2.3) is well-defined. We say that $[m, n]^{**}$ in (2.3) is the pseudo-LCUM of m and n. If the LCUM exists, then it is equal to the pseudo-LCUM. There exist also extensions of the LCUM other than the pseudo-LCUM in the literature [18].

The unitary convolution of arithmetical functions f and g is defined as

$$(f \oplus g)(n) = \sum_{d \| n} f(d)g(n/d).$$

The function δ also serves as the identity under the unitary convolution. The unitary analog of the Möbius function is the inverse of the constant function 1 under the unitary convolution and it is denoted by μ^*. The function μ^* is the multiplicative function such that $\mu^*(p^k) = -1$ for all prime powers p^k ($k \geq 1$).

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and define

$$B_f^*(x_i) = \sum_{\substack{d \| x_i \\ d \nparallel x_t \\ t<i}} \phi_f^*(d) \tag{2.4}$$

for all $i = 1, 2, \ldots, n$, where

$$\phi_f^*(x) = \sum_{d \| x} f(d)\mu^*(x/d) = (f \oplus \mu^*)(x). \tag{2.5}$$

If S is unitary divisor (UD) -closed, then $B_f^*(x_i) = \phi_f^*(x_i)$. If $f(x) = x$ for all x, then $\phi_f^* = \phi^*$, the unitary analog of Euler's totient function ϕ, see [10, 32]. If $f(x) = x^m$ for all x, then $\phi_f^* = J_m^*$, the unitary analog of Jordan's totient function J_m, see [32, 35].

2.3 Quasimultiplicative functions

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}_+$ with $(m, n) = 1$, and an arithmetical function f is said to be quasimultiplicative if $f(1) \neq 0$ and

$$f(1)f(mn) = f(m)f(n) \tag{2.6}$$

for all $m, n \in \mathbb{N}_+$ with $(m, n) = 1$. A quasimultiplicative function f is multiplicative if and only if $f(1) = 1$. An arithmetical function f with $f(1) \neq 0$ is quasimultiplicative if and only if $f/f(1)$ is multiplicative. Completely multiplicative functions are multiplicative functions satisfying $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}_+$. General accounts on multiplicative functions are presented in [31, 36].

2.4 GCD type matrices and their unitary analogs

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. The GCD matrix $(S)_f$ and the LCM matrix $[S]_f$ are defined in Section 1. Their unitary analogs go as follows. The $n \times n$ matrix having $f((x_i, x_j)^{**})$ as its ij entry is denoted as $(S^{**})_f$, and similarly the $n \times n$ matrix having $f([x_i, x_j]^{**})$ as its ij entry is denoted as $[S^{**}]_f$. We say that these matrices are the GCUD and the pseudo-LCUM matrices on S with respect to f. For $f(x) = x$ we obtain the usual GCUD and pseudo-LCUM matrices (S^{**}) and $[S^{**}]$ on S, and for $f(x) = x^m$ we obtain the power GCUD and power-pseudo-LCUM matrices on S. For general accounts on GCD type matrices see [21, 34].

3 Factorizations

We first review some factorizations presented in [20].

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a GCD-closed set of distinct positive integers, and let f be any arithmetical function. Then

$$(S)_f = E\Delta E^T, \tag{3.1}$$

where E and $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ are the $n \times n$ matrices defined by

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \mid x_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_i = A_f(x_i) = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ x_t < x_i}} (f \star \mu)(d).$$

Further, if f is a quasi-multiplicative function such that $f(x) \neq 0$ for all x, then

$$[S]_f = \Lambda E \Delta' E^T \Lambda, \tag{3.2}$$

where Λ and Δ' are the $n \times n$ diagonal matrices, whose diagonal elements are $\lambda_i = f(x_i)$ and

$$\delta'_i = A_{1/f}(x_i) = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ x_t < x_i}} (\frac{1}{f} \star \mu)(d).$$

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a GCUD-closed set of distinct positive integers, and let f be any arithmetical function. Then

$$(S^{**})_f = U\,\Gamma\,U^T, \tag{3.3}$$

where U and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ are the $n \times n$ matrices defined by

$$u_{ij} = \begin{cases} 1 & \text{if } x_j \,\|\, x_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\gamma_i = B_f^*(x_i) = \sum_{\substack{d \| x_i \\ d \not\| x_t \\ x_t < x_i}} (f \oplus \mu^*)(d).$$

Further, if f is a completely multiplicative function such that $f(x) \neq 0$ for all x, then

$$[S^{**}]_f = \Lambda\,U\,\Gamma'\,U^T\,\Lambda, \tag{3.4}$$

where Γ' is the $n \times n$ diagonal matrix defined by

$$\gamma_i' = B_{1/f}^*(x_i) = \sum_{\substack{d \| x_i \\ d \not\| x_t \\ x_t < x_i}} (\frac{1}{f} \oplus \mu^*)(d).$$

The following factorizations may be considered well known [5, 7, 28].

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let $S_d = \{w_1, w_2, \ldots, w_r\}$ be the set of all (positive) divisors of the elements of S (that is, S_d is the divisor closure of S). Let f be an arithmetical function such that $(f \star \mu)(w_i) > 0$ for all $w_i \in S_d$. Then

$$(S)_f = NN^T, \tag{3.5}$$

where N is the $n \times r$ matrix defined by

$$n_{ij} = \begin{cases} \sqrt{(f \star \mu)(w_j)} & \text{if } w_j \mid x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let f be a quasi-multiplicative function such that $f(w_i) \neq 0$ and $((1/f) \star \mu)(w_i) > 0$ for all $w_i \in S_d$. Then

$$[S]_f = N'(N')^T, \tag{3.6}$$

where N' is the $n \times r$ matrix defined by

$$n'_{ij} = \begin{cases} f(x_i)\sqrt{((1/f) \star \mu)(w_j)} & \text{if } w_j \,|\, x_i, \\ 0 & \text{otherwise.} \end{cases}$$

We next present some new factorizations; although the ideas are well known [28].

