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Non-smooth Atomic Decompositions for Generalized Orlicz–Morrey Spaces of the Third Kind

Vagif S. Guliyev[1,2] · **Sabir G. Hasanov**[3] · **Yoshihiro Sawano**[4] · **Takahiro Noi**[4]

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Abstract We deal with the generalized Orlicz–Morrey space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ of the third kind and consider the decomposition method. Also we characterize its predual space. Some maximal estimates for generalized Orlicz–Morrey spaces of the third kind are also obtained by using the weighted Hardy operators. As an application, we consider the Olsen inequality, which is a bilinear estimate on the fractional integral operator. As an appendix, we consider a general form of the vector-valued boundedness of the Hardy–Littlewood maximal operator, where ϕ in the definition of $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ depends on x as well. This paper contains a remedy for the mistake in the proof of the Olsen inequality of the 2014 paper by the second author (Iida et al. in Z. Anal. Anwend. 33(2):149–170, 2014).

Keywords Orlicz–Morrey spaces · Atomic decomposition · Maximal operators

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✉ Y. Sawano
yoshihiro-sawano@celery.ocn.ne.jp

V.S. Guliyev
vagif@guliyev.com

S.G. Hasanov
sabirhasanov@gmail.com

T. Noi
taka.noi.hiro@gmail.com

1 Department of Mathematics, Ahi Evran University, Kirsehir, Turkey

2 Institute of Mathematics and Mechanics, Baku, Azerbaijan

3 Ganja State University, Ganja, Azerbaijan

4 Department of Mathematics and Information Science, Tokyo Metropolitan University, Hachioji, Japan

1 Introduction

In this paper, we are oriented to the decomposition of generalized Orlicz–Morrey spaces of the third kind and its applications. Generalized Orlicz–Morrey spaces of the third kind are equipped with two functions Φ and ϕ; $\Phi : [0,\infty) \to [0,\infty)$ is a convex bijection and $\phi : (0,\infty) \to (0,\infty)$ belongs to a class of $\mathcal{G}_\Phi$, which consists of all decreasing functions $\phi : (0,\infty) \to (0,\infty)$ such that $\mu : (0,\infty) \ni t \mapsto \Phi^{-1}(t^{-n})\phi(t)^{-1} \in (0,\infty)$ is almost decreasing, that is, there exists a constant $C > 0$ such that $\mu(t) \le C\mu(s)$ for all $0 < s < t < \infty$. Denote by Δ_2 the set of all convex bijections $\Phi : [0,\infty) \to [0,\infty)$ such that the *doubling condition*:

$$\Phi(2t) \le C\Phi(t) \quad (t \ge 0) \tag{1.1}$$

holds for some constant $C \ge 2$, which is called the doubling constant, and by ∇_2 the set of all convex functions $\Phi : [0,\infty) \to [0,\infty]$ such that the ∇_2-*condition*:

$$2C'\Phi(t) \le \Phi(2t) \quad (t \ge 0) \tag{1.2}$$

holds for some $C' > 1$. Note that C in (1.1) exceeds 2 when $\Phi \in \Delta_2 \cap \nabla_2$ due to (1.2). Recall also that the *conjugate function* Ψ of Φ is defined by:

$$\Psi(t) \equiv \sup\{st - \Phi(s) : s \ge 0\} \quad (t \ge 0). \tag{1.3}$$

Let Φ be a Young function. Recall that the *Orlicz norm* $\|f\|_{L^\Phi(E)}$ over a measurable set E in $\mathbb{R}^n$ is defined by:

$$\|f\|_{L^\Phi(E)} \equiv \inf\left\{\lambda > 0 : \int_E \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}. \tag{1.4}$$

Define $L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ as the set of all measurable functions f for which $f \in L^\Phi(K)$ for all compact sets K in $\mathbb{R}^n$.

We now define generalized Orlicz–Morrey spaces of the third kind. All the "cubes" in $\mathbb{R}^n$ are assumed to have their sides parallel to the coordinate axes. Denote by $\mathcal{Q}$ the set of all cubes. For a cube $Q \in \mathcal{Q}$, the symbol $\ell(Q)$ stands for the *side-length* of the cube Q; $\ell(Q) \equiv |Q|^{\frac{1}{n}}$, where $|\cdot|$ stands for the Lebesgue measure. The *generalized Orlicz–Morrey space* $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|f\|_{L^\Phi(Q)}$$

is finite. Write $\mathcal{G}_p \equiv \mathcal{G}_\Phi$ when $\Phi(t) = t^p$ with $1 \le p < \infty$. For $\kappa \in (1,\infty)$, the *harmonic conjugate* is defined by: $\kappa' \equiv \kappa(\kappa-1)^{-1}$. We use the *powered Hardy–Littlewood maximal operator* $M^{(\kappa)}$ given by: for a measurable function $g : \mathbb{R}^n \to \mathbb{C}$

$$M^{(\kappa)}g(x) \equiv \sup_{Q\in\mathcal{Q}}\left(\frac{\chi_Q(x)}{|Q|}\int_Q |g(y)|^\kappa dy\right)^{\frac{1}{\kappa}}. \tag{1.5}$$

The following result is one of the fundamental theorems in this paper:

Theorem 1.1 *Let $\kappa \in (1,\infty)$ and $\Phi \in \Delta_2 \cap \nabla_2$. Define Ψ by (1.3). Assume in addition that the quotient ϕ/η of $\phi \in \mathcal{G}_\Phi$ and $\eta \in \mathcal{G}_{\kappa'}$ satisfies the integral condition*:

$$\int_r^\infty \frac{\phi(s)}{\eta(s)s}\,ds \le C\frac{\phi(r)}{\eta(r)} \quad (r>0) \tag{1.6}$$

and that $M^{(\kappa)}$ is bounded on $L^\Psi(\mathbb{R}^n)$: for all $g \in L^\Psi(\mathbb{R}^n)$

$$\left\| M^{(\kappa)} g \right\|_{L^\Psi} \le C\|g\|_{L^\Psi}. \tag{1.7}$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^{\kappa'}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill the following three conditions:

1. *The support condition on a_j*:

$$\mathrm{supp}(a_j) \subset Q_j, \tag{1.8}$$

2. *The size condition on a_j*:

$$\|a_j\|_{\mathcal{M}^\eta_{\kappa'}} := \sup_{Q\in\mathcal{Q}} \frac{1}{\eta(\ell(Q))}\left(\frac{1}{|Q|}\int_Q \left|a_j(y)\right|^{\kappa'} dy\right)^{\frac{1}{\kappa'}} \le \frac{1}{\eta(\ell(Q_j))}. \tag{1.9}$$

3. *The coefficient condition on $\{\lambda_j\}_{j=1}^\infty$*:

$$\left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}} < \infty. \tag{1.10}$$

Then

$$f \equiv \sum_{j=1}^\infty \lambda_j a_j \tag{1.11}$$

converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^\Phi_{\rm loc}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \le C\left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}. \tag{1.12}$$

A tacit understanding is that we use the convention that the meaning of C can change from line to line.

Now we state the next theorem. To this end we fix some notation. We denote by $\mathcal{P}_d(\mathbb{R}^n)$ the set of all polynomials in variables $x_1, x_2, \ldots, x_n$ with degree less than or equal to d. Let $\mathbb{N}_0 \equiv \{0,1,\ldots\}$. The space $\mathcal{P}_L^\perp(\mathbb{R}^n)$ is the set of all measurable functions f such that

$$\int_{\mathbb{R}^n} \left|f(x)\right|\left(1+|x|\right)^L dx < \infty, \qquad \int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le L$. Denote by $L^\infty_{\rm comp}(\mathbb{R}^n)$ the set of all compactly supported functions which are essentially bounded.

In the present paper we also seek to prove the following decomposition result for the functions in generalized Orlicz–Morrey spaces of the third kind:

Theorem 1.2 *Let $L \in \mathbb{N}_0$, $\Phi \in \Delta_2 \cap \nabla_2$ and $\phi \in \mathcal{G}_\Phi$. Assume that ϕ satisfies the integral condition*:

$$\int_r^\infty \phi(s)\frac{ds}{s} \le C\phi(r) \quad (r > 0). \tag{1.13}$$

Let $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Then there exist $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^\infty_{\mathrm{comp}}(\mathbb{R}^n) \cap \mathcal{P}_L^\perp(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that, for all $v > 0$

$$|a_j| \le \chi_{Q_j} \quad (j = 1, 2, \ldots), \qquad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_{\phi,\Phi}} \le C_v \|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{1.14}$$

Here the constant $C_v > 0$ is independent of f.

In view of Theorems 1.1 and 1.2 and the results in [40, Sect. 4], the decomposition of the functions requires us the vector-valued inequality for the Hardy–Littlewood maximal operator and the synthesis of the functions requires us the duality. As we did in [40, 41], if we combine these theorems, we can prove that the singular integral operators are bounded in $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

As is mentioned in [57, p. 185], these theorems date back to the papers by Uchiyama and Jones. Note that $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ covers many classical function spaces.

Example 1.3 Let $1 \le q \le p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L^p(\mathbb{R}^n)$, the *classical Morrey space* $\mathcal{M}_q^p(\mathbb{R}^n)$, the generalized Morrey space $\mathcal{M}_p^\phi(\mathbb{R}^n)$ and the Orlicz space $L^\Phi(\mathbb{R}^n)$ with norm coincidence:

1. If $\Phi(t) = t^p$ and $\phi(t) = t^{-\frac{n}{p}}$, then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with norm equivalence.
2. If $\Phi(t) = t^q$ and $\phi(t) = t^{-\frac{n}{p}}$, then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, which is denoted by $\mathcal{M}_q^p(\mathbb{R}^n)$, is the classical Morrey space.
3. If $\Phi(t) = t^p$, then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) = \mathcal{M}_p^\phi(\mathbb{R}^n)$ is the generalized Morrey space which were discussed in [16–18, 35, 54]. See [1–3] for local Morrey spaces.
4. If $\phi(t) = \Phi^{-1}(t^{-n})$, then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) = L^\Phi(\mathbb{R}^n)$, which is beyond the reach of generalized Orlicz–Morrey spaces of the second kind defined in [54] according to an example constructed in [14]; see Definition 9.1 for its definition.
5. As a particular case, by letting $\Phi(t) = t^{p_1} + t^{p_2}$ for some $1 < p_1, p_2 < \infty$, we can recover the intersection space $L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$.
6. Another example is $\Phi(t) = t^p(\log(3+t))^q$ with $p > 1$ and $q \in \mathbb{R}$. In general Φ is not convex but we can replace Φ with a convex function equivalent to Ψ.

Other definitions of generalized Orlicz–Morrey spaces can be found in [36, 38, 39, 54]; see Definition 9.1 in the present paper. Therefore, our definition of generalized Orlicz–Morrey spaces here is named "third kind".

By combining the idea of Theorem 1.1 with Example 1.3, we have the following estimate for Orlicz spaces, whose proof is similar to Theorem 1.1:

Corollary 1.4 *Let $\kappa \in (1,\infty)$ and $\Phi \in \Delta_2 \cap \nabla_2$ satisfy (1.7). Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^{\kappa'}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill the following three conditions*:

1. *The support condition on* a_j:

$$\operatorname{supp}(a_j) \subset Q_j. \tag{1.15}$$

2. *The size condition on* a_j:

$$\|a_j\|_{L^{\kappa'}} \le |Q_j|^{1/\kappa'}. \tag{1.16}$$

3. *The coefficient condition on* $\{\lambda_j\}_{j=1}^{\infty}$:

$$\left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^\Phi} < \infty. \tag{1.17}$$

Then $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $L^\Phi(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{L^\Phi} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^\Phi}. \tag{1.18}$$

Our strategy of the proof of Theorems 1.1 and 1.2 is as follows: The proof of Theorem 1.1 hinges upon the duality. Meanwhile, to prove Theorem 1.2 we shall convert the generalized Orlicz–Morrey space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ of the third kind to the generalized Hardy–Orlicz–Morrey space $H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ of the third kind as we do in (5.1). In fact, our experience shows that the Hardy space $H^p(\mathbb{R}^n)$ with $0 < p < \infty$ can be more informative than the Lebesgue space $L^p(\mathbb{R}^n)$ with $0 < p < \infty$ when we discuss the boundedness of some operators. See [57] and references therein for more information on the Hardy space $H^p(\mathbb{R}^n)$.

We adopt the following notation, although some of them duplicate:

1. Let $A, B \ge 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \le CB$ and $A \sim B$ stands for $A \lesssim B \lesssim A$, where C depends only on the parameters of importance.
2. The symbol $B(x, r)$ stands for the *open ball* centered at x or radius $r > 0$.
3. By a "cube" we mean a compact cube whose edges are parallel to the coordinate axes, namely, the metric ball defined by ℓ^∞ is called a *cube*. If a cube has center x and radius r, we denote it by $Q(x, r)$. From the definition of $Q(x, r)$, its volume is $(2r)^n$. We write $Q(r)$ instead of $Q(0, r)$, where 0 denotes the origin. Given a cube Q, we denote by $c(Q)$ *the center of* Q and by $\ell(Q)$ the *sidelength of* Q: $\ell(Q) = |Q|^{\frac{1}{n}}$. Given a cube Q and $k > 0$, its k*-times expansion* kQ means the *cube concentric to* Q *with sidelength* $k\ell(Q)$.
4. By a *dyadic cube*, we mean a set of the form $2^{-j}m + [0, 2^{-j})^n$ for some $m \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. Note that dyadic cubes are not open nor closed but that cubes are closed.
5. In the whole paper, we adopt the following definition of the Hardy–Littlewood maximal operator to estimate some integrals: The *Hardy–Littlewood maximal operator* M is defined by:

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)|\, dy \quad \left(x \in \mathbb{R}^n\right), \tag{1.19}$$

for a locally integrable function f on $\mathbb{R}^n$.
6. Let $0 < \alpha < n$. We define the *fractional integral operator* I_α by:

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}\, dy \quad \left(x \in \mathbb{R}^n\right)$$

for all suitable functions f on $\mathbb{R}^n$.