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let $S_{ud} = \{w_1, w_2, \ldots, w_r\}$ be the set of all unitary divisors of the elements of S (that is, S_{ud} is the unitary divisor closure of S). Let f be an arithmetical function such that $(f \oplus \mu^*)(w_i) > 0$ for all $w_i \in S_{ud}$. Then

$$(S^{**})_f = MM^T, \tag{3.7}$$

where M is the $n \times r$ matrix defined by

$$m_{ij} = \begin{cases} \sqrt{(f \oplus \mu^*)(w_j)} & \text{if } w_j \,\|\, x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let f be a completely multiplicative function such that $f(w_i) \neq 0$ and $((1/f) \oplus \mu^*)(w_i) > 0$ for all $w_i \in S_{ud}$. Then

$$[S^{**}]_f = M'(M')^T, \tag{3.8}$$

where M' is the $n \times r$ matrix defined by

$$m'_{ij} = \begin{cases} f(x_i)\sqrt{((1/f) \oplus \mu^*)(w_j)} & \text{if } w_j \,\|\, x_i, \\ 0 & \text{otherwise.} \end{cases}$$

4 Inertia

The inertia of a Hermitian matrix H is the triple $(i_+(H), i_-(H), i_0(H))$, where $i_+(H)$, $i_-(H)$ and $i_0(H)$ are the numbers of positive, negative and zero eigenvalues of the matrix H, counting multiplicities [25]. In this paper we consider real matrices. Sylvester's law of inertia [25, p. 223] for real matrices says that two ^{T}congruent symmetric matrices have the same inertia. Utilizing this law with the factorizations in Section 3 we examine the inertias of the matrices $(S)_f$, $[S]_f$, $(S^{**})_f$ and $[S^{**}]_f$. Inertia of MIN and MAX matrices is considered in [30].

Theorem 4.1. *Let $f(x) = x^m$, where $m > 0$.*

(a) *If S is any set of n distinct positive integers, then*

$i_+((S)_f) = n,\ i_-((S)_f) = i_0((S)_f) = 0.$

(b) *If S is factor-closed, then*

$i_+([S]_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is even,*

$i_-([S]_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is odd,*

$i_0([S]_f) = 0.$

Here $\omega(x_i)$ *is the number of distinct prime factors of* x_i *with* $\omega(1) = 0$.

Proof. (a) Let $S_d = \{w_1, w_2, \ldots, w_r\}$ be the set of all (positive) divisors of the elements of S. Then

$$(f \star \mu)(w_i) = J_m(w_i) = \sum_{d|w_i} d^m \mu(w_i/d).$$

The values of Jordan's totient J_m at prime powers are given as

$$J_m(p^a) = p^{am} - p^{(a-1)m} > 0,$$

and since J_m is multiplicative, all the values of J_m are positive. Thus we may apply Equation (3.5), which shows that $(S)_f$ is positive definite. Thus all eigenvalues are positive.

(b) Let S be factor-closed. Factorization (3.2) implies that $[S]_f$ is ^{T}congruent with the diagonal matrix Δ'. By Sylvester's law, ^{T}congruence preserves inertia; hence it suffices to consider the diagonal matrix Δ', whose diagonal elements in the case of factor-closed set are

$$A_{1/f}(x_i) = ((1/f) \star \mu)(x_i) = J_{-m}(x_i) = \sum_{d|x_i} d^{-m} \mu(x_i/d).$$

The values of J_{-m} at prime powers are given as

$$J_{-m}(p^a) = p^{-am} - p^{-(a-1)m} < 0.$$

Since in this case J_{-m} is multiplicative, we conclude that $J_{-m}(x_i)$ is positive if and only if $\omega(x_i)$ is even, and $J_{-m}(x_i)$ is negative if and only if $\omega(x_i)$ is odd. This completes the proof. □

Remark 4.1. Theorem 4.1 holds for multiplicative functions f with $f(p^a) > f(p^{a-1})$ for all prime powers $p^a > 1$.

Theorem 4.2. *Let $f(x) = x^{-m}$, where $m > 0$.*

(a) *If S is any set of n distinct positive integers, then*

$i_+([S]_f) = n$, $i_-([S]_f) = i_0([S]_f) = 0$.

(b) *If S is factor-closed, then*

$i_+((S)_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is even,*

$i_-((S)_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is odd,*

$i_0((S)_f) = 0$.

Proof is similar to that of Theorem 4.1, and we omit the details.

Remark 4.2. Theorem 4.2 holds for multiplicative functions f with $0 \neq f(p^a) < f(p^{a-1})$ for all prime powers $p^a > 1$.

Theorem 4.3. *Let $f(x) = x^m$, where $m > 0$.*

(a) *If S is any set of n distinct positive integers, then*

$i_+((S^{**})_f) = n$, $i_-((S^{**})_f) = i_0((S^{**})_f) = 0$.

(b) *If S is UD-closed, then*

$i_+([S^{**}]_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is even,*

$i_-([S^{**}]_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is odd,*

$i_0([S^{**}]_f) = 0$.

Proof. Proof is similar to that of Theorem 4.1. We, however, present the details.

(a) Let $S_{ud} = \{w_1, w_2, \ldots, w_r\}$ be the set of all unitary divisors of the elements of S. Then

$$(f \oplus \mu^*)(w_i) = J_m^*(w_i) = \sum_{d \| w_i} d^m \mu^*(w_i/d).$$

The values of J_m^* at prime powers are given as

$$J_m^*(p^a) = p^{am} - 1 > 0,$$

and since J_m^* is multiplicative, all the values of J_m^* are positive. Thus we may apply Equation (3.7), which shows that $(S^{**})_f$ is positive definite. Thus all eigenvalues are positive.

(b) Factorization (3.4) implies that $[S^{**}]_f$ is ^{T}congruent with the diagonal matrix Γ'. By Sylvester's law, ^{T}congruence preserves inertia; hence it suffices to consider the diagonal matrix Γ', whose diagonal elements in the case of UD-closed set are

$$B^*_{1/f}(x_i) = ((1/f) \oplus \mu^*)(x_i) = J^*_{-m}(x_i) = \sum_{d \| x_i} d^{-m} \mu^*(x_i/d).$$

The values of J^*_{-m} at prime powers in this case are given as

$$J^*_{-m}(p^a) = p^{-am} - 1 < 0.$$

Since in this case J^*_{-m} is multiplicative, we see that $J^*_{-m}(x_i)$ is positive if and only if $\omega(x_i)$ is even, and $J^*_{-m}(x_i)$ is negative if and only if $\omega(x_i)$ is odd. This completes the proof. □

Remark 4.3. Theorem 4.3 holds for all multiplicative functions f with $f(p^a) > 1$ for all prime powers $p^a > 1$.

Theorem 4.4. *Let $f(x) = x^{-m}$, where $m > 0$.*

(a) *If S is any set of n distinct positive integers, then*
   $i_+([S^{**}]_f) = n$, $i_-([S^{**}]_f) = i_0([S^{**}]_f) = 0$.

(b) *If S is UD-closed, then*
   $i_+((S^{**})_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is even,*
   $i_-((S^{**})_f)$ *is the number of* $x_i \in S$ *such that* $\omega(x_i)$ *is odd,*
   $i_0((S^{**})_f) = 0$.

Proof is similar to that of Theorem 4.3, and we omit the details.

Remark 4.4. Theorem 4.4 holds for multiplicative functions f with $0 \neq f(p^a) < 1$ for all prime powers $p^a > 1$.

5 Positive definite matrices

Factorizations in Section 3 and Sylvester's law of inertia make it possible to easily consider positive definiteness of GCD type matrices.

Theorem 5.1. *If S is GCD-closed and f is any arithmetical function, then $(S)_f$ is positive definite if and only if $A_f(x_i) > 0$ for all $i = 1, 2, \ldots, n$.*

Proof. Factorization (3.1) and Sylvester's law show that $(S)_f$ is positive definite if and only if the diagonal matrix Δ is positive definite, which holds exactly when the diagonal elements $A_f(x_i)$ are positive. □

Theorem 5.2. *If S is GCD-closed and f is a quasi-multiplicative function with $f(x) \neq 0$ for all x, then $[S]_f$ is positive definite if and only if $A_{1/f}(x_i) > 0$ for all $i = 1, 2, \ldots, n$.*

Proof is similar to that of Theorem 5.1 and utilizes factorization (3.2). We omit the details.

Remark 5.1. Theorems 5.1 and 5.2 are known for meet and join matrices [29].

Theorem 5.3. *Let S be any set of n distinct positive integers, and let $f(x) = x^m$, where $m > 0$. Then*

(a) *$(S)_f$ is positive definite,*

(b) *$[S]_f$ is indefinite for $n \geq 2$.*

Proof. (a) This is shown in Theorem 4.1 (a).

(b) The first leading principal minor of $[S]_f$ is $x_1^m > 0$, and the second leading principal minor is $x_1^m x_2^m - ([x_1, x_2])^{2m} < 0$. This shows that $[S]_f$ is indefinite for $n \geq 2$. □

Remark 5.2. Theorem 5.3(a) holds for all arithmetical functions f with $(f \star \mu)(w_i) > 0$ for all $w_i \in S_d$. Theorem 5.3(b) holds for all strictly increasing arithmetical functions f.

Bhatia [4] says that a positive semidefinite matrix H with $h_{ij} \geq 0$ for all i, j is *infinitely divisible* if the mth Hadamard (or entrywise) power of H is positive semidefinite for all $m \geq 0$.

Corollary 5.1. *The matrix (S) is infinitely divisible.*

Remark 5.3. Theorem 5.3(a) is a known result [5, 6]. Corollary 5.1 is also known [4]. Theorem 5.3(b) is known for $m = 1$ [33].

Theorem 5.4. *Let S be any set of n distinct positive integers, and let $f(x) = x^{-m}$, where $m > 0$. Then*

(a) *$[S]_f$ is positive definite,*

(b) *$(S)_f$ is indefinite for $n \geq 2$.*

Proof is similar to that of Theorem 5.3 and utilizes Theorem 4.2. We omit the details.

Remark 5.4. Theorem 5.4(a) holds for all quasi-multiplicative functions with $f(w_i) \neq 0$ and $((1/f)\star\mu)(w_i) > 0$ for all $w_i \in S_d$. Theorem 5.4(b) holds for all strictly decreasing arithmetical functions f.

Remark 5.5. Theorem 5.4(a) is a known result [7, 24].

Corollary 5.2. *The Hadamard inverse of* $[S]$ *is infinitely divisible.*

Theorem 5.5. *If* S *is GCUD-closed and* f *is any arithmetical function, then* $(S^{**})_f$ *is positive definite if and only if* $B_f^*(x_i) > 0$ *for all* $i = 1, 2, \ldots, n$.

Proof. Factorization (3.3) and Sylvester's law show that $(S^{**})_f$ is positive definite if and only if the diagonal matrix Γ is positive definite, which holds exactly when the diagonal elements $B_f^*(x_i)$ are positive. □

Remark 5.6. Theorem 5.5 is known for meet matrices [29].

Theorem 5.6. *If* S *is GCUD-closed and* f *is a completely multiplicative function with* $f(x) \neq 0$ *for all* x, *then* $[S^{**}]_f$ *is positive definite if and only if* $B_{1/f}^*(x_i) > 0$ *for all* $i = 1, 2, \ldots, n$.

Proof is similar to that of Theorem 5.5 and utilizes factorization (3.4). We omit the details.

Theorem 5.7. *Let* S *be any set of* n *distinct positive integers, and let* $f(x) = x^m$, *where* $m > 0$. *Then*

(a) $(S^{**})_f$ *is positive definite,*

(b) $[S^{**}]_f$ *is indefinite for* $n \geq 2$.

Proof. (a) This is shown in Theorem 4.3(a).

(b) The first leading principal minor of $[S^{**}]_f$ is $x_1^m > 0$, and the second leading principal minor is $x_1^m x_2^m - ([x_1, x_2]^{**})^{2m} < 0$. This shows that $[S^{**}]_f$ is indefinite for $n \geq 2$. □

Corollary 5.3. *The matrix* (S^{**}) *is infinitely divisible.*

Remark 5.7. Theorem 5.7(a) holds for all arithmetical functions f with $(f \oplus \mu^*)(w_i) > 0$ for all $w_i \in S_{ud}$. Theorem 5.7(b) holds for all strictly increasing arithmetical functions f.

Theorem 5.8. *Let* S *be any set of* n *distinct positive integers, and let* $f(x) = x^{-m}$, *where* $m > 0$. *Then*

(a) $[S^{**}]_f$ *is positive definite,*

(b) $(S^{**})_f$ *is indefinite for* $n \geq 2$.

Proof is similar to that of Theorem 5.7. We omit the details.

Corollary 5.4. *The Hadamard inverse of* $[S^{**}]$ *is infinitely divisible.*

Remark 5.8. Theorem 5.8(a) holds for all completely multiplicative function with $f(w_i) \neq 0$ and $((1/f) \oplus \mu^*)(w_i) > 0$ for all $w_i \in S_{ud}$. Theorem 5.8(b) holds for all strictly decreasing arithmetical functions f.

6 ℓ_p norms

Norms of GCD and LCM matrices have not been studied much in the literature. Some results are obtained in [1, 2, 8, 15, 16, 17, 38, 39, 40, 44].