7. Let $\mathfrak{M}(0,\infty)$ be the set of all measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all non-decreasing functions in $\mathfrak{M}^+(0,\infty)$. The subset $\mathbb{A}$ of $\mathfrak{M}^+(0,\infty;\uparrow)$ is define by:

$$\mathbb{A} \equiv \left\{\phi \in \mathfrak{M}^+(0,\infty;\uparrow) : \inf \phi = 0\right\}. \tag{1.20}$$

Other definitions are given when needed.

To conclude this section, we describe how we organize the remaining part of this paper. In Sect. 2 we collect some preliminary facts such as the Hölder inequality for our function spaces as well as preduality. As an auxiliary step, in Sect. 3 we shall prove the boundedness of the Hardy–Littlewood maximal operator on generalized Orlicz–Morrey spaces of the third kind. We prove Theorems 1.1 and 1.2 in Sects. 4 and 5, respectively. A counterpart of Theorems 1.1 and 1.2 to Hardy spaces with variable exponents and to Hardy–Orlicz spaces are proved in [40] and [41], respectively. However, as is the case with classical Morrey spaces [25], we need to be careful when we prove Theorem 1.2. In fact, for a sequence $F = \{f_j\}_{j=1}^{\infty}$ of measurable functions satisfying $0 \le f_1 \le f_2 \le \cdots \to f$ and $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ does not imply $\|f - f_j\|_{\mathcal{M}_{\phi,\Phi}} \to 0$ as is seen from the example of $f_j(x) \equiv |x|^{-n/p}(1-\chi_{B(0,j^{-1})})(x)$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$. This difficulty will be overcome by the use of Lemma 5.6. Applications are taken up in Sects. 6 and 7. We apply Theorems 1.1 and 1.2 to prove that the singular integral operators are bounded on generalized Orlicz–Morrey spaces of the third kind in Sect. 6. As another application of Theorems 1.1 and 1.2, we consider a bilinear operator $(f,g) \mapsto g \cdot I_\alpha f$ in Sect. 7. In Sect. 8 we refine what we obtained in Sect. 3; we reconsider the case when ϕ depends on x. Finally, in Sect. 9 we first compare three generalized-Orlicz–Morrey spaces and then we discuss the assumptions in Theorems 1.1 and 1.2.

2 Preliminaries

2.1 Fundamental Structure of Generalized Orlicz–Morrey Spaces of the Third Kind

Here and below, by a Young function we mean a convex bijection on $[0,\infty)$. The main structure of the generalized Morrey space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ of the third kind is as follows:

Proposition 2.1 *Let $\phi \in \mathfrak{M}^+(0,\infty)$ be a decreasing function and Φ a Young function. Write*

$$\mu(t) = \frac{1}{\phi(t)}\Phi^{-1}\left(t^{-n}\right) \quad (t > 0).$$

Then for all $a > 0$, we have

$$\|\chi_{Q(a)}\|_{\mathcal{M}_{\phi,\Phi}} = \sup_{r \ge 2a} \frac{\mu(r)}{\Phi^{-1}((2a)^{-n})}.$$

In particular, the following are equivalent;

1. $\phi \in \mathcal{G}_\Phi$;
2. *for all $a > 0$, we have $\chi_{Q(a)} \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ and*

$$\frac{1}{\phi(2a)} \le \|\chi_{Q(a)}\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \frac{1}{\phi(2a)}. \tag{2.1}$$

Proof We calculate that

$$
\begin{aligned}
\|\chi_{Q(a)}\|_{\mathcal{M}_{\phi,\Phi}} &= \sup_{r>0} \mu(2r)\|\chi_{Q(a)}\|_{L^\Phi(Q(r))} \\
&= \sup_{r>0} \mu(2r)\|1\|_{L^\Phi(Q(\min(a,r)))} \\
&= \max\left(\sup_{0<r\le a} \mu(2r)\|1\|_{L^\Phi(Q(r))}, \sup_{r>a} \mu(2r)\|1\|_{L^\Phi(Q(a))} \right) \\
&= \max\left(\sup_{0<r\le a} \frac{1}{\phi(2r)}, \sup_{r>a} \frac{\mu(2r)}{\Phi^{-1}((2a)^{-n})} \right).
\end{aligned}
$$

Since we are assuming that ϕ is decreasing, we have

$$
\|\chi_{Q(a)}\|_{\mathcal{M}_{\phi,\Phi}} = \max\left(\frac{1}{\phi(2a)}, \sup_{r>a} \frac{\Phi^{-1}(r^{-n})}{\phi(r)\Phi^{-1}((2a)^{-n})} \right) = \sup_{r\ge a} \frac{\Phi^{-1}(r^{-n})}{\phi(r)\Phi^{-1}((2a)^{-n})},
$$

as was to be shown. □

We also need the following scaling law for later consideration:

Lemma 2.2 *Let $0 < b < 1$ and Φ be a Young function. Define $\tilde{\Phi}(t) = \Phi(t^{\frac{1}{b}})$ for $t \ge 0$. Then for all cubes Q and all measurable functions f on Q, we have*

$$
\left\| |f|^{\frac{1}{b}} \right\|_{L^\Phi(Q)} = \left(\|f\|_{L^{\tilde{\Phi}}(Q)} \right)^{\frac{1}{b}}.
$$

Proof The proof is straightforward; just use the definition (1.4). We omit the detail. □

2.2 The Conjugate of Φ and Related Hölder's Inequality

Let Φ be a function satisfying the doubling condition (1.1) and the ∇_2-condition (1.2). Then define the conjugate function Ψ of Φ by (1.3). Note that Ψ satisfies the same condition as Φ; $\Psi \in \Delta_2 \cap \nabla_2$. Observe also that

$$
st \le \Phi(s) + \Psi(t) \tag{2.2}
$$

for all $s, t \ge 0$ from the definition of Ψ.

We recall the following Hölder inequality for Orlicz spaces:

Lemma 2.3 *Let $\Phi, \Psi : [0, \infty) \to [0, \infty)$ be Young function that are conjugate each other. Let f be a measurable function defined on a measurable set E.*

1. *For all measurable functions $g : E \to \mathbb{C}$,*

$$
\|f \cdot g\|_{L^1(E)} \le 2\|f\|_{L^\Phi(E)}\|g\|_{L^\Psi(E)}. \tag{2.3}
$$

2. *We can find a measurable function $g : E \to \mathbb{C}$ with norm 1 such that*

$$
\int_E f(x)g(x)\,dx \ge 0, \qquad C^{-1}\|f\|_{L^\Phi(E)} \le \int_E f(x)g(x)\,dx \le 2\|f\|_{L^\Phi(E)}, \tag{2.4}
$$

where $C > 1$ is a constant depending on Φ.

3. *For any* $a > 0$,

$$a \le \Phi^{-1}(a)\Psi^{-1}(a) \le 2a. \tag{2.5}$$

Proof For the sake of convenience we recall the proof of (2.5). We refer to [44] for (2.3) and to [55, Lemma 2.4] for an estimate similar to (2.4).

From inequality (2.2) if we take $s = \Phi^{-1}(a)$ and $t = \Psi^{-1}(a)$ we have

$$\Phi^{-1}(a)\Psi^{-1}(a) \le \Phi\big(\Phi^{-1}(a)\big) + \Psi\big(\Psi^{-1}(a)\big) = a + a = 2a. \tag{2.6}$$

On the other hand for every $t > 0$

$$\begin{aligned}\Psi\left(\frac{\Phi(t)}{t}\right) &= \sup_{s>0}\left(s\frac{\Phi(t)}{t} - \Phi(s)\right)\\ &= \sup_{s\in(0,t)}\left(s\frac{\Phi(t)}{t} - \Phi(s)\right)\\ &\le \sup_{s\in(0,t)} s\frac{\Phi(t)}{t}\\ &= \Phi(t).\end{aligned}$$

Replacing $\Phi(t)$ by t in this inequality, we obtain the following equivalent inequalities:

$$\begin{aligned}\Psi\left(\frac{a}{\Phi^{-1}(a)}\right) \le a \quad &\Longleftrightarrow \quad \frac{a}{\Phi^{-1}(a)} \le \Psi^{-1}(a)\\ &\Longleftrightarrow \quad a \le \Phi^{-1}(a)\Psi^{-1}(a).\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we obtain (2.5). Thus, the proof is complete. □

We also use the following absolute continuity of the Orlicz-norm:

Lemma 2.4 *Let* $\Phi : \mathbb{R}^n \to [0, \infty)$ *be a Young function and let* $f : \mathbb{R}^n \to \mathbb{C}$ *be a measurable function such that* $\Phi(|f|) \in L^1(\mathbb{R}^n)$. *Define* $f^j \equiv f\chi_{\{|f|\le j\}}$. *Then we have*

$$\lim_{j\to\infty} \|f - f^j\|_{L^\Phi(Q)} = 0 \tag{2.8}$$

for any cube Q.

Proof The proof is similar to [55, Proposition 2.7]. We omit the detail. □

The following embedding relation is useful when we want to consider the functions in $\mathcal{M}^{\eta,\Phi}(\mathbb{R}^n)$:

Proposition 2.5 *Let* $\kappa \in (1, \infty)$, $\Phi \in \Delta_2 \cap \nabla_2$ *and* $\eta \in \mathcal{G}_\Phi \cap \mathcal{G}_{\kappa'}$. *Define* Ψ *by* (1.3). *Assume in addition that* Ψ *satisfies* (1.7). *Then*

$$\|f\|_{\mathcal{M}_{\eta,\Phi}} \lesssim \|f\|_{\mathcal{M}^\eta_{\kappa'}} \tag{2.9}$$

for all $f \in \mathcal{M}^\eta_{\kappa'}(\mathbb{R}^n)$.

Proof By (2.4), we can find $g \in L^{\Psi}(\mathbb{R}^n)$ with norm 1 such that

$$\frac{1}{\eta(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|f\|_{L^{\Phi}(Q)} \lesssim \frac{1}{\eta(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\int_Q f(x)g(x)\,dx.$$

Hölder's inequality yields

$$\frac{1}{\eta(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|f\|_{L^{\Phi}(Q)} \lesssim \frac{1}{\eta(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|f\|_{L^{\kappa'}(Q)}\|g\|_{L^{\kappa}(Q)}.$$

By the definition of the norm, we obtain

$$\frac{1}{\eta(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|f\|_{L^{\Phi}(Q)} \lesssim \|f\|_{\mathcal{M}^{\eta}_{\kappa'}}\Phi^{-1}\left(\frac{1}{|Q|}\right)|Q|^{1/\kappa'}\|g\|_{L^{\kappa}(Q)}.$$

Thanks to (1.7) and (2.5), we have

$$\begin{aligned}\frac{1}{\eta(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|f\|_{L^{\Phi}(Q)} &\lesssim \|f\|_{\mathcal{M}^{\eta}_{\kappa'}}\Phi^{-1}\left(\frac{1}{|Q|}\right)\Psi^{-1}\left(\frac{1}{|Q|}\right)|Q|\cdot\left\|M^{(\kappa)}g\right\|_{L^{\Psi}}\\ &\lesssim \|f\|_{\mathcal{M}^{\eta}_{\kappa'}}\Phi^{-1}\left(\frac{1}{|Q|}\right)\Psi^{-1}\left(\frac{1}{|Q|}\right)|Q|\\ &\lesssim \|f\|_{\mathcal{M}^{\eta}_{\kappa'}}.\end{aligned}$$

Thus, the proof is complete. □

For later consideration, we formulate and prove the following Hölder inequality for $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$:

Theorem 2.6 *Suppose that we are given functions $\Phi_i \in \Delta_2 \cap \nabla_2$ and $\phi_i \in \mathcal{G}_{\Phi_i}$ for $i = 1, 2, 3$. If these functions satisfy*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \le \Phi_3^{-1}(t) \quad (t \ge 0)$$

and

$$\phi_3(t) \le \phi_1(t)\phi_2(t) \quad (t > 0),$$

then for every $f \in \mathcal{M}_{\phi_1,\Phi_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{\phi_2,\Phi_2}(\mathbb{R}^n)$, we have

$$\|f \cdot g\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le 2\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}}\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}. \tag{2.10}$$

Proof Since $\|f \cdot g\|_{L^{\Phi_3}} \le 2\|f\|_{L^{\Phi_1}}\|g\|_{L^{\Phi_2}}$; see (2.3), we have

$$\phi_3\big(\ell(Q)\big)\|f \cdot g\|_{L^{\Phi_3}} \le 2\phi_1\big(\ell(Q)\big)\|f\|_{L^{\Phi_1}}\phi_2\big(\ell(Q)\big)\|g\|_{L^{\Phi_2}} \le 2\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}}\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}.$$

Hence (2.10) follows. □

See [58, Lemma 2.4] for the classical case of generalized Morrey spaces.
We conclude this section with the following estimate:

Lemma 2.7 *Let Ψ be a doubling Young function. Write $p := \log_2 C_\Psi$, where $C_\Psi \geq 2$ is the doubling constant for Ψ, so that $\Psi(2t) \leq C_\Psi \Psi(t)$ for $t \geq 0$. Then we have*

$$\|g\|_{L^\Psi(Q)} \leq C_{Q,p}\|g\|_{L^p(Q)} \tag{2.11}$$

for any cube Q and any measurable function g on Q, where $C_{Q,p}$ depends on Q.