In this section we provide asymptotic formulas for the ℓ_p norms of the GCD matrix $((i,j))_{n\times n}$, the LCM matrix $([i,j])_{n\times n}$, the GCUD matrix $((i,j)^{**})_{n\times n}$, the pseudo-LCUM matrix $([i,j]^{**})_{n\times n}$ and the matrix $((i,j)^{*})_{n\times n}$. Here $(i,j)^*$ stands for the semi-unitary greatest common divisor (SUGCD), being the greatest divisor of i which is a unitary divisor of j. See, e.g., [20]. We utilize known asymptotic formulas for arithmetical functions.

Let $p \in \mathbb{N}^+$. The ℓ_p norm of an $n \times n$ matrix M is defined as

$$\|M\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |m_{ij}|^p\right)^{1/p}.$$

6.1 Norms of GCD matrices

It is known that

$$\sum_{i,j\leq x} (i,j) = Ax^2 \log x + Bx^2 + O(x^{1+\theta+\epsilon})$$

for every $\epsilon > 0$, where $A := 1/\zeta(2)$,

$$B := \frac{1}{\zeta(2)}\left(2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)}\right)$$

and θ is the exponent in Dirichlet's divisor problem. (Here γ is Euler's constant and ζ is the Riemann ζ-function.) This asymptotic formula is equivalent to that deduced in [9] for the sum $\sum_{i\leq j\leq x}(i,j)$. See also [22, 43].

This means that for $p = 1$,

$$\|((i,j))\|_1 = An^2 \log n + Bn^2 + O(n^{1+\theta+\epsilon}). \tag{6.1}$$

In [15] it is shown a more rough result, namely

$$\|((i,j))\|_1 = O(n^2 \log n).$$

Theorem 6.1. *Let $p \geq 2$ be a fixed integer. Then*

$$\sum_{i,j \leq x} (i,j)^p = C_p x^{p+1} + O(E_p(x)), \tag{6.2}$$

where

$$C_p := \frac{2\zeta(p) - \zeta(p+1)}{(p+1)\zeta(p+1)}$$

and $E_p(x) = x^p$ for $p > 2$ and $E_2(x) = x^2 \log x$.

This formula can be obtained from general results of Cohen [11, 13] established for sums $\sum_{a,b \leq x} f((a,b))$, where f is a certain arithmetic function. However, we offer here an alternative approach to proof, which will be used for the next theorems, as well.

Proof. Consider the Jordan function J_p. We have

$$S_p(x) := \sum_{i,j \leq x} (i,j)^p = \sum_{i,j \leq x} \sum_{d|(i,j)} J_p(d) = \sum_{\substack{da \leq x \\ db \leq x}} J_p(d).$$

Writing this into

$$S_p(x) = \sum_{d \leq x} J_p(d) \left(\sum_{a \leq x/d} 1 \right)^2,$$

and by applying usual estimates, we only obtain that $S_p(x) = O(x^{p+1})$, the main term being absorbed by the error term. The idea is to change the order of summation:

$$S_p(x) = \sum_{a,b \leq x} \sum_{d \leq x/M} J_p(d),$$

where $M := \max(a,b)$. By using the well known [31, Th. 6.4] formula

$$\sum_{n \leq x} J_p(n) = \frac{1}{(p+1)\zeta(p+1)} x^{p+1} + O(x^p), \tag{6.3}$$

valid for any fixed $p \geq 2$, we obtain

$$S_p(x) = \sum_{a,b \leq x} \left(\frac{1}{(p+1)\zeta(p+1)} (x/M)^{p+1} + O((x/M)^p) \right)$$

$$= \frac{x^{p+1}}{(p+1)\zeta(p+1)} \sum_{a,b\le x} \frac{1}{M^{p+1}} + O\left(x^p \sum_{a,b\le x} \frac{1}{M^p} \right). \tag{6.4}$$

Here the first sum is

$$\sum_{a,b\le x} \frac{1}{M^{p+1}} = 2 \sum_{a\le b\le x} \frac{1}{b^{p+1}} - \sum_{a=b\le x} \frac{1}{b^{p+1}}$$

$$= 2 \sum_{b\le x} \frac{1}{b^{p+1}} \sum_{a\le b} 1 - \sum_{b\le x} \frac{1}{b^{p+1}} = 2 \sum_{b\le x} \frac{1}{b^p} - \sum_{b\le x} \frac{1}{b^{p+1}}$$

$$= 2 \left(\zeta(p) + O(\frac{1}{x^{p-1}}) \right) - \left(\zeta(p+1) + O(\frac{1}{x^p}) \right) = 2\zeta(p) - \zeta(p+1) + O(\frac{1}{x^{p-1}}).$$

Similarly,

$$\sum_{a,b\le x} \frac{1}{M^p} = 2 \sum_{a\le b\le x} \frac{1}{b^p} - \sum_{a=b\le x} \frac{1}{b^p} \ll \sum_{b\le x} \frac{1}{b^{p-1}},$$

which is $\ll \log x$ for $p = 2$ and is $\ll 1$ for $p > 2$, see [3, p. 70]. Inserting into (6.4) completes the proof (valid for any real $p \ge 2$). □

By applying Newton's generalized binomial theorem we obtain

$$(\sum_{i,j\le x} (i,j)^p)^{1/p} = (C_p x^{p+1})^{1/p} + O((x^{p+1})^{(1/p)-1} E_p(x))$$

$$= C_p^{1/p} x^{1+(1/p)} + O((x^{(1/p)-p} E_p(x)).$$

Thus the ℓ_p norm of the $n\times n$ GCD matrix $((i,j))$ possesses the asymptotic formula given in the next Corollary.

Corollary 6.1. *Let $p \ge 2$ be an integer. Then*

$$\|((i,j))\|_p = C_p^{1/p} n^{1+(1/p)} + O((n^{(1/p)-p} E_p(n)). \tag{6.5}$$

In [15] it is shown a more rough result

$$\|((i,j))\|_p = O(n^{1+(1/p)})$$

for $p \ge 2$.

6.2 Norms of LCM matrices

It is known that for every integer $p \geq 1$ one has

$$\sum_{i,j\leq x} [i,j]^p = D_p x^{2(p+1)} + O(x^{2p+1}(\log x)^{2/3}(\log\log x)^{4/3}), \qquad (6.6)$$

where

$$D_p := \frac{\zeta(p+2)}{(p+1)^2\zeta(p)},$$

deduced in [26, Th. 2]. Applying Newton's generalized binomial theorem we obtain

$$(\sum_{i,j\leq n} [i,j]^p)^{1/p} = (D_p n^{2(p+1)})^{1/p} + O((n^{2(p+1)})^{(1/p)-1} n^{2p+1} (\log n)^{2/3}(\log\log n)^{4/3})$$

$$= D_p^{1/p} n^{2+(2/p)} + O((n^{(2/p)+1}(\log n)^{2/3}(\log\log n)^{4/3}).$$

Thus the ℓ_p norm of the $n\times n$ LCM matrix $([i,j])$ possesses the asymptotic formula

$$\|([i,j])\|_p = D_p^{1/p} n^{2+(2/p)} + O((n^{(2/p)+1}(\log n)^{2/3}(\log\log n)^{4/3}) \qquad (6.7)$$

for $p \geq 1$. In [15] it is shown a more rough result

$$\|([i,j])\|_p = O(n^{2+(2/p)})$$

for $p \geq 1$.