Proof Observe that $\Psi(t) \leq 2^p \Psi(1) t^p$, $t \geq 1$ and that $\Psi(0) = 0$. As a consequence we may suppose

$$\Psi(t) \leq \frac{1}{2|Q|} + C'_Q t^p \tag{2.12}$$

for all $t \geq 0$ by replacing constant C'_Q with a large one. Hence we have

$$\begin{aligned}
\|g\|_{L^\Psi(Q)} &= \inf\left\{\lambda > 0 : \int_Q \Psi\left(\frac{|g(x)|}{\lambda}\right) dx \leq 1\right\} \\
&\leq \inf\left\{\lambda > 0 : \int_Q \left(\frac{1}{2|Q|} + C'_Q\left(\frac{|g(x)|}{\lambda}\right)^p\right) dx \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : C'_Q \int_Q \left(\frac{|g(x)|}{\lambda}\right)^p dx \leq \frac{1}{2}\right\} \\
&= \left(2C'_Q\right)^{\frac{1}{p}} \|g\|_{L^p(Q)},
\end{aligned}$$

as was to be shown. □

2.3 Predual Space

In this section, we characterize the predual space of $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ by the method of Zorko [72].

Definition 2.8 Let (Φ, Ψ) be a *complementary pair* of Young functions; Φ and Ψ are related by (1.3). Let also $\phi \in \mathcal{G}_\Phi$. A (ϕ, Ψ)*-block of the third kind* is a measurable function A supported on a cube Q satisfying

$$\|A\|_{L^\Psi(Q)} \leq \frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right). \tag{2.13}$$

A function f is said to belong to $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ if there exist a sequence $\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^1(\mathbb{N})$ and a sequence $\{A_j\}_{j=1}^\infty$ of (ϕ, Ψ)-blocks of the third kind such that

$$f(x) = \sum_{j=1}^\infty \lambda_j A_j(x) \tag{2.14}$$

for almost every $x \in \mathbb{R}^n$, where the convergence takes place absolutely. The norm of f is given by;

$$\|f\|_{\mathcal{B}_{\phi,\Psi}} \equiv \inf \|\lambda\|_{\ell^1},$$

where the infimum is taken over all admissible expressions (2.14).

Example 2.9 When Q is a cube and $g \in L^\Psi(\mathbb{R}^n) \setminus \{0\}$, g^Q is a (ϕ, Ψ)-block of the third kind, where

$$g^Q \equiv \frac{1}{\|g\|_{L^\Psi}\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\chi_Q g. \tag{2.15}$$

In fact, we check that $\mathrm{supp}(g^Q) \subset Q$ and that

$$\|g^Q\|_{L^\Psi(Q)} = \frac{1}{\|g\|_{L^\Psi}\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\|g\|_{L^\Psi(Q)} \le \frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right),$$

which implies that g^Q is a (ϕ, Ψ)-block of the third kind. Thus, in particular it follows that

$$\|g^Q\|_{\mathcal{B}_{\phi,\Psi}} \le 1. \tag{2.16}$$

Lemma 2.10 *Let $\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^1(\mathbb{N})$ and let $\{A_j\}_{j=1}^\infty$ be a sequence of (ϕ, Ψ)-blocks of the third kind. Then for all $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, we have*

$$\left\| f \cdot \sum_{j=1}^\infty |\lambda_j A_j| \right\|_{L^1} \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}}\|\lambda\|_{\ell^1}. \tag{2.17}$$

Proof Let $Q_j \in \mathcal{Q}$ satisfy $\mathrm{supp}(A_j) \subset Q_j$ and that $\|A_j\|_{L^\Psi(Q)} \le \frac{1}{\phi(\ell(Q_j))}\Phi^{-1}(\frac{1}{|Q_j|})$. Then we have

$$\|f \cdot A_j\|_{L^1} \le 2\|f\|_{L^\Phi(Q_j)}\|A_j\|_{L^\Psi(Q_j)} \le 2\frac{\|f\|_{L^\Phi(Q_j)}}{\phi(\ell(Q_j))}\Phi^{-1}\left(\frac{1}{|Q_j|}\right) \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}}$$

from (2.13). As a result,

$$|\lambda_j| \cdot \|f \cdot A_j\|_{L^1} \le 2|\lambda_j| \cdot \|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{2.18}$$

If we add (2.18) over $j \in \mathbb{N}$ then we obtain (2.17). □

Corollary 2.11 *Let (Φ, Ψ) be a complementary Young pair and $\phi \in \mathcal{G}_\Phi$.*

1. *Let $\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^1(\mathbb{N})$ and $\{A_j\}_{j=1}^\infty$ be a sequence of (ϕ, Ψ)-blocks of the third kind. Then the series*

$$f(x) = \sum_{j=1}^\infty \lambda_j A_j(x) \tag{2.19}$$

   converges absolutely for almost all $x \in \mathbb{R}^n$.
2. *Let $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Then the mapping*

$$\mathfrak{L}_f : \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n) \ni g \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{C} \tag{2.20}$$

   defines a bounded linear functional on $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$.
3. *Let $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then the mapping*

$$\mathfrak{M}_g : \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \ni f \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{C}$$

   defines a bounded linear functional on $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Proof The convergence of the right-hand side of (2.19) is trivial from (2.17). Let $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then we have an expression;

$$g = \sum_{j=1}^{\infty} \lambda_j a_j, \qquad \|\lambda\|_{\ell^1} \le 2\|g\|_{\mathcal{B}_{\phi,\Psi}},$$

where each a_j is a (ϕ, Ψ)-block of the third kind, $\lambda = \{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ and the convergence of the sum takes place in the sense of almost everywhere. Then we have

$$\|f \cdot g\|_{L^1} \le \sum_{j=1}^{\infty} |\lambda_j| \cdot \|f \cdot a_j\|_{L^1} \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}} \|\lambda\|_{\ell^1} \le 4\|f\|_{\mathcal{M}_{\phi,\Phi}} \|g\|_{\mathcal{B}_{\phi,\Psi}}$$

from (2.18). Thus, we conclude $\|f \cdot g\|_{L^1} \le 4\|f\|_{\mathcal{M}_{\phi,\Phi}} \|g\|_{\mathcal{B}_{\phi,\Psi}}$. Note that (2.11) implies that $f \cdot g \in L^1(\mathbb{R}^n)$ and that

$$\left| \int_{\mathbb{R}^n} f(x) g(x)\, dx \right| \le 4\|f\|_{\mathcal{M}_{\phi,\Phi}} \|g\|_{\mathcal{B}_{\phi,\Psi}}.$$

Thus, we see that $\mathfrak{L}_f$ and $\mathfrak{M}_g$ define bounded linear functionals. □

Proposition 2.12 *Let (Φ, Ψ) be a complementary Young pair and $\phi \in \mathcal{G}_\Phi$. Then $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$.*

Proof Let $f \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then f has an expression:

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where each a_j is a (ϕ, Ψ)-block of the third kind and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$. Set $f_k \equiv \sum_{j=1}^{k} \lambda_j a_j$ for $k \in \mathbb{N}$. Then

$$\|f - f_k\|_{\mathcal{B}_{\phi,\Psi}} \le \sum_{j=k+1}^{\infty} |\lambda_j| \to 0$$

as $k \to \infty$. This means that we can suppose that f is expressed as a finite linear combination of (ϕ, Ψ)-blocks of the third kind or even that f itself is a (ϕ, Ψ)-block of the third kind.

Let f be a (ϕ, Ψ)-block of the third kind; for some $Q \in \mathcal{Q}$,

$$\mathrm{supp}(f) \subset Q, \qquad \|f\|_{L^\Psi(Q)} \le \frac{1}{\phi(\ell(Q))} \Phi^{-1}\left(\frac{1}{|Q|}\right).$$

Let us set $f^j \equiv f\chi_{\{|f| \le j\}}$. Then by virtue of Lemma 2.4

$$\limsup_{j\to\infty} \|f - f^j\|_{\mathcal{B}_{\phi,\Psi}} \le \frac{\phi(\ell(Q))}{\Phi^{-1}(\frac{1}{|Q|})} \limsup_{j\to\infty} \|f - f^j\|_{L^\Psi(Q)} = 0.$$

Since $f^j \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, we conclude that $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. □

Theorem 2.13 *Let* (Φ, Ψ) *be complementary Young pair and* $\Phi \in \Delta_2$. *Let* $\mathfrak{L}_f$ *be a linear mapping defined by* (2.20) *for* $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. *Then any bounded linear functional* $\mathfrak{L}$ *on* $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ *is realized as* $\mathfrak{L} = \mathfrak{L}_f$ *with some* $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Proof We fix a cube $Q \in \mathcal{Q}$ for the time being. Let $p := \log_2 C_\Psi$ be the constant in (2.11), so that $p > 1$ due to the assumption $\Phi \in \Delta_2$.

From (2.11), for each cube Q, we can define a bounded linear functional

$$L^p\left(\mathbb{R}^n\right) \ni g \mapsto \mathfrak{L}^Q(g) \equiv \mathfrak{L}(\chi_Q g) \in \mathbb{C}.$$

In fact, when $g \in L^p(\mathbb{R}^n) \setminus \{0\}$, $\chi_Q g \in L^p(\mathbb{R}^n)$ and g^Q, defined by (2.15), is a (ϕ, Ψ)-block of the third kind from Example 2.9. Since $\mathfrak{L}$ is bounded on $\mathcal{B}_{\phi,\psi}(\mathbb{R}^n)$, we see that

$$\left|\mathfrak{L}\left(\frac{1}{\phi(\ell(Q))\|g\|_{L^\Psi}}\Phi^{-1}\left(\frac{1}{|Q|}\right)\chi_Q g\right)\right| \leq \|\mathfrak{L}\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}$$

from (2.16), which together with (2.11) implies

$$\left|\mathfrak{L}(\chi_Q \cdot g)\right| \leq \frac{\phi(\ell(Q))}{\Phi^{-1}(\frac{1}{|Q|})}\|\mathfrak{L}\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}\|g\|_{L^\Psi} \leq C_{Q,p}\frac{\phi(\ell(Q))}{\Phi^{-1}(\frac{1}{|Q|})}\|\mathfrak{L}\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}\|g\|_{L^p}, \tag{2.21}$$

whenever $g \in L^p(\mathbb{R}^n) \setminus \{0\}$. Note that (2.21) is trivially valid for $g = 0$. By the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$, we can find $f_Q \in L^{p'}(\mathbb{R}^n)$ such that

$$\operatorname{supp}(f_Q) \subset Q, \qquad \mathfrak{L}^Q(g) = \int_{\mathbb{R}^n} g(x) f_Q(x)\,dx \tag{2.22}$$

for all $g \in L^p(\mathbb{R}^n)$ and that

$$\|f_Q\|_{L^{p'}} \leq C_{Q,p}\frac{\phi(\ell(Q))}{\Phi^{-1}(\frac{1}{|Q|})}\|\mathfrak{L}\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}.$$

Let Q be a cube and R a cube that engulfs Q. The left-hand side of the second expression in (2.22) vanishes if g is supported outside Q. In addition, we have

$$\mathfrak{L}^Q(g) = \mathfrak{L}(\chi_Q g) = \mathfrak{L}(\chi_Q \chi_R g) = \int_{\mathbb{R}^n} \chi_Q(x) f_R(x) g(x)\,dx.$$

Thus, $\chi_Q(x) f_R(x) = f_Q(x)$ for almost every $x \in \mathbb{R}^n$. This implies that the limit $f(x) \equiv \lim_{j\to\infty} f_{Q(j)}(x)$ exists for almost every $x \in \mathbb{R}^n$.

Let us check that the function f belongs to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. To this end, we fix a cube Q once again. Then we have

$$C^{-1}\|f\|_{L^\Phi(Q)} \leq \int_Q f(x)g(x)\,dx, \qquad \operatorname{supp}(g) \subset Q$$

for some measurable function $g \in L^\Psi(\mathbb{R}^n)$ with norm 1, where C is a constant from Lemma 2.3.

Note that $\frac{1}{\phi(\ell(Q))}\Phi^{-1}(\frac{1}{|Q|})\chi_Q g$ is a (ϕ, Ψ)-block of the third kind from Example 2.9. Thus, keeping in mind

$$\int_Q f(x)g(x)\,dx = \mathfrak{L}^Q(g),$$

we obtain

$$\|f\|_{L^\Phi(Q)} \lesssim \mathfrak{L}^Q(g) \le \|\chi_Q g\|_{\mathcal{B}_{\phi,\Psi}} \|\mathfrak{L}\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}} \le \frac{\phi(\ell(Q))\|\mathfrak{L}\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}}{\Phi^{-1}(\frac{1}{|Q|})}.$$

Hence we have $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Finally, let us check $\mathfrak{L} = \mathfrak{L}_f$. To this end, we need to prove $\mathfrak{L}(g) = \mathfrak{L}_f(g)$ for all $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ but by Proposition 2.12, we can suppose that $g \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. Since g is compactly supported, we can take $j_0 \in \mathbb{N}$ so that $\mathrm{supp}(g) \subset Q(j_0)$. Then

$$\mathfrak{L}(g) = \int_{\mathbb{R}^n} f_{Q(j_0)}(x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(x)\,dx = \mathfrak{L}_f(g).$$

As a consequence we have $\mathfrak{L} = \mathfrak{L}_f$. □

Remark 2.14 See [15, 28] for other constructions of predual spaces of classical Morrey spaces.