In a similar way, having an asymptotic formula of type (6.2) or (6.6), the ℓ_p norm of the corresponding matrix can be easily estimated.

6.3 Norms of SUGCD matrices

Now consider the SUGCD matrix $((i,j)^*)_{n\times n}$, where $(i,j)^*$ is the greatest divisor of i which is a unitary divisor of j.

Theorem 6.2.

$$\sum_{i,j\leq x} (i,j)^* = Gx^2 \log x + O(x^2),$$

where

$$G := \zeta(2)^{-1} \prod_{q\in\mathbb{P}} \left(1 - \frac{1}{(q+1)^2}\right),$$

the product being over the primes q.

Proof. We use that $d \mid\mid (i,j)^*$ if and only if $d \mid i$ and $d \mid\mid j$. By the property of the unitary Euler function $J_1^* = \phi^*$,

$$S^*(x) := \sum_{i,j\le x} (i,j)^* = \sum_{i,j\le x} \sum_{d\mid\mid(i,j)^*} \phi^*(d)$$

$$= \sum_{i,j\le x} \sum_{\substack{d\mid i\\ d\mid\mid j}} \phi^*(d) = \sum_{\substack{i=da\le x\\ j=db\le x\\ (d,b)=1}} \phi^*(d)$$

$$= \sum_{d\le x} \phi^*(d) \sum_{a\le x/d} 1 \sum_{\substack{b\le x/d\\ (b,d)=1}} 1.$$

Let $\sigma_s(n) = \sum_{d\mid n} d^s$. According to [41, Lemma 2.1], for any fixed k and any $\varepsilon > 0$,

$$\sum_{\substack{m\le x\\ (m,k)=1}} 1 = x\frac{\phi(k)}{k} + O(x^\varepsilon \sigma_{-\varepsilon}(k)).$$

We deduce that

$$S^*(x) = \sum_{d\le x} \phi^*(d) \left(\frac{x}{d} + O(1)\right) \left(\frac{x}{d} \cdot \frac{\phi(d)}{d} + O((\frac{x}{d})^\varepsilon \sigma_{-\varepsilon}(d))\right)$$

$$= x^2 \sum_{d\le x} \frac{\phi(d)\phi^*(d)}{d^3} + O(x \sum_{d\le x} 1) + O(x^{1+\varepsilon} \sum_{d\le x} \frac{\sigma_{-\varepsilon}(d)}{d^\varepsilon})$$

$$= x^2(G \log x + O(1)) + O(x^2) + O(x^{1+\varepsilon} x^{1-\varepsilon})$$

$$= Gx^2 \log x + O(x^2),$$

by using [41, Lemmas 2.2, 3.4]. □

Remark 6.1. Let

$$P^*(n) = \sum_{k=1}^{n} (k,n)^*$$

be the unitary gcd-sum function. It is known ([41, Th. 3.2]) that

$$\sum_{n\le x} P^*(n) = \frac{G}{2} x^2 \log x + O(x^2).$$

It follows that

$$\sum_{m,n\le x} (m,n)^* \sim 2 \sum_{n\le x} P^*(n) \sim Gx^2 \log x, \quad x \to \infty.$$

On the other hand,

$$\sum_{m,n\le x}(m,n)^* = \sum_{m\le n\le x}(m,n)^* + \sum_{n\le m\le x}(m,n)^* - \sum_{n\le x}(n,n)^*$$

$$= \sum_{n\le x}P^*(n) + \sum_{m\le x}P_1^*(m) - \sum_{n\le x}n,$$

where

$$P_1^*(n) = \sum_{k=1}^{n}(n,k)^*$$

is the "dual" unitary gcd-sum function. Hence

$$\sum_{n\le x}P_1^*(n) \sim \frac{G}{2}x^2\log x, \quad x\to\infty.$$

Theorem 6.3. *Let $p\ge 2$ be a fixed integer. Then*

$$\sum_{i,j\le x}((i,j)^*)^p = C_p^* x^{p+1} + O(E_p^*(x)),$$

where

$$C_p^* := \frac{\zeta(p+1)}{p+1}D_p^*\sum_{n=1}^{\infty}\frac{(n-1)g_p(n)+G_p(n)}{n^{p+1}},$$

$$D_p^* := \prod_{q\in\mathbb{P}}\left(1-\frac{2}{q^{p+1}}+\frac{1}{q^{p+2}}\right), \tag{6.8}$$

$$g_p(n) = \prod_{q|n}\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^{p+1}}\right)\left(1-\frac{2}{q^{p+1}}+\frac{1}{q^{p+2}}\right)^{-1}, \tag{6.9}$$

the product being over the prime divisors q of n,

$$G_p(n) = \sum_{k=1}^{n}g_p(k),$$

and $E_p^*(x) = x^p$ for $p>2$ and $E_2^*(x) = x^2(\log x)^2$.

Proof. We use the method of the proof of Theorem 6.1, namely summation in reverse order. Consider the unitary Jordan function J_p^*, already defined in Section 4. We have

$$S_p^*(x) := \sum_{i,j\le x}((i,j)^*)^p = \sum_{i,j\le x}\sum_{d||(i,j)^*}J_p^*(d)$$

$$= \sum_{i,j\le x} \sum_{\substack{d|i\\ d||j}} J_p^*(d) = \sum_{\substack{da\le x\\ db\le x\\ (d,b)=1}} J_p^*(d)$$

$$= \sum_{a,b\le x} \sum_{\substack{d\le x/M\\ (d,b)=1}} J_p^*(d),$$

where $M := \max(a,b)$. We use the formula

$$\sum_{\substack{n\le x\\ (n,k)=1}} J_p^*(n) = \frac{\zeta(p+1)}{p+1} D_p^* x^{p+1} g_p(k) + O(x^p \tau(k)), \tag{6.10}$$

valid for any fixed $p \ge 2$, $k \in \mathbb{N}_+$, where $\tau(k) = \sum_{d|k} 1$. The proof of (6.10) is similar to the proof of (6.3). See also [12]. Note that $0 < g_p(k) < 1$ holds for every $k \in \mathbb{N}_+$.

We obtain

$$S_p^*(x) = \sum_{a,b\le x} \left(\frac{\zeta(p+1)}{p+1} D_p^* g_p(b)(x/M)^{p+1} + O((x/M)^p \tau(b)) \right)$$

$$= x^{p+1} \frac{\zeta(p+1)}{p+1} D_p^* \sum_{a,b\le x} \frac{g_p(b)}{M^{p+1}} + O\left(x^p \sum_{a,b\le x} \frac{\tau(b)}{M^p} \right). \tag{6.11}$$

Here

$$\sum_{a,b\le x} \frac{g_p(b)}{M^{p+1}} = \sum_{a\le b\le x} \frac{g_p(b)}{b^{p+1}} + \sum_{b\le a\le x} \frac{g_p(b)}{a^{p+1}} - \sum_{a=b\le x} \frac{g_p(b)}{b^{p+1}} =: S_1 + S_2 - S_3,$$

say. We deduce

$$S_1 = \sum_{b\le x} \frac{g_p(b)}{b^{p+1}} \sum_{a\le b} 1 = \sum_{b\le x} \frac{g_p(b)}{b^p} = \sum_{b=1}^{\infty} \frac{g_p(b)}{b^p} + O(\frac{1}{x^{p-1}}),$$

the series being convergent since $0 < g_p(k) < 1$ for every $k \in \mathbb{N}_+$.

$$S_2 = \sum_{a\le x} \frac{1}{a^{p+1}} \sum_{b\le a} g_p(b) = \sum_{a\le x} \frac{G_p(a)}{a^{p+1}} = \sum_{a=1}^{\infty} \frac{G_p(a)}{a^{p+1}} + O(\frac{1}{x^{p-1}}),$$

using that $0 < G_p(k) < k$ for every $k \in \mathbb{N}_+$.

Also,

$$S_3 = \sum_{b=1}^{\infty} \frac{g_p(b)}{b^{p+1}} + O(\frac{1}{x^p}).$$

For the error term in (6.11),

$$\sum_{a,b\le x}\frac{\tau(b)}{M^p}<\sum_{a\le b\le x}\frac{\tau(b)}{b^p}+\sum_{b\le a\le x}\frac{\tau(b)}{a^p}=\sum_{b\le x}\frac{\tau(b)}{b^{p-1}}+\sum_{a\le x}\frac{1}{a^p}\sum_{b\le a}\tau(b)$$

$$\ll\sum_{b\le x}\frac{\tau(b)}{b^{p-1}}+\sum_{a\le x}\frac{\log a}{a^{p-1}}$$

which is $\ll(\log x)^2$ for $p=2$ and is $\ll 1$ for $p>2$. Inserting into (6.11) completes the proof (valid for any real $p\ge 2$). □

6.4 Norms of GCUD matrices

Next consider the GCUD matrix $((i,j)^{**})_{n\times n}$.

Theorem 6.4.

$$\sum_{i,j\le x}(i,j)^{**}=Fx^2\log x+O(x^2),\tag{6.12}$$

where

$$F:=\zeta(2)\prod_{q\in\mathbb{P}}\left(1-\frac{4}{q^2}+\frac{4}{q^3}-\frac{1}{q^4}\right).$$

Proof. Let

$$P^{**}(n)=\sum_{k=1}^{n}(k,n)^{**}$$

be the bi-unitary gcd-sum function. It is known ([42, Th. 3]) that

$$\sum_{n\le x}P^{**}(n)=\frac{F}{2}x^2\log x+O(x^2).\tag{6.13}$$

We have

$$\sum_{i,j\le x}(i,j)^{**}=2\sum_{i\le j\le x}(i,j)^{**}-\sum_{j\le x}(j,j)^{**}$$

$$=2\sum_{j\le x}P^{**}(j)-\sum_{j\le x}j,$$

and (6.12) is a direct consequence of (6.13). □

Theorem 6.5. *Let $p\ge 2$ be a fixed integer. Then*

$$\sum_{i,j\le x}((i,j)^{**})^p=C_p^{**}x^{p+1}+O(E_p^{**}(x)),$$

where

$$C_p^{**} := \frac{\zeta(p+1)}{p+1} D_p^* \sum_{n=1}^{\infty} \frac{2\overline{G}_p(n) - g_p(n^2)}{n^{p+1}},$$

D_p^* and $g_p(n)$ are defined by (6.8) *and* (6.9)*, respectively,*

$$\overline{G}_p(n) = \sum_{k=1}^{n} g_p(kn),$$

and $E_p^{**}(x) = x^p \log x$ for $p > 2$ and $E_2^{**}(x) = x^2(\log x)^3$.

Proof. Similar to the proofs of Theorems 6.1 and 6.3. We use that $d \mid\mid (i,j)^{**}$ if and only if $d \mid\mid i$ and $d \mid\mid j$.

$$S_p^{**}(x) := \sum_{i,j \le x} ((i,j)^{**})^p = \sum_{i,j \le x} \sum_{d \mid\mid (i,j)^{**}} J_p^*(d)$$

$$= \sum_{i,j \le x} \sum_{\substack{d \mid\mid i \\ d \mid\mid j}} J_p^*(d) = \sum_{\substack{da \le x \\ db \le x \\ (d,ab)=1}} J_p^*(d)$$

$$= \sum_{a,b \le x} \sum_{\substack{d \le x/M \\ (d,ab)=1}} J_p^*(d),$$

where $M := \max(a,b)$. We use formula (6.10) and obtain that

$$S_p^{**}(x) = \sum_{a,b \le x} \left(\frac{\zeta(p+1)}{p+1} D_p^* g_p(ab)(x/M)^{p+1} + O((x/M)^p \tau(ab)) \right)$$

$$= x^{p+1} \frac{\zeta(p+1)}{p+1} D_p^* \sum_{a,b \le x} \frac{g_p(ab)}{M^{p+1}} + O\left(x^p \sum_{a,b \le x} \frac{\tau(ab)}{M^p} \right). \tag{6.14}$$