3 Boundedness of the Maximal Operator

In this section we aim to recall a boundedness criteria of the Hardy–Littlewood operator M defined in (1.19) and to extend the Fefferman–Stein vector-valued inequality from $L^p(\ell^q, \mathbb{R}^n)$ to $\mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$. Recall that

$$\|Mf\|_{L^p} \lesssim \|f\|_{L^p}, \tag{3.1}$$

for $1 < p \le \infty$ and

$$\left|\left\{x \in \mathbb{R}^n : Mf(x) > \lambda\right\}\right| \lesssim \frac{\|f\|_{L^1}}{\lambda} \tag{3.2}$$

for all $\lambda > 0$, which is obtained in [22]. We define $L^p(\ell^q, \mathbb{R}^n)$ to the set of all sequences of measurable functions $F = \{f_j\}_{j=1}^\infty$ such that

$$\|F\|_{L^p(\ell^q)} \equiv \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p}$$

is finite. Here a natural modification is made when $q = \infty$. We write $MF \equiv \{Mf_j\}_{j=1}^\infty$ when $F = \{f_j\}_{j=1}^\infty$. Recall that for $1 < p < \infty$, $1 < q \le \infty$ and a sequence $F = \{f_j\}_{j=1}^\infty$ of measurable functions,

$$\|MF\|_{L^p(\ell^q)} \lesssim \|F\|_{L^p(\ell^q)}, \tag{3.3}$$

which is obtained in [10].

3.1 $L^\Phi(\mathbb{R}^n)$-Boundedness

Recall that the Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined by the norm (1.4). The *weak Orlicz space* $\mathrm{W}L^\Phi(\mathbb{R}^n)$ is the set of all measurable functions f for which the quasi-norm

$$\|f\|_{\mathrm{W}L^\Phi} := \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{L^\Phi}$$

is finite. We invoke the following theorem:

Theorem 3.1 [29, 30] *Let $\Phi \in \Delta_2$ be a Young function. Then;*

1. *the maximal operator M is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{W}L^\Phi(\mathbb{R}^n)$;*
2. *the maximal operator M is bounded on $L^\Phi(\mathbb{R}^n)$ provided $\Phi \in \nabla_2$;*

$$\|Mf\|_{L^\Phi} \lesssim \|f\|_{L^\Phi} \tag{3.4}$$

for all $f \in L^\Phi(\mathbb{R}^n)$.

Corollary 3.2 *Let $\Phi \in \nabla_2$ and $\phi \in \mathcal{G}_\Phi$. Then for all $\tau \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$,*

$$\|\tau \cdot f\|_{L^1} \lesssim \|f\|_{\mathcal{M}_{\phi,\Phi}} \sup_{x\in\mathbb{R}^n} \big(1+|x|\big)^{2n+1} \big|\tau(x)\big|. \tag{3.5}$$

Proof We proceed as in [55, Lemma 2.5] by using (3.4). We omit the detail. □

We define the space $L^\Phi(\ell^q, \mathbb{R}^n)$ to be the set of all sequences $F = \{f_j\}_{j=1}^\infty$ of measurable functions for which

$$\|F\|_{L^\Phi(\ell^q)} \equiv \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^\Phi}$$

is finite.

The proof of estimate (3.6) can be found in [5].

Lemma 3.3 *If $\Phi \in \Delta_2 \cap \nabla_2$ and $1 < q \le \infty$, then we have*

$$\|MF\|_{L^\Phi(\ell^q)} \le C_{q,\Phi} \|F\|_{L^\Phi(\ell^q)} \tag{3.6}$$

for all $F = \{f_j\}_{j=1}^\infty \in L^\Phi(\ell^q, \mathbb{R}^n)$. Here $C_{q,\Phi}$ depends only on q and Φ.

See [36, 41] as well for related estimates.

3.2 $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$-Boundedness

The *vector-valued generalized Orlicz–Morrey space $\mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$ of the third kind* is the set of all sequences $F = \{f_j\}_{j=1}^\infty$ of measurable functions for which

$$\|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)} \equiv \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{M}_{\phi,\Phi}} < \infty.$$

We start with the following well-known estimate; for example, see [26] for the proof.

Lemma 3.4 *For any cube Q and a measurable function f, we have*

$$M[\chi_{\mathbb{R}^n\setminus 3Q} f](x) \lesssim \sum_{k=1}^\infty \frac{1}{|2^k Q|} \int_{2^k Q} \big|f(y)\big|\, dy \quad (x \in Q).$$

The next theorem is the Fefferman–Stein vector-valued inequality for generalized Orlicz–Morrey spaces of the third kind.

Theorem 3.5 *Let $\Phi \in \Delta_2 \cap \nabla_2$ and $1 < q \le \infty$. Assume that ϕ satisfies the integral condition* (1.13). *Then*

$$\|MF\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)} \lesssim \|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)}$$

for all $F = \{f_j\}_{j=1}^{\infty} \in \mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$.

Proof Let Q be a fixed cube. We need to establish:

$$\frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\left\|\{\chi_Q M f_j\}_{j=1}^{\infty}\right\|_{L^{\Phi}(\ell^q)} \lesssim \|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)} \tag{3.7}$$

with the implicit constant independent of Q and dependent on the ∇_2 and Δ_2 constants, the implicit constant in (1.13) and q.

To simplify, we normalize the right-hand side; $\|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)} = 1$.

We decompose (3.7) into (3.8) and (3.9), where

$$\mathrm{I} \equiv \frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\left\|\left\{\chi_Q M[\chi_{3Q} f_j]\right\}_{j=1}^{\infty}\right\|_{L^{\Phi}(\ell^q)} \lesssim 1 \tag{3.8}$$

and

$$\mathrm{II} \equiv \frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\left\|\left\{\chi_Q M[\chi_{\mathbb{R}^n\setminus 3Q} f_j]\right\}_{j=1}^{\infty}\right\|_{L^{\Phi}(\ell^q)} \lesssim 1. \tag{3.9}$$

As for (3.8), we use the vector-valued maximal inequality (see Lemma 3.3) and proceed as in [49, 61]. We omit the detail.

As for (3.9), we use Lemma 3.4. We can find $\{a_j\}_{j=1}^{\infty} \in \ell^{q'}(\mathbb{N})$ such that

$$\begin{aligned}\mathrm{II} &\lesssim \frac{1}{\phi(\ell(Q))}\sum_{j,k=1}^{\infty}\frac{a_j}{|2^k Q|}\int_{2^k Q}|f_j(y)|\,dy\\ &= \frac{1}{\phi(\ell(Q))}\sum_{k=1}^{\infty}\frac{1}{|2^k Q|}\int_{2^k Q}\sum_{j=1}^{\infty}a_j|f_j(y)|\,dy,\end{aligned}$$

and that

$$\sum_{j=1}^{\infty} a_j^{q'} = 1. \tag{3.10}$$

Meanwhile,

$$\sum_{k=1}^{\infty}\phi\big(\ell(2^k Q)\big) \lesssim \int_{\ell(Q)}^{\infty}\phi(t)\frac{dt}{t}$$

from the doubling property of ϕ. Thus, by (1.13) and (3.10),

$$\mathrm{II} \lesssim \sum_{k=1}^{\infty}\frac{1}{\phi(\ell(Q))|2^k Q|}\int_{2^k Q}\left(\sum_{j=1}^{\infty}|f_j(y)|^q\right)^{\frac{1}{q}}dy \lesssim \sum_{k=1}^{\infty}\frac{\phi(\ell(2^k Q))}{\phi(\ell(Q))} \lesssim 1,$$

implying (3.9). Thus, the proof is complete. □

4 A Norm Estimate

As a model case, we prove Corollary 1.4 before Theorem 1.1.

4.1 Proof of Corollary 1.4

Let Ψ be a function conjugate to Φ. Then we have (2.4) for some $g \in L^{\Psi}(\mathbb{R}^n)$ with norm 1. If we use the definition of f and use the Hölder inequality for the couples $(L^{\kappa'}(\mathbb{R}^n), L^{\kappa}(\mathbb{R}^n))$ and $(L^{\Phi}(\mathbb{R}^n), L^{\Psi}(\mathbb{R}^n))$, then we have

$$\|f \cdot g\|_{L^1} \le \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(x) M^{(\kappa)} g(x)\, dx \le 2 \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{\Phi}} \left\| M^{(\kappa)} g \right\|_{L^{\Psi}}. \tag{4.1}$$

Putting together (1.7), (2.4) and (4.1), we obtain the desired result.

4.2 Proof of Theorem 1.1

By decomposing Q_j into cubes of equivalent length, we may suppose each Q_j is dyadic. In fact, we know that Q_j is covered by 3^n dyadic cubes of equivalent length.

To prove (1.12), we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let us assume in addition that the a_j's are non-negative without loss of generality. Fix a non-negative (ϕ, Ψ)-block $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ of the third kind with the associated cube Q, namely, g is supported on the cube Q and g satisfies the size condition:

$$\|g\|_{L^{\Psi}(Q)} \le \frac{1}{\phi(\ell(Q))} \Phi^{-1}\left(\frac{1}{|Q|}\right). \tag{4.2}$$

We claim;

$$\|f \cdot g\|_{L^1} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}. \tag{4.3}$$

Let us admit (4.3) and complete the proof of Theorem 1.1. Let $h \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then $h = \sum_{j=1}^{\infty} \tilde{\lambda}_j g_j$, where each g_j is a (ϕ, Ψ)-block of the third kind and $\tilde{\lambda} = \{\tilde{\lambda}_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ satisfies $\|\tilde{\lambda}\|_{\ell^1(\mathbb{N})} \le 2\|h\|_{\mathcal{B}_{\phi,\Psi}}$. Thus, from (4.3), we obtain

$$\|f \cdot h\|_{L^1} \le \sum_{j=1}^{\infty} |\tilde{\lambda}_j| \cdot \|f \cdot g_j\|_{L^1} \lesssim \|h\|_{\mathcal{B}_{\phi,\Psi}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}},$$

or equivalently,

$$\|\mathfrak{L}_f\|_{\mathcal{B}_{\phi,\Psi} \to \mathbb{C}} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Since $\|f\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \|\mathfrak{L}_f\|_{\mathcal{B}_{\phi,\Psi} \to \mathbb{C}}$ due to Theorem 2.13, we have the desired result. So, let us prove (4.3).

By decomposing Q, we can assume that Q is a dyadic cube as well. We need to distinguish three cases;

1. each Q_j contains Q as a proper subset;
2. Q contains each Q_j as a proper subset;
3. none of the above happens.

We can handle the third case by mixing the remaining cases. So, we do not consider the third case. Assume first that each Q_j contains Q as a proper subset. If we group the j's such that Q_j are identical, we can assume that Q_j is a dyadic cube containing Q and satisfying $|Q_j| = 2^{jn}|Q|$ for each $j \in \mathbb{N}$. Then we have

$$\|f \cdot g\|_{L^1} = \sum_{j=1}^{\infty} \lambda_j \|a_j \cdot g\|_{L^1(Q)} \leq 2 \sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^{\Phi}(Q)} \|g\|_{L^{\Psi}(Q)}. \tag{4.4}$$

By the size conditions (1.9) and (4.2) of a_j and g, we obtain

$$\begin{aligned}
\|a_j\|_{L^{\Phi}(Q)} \|g\|_{L^{\Psi}(Q)} &= \frac{\eta(\ell(Q))}{\Phi^{-1}(\frac{1}{|Q|})} \left(\frac{1}{\eta(\ell(Q))} \|a_j\|_{L^{\Phi}(Q)} \Phi^{-1}\left(\frac{1}{|Q|}\right) \|g\|_{L^{\Psi}(Q)} \right) \\
&\leq \frac{\eta(\ell(Q))}{\Phi^{-1}(\frac{1}{|Q|})} \|a_j\|_{\mathcal{M}_{\eta,\Phi}(Q)} \|g\|_{L^{\Psi}(Q)} \\
&\leq \frac{\eta(\ell(Q))}{\eta(\ell(Q_j))\phi(\ell(Q))}.
\end{aligned}$$

If we insert this estimate into (4.4), then we obtain

$$\|f \cdot g\|_{L^1} \leq 2 \sum_{j=1}^{\infty} \frac{\lambda_j \eta(\ell(Q))}{\eta(\ell(Q_j))\phi(\ell(Q))}. \tag{4.5}$$

Meanwhile, Proposition 2.1 yields

$$\left\| \sum_{k=1}^{\infty} \lambda_k \chi_{Q_k} \right\|_{\mathcal{M}_{\phi,\Phi}} \geq \|\lambda_j \chi_{Q_j}\|_{\mathcal{M}_{\phi,\Phi}} \geq \frac{1}{\phi(\ell(Q_j))} \lambda_j \tag{4.6}$$

for each $j \in \mathbb{N}$. Consequently, it follows from (1.6), (4.5) and (4.6) that

$$\|f \cdot g\|_{L^1} \leq 2 \sum_{j=1}^{\infty} \frac{\phi(\ell(Q_j))\eta(\ell(Q))}{\eta(\ell(Q_j))\phi(\ell(Q))} \left\| \sum_{k=1}^{\infty} \lambda_k \chi_{Q_k} \right\|_{\mathcal{M}_{\phi,\Phi}} \leq 2 \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Conversely, assume that Q contains each Q_j. Then we have

$$\|f \cdot g\|_{L^1} = \sum_{j=1}^{\infty} \lambda_j \|a_j \cdot g\|_{L^1(Q_j)} \leq 2 \sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^{\kappa'}(Q_j)} \|g\|_{L^{\kappa}(Q_j)}. \tag{4.7}$$

By the size condition (1.9) of a_j, we obtain

$$\|f \cdot g\|_{L^1} = \sum_{j=1}^{\infty} \lambda_j \|a_j \cdot g\|_{L^1(Q_j)} \leq 2 \sum_{j=1}^{\infty} \lambda_j \frac{\eta(\ell(Q))}{\eta(\ell(Q_j))} \left(\frac{\|g\|_{L^{\kappa}(Q_j)}}{|Q_j|^{\frac{1}{\kappa}}} \right) |Q_j|.$$

Thus, assuming $Q \supset Q_j$, we obtain

$$\begin{aligned}\|f\cdot g\|_{L^1} &\le 2\sum_{j=1}^{\infty}\lambda_j|Q_j|\Big(\inf_{y\in Q_j} M^{(\kappa)}g(y)\Big)\\ &= 2\int_Q\left(\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}(y)\right)\inf_{z\in Q_j} M^{(\kappa)}g(z)\,dy\\ &\le 2\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{L^{\Phi}(Q)}\left\|M^{(\kappa)}g\right\|_{L^{\Psi}(Q)}.\end{aligned}$$

Using (1.7) and (4.2), we obtain

$$\begin{aligned}\|f\cdot g\|_{L^1} &\lesssim \left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{L^{\Phi}(Q)}\|g\|_{L^{\Psi}(Q)}\\ &\le \left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{L^{\Phi}(Q)}\frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{|Q|}\right)\\ &\le \left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}.\end{aligned}$$

Thus, (4.3) is proved.