Here (having again symmetry in the variables a and b),

$$\sum_{a,b \le x} \frac{g_p(ab)}{M^{p+1}} = 2 \sum_{a \le b \le x} \frac{g_p(ab)}{b^{p+1}} - \sum_{a=b \le x} \frac{g_p(b^2)}{b^{p+1}}$$

$$= 2 \sum_{b \le x} \frac{\overline{G}_p(b)}{b^{p+1}} - \sum_{b \le x} \frac{g_p(b^2)}{b^{p+1}} = 2 \sum_{b=1}^{\infty} \frac{\overline{G}_p(b)}{b^{p+1}} - \sum_{b=1}^{\infty} \frac{g_p(b^2)}{b^{p+1}} + O(\frac{1}{x^{p-1}}).$$

For the error term in (6.14), use that $\tau(ab) \le \tau(a)\tau(b)$ for any $a, b \in \mathbb{N}_+$.

$$\sum_{a,b \le x} \frac{\tau(ab)}{M^p} \ll \sum_{a \le b \le x} \frac{\tau(a)\tau(b)}{b^p} = \sum_{b \le x} \frac{\tau(b)}{b^p} \sum_{a \le b} \tau(a)$$

$$\ll \sum_{b\le x} \frac{\tau(b)}{b^p} b\log b \ll (\log x) \sum_{b\le x} \frac{\tau(b)}{b^{p-1}},$$

which is $\ll (\log x)^3$ for $p=2$ and is $\ll \log x$ for $p>2$, see [3, p. 70]. The proof works for any real $p \ge 2$. □

6.5 Norms of pseudo-LCUM matrices

Finally, consider the pseudo-LCUM matrix $([i,j]^{**})_{n\times n}$.

Theorem 6.6. *Let $p\ge 1$ be an integer. Then*

$$\sum_{i,j\le x} ([i,j]^{**})^p = \frac{\beta_p}{(p+1)^2} x^{2(p+1)} + O(x^{2p+1}(\log x)^2),$$

where

$$\beta_p := \zeta(2)\zeta(p+2) \prod_{q\in\mathbb{P}} \left(1 - \frac{2}{q^2} + \frac{2}{q^3} - \frac{1}{q^4} - \frac{2}{q^{p+3}} + \frac{2}{q^{p+4}}\right).$$

Proof. Let $\mathrm{id}_s(n) = n^s$.

$$U_p(x) := \sum_{i,j\le x} ([i,j]^{**})^p = \sum_{i,j\le x} \left(\frac{ij}{(i,j)^{**}}\right)^p = \sum_{i,j\le x} (ij)^p \sum_{d\|(i,j)^{**}} (\mu^* \oplus \mathrm{id}_{-p})(d).$$

Let denote $h_p(n) = (\mu^*\oplus \mathrm{id}_{-p})(n)$, which is multiplicative and $h_p(q^\nu) = 1/q^{\nu p} - 1$ for every prime power q^ν ($\nu \ge 1$). Hence $|h_p(n)| \le 1$ for every $n \in \mathbb{N}_+$ (and every real $p>0$). We have

$$U_p(x) = \sum_{\substack{da\le x\\ db\le x\\ (d,ab)=1}} (d^2ab)^p h_p(d) = \sum_{d\le x} d^{2p} h_p(d) \left(\sum_{\substack{a\le x/d\\ (a,d)=1}} a^p \right)^2.$$

We use the known [41, Lemma 2.1] formula

$$\sum_{\substack{n\le x\\ (n,k)=1}} n^p = \frac{x^{p+1}}{p+1}\frac{\phi(k)}{k} + O(x^p\tau(k)),$$

valid for every real $p\ge 0$ and $k\in\mathbb{N}_+$, and obtain

$$U_p(x) = \sum_{d\le x} d^{2p} h_p(d) \left(\frac{(x/d)^{p+1}}{p+1} \cdot \frac{\phi(d)}{d} + O((x/d)^p\tau(d))\right)^2$$

$$= \sum_{d\le x} d^{2p} h_p(d) \left(\frac{(x/d)^{2(p+1)}}{(p+1)^2} \cdot \frac{\phi^2(d)}{d^2} + O((x/d)^{2p+1}\tau(d)) \right)$$

$$= \frac{x^{2(p+1)}}{(p+1)^2} \sum_{d\le x} \frac{h_p(d)\phi^2(d)}{d^4} + O\left(x^{2p+1} \sum_{d\le x} \frac{|h_p(d)|\tau(d)}{d} \right). \tag{6.15}$$

The sum of the main term in (6.15) can be written as

$$\sum_{d=1}^{\infty} \frac{h_p(d)\phi^2(d)}{d^4} + O(\frac{1}{x}) = \beta_p + O(\frac{1}{x}),$$

the series being convergent since $h_p(n)$ is bounded, where β_p can be easily computed by the Euler product formula. The error term in (6.15) is $\ll x^{2p+1}(\log x)^2$, see [3, p. 70]. Note that the proof is valid for any positive real p. □

Concluding Remarks. Our main themes are inertia and ℓ_p norm of GCD type matrices. We considered inertia utilizing matrix factorizations and Sylvester's law. We obtained the inertia of the power GCD matrix $((x_i, x_j)^m)_{n\times n}$ with $m > 0$ and the power LCM matrix $([x_i, x_j]^m)_{n\times n}$ with $m < 0$ on *any* set S of n distinct positive integers, and the inertia of the power GCD matrix $((x_i, x_j)^m)_{n\times n}$ with $m < 0$ and the power LCM matrix $([x_i, x_j]^m)_{n\times n}$ with $m > 0$ on *factor-closed* sets S. We also obtained similar results for the unitary analogs of these matrices. Further, we applied the same kind of methods to examine positive definiteness of the GCD matrix $(f((x_i, x_j)))_{n\times n}$ and the LCM matrix $(f([x_i, x_j]))_{n\times n}$ and their unitary analogs with respect to various classes of arithmetical functions f and sets S.

We provided asymptotic formulas with O-terms for the ℓ_p norms of the classical GCD matrix $((i, j))_{n\times n}$, the classical LCM matrix $([i, j])_{n\times n}$ and their unitary analogs. We applied convolutional methods and known asymptotic results for certain arithmetical functions.

A number of questions remain, some of which are listed below.

- Consider the inertia of the power GCD matrix $((x_i, x_j)^m)_{n\times n}$ with $m < 0$ and the power LCM matrix $([x_i, x_j]^m)_{n\times n}$ with $m > 0$ on GCD-closed sets S or even on arbitrary sets S.
- Consider the inertia of the GCD matrix $(f((x_i, x_j)))_{n\times n}$ and the LCM matrix $(f([x_i, x_j]))_{n\times n}$ when f is an arithmetical function other than the power function.
- Improve the O-terms in the asymptotic formulas for the ℓ_p norms.

- Provide similar results for matrix norms other than the ℓ_p norm.