A (weak) variant of Theorem 1.1 is as follows;

Corollary 4.1 *Let Φ be a Young function and $\phi \in \mathcal{G}_{\Phi}$. Assume also that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}$, $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ fulfill*

$$|a_j| \le \chi_{Q_j} \quad (j = 1, 2, \ldots) \tag{4.8}$$

and

$$\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}} < \infty.$$

Then $f \equiv \sum_{j=1}^{\infty}\lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \le \left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}. \tag{4.9}$$

Proof Observe

$$\big|f(x)\big| \le \sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}(x)$$

for almost every $x \in \mathbb{R}^n$. □

Remark 4.2 In comparison with the condition (4.8) in Corollary 4.1, the conditions (1.9) and (1.16) on a_j in Theorem 1.1 are very weak. One can relax the assumption on a_j to (1.9) and (1.16) starting from (4.8) by the use of the duality as was done in (4.7).

5 Non-smooth Decomposition of Functions

In Sect. 4, we considered a synthesis result. Here we obtain an analysis result by proving Theorem 1.2.

Let $t > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then define the *heat extension* of f by:

$$e^{t\Delta} f(x) \equiv \left\langle \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-\cdot|^2}{4t}\right), f \right\rangle \quad \left(x \in \mathbb{R}^n\right).$$

We say that $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, the *generalized Hardy–Orlicz–Morrey space of the third kind* if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and it satisfies $\sup_{t>0} |e^{t\Delta} f| \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. We define

$$\|f\|_{H\mathcal{M}_{\phi,\Phi}} \equiv \left\| \sup_{t>0} \left| e^{t\Delta} f \right| \right\|_{\mathcal{M}_{\phi,\Psi}}. \tag{5.1}$$

The next proposition characterizes the space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Proposition 5.1 *Let $\Phi \in \Delta_2 \cap \nabla_2$ and suppose that $\phi \in \mathcal{G}_\Phi$ satisfies inequality* (1.13).

1. *If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*
2. *If $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then f is represented by a measurable function g which belongs to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, namely,*

$$\langle f, \zeta \rangle = \int_{\mathbb{R}^n} g(x)\zeta(x)\, dx \quad \left(\zeta \in \mathcal{S}\left(\mathbb{R}^n\right)\right).$$

Furthermore, $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then

$$\|f\|_{H\mathcal{M}_{\phi,\Phi}} \sim \|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{5.2}$$

Proof

1. We can easily verify that $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Also, we have

$$\sup_{t>0} 2\left| e^{t\Delta} f \right| \lesssim Mf$$

by virtue of [7, Proposition 2.7]. Again thanks to Theorem 3.1, we see that $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ with the estimate $\|f\|_{H\mathcal{M}_{\phi,\Phi}} \lesssim \|f\|_{\mathcal{M}_{\phi,\Phi}}$.

2. Recall that the dual of $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ is isomorphic to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ as we have established in Theorem 2.13. Let $\mathfrak{L} : \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \ni f \mapsto \mathfrak{L}_f \in (\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n))^*$ be the isomorphism in Theorem 2.13. By assumption $\{e^{t\Delta} f\}_{t>0}$ forms a bounded set in $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Consider any sequence $\{t_j\}_{j=1}^\infty$ which decreases to 0. Then $\{\mathfrak{L}_{e^{t_j\Delta} f}\}_{j=1}^\infty$ forms a bounded set in $(\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n))^*$. Thus, by the Banach–Alaoglu theorem, there exists a positive sequence $\{t_j\}_{j=1}^\infty$ which decreases to 0 such that $\mathfrak{L}_{e^{t_j\Delta} f}$ is convergent to $G = \mathfrak{L}_g \in (\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n))^*$ for some $g \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ in the weak-* sense. Meanwhile $e^{t_j\Delta} f$ is convergent to $f \in \mathcal{S}'(\mathbb{R}^n)$. Thus, we conclude $\mathcal{S}'(\mathbb{R}^n) \ni f = g \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Finally we have $\|f\|_{H\mathcal{M}_{\phi,\Phi}} \gtrsim \|f\|_{\mathcal{M}_{\phi,\Phi}}$ for $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ which is a direct consequence of

$$\left|f(x)\right| \le \sup_{t>0}\left|e^{t\Delta} f(x)\right|$$

for almost all $x \in \mathbb{R}^n$ thanks to the Lebesgue differentiation theorem. Thus, the proof is complete. □

Definition 5.2 Define the topology on $\mathcal{S}(\mathbb{R}^n)$ with the norm $\{\rho_N\}_{N\in\mathbb{N}}$ which is given by the following formula:

$$\rho_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} \left(1+|x|\right)^N \left|\partial^\alpha \varphi(x)\right| \quad \left(\varphi \in \mathcal{S}\left(\mathbb{R}^n\right)\right).$$

Write $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \rho_N(\varphi) \le 1\}$.

Definition 5.3 Let $N \gg 1$. The grand maximal operator $\mathcal{M}$ is defined by:

$$\mathcal{M} f(x) \equiv \sup\left\{\left|t^{-n}\varphi\left(t^{-1}\cdot\right) * f(x)\right| : t > 0, \varphi \in \mathcal{F}_N\right\}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

The next proposition characterizes the space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ analogously to Proposition 5.1.

Proposition 5.4 *Let $\Phi \in \Delta_2 \cap \nabla_2$ and suppose $\phi \in \mathcal{G}_\Phi$ satisfies inequality* (1.13). *Let $N \in \mathbb{N}$ in Definition* 5.3 *be sufficiently large.*

1. *If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then $\mathcal{M} f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*
2. *If $f \in \mathcal{S}'(\mathbb{R}^n)$ is such that $\mathcal{M} f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then f is represented by a measurable function g which belongs to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*

Furthermore, if $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then

$$\|\mathcal{M} f\|_{\mathcal{M}_{\phi,\Phi}} \sim \|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{5.3}$$

Proof If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, again by virtue of [7, Proposition 2.7] we have

$$\mathcal{M} f(x) \lesssim M f(x). \tag{5.4}$$

So, if we go through the same argument as Proposition 5.1, then we learn $\mathcal{M} f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{M} f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then a pointwise estimate

$$\sup_{t>0}\left|e^{t\Delta} f(x)\right| \lesssim \mathcal{M} f(x) \quad \left(x \in \mathbb{R}^n\right) \tag{5.5}$$

allows us to invoke Proposition 5.1; we first obtain

$$\sup_{t>0}\left|e^{t\Delta} f\right| \in \mathcal{M}_{\phi,\Phi}\left(\mathbb{R}^n\right)$$

and then Proposition 5.1 yields $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Note also that (5.3) is a consequence of Theorem 3.5, (5.2), (5.4) and (5.5). □

Define $C^\infty_{\mathrm{comp}}(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n) \cap L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. We invoke the following lemma; see [57] for the proof.

Lemma 5.5 *Let $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $d \in \mathbb{N}_0$ and $j \in \mathbb{Z}$. Then there exist collections of cubes $\{Q_{j,k}\}_{k\in K_j}$ and functions $\{\eta_{j,k}\}_{k\in K_j} \subset C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, which are all indexed by a set K_j, and a decomposition*

$$f = g_j + b_j, \quad b_j = \sum_{k\in K_j} b_{j,k},$$

such that:

(0) $g_j, b_j, b_{j,k} \in \mathcal{S}'(\mathbb{R}^n)$.

(i) *Define*

$$\mathcal{O}_j \equiv \left\{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2^j\right\} \tag{5.6}$$

and consider its Whitney decomposition:

$$\mathcal{O}_j = \bigcup_{k\in K_j} Q_{j,k}, \tag{5.7}$$

where the cubes $\{Q_{j,k}\}_{k\in K_j}$ have the bounded intersection property.

(ii) *Consider the partition of unity $\{\eta_{j,k}\}_{k\in K_j}$ with respect to $\{Q_{j,k}\}_{k\in K_j}$. Then each function $\eta_{j,k}$ is supported in $Q_{j,k}$ and*

$$\sum_{k\in K_j} \eta_{j,k} = \chi_{\mathcal{O}_j}, \quad 0 \le \eta_{j,k} \le 1.$$

(iii) *The function g_j belongs to $L^\infty(\mathbb{R}^n)$ and it satisfies the inequality*:

$$\|g_j\|_{L^\infty} \lesssim 2^{-j}. \tag{5.8}$$

(iv) *Let $x_{j,k} \equiv c(Q_{j,k})$ and $\ell_{j,k} \equiv \ell(Q_{j,k})$. Then each distribution $b_{j,k}$ is given by*: $b_{j,k} = (f - c_{j,k})\eta_{j,k}$ *with a certain polynomial $c_{j,k} \in \mathcal{P}_d(\mathbb{R}^n)$ satisfying*

$$\langle f - c_{j,k}, q \cdot \eta_{j,k}\rangle = 0$$

for all $q \in \mathcal{P}_d(\mathbb{R}^n)$, and

$$\mathcal{M}b_{j,k}(x) \lesssim \mathcal{M}f(x)\chi_{Q_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n\setminus Q_{j,k}}(x) \tag{5.9}$$

for all $x \in \mathbb{R}^n$.

In the above the implicit constants are dependent only on n.

For each $j \in \mathbb{Z}$, consider the level set $\mathcal{O}_j$ given by (5.6). Then it follows immediately from the definition that

$$\mathcal{O}_j \supset \mathcal{O}_{j+1} \supset \mathcal{O}_{j+2} \supset \cdots \to \emptyset. \tag{5.10}$$

To handle generalized Orlicz–Morrey spaces of the third kind, we need the following lemma:

Lemma 5.6 *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Keep to the same notation as Lemma 5.5. Then we have*

$$\left|\langle b_j, \varphi\rangle\right| \le C_\varphi \left\{\sum_{l=0}^\infty \left(\frac{1}{2^{ln}} \|\mathcal{M}f \cdot \chi_{\mathcal{O}_j}\|_{L^1(Q(2^l))}\right)^{\frac{n+d+1}{n}}\right\}^{\frac{n}{n+d+1}}, \tag{5.11}$$

where the constant C_φ in (5.11) depends on φ but not on j or k.

Proof Similar to [25, Lemma 3.3]. We omit the detail. □

The key observation for the proof of Theorem 1.2 is the following, which is based on Lemma 5.6:

Lemma 5.7 *Assume* (1.13). *In Lemma* 5.5, *in the topology of* $\mathcal{S}'(\mathbb{R}^n)$, *we have* $g_j \to 0$ *as* $j \to -\infty$ *and* $b_j \to 0$ *as* $j \to \infty$. *In particular,*

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j) \tag{5.12}$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$.

Proof Let us show that $b_j \to 0$ as $j \to \infty$ in $\mathcal{S}'(\mathbb{R}^n)$. Once this is proved, we have $f = \lim_{j\to\infty} g_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Let us choose a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Let $j \geq 0$. Observe that

$$\begin{aligned}
&\left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\|(\mathcal{M}f)\cdot\chi_{\mathcal{O}_j}\|_{L^1(Q(2^l))}\right)^{\frac{n+d+1}{n}}\right\}^{\frac{n}{n+d+1}}\\
&\lesssim \left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\|(\mathcal{M}f)\cdot\chi_{\mathcal{O}_0}\|_{L^1(Q(2^l))}\right)^{\frac{n+d+1}{n}}\right\}^{\frac{n}{n+d+1}}\\
&\lesssim \left(\sum_{l=0}^{\infty}\frac{1}{\phi(2^l)^{\frac{n+d+1}{n}}}\right)^{\frac{n}{n+d+1}}\|f\|_{H\mathcal{M}_{\phi,\Phi}}\\
&\leq \left(\sum_{l=0}^{\infty}\frac{1}{\phi(2^l)}\right)\|f\|_{H\mathcal{M}_{\phi,\Phi}}\\
&\sim \|\mathcal{M}f\|_{\mathcal{M}_{\phi,\Phi}}\\
&\sim \|f\|_{\mathcal{M}_{\phi,\Phi}}.
\end{aligned}$$

Hence it follows from (5.11) that $\langle b_j, \varphi\rangle \to 0$ as $j \to \infty$. Likewise by using (5.8), we obtain $g_j \to 0$ as $j \to -\infty$. Consequently, (5.4) follows. □

Proof of Theorem 1.2 If we invoke Lemma 5.5, then f can be decomposed;

$$f = g_j + b_j, \quad b_j = \sum_{k\in K_j} b_{j,k}, \; b_{j,k} = (f - c_{j,k})\eta_{j,k}$$

where each $b_{j,k}$ is supported in a cube $Q_{j,k}$ as is described in Lemma 5.5.

We have (5.4) from Lemma 5.7. Here, going through the same argument as the one in [57, pp. 108–109], we have an expression;

$$f = \sum_{j=-\infty}^{\infty}\sum_{k\in K_j} A_{j,k}, \qquad g_{j+1} - g_j = \sum_{k\in K_j} A_{j,k} \quad (j \in \mathbb{Z}) \tag{5.13}$$

in the sense of distributions, where each $A_{j,k}$ belongs to $L^\infty_{\rm comp}(\mathbb{R}^n) \cap \mathcal{P}_L^\perp(\mathbb{R}^n)$, supported in $Q_{j,k}$, satisfies the pointwise estimate

$$\|A_{j,k}\|_{L^\infty} \le C_0 2^j \tag{5.14}$$

for some universal constant C_0. With these observations in mind, let us set

$$a_{j,k} \equiv \frac{A_{j,k}}{C_0 2^j} \in L^\infty_{\rm comp}(\mathbb{R}^n) \cap \mathcal{P}_L^\perp(\mathbb{R}^n), \qquad \kappa_{j,k} \equiv C_0 2^j \in [0,\infty),$$

where C_0 is from (5.14). Then we automatically obtain that each $a_{j,k}$ satisfies $|a_{j,k}| \le \chi_{Q_{j,k}}$ and that $f = \sum_{j,k} \kappa_{j,k} a_{j,k}$ in the topology of $H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ thanks to Corollary 4.1, once we prove the estimate on the coefficients. Rearrange $\{a_{j,k}\}$ and so on to obtain $\{a_j\}$ and so on.

To establish (1.14) we need to estimate $\|\{\lambda_j \chi_{Q_j}\}_{j=-\infty}^\infty\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)}$. It follows from the definition of the sequences that $\{(\kappa_{j,k}; Q_{j,k})\}_{j,k} = \{(\lambda_j; Q_j)\}_{j=-\infty}^\infty$. Thus we have $\|\{\lambda_j \chi_{Q_j}\}_{j=-\infty}^\infty\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)} = \|\{\kappa_{j,k}\chi_{Q_{j,k}}\}_{j\in\mathbb{Z},k\in K_j}\|_{\mathcal{M}_{\phi,\Phi}}$. If we insert the definition of $\kappa_{j,k}$ into this expression, then we have

$$\begin{aligned}\left\|\{\lambda_j \chi_{Q_j}\}_{j=-\infty}^\infty\right\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)} &= C_0 \left\|\left\{2^j \chi_{Q_{j,k}}\right\}_{j\in\mathbb{Z},k\in K_j}\right\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)} \\ &= C_0 \left\|\left(\sum_{j=-\infty}^\infty 2^{jv} \sum_{k\in K_j} \chi_{Q_{j,k}}\right)^{\frac{1}{v}}\right\|_{\mathcal{M}_{\phi,\Phi}}.\end{aligned}$$

We deduce $\sum_{k\in K_j} \chi_{Q_{j,k}} \sim \chi_{\mathcal{O}_j}$ from (5.7) and the bounded intersection property of $\{Q_{j,k}\}_{k\in K_j}$. Thus, we have $\|\{\lambda_j \chi_{Q_j}\}_{j=-\infty}^\infty\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)} \lesssim \|\{2^j \chi_{\mathcal{O}_j}\}_{j=-\infty}^\infty\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)}$. Recall that $\mathcal{O}_j \supset \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}$. Consequently we have

$$\sum_{j=-\infty}^\infty (2^j \chi_{\mathcal{O}_j})^v \sim \left(\sum_{j=-\infty}^\infty 2^j \chi_{\mathcal{O}_j}\right)^v \sim \left(\sum_{j=-\infty}^\infty 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}\right)^v$$

from (5.10). Thus, we obtain

$$\left\|\{\lambda_j \chi_{Q_j}\}_{j=-\infty}^\infty\right\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)} \lesssim \left\|\sum_{j=-\infty}^\infty 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}\right\|_{\mathcal{M}_{\phi,\Phi}}.$$

It follows from the definition of $\mathcal{O}_j$ that $2^j < \mathcal{M}f(x)$ for all $x \in \mathcal{O}_j$. Hence we have

$$\left\|\{\lambda_j \chi_{Q_j}\}_{j=-\infty}^\infty\right\|_{\mathcal{M}_{\phi,\Phi}(\ell^v)} \lesssim \left\|\sum_{j=-\infty}^\infty \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} \mathcal{M}f\right\|_{\mathcal{M}_{\phi,\Phi}} = \|\mathcal{M}f\|_{\mathcal{M}_{\phi,\Phi}}.$$

This is the desired result. □

Before we conclude this section, a helpful remark may be in order.

Remark 5.8 If $f \in L^v(\mathbb{R}^n)$ for some $1 < v < \infty$, then the convergence in (1.11) takes place in the topology of $L^v(\mathbb{R}^n)$.

6 Applications to Singular Integral Operators

Going through the same argument as [40, Theorem 5.5] and [41, Theorem 5.5], we can prove the following theorem:

Theorem 6.1 *Let Φ be a Young function and let $\phi \in \mathcal{G}_\Phi$. Let also $k \in \mathcal{S}(\mathbb{R}^n)$. Write*

$$A_m \equiv \sup_{x \in \mathbb{R}^n} |x|^{n+m} \left|\nabla^m k(x)\right| \quad (m \in \mathbb{N}_0).$$

Define a convolution operator T by:

$$Tf(x) \equiv k * f(x) \quad \left(f \in \mathcal{S}'\left(\mathbb{R}^n\right)\right).$$

Then T, restricted to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, is an $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$-bounded operator and the norm depends only on $\|\mathcal{F}k\|_{L^\infty}$ and a finite number of collections $A_1, A_2, \ldots, A_N$ with N depending only on Φ.

Once Theorem 6.1 is proved, we can obtain the Littlewood–Paley decomposition in the same way as [40, Theorem 5.7] and [41, Theorem 5.10].

Theorem 6.2 *Let Φ be a Young function and let $\phi \in \mathcal{G}_\Phi$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function which is supported on $Q(4) \setminus Q(1/4)$ and satisfies*

$$\sum_{j=-\infty}^{\infty} \left|\varphi\left(2^{-j}\xi\right)\right|^2 > 0$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$. Then the following norm is an equivalent norm of $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$:

$$\|f\|_{\dot{\mathcal{E}}^0_{\phi,\Phi,2}} \equiv \left\| \left(\sum_{j=-\infty}^{\infty} \left|\mathcal{F}^{-1}\left[\varphi\left(2^{-j}\cdot\right)\mathcal{F}f\right]\right|^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}_{\phi,\Phi}} \quad \left(f \in \mathcal{M}_{\phi,\Phi}\left(\mathbb{R}^n\right)\right). \tag{6.1}$$

Once we obtain Theorem 6.2, we can establish the wavelet decomposition and the smooth atomic decomposition as in [46, 50]. Further details are omitted.

7 Olsen Inequality

We shall prove the Olsen inequality on generalized Orlicz–Morrey spaces of the third kind. This is a bilinear estimate of I_α, which is nowadays called the Olsen inequality [43]. Recall that we define the fractional integral operator I_α with $0 < \alpha < n$ by;

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for all suitable functions f on $\mathbb{R}^n$. Olsen's inequality is the inequality of the form

$$\|g \cdot I_\alpha f\|_Z \lesssim \|f\|_X \|g\|_Y,$$

where X, Y, Z are suitable Banach spaces. There is a vast amount of literatures on the Olsen inequalities; see [8, 52–54, 56, 58–60, 63] for theoretical aspects and [11–13] for applications to PDEs.

7.1 Boundedness of the Fractional Integral Operator

We shall prove the boundedness of the fractional integral operator I_α and the Olsen inequality by using the Hölder inequality on generalized Orlicz–Morrey spaces of the third kind as follows:

Theorem 7.1 *Let* $0 < \alpha < n$, $\phi \in \mathcal{G}_\Phi$ *and* $\Phi \in \Delta_2 \cap \nabla_2$. *Assume that* ϕ *satisfies*

$$r^\alpha \phi(r) + \int_r^\infty t^{\alpha-1}\phi(t)\,dt \lesssim \phi(b) \quad (r > 0) \tag{7.1}$$

for some $b \in (0, 1)$. *Define*

$$\tilde{\phi}(t) \equiv \phi(t)^b \quad (t > 0) \tag{7.2}$$

and

$$\tilde{\Phi}(t) \equiv \Phi\left(t^{\frac{1}{b}}\right) \quad (t \geq 0). \tag{7.3}$$

Then $\tilde{\phi} \in \mathcal{G}_{\tilde{\Phi}}$, *the integral defining* $I_\alpha f(x)$ *converges for almost every* $x \in \mathbb{R}^n$ *and*

$$\|I_\alpha f\|_{\mathcal{M}_{\tilde{\phi},\tilde{\Phi}}} \lesssim \|f\|_{\mathcal{M}_{\phi,\Phi}} \tag{7.4}$$

for every $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Proof The proof of the fact that $\tilde{\phi} \in \mathcal{G}_{\tilde{\Phi}}$ is direct; we omit the proof.

We may assume that f is non-negative, since the integral kernel of I_α is positive. Let Q be any cube centered at $x \in \mathbb{R}^n$. We define

$$\mathrm{I}_1(x) \equiv \int_{2Q} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \quad \text{and} \quad \mathrm{I}_2(x) \equiv \int_{\mathbb{R}^n \setminus 2Q} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Using the definition of the Hardy–Littlewood maximal operator and inequality (7.1), we have

$$\begin{aligned}
\mathrm{I}_1(x) &\leq \sum_{k=-\infty}^{0} \int_{2^{k+1}Q \setminus 2^k Q} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \\
&\leq \sum_{k=-\infty}^{0} \frac{1}{(2^k \ell(Q))^{n-\alpha}} \left|2^{k+1}Q\right| Mf(x) \\
&= 2^n \ell(Q)^\alpha Mf(x) \sum_{k=-\infty}^{0} 2^{k\alpha} \\
&\lesssim \phi\big(\ell(Q)\big)^{b-1} Mf(x).
\end{aligned}$$

In total, we obtain

$$\mathrm{I}_1(x) \lesssim \phi\big(\ell(Q)\big)^{b-1} Mf(x). \tag{7.5}$$

Let Ψ be the conjugate of Φ. We use Lemma 2.3 to obtain

$$\begin{aligned}
\mathrm{I}_2(x) &\le \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \\
&\le \sum_{k=1}^{\infty} \frac{1}{(2^k\ell(Q))^{n-\alpha}} \int_{2^{k+1}Q} f(y)\,dy \\
&\le 2\sum_{k=1}^{\infty} \frac{(2^k\ell(Q))^{\alpha}}{|2^kQ|} \cdot \|f\|_{L^{\Phi}(2^{k+1}Q)} \|1\|_{L^{\Psi}(2^{k+1}Q)} \\
&\le 4\|f\|_{\mathcal{M}_{\phi,\Phi}} \sum_{k=1}^{\infty} (2^{k+1}\ell(Q))^{\alpha}\phi(2^{k+1}\ell(Q)).
\end{aligned}$$

Now we use our assumption on ϕ;

$$\begin{aligned}
\mathrm{I}_2(x) &\lesssim \|f\|_{\mathcal{M}_{\phi,\Phi}} \sum_{k=1}^{\infty} \int_{2^k\ell(Q)}^{2^{k+1}\ell(Q)} t^{\alpha-1}\phi(t)\,dt \\
&= \|f\|_{\mathcal{M}_{\phi,\Phi}} \int_{2\ell(Q)}^{\infty} t^{\alpha-1}\phi(t)\,dt \lesssim \phi(\ell(Q))^b \|f\|_{\mathcal{M}_{\phi,\Phi}}. \qquad (7.6)
\end{aligned}$$

By combining (7.5) and (7.6), we have

$$I_\alpha f(x) \lesssim \phi(\ell(Q))^{b-1} Mf(x) + \phi(\ell(Q))^b \|f\|_{\mathcal{M}_{\phi,\Phi}}. \qquad (7.7)$$

Note that $Mf(x) > \frac{2\|f\|_{\mathcal{M}_{\phi,\Phi}}}{\Psi^{-1}(1)} \inf_{r>0} \frac{1}{\phi(r)} = 0$ from (7.1). Let Q' be any cube that contain x, then

$$\begin{aligned}
\frac{1}{|Q'|}\int_{Q'} f(y)\,dy &\le \frac{2}{|Q'|}\|f\|_{L^{\Phi}(Q')}\|1\|_{L^{\Psi}(Q')} \le \frac{2\phi(\ell(Q'))\|f\|_{\mathcal{M}_{\phi,\Phi}}}{\Psi^{-1}(1)} \\
&\le \frac{4\|f\|_{\mathcal{M}_{\phi,\Phi}}}{\Psi^{-1}(1)} \sup_{r>0}\phi(r).
\end{aligned}$$

Hence it follows that

$$Mf(x) \le \frac{4\|f\|_{\mathcal{M}_{\phi,\Phi}}}{\Psi^{-1}(1)} \sup_{r>0}\phi(r).$$

Consequently,

$$0 = \sup_{r>0}\phi(r) \le \frac{\Psi^{-1}(1)Mf(x)}{4\|f\|_{\mathcal{M}_{\phi,\Phi}}} \le \frac{1}{4}\sup_{r>0}\phi(r) < \sup_{r>0}\phi(r).$$