Acknowledgement. The authors are grateful to the referee for his/her careful reading and helpful suggestion.

References

[1] E. Altinisik, On the matrix norms of a GCD related matrix, *Math. Inequal. Appl.* **11.4** (2008), 635–646.

[2] E. Altinisik, N. Tuglu and P. Haukkanen, A note on bounds for norms of the reciprocal LCM matrix, *Math. Inequal. Appl.* **7.4** (2004), 491–496.

[3] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.

[4] R. Bhatia, Infinitely divisible matrices, *Amer. Math. Monthly* **113** (2006), 221–235.

[5] K. Bourque and S. Ligh, Matrices associated with arithmetical functions. *Linear and Multilinear Algebra* **34** (1993), 261–267.

[6] K. Bourque and S. Ligh, Matrices associated with classes of arithmetical functions, *J. Number Theory* **45** (1993), 367–376.

[7] K. Bourque and S. Ligh, Matrices associated with multiplicative functions, *Linear Algebra Appl.* **216** (1995), 267–275.

[8] D. Bozkurt and S. Solak, On the norms of GCD matrices, *Math. Comput. Appl.* **7** (2002), no. 3, 205–210.

[9] J. Chidambaraswamy and R. Sitaramachandrarao, Asymptotic results for a class of arithmetical functions, *Monatsh. Math.* **99** (1985), 19–27.

[10] E. Cohen, Arithmetical functions associated with the unitary divisors of an integer, *Math. Z.* **74** (1960), 66–80.

[11] E. Cohen, Arithmetical functions of a greatest common divisor, I., *Proc. Amer. Math. Soc.* **11** (1960), 164–171.

[12] E. Cohen, An elementary method in the asymptotic theory of numbers, *Duke Math. J.* **28** (1961), 183–192.

[13] E. Cohen, Arithmetical functions of a greatest common divisor, II. An alternative approach, *Boll. Un. Mat. Ital.* **17** (1962), 349–356.

[14] R. T. Hansen and L. G. Swanson, Unitary divisors, *Math. Mag.* **52** (1979), 217–222.

[15] P. Haukkanen, On the ℓ_p norm of GCD and related matrices, *J. Inequal. Pure Appl. Math.* **5.3** (2004), Article 61.

[16] P. Haukkanen, An upper bound for the ℓ_p norm of a GCD related matrix, *J. Inequal. Appl.* (2006), Article ID 25020, 6 p.

[17] P. Haukkanen, On the maximum row and column sum norm of GCD and related matrices. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 4, Article 97.

[18] P. Haukkanen, P. Ilmonen, A. Nalli, Ayse, J. Sillanpää, On unitary analogs of GCD reciprocal LCM matrices. *Linear Multilinear Algebra* **58** (2010), 599–616.

[19] P. Haukkanen, I. Korkee, Notes on the divisibility of GCD and LCM matrices. *Int. J. Math. Math. Sci.* 2005, no. 6, 925–935.

[20] P. Haukkanen and J. Sillanpää, Some analogues of Smith's determinant. *Linear Multilinear Algebra* **41**, no. 3 (1996), 233–244.

[21] P. Haukkanen, J. Wang and J. Sillanpää, On Smith's determinant, *Linear Algebra Appl.* **258** (1997), 251–269.

[22] T. Hilberdink and L. Tóth, On the average value of the least common multiple of k positive integers, *J. Number Theory* **169** (2016), 327–341.

[23] S. Hong, Gcd-closed sets and determinants of matrices associated with arithmetical functions, *Acta Arith.* **101** (2002), 321–332.

[24] S. Hong and K. S. Enoch Lee, Asymptotic behavior of eigenvalues of reciprocal power LCM matrices, *Glasg. Math. J.* **50** (2008), 163–174.

[25] R. A. Horn and C. R. Johnson, Matrix Analysis, 1st ed., Cambridge University Press, 1985.

[26] S. Ikeda and K. Matsuoka, On the lcm-sum function, *J. Integer Seq.* **17** (2014), Article 14.1.7.

[27] V. Kaarnioja, On sparse tensor structures in lattice theory and applications of the polynomial collocation method based on sparse grids, Doctoral Dissertations, Aalto University publication series, Doctoral Dissertations 130 (2017).

[28] I. Korkee and P. Haukkanen, On meet and join matrices associated with incidence functions, *Linear Algebra Appl.* **372** (2003), 127–153.

[29] M. Mattila and P. Haukkanen, On the positive definiteness and eigenvalues of meet and join matrices. *Discrete Math.* **326** (2014), 9–19.

[30] M. Mattila and P. Haukkanen, Studying the various properties of MIN and MAX matrices–elementary vs. more advanced methods. *Special Matrices* **4** (2016), 101–109.

[31] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer-Verlag, 1986.

[32] K. Nageswara Rao, On the unitary analogues of certain totients, *Monatsh. Math.* **70** (1966), 149–154.

[33] J. S. Ovall, An analysis of GCD and LCM matrices via the LDL^T-factorization, *Electron. J. Linear Algebra* **11** (2004), 51–58.

[34] J. Sándor and B. Crstici, *Handbook of Number Theory II*, Kluwer Academic, 2004.

[35] J. Sándor and L. Tóth, On certain number-theoretic inequalities, *Fibonacci Q.* **28** (1990), 255–258.

[36] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Marcel Dekker, Inc., 1989.

[37] H. J. S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc.* **7** (1875/76), 208-212.

[38] S. Solak, R. Türkmen and D. Bozkurt, On GCD, LCM and Hilbert matrices and their applications, *Appl. Math. Comput.* **146** (2003), no. 2-3, 595–600.

[39] S. Solak, R. Türkmen and D. Bozkurt, On the norms of GCD, Toeplitz and Hankel matrices related to Fibonacci numbers, *Int. Math. J.* **3** (2003), no. 2, 195–200.

[40] D. Taşcı, The bounds for Perron roots of GCD, GMM, and AMM matrices. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **46** (1997), no. 1–2, 165–171.

[41] L. Tóth, The unitary analogue of Pillai's arithmetical function, *Collect. Math.* **40** (1989), 19–30.

[42] L. Tóth, On the bi-unitary analogues of Euler's arithmetical function and the gcd-sum function, *J. Integer Seq.* **12** (2009), Article 09.5.2.

[43] L. Tóth, A survey of gcd-sum functions, *J. Integer Seq.* **13** (2010), Article 10.8.1.

[44] R. Türkmen and D. Bozkurt, A note on the norms of the GCD matrix. *Math. Comput. Appl.* **9** (2004), no. 2, 303–308.