We can choose $j_0 \in \mathbb{Z}$ such that

$$\phi(2^{j_0}) \le \frac{\Psi^{-1}(1)Mf(x)}{4\|f\|_{\mathcal{M}_{\phi,\Phi}}} \le C_0\phi(2^{j_0}) \qquad (7.8)$$

where C_0 is the doubling constant of ϕ; $\phi(2t) \le C_0\phi(t)$ for $t \ge 0$. Using the inequalities (7.7) and (7.8), we obtain

$$I_\alpha f(x) \lesssim \phi(2^{j_0})\phi(2^{j_0})^{b-1}\|f\|_{\mathcal{M}_{\phi,\Phi}} + \frac{\|f\|_{\mathcal{M}_{\phi,\Phi}}}{\phi(2^{j_0})^b} \lesssim \frac{Mf(x)^b}{\|f\|_{\mathcal{M}_{\phi,\Phi}}^{b-1}} = Mf(x)^b\|f\|_{\mathcal{M}_{\phi,\Phi}}^{1-b}. \quad (7.9)$$

Let $S \in \mathcal{Q}$. Using (7.9), we have

$$\|I_\alpha f\|_{L^{\tilde{\Phi}}(S)} \lesssim \left\|(Mf)^b\right\|_{L^{\tilde{\Phi}}(S)}\|f\|_{\mathcal{M}_{\phi,\Phi}}^{1-b}.$$

Thanks to (7.3) and Lemma 2.2, we have $\|(Mf)^b\|_{L^{\tilde{\Phi}}(S)} = \|Mf\|_{L^\Phi(S)}^b$. Consequently,

$$\|I_\alpha f\|_{L^{\tilde{\Phi}}(S)} \lesssim \|Mf\|_{L^\Phi(S)}^b\|f\|_{\mathcal{M}_{\phi,\Phi}}^{1-b}.$$

Therefore,

$$\tilde{\phi}\big(\ell(S)\big)\|I_\alpha f\|_{L^{\tilde{\Phi}}(S)} \lesssim \big(\phi\big(\ell(S)\big)\|Mf\|_{L^\Phi(S)}\big)^b\|f\|_{\mathcal{M}_{\phi,\Phi}}^{1-b} \lesssim \|Mf\|_{\mathcal{M}_{\phi,\Phi}}^b\|f\|_{\mathcal{M}_{\phi,\Phi}}^{1-b}.$$

Using Corollary 8.3 and the boundedness of the maximal operator (see [6]), we obtain

$$\tilde{\phi}\big(\ell(S)\big)\|I_\alpha f\|_{L^{\tilde{\Phi}}(S)} \lesssim \|f\|_{\mathcal{M}_{\phi,\Phi}}.$$

By taking the supremum of all cubes $S \in \mathcal{Q}$, we obtain inequality (7.4). □

Remark 7.2 Theorem 7.1 is a counterpart to the weak type estimate obtained in [21, Theorem 3.1], which was obtained in a fashion similar to that used for [9].

Theorem 7.3 *Let $b \in (0,1)$ and $0 < \alpha < n$. Suppose that we are given functions $\Phi_i \in \Delta_2 \cap \nabla_2$ and $\phi_i \in \mathcal{G}_{\Phi_i}$ for $i = 1, 2, 3$. Assume that*

$$r^\alpha\phi_1(r) + \int_r^\infty t^{\alpha-1}\phi_1(t)\,dt \lesssim \phi_1(r)^b \quad (7.10)$$

for every $r > 0$. If these functions satisfy

$$\big(\Phi_1^{-1}(t)\big)^b\Phi_2^{-1}(t) \le \Phi_3^{-1}(t) \quad (t \ge 0) \quad (7.11)$$

and

$$\phi_3(t) \ge \phi_1(t)^b\phi_2(t) \quad (t > 0), \quad (7.12)$$

then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \lesssim \|f\|_{\mathcal{M}_{\phi_1,\Phi_1}}\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \quad (7.13)$$

for every $f \in \mathcal{M}_{\phi_1,\Phi_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Proof Define $\tilde{\phi}_1(t) = \phi_1(t)^b$ for $t > 0$ and $\tilde{\Phi}_1(t) = \Phi_1(t^{\frac{1}{b}})$ for $t \ge 0$. Assumption (7.12) reads;

$$\phi_3(t) \ge \tilde{\phi}_1(t)\phi_2(t) \quad (t > 0).$$

Let $\Theta(t) = (\Phi_1^{-1}(t))^b$ for $t \geq 0$. Then

$$\tilde{\Phi}_1 \circ \Theta(t) = \tilde{\Phi}_1\left(\left(\Phi_1^{-1}(t)\right)^b\right) = \Phi_1\left(\Phi_1^{-1}(t)\right) = t$$

and

$$\Theta \circ \tilde{\Phi}_1(t) = \Theta\left(\Phi_1\left(t^{\frac{1}{b}}\right)\right) = \left(\Phi_1^{-1}\left(\Phi_1\left(t^{\frac{1}{b}}\right)\right)\right)^b = \left(t^{1/b}\right)^b = t,$$

which implies Θ and $\tilde{\Phi}_1$ are inverse to each other. Thus, $\tilde{\Phi}_1^{-1} = \Theta = (\Phi_1^{-1})^b$. By virtue of (7.11), we have $\Phi_2^{-1}(t)\tilde{\Phi}_1^{-1}(t) \leq \Phi_3^{-1}(t)$. Using Theorem 2.6, we obtain $\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \leq 2\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \|I_\alpha f\|_{\mathcal{M}_{\tilde{\phi}_1,\tilde{\Phi}_1}}$. Finally, using Theorem 7.1, we obtain (7.13). □

7.2 Olsen's Inequality Revisited

As an application of Theorems 1.1 and 1.2 we can prove the following theorem:

Theorem 7.4 *Let $\alpha \in (0, n)$, $\kappa \in (1, \infty)$ be such that $\kappa' \leq n/\alpha$. Let $\Phi \in \Delta_2 \cap \nabla_2$ and $\phi \in \mathcal{G}_\Phi$ satisfy (7.1). Define the conjugate function Ψ of Φ by (1.3). Assume that*

$$\int_r^\infty s^{\alpha-1}\phi(s)\,ds \lesssim r^\alpha \phi(r) \quad (r > 0) \tag{7.14}$$

and that Ψ satisfies (1.7). Then we have

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \|g\|_{\mathcal{M}^{n/\alpha}_{\kappa'}} \|f\|_{\mathcal{M}_{\phi,\Phi}} \tag{7.15}$$

for all $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ and $g \in \mathcal{M}^{n/\alpha}_{\kappa'}(\mathbb{R}^n)$.

Before the proof of Theorem 7.4, a couple of remarks on its assumptions may be in order.

Remark 7.5 Let $\eta(t) = t^{-\alpha}$ for $t > 0$ and $1 < \theta, q < \infty$. Set

$$\tilde{\phi}(t) = \phi(t)^\theta \quad (t > 0), \qquad \tilde{\Phi}(t) = \Phi\left(t^\theta\right) \quad (t \geq 0).$$

1. The assumptions (1.7), (7.1) and (7.14) are stronger than those of Theorems 1.1 and 1.2 and justify the definition of $I_\alpha f$, since;
   (a) the condition (7.1) together with Theorem 7.1 guarantees that $I_\alpha f$ makes sense;
   (b) the condition (7.14) corresponds to (1.6) and covers the integral condition (1.13).
2. The function $\tilde{\Phi}$ belongs to ∇_2. The function $\tilde{\phi}$ satisfies the integral condition as is seen from [42, Proposition 2.7]. As a result, due to Theorem 3.5, the vector-valued inequality

$$\|MF\|_{\mathcal{M}^{\tilde{\phi}}_{\tilde{\Phi}}(\ell^q)} \lesssim \|F\|_{\mathcal{M}^{\tilde{\phi}}_{\tilde{\Phi}}(\ell^q)}$$

   holds for all $F = \{f_j\}_{j=1}^\infty \in \mathcal{M}^{\tilde{\phi}}_{\tilde{\Phi}}(\ell^q)$.

Proof The proof is analogous to that of [25, Theorem 1.7], however, the proof of [25, Theorem 1.7] was not correct; it contains a small mistake. To indicate what it is, we will give a detailed proof together with a remedy of the proof of [25, Theorem 1.7] in the last paragraph of this proof.

Let us modify the proof of [25, Theorem 1.7] to our setting.

We decompose f according to Theorem 1.2 with $L \gg 1$;

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{7.16}$$

where $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}_{\rm comp}(\mathbb{R}^n) \cap \mathcal{P}_L^{\perp}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ fulfill (1.15)–(1.18).

We concluded in [25] the wrong conclusion;

$$\big|g(x) I_\alpha f(x)\big| \le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\lambda_j}{2^{1+L+n-\alpha}} \big(\ell(Q_j)^\alpha \big|g(x)\big| \chi_{2^k Q_j}(x)\big) \quad \big(x \in \mathbb{R}^n\big) \tag{7.17}$$

by using [25, Lemma 4.2];

$$\big|I_\alpha f(x)\big| \le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\lambda_j}{2^{1+L+n-\alpha}} \big(\ell(Q_j)^\alpha \big|g(x)\big| \chi_{2^k Q_j}(x)\big) \quad \big(x \in \mathbb{R}^n\big). \tag{7.18}$$

It seems that we can justify (7.18) to a large extent. However, in [25], we are led to thinking that

$$I_\alpha f = \sum_{j=1}^{\infty} \lambda_j I_\alpha a_j \tag{7.19}$$

almost everywhere or in $\mathcal{M}_{\tilde{\phi},\tilde{\Phi}}(\mathbb{R}^n)$ but this is not the case because (7.16) does not take place in $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

We can justify (7.18) in the last paragraph of the proof. For the time being, let us suppose that (7.18) is justified and conclude the proof.

From (7.17), we deduce

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \left\| \sum_{j,k=1}^{\infty} \frac{\lambda_j}{2^{k(1+L+n)}} \big(\ell\big(2^k Q_j\big)^\alpha |g| \chi_{2^k Q_j}\big) \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Write $b_{jk} \equiv \ell(2^k Q_j)^\alpha |g| \chi_{2^k Q_j}$. Then we have

$$\|b_{jk}\|_{\mathcal{M}^{n/\alpha}_{\kappa'}} \le \ell\big(2^k Q_j\big)^\alpha \|g\|_{\mathcal{M}^{n/\alpha}_{\kappa'}}.$$

Thus, it follows that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \|g\|_{\mathcal{M}^{n/\alpha}_{\kappa'}} \left\| \sum_{j,k=1}^{\infty} \frac{\lambda_j}{2^{k(1+L+n)}} \chi_{2^k Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Observe that $\chi_{2^k Q_j} \le 2^{kn} M \chi_{Q_j}$. Hence if we choose $1 < \theta < \infty$ so that

$$L + n + 1 - \theta n > 0,$$

then we have

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \|g\|_{\mathcal{M}^{n/\alpha}_{\kappa'}} \left\| \sum_{j=1}^{\infty} \lambda_j \big(M\chi_{Q_j}\big)^\theta \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

By the Fefferman–Stein inequality for generalized Morrey spaces of the third kind; see Theorem 3.5, we can remove the maximal operator and we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi,\Phi}} \lesssim \|g\|_{\mathcal{M}^{n/\alpha}_{\kappa'}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

It remains to justify (7.18). Since the kernel of I_α is non-negative, we may assume that $f \in L^v(\mathbb{R}^n)$ for some $v \in (1, n/\alpha)$ by a suitable truncation of f. Then we are in the position of applying Remark 5.8 to have (7.19) in $L^u(\mathbb{R}^n)$, where u is given by:

$$\frac{1}{u} = \frac{1}{v} - \frac{\alpha}{n}.$$

With this remark, we can use (7.18) and we have (7.15). □

8 More General form of the Boundedness of the Maximal Operator

In this section, we consider the case when ϕ is dependent of x as well. So we are given a function $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ as well as the Young function $\Phi : [0, \infty) \to [0, \infty)$. Recall that for a cube Q, $c(Q)$ stands for its center and $\ell(Q)$ stands for its side-length; $\ell(Q) = |Q|^{\frac{1}{n}}$. In this case the *generalized Orlicz–Morrey space* $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \equiv \sup_{Q \in \mathcal{Q}} \frac{1}{\phi(c(Q), \ell(Q))} \Phi^{-1}\left(\frac{1}{|Q|}\right) \|f\|_{L^\Phi(Q)}$$

is finite. Likewise the *weak generalized Orlicz–Morrey space* $\mathrm{W}\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{\mathrm{W}\mathcal{M}_{\phi,\Phi}} \equiv \sup_{Q \in \mathcal{Q}, \gamma > 0} \frac{\gamma}{\phi(c(Q), \ell(Q))} \Phi^{-1}\left(\frac{1}{|Q|}\right) \|\chi_{\{|f| > \gamma\}}\|_{L^\Phi(Q)}$$

is finite. Let $1 \le q \le \infty$. Denote by $\mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$ the set of all sequences $F = \{f_j\}_{j=1}^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$\|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)} \equiv \left\| \left\| \{f_j(\cdot)\}_{j=1}^\infty \right\|_{\ell^q} \right\|_{\mathcal{M}_{\phi,\Phi}} < \infty.$$

Similarly denote by $\mathrm{W}\mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$ the set of all sequences $F = \{f_j\}_{j=1}^\infty$ such that

$$\|F\|_{\mathrm{W}\mathcal{M}_{\phi,\Phi}(\ell^q)} \equiv \left\| \left\| \{f_j(\cdot)\}_{j=1}^\infty \right\|_{\ell^q} \right\|_{\mathrm{W}\mathcal{M}_{\phi,\Phi}} < \infty.$$

The spaces $\ell^q(\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n))$ and $\ell^q(\mathrm{W}\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n))$ can be also similarly defined by the (quasi-)norms:

$$\left\| \{f_j\}_{j=1}^\infty \right\|_{\ell^q(\mathcal{M}_{\phi,\Phi})} \equiv \left\| \left\{ \|f_j\|_{\mathcal{M}_{\phi,\Phi}} \right\}_{j=1}^\infty \right\|_{\ell^q} < \infty$$

and

$$\left\| \{f_j\}_{j=1}^\infty \right\|_{\ell^q(\mathrm{W}\mathcal{M}_{\phi,\Phi})} \equiv \left\| \left\{ \|f_j\|_{\mathrm{W}\mathcal{M}_{\phi,\Phi}} \right\}_{j=1}^\infty \right\|_{\ell^q} < \infty,$$

respectively.

Denote by $\mathcal{G}_\Phi$ the set of all functions $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that $\phi(x,t) \le \phi(x,s)$ for all $t > s > 0$ and that $\mu_x : (0,\infty) \ni t \mapsto \Phi^{-1}(x, t^{-n})\phi(t)^{-1} \in (0,\infty)$ is almost decreasing, that is, there exists a constant $C > 0$ independent of x such that $\mu_x(t) \le C\mu_x(s)$ for all $0 < s < t < \infty$. Here $\Phi^{-1}(x,\cdot)$ is the inverse of $\Phi(x,\cdot)$.

The following theorem was proved in [6, Theorem 4.6]:

Theorem 8.1 *Let Φ be a Young function and $\phi_1, \phi_2 \in \mathcal{G}_\Phi$. Suppose that the functions ϕ_1, ϕ_2 and Φ satisfy the condition*;

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\phi_1(x,s)}{\Phi^{-1}(s^{-n})} \lesssim \phi_2(x,r), \tag{8.1}$$

where the implicit constant does not depend on x and r. Then;

1. *the maximal operator M is bounded from $\mathcal{M}_{\phi_1,\Phi}(\mathbb{R}^n)$ to $\mathcal{M}_{\phi_2,\Phi}(\mathbb{R}^n)$ if $\Phi \in \nabla_2$*;
2. *the maximal operator M is bounded from $\mathcal{M}_{\phi_1,\Phi}(\mathbb{R}^n)$ to $\mathrm{W}\mathcal{M}_{\phi_2,\Phi}(\mathbb{R}^n)$.*

Using the boundedness of M on $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ (see Theorem 8.1 to follow), we obtain the following result:

Lemma 8.2 *Let $\Phi \in \nabla_2$ and $\phi \in \mathcal{G}_\phi$. Assume that ϕ, Φ satisfy*

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\phi(x,s)}{\Phi^{-1}(s^{-n})} \lesssim \phi(x,r), \tag{8.2}$$

for every $r > 0$. Then for all $\tau \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$,

$$\|\tau \cdot f\|_{L^1} \lesssim \|f\|_{\mathcal{M}_{\phi,\Phi}} \sup_{x\in\mathbb{R}^n} \big(1+|x|\big)^{2n+1}\big|\tau(x)\big|. \tag{8.3}$$

Proof We proceed as in [55, Lemma 2.5]. We omit the detail. □

Corollary 8.3 *Let Φ be a Young function and let $\phi \in \mathcal{G}_\Phi$. Then*;

1. *the maximal operator M is bounded on $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ if $\Phi \in \nabla_2$*;
2. *the maximal operator M is bounded from $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ to $\mathrm{W}\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*

Corollary 8.4 *Let Φ be a Young function, $\phi_1, \phi_2 \in \mathcal{G}_\Phi$ and $1 \le q \le \infty$. Suppose the functions ϕ_1, ϕ_2 and Φ satisfy condition (8.1). Then*;

1. *the maximal operator M is bounded on $\ell^q(\mathcal{M}_{\phi_1,\Phi}(\mathbb{R}^n))$ if $\Phi \in \nabla_2$*;
2. *the maximal operator M is bounded from $\ell^q(\mathcal{M}_{\phi_1,\Phi}(\mathbb{R}^n))$ to $\ell^q(\mathrm{W}\mathcal{M}_{\phi_2,\Phi}(\mathbb{R}^n))$.*

We can use Corollary 8.4 to consider generalized Besov–Orlicz–Morrey spaces of the third kind, as we did in [23, 33, 45, 62].

Let $v \in \mathfrak{M}^+(0,\infty)$. We denote by $L^{\infty,v}(0,\infty)$ the space of all functions $g \in \mathfrak{M}(0,\infty)$ with finite norm;

$$\|g\|_{L^{\infty,v}(0,\infty)} \equiv \sup_{t>0} v(t)\big|g(t)\big|.$$

Let $u \in \mathfrak{M}^+(0,\infty)$. We define the *supremal operator* $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by:

$$\overline{S}_u g(t) \equiv \|ug\|_{L^\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [4]:

Theorem 8.5 [4] *Let* v_1, v_2, $u \in \mathfrak{M}^+(0,\infty)$ *satisfy* $0 < \|v_1\|_{L^\infty(t,\infty)} < \infty$ *for any* $t > 0$. *Then the supremal operator* $\overline{S}_u$ *is bounded from* $L^{\infty,v_1}(0,\infty)$ *to* $L^{\infty,v_2}(0,\infty)$ *on the cone* $\mathbb{A}$, *defined by* (1.20), *if and only if*

$$\left\| v_2 \overline{S}_u \left(\frac{1}{\|v_1\|_{L^\infty(\cdot,\infty)}} \right) \right\|_{L^\infty(0,\infty)} < \infty. \tag{8.4}$$

Here and below by a "weight" we mean a measurable function which is finite and positive almost everywhere. We will use the following statement on the boundedness of the weighted Hardy operator

$$g \mapsto H_w^* g(t) \equiv \int_t^\infty g(s)w(s)\,ds, \quad 0 < t < \infty,$$

where $w \in \mathfrak{M}^+(0,\infty)$.

Theorem 8.6 [19, Theorem 3.1] *Let* v_1, v_2 *and* $w \in \mathfrak{M}^+(0,\infty)$. *Assume that* $v_1 : (0,\infty) \to (0,\infty)$ *is bounded outside a neighborhood of the origin. Then the inequality*

$$\|H_w^* g\|_{L^{\infty,v_2}(0,\infty)} \le C \|g\|_{L^{\infty,v_1}(0,\infty)} \tag{8.5}$$

holds for all $g \in \mathfrak{M}^+(0,\infty;\uparrow)$ *if and only if*

$$B \equiv \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{8.6}$$

Moreover, the value $C = B$ *is the best constant for* (8.5).

Before we go further, a couple of remarks may be in order.

Remark 8.7

1. Theorem 8.6 in the case $w = 1$ was proved in [4, Theorem 5.3].
2. In (8.6) it will be understood that $\frac{1}{\infty} = 0 \cdot \infty = 0$.
3. See [20, Theorem 1] as well for an application of this result.

The following lemma is true:

Lemma 8.8 *Let* Φ *be a Young function and* $1 < q \le \infty$.

1. *If* $\Phi \in \Delta_2 \cap \nabla_2$, *then*

$$\begin{aligned} &\left\| \|MF(\cdot)\|_{\ell^q} \right\|_{L^\Phi(B(x,r))} \\ &\quad \lesssim \left\| \|F(\cdot)\|_{\ell^q} \right\|_{L^\Phi(B(x,3r))} + \frac{1}{\Phi^{-1}(r^{-n})} \int_r^\infty \frac{\| \|F(\cdot)\|_{\ell^q} \|_{L^1(B(x,t))}}{t^{n+1}}\,dt \end{aligned} \tag{8.7}$$

holds for any $F = \{f_j\}_{j=0}^\infty \subset L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ *and for any ball* $B(x,r)$.

2. *If $\Phi \in \Delta_2$, then*

$$\big\| \|MF(\cdot)\|_{\ell^q} \big\|_{\mathrm{W}L^\Phi(B(x,r))} \lesssim \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,3r))} + \frac{1}{\Phi^{-1}(r^{-n})} \int_r^\infty \frac{\| \|F\|_{\ell^q} \|_{L^1(B(x,t))}}{t^{n+1}}\,dt \tag{8.8}$$

holds for any $F = \{f_j\}_{j=0}^\infty \subset L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ and for any ball $B(x,r)$.

Proof We split $F = \{f_j\}_{j=1}^\infty$ with

$$\begin{aligned} &F = F_1 + F_2, \qquad F_1 \equiv \{f_{j,1}\}_{j=1}^\infty, \qquad F_2 \equiv \{f_{j,2}\}_{j=1}^\infty, \\ &f_{j,1}(y) \equiv f_j(y)\chi_{B(x,3r)}(y), \qquad f_{j,2}(y) \equiv f_j(y) - f_{j,1}(y). \end{aligned} \tag{8.9}$$

1. Assume $\Phi \in \Delta_2 \cap \nabla_2$. It is obvious that

$$\big\| \|MF(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))} \le \big\| \|MF_1(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))} + \big\| \|MF_2(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))}.$$

First, we estimate $\| \|MF_1(\cdot)\|_{\ell^q} \|_{L^\Phi(B(x,r))}$. By Lemma 3.6, we have

$$\begin{aligned} \big\| \|MF_1(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))} &\le \big\| \|MF_1(\cdot)\|_{\ell^q} \big\|_{L^\Phi} \\ &\lesssim \big\| \|F_1(\cdot)\|_{\ell^q} \big\|_{L^\Phi} \\ &= \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,3r))}, \end{aligned} \tag{8.10}$$

where the implicit constant is independent of the vector-valued function F.

On the other hand, the estimate for MF_2 is valid from an estimate similar to Lemma 3.4. Thus we obtain (8.7) from (8.10).

2. Let $\Phi \in \Delta_2$. It is obvious that

$$\begin{aligned} &\big\| \|MF(\cdot)\|_{\ell^q} \big\|_{\mathrm{W}L^\Phi(B(x,r))} \\ &\quad \le 2\big\| \|MF_1(\cdot)\|_{\ell^q} \big\|_{\mathrm{W}L^\Phi(B(x,r))} + 2\big\| \|MF_2(\cdot)\|_{\ell^q} \big\|_{\mathrm{W}L^\Phi(B(x,r))}. \end{aligned}$$

By the weak-type vector valued maximal inequality for Orlicz spaces (see [30]) we have

$$\begin{aligned} \big\| \|MF_1(\cdot)\|_{\ell^q} \big\|_{\mathrm{W}L^\Phi(B(x,r))} &\le \big\| \|MF_1(\cdot)\|_{\ell^q} \big\|_{\mathrm{W}L^\Phi} \\ &\lesssim \big\| \|F_1(\cdot)\|_{\ell^q} \big\|_{L^\Phi} \\ &= \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,3r))}, \end{aligned} \tag{8.11}$$

where the implicit constant is independent of the vector-valued function F. On the other hand, the estimate for MF_2 is again valid from an estimate similar to Lemma 3.4. Thus by (8.11) we obtain inequality (8.8). □

Lemma 8.9 *Let Φ be a Young function and $1 < q \le \infty$.*

1. *If $\Phi \in \Delta_2 \cap \nabla_2$, then*

$$\big\| \|MF(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))} \lesssim \int_r^\infty \frac{\Phi^{-1}(t^{-n})}{\Phi^{-1}(r^{-n})} \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,t))} \frac{dt}{t} \tag{8.12}$$

for any ball $B(x,r)$ and for any $F = \{f_j\}_{j=0}^\infty \subset L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$.

2. *If $\Phi \in \Delta_2$, then*

$$\left\| \left\| MF(\cdot) \right\|_{\ell^q} \right\|_{\mathrm{W}L^{\Phi}(B(x,r))} \lesssim \int_r^\infty \frac{\Phi^{-1}(t^{-n})}{\Phi^{-1}(r^{-n})} \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \frac{dt}{t} \tag{8.13}$$

for any ball $B(x,r)$ and for any $F = \{f_j\}_{j=0}^\infty \subset L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $\Phi \in \Delta_2 \cap \nabla_2$. Write

$$\mathrm{I} \equiv \int_r^\infty \frac{\| \|F(\cdot)\|_{\ell^q} \|_{L^1(B(x,t))}}{\Phi^{-1}(r^{-n}) t^{n+1}} \, dt, \qquad \mathrm{II} \equiv \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,2r))}.$$

Let Ψ be the conjugate of Φ. Applying Hölder's inequality and inequality (2.5) we obtain

$$\begin{aligned}
\mathrm{I} &\lesssim \frac{1}{\Phi^{-1}(r^{-n})} \int_r^\infty \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \|1\|_{L^{\Psi}(B(x,t))} \frac{dt}{t^{n+1}} \\
&= \frac{1}{\Phi^{-1}(r^{-n})} \int_r^\infty \frac{t^{-n}}{\Psi^{-1}\left(\frac{1}{|B(x,t)|}\right)} \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \frac{dt}{t} \\
&\lesssim \frac{1}{\Phi^{-1}(r^{-n})} \int_r^\infty \Phi^{-1}\left(t^{-n}\right) \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \frac{dt}{t}.
\end{aligned}$$

On the other hand, assuming that $\Phi \in \Delta_2$, we have

$$\begin{aligned}
\mathrm{II} &\sim \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,2r))} \frac{\Phi^{-1}((2r)^{-n})}{(2r)^{-n} \Phi^{-1}(r^{-n})} \int_{2r}^\infty \frac{dt}{t^{n+1}} \\
&\lesssim \frac{1}{\Phi^{-1}(r^{-n}) \Psi^{-1}((2r)^{-n})} \int_{2r}^\infty \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \frac{dt}{t^{n+1}} \\
&\le \frac{1}{\Phi^{-1}(r^{-n})} \int_{2r}^\infty \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \frac{t^{-n}}{\Psi^{-1}(t^{-n})} \frac{dt}{t} \\
&\le \frac{1}{\Phi^{-1}(r^{-n})} \int_{2r}^\infty \left\| \left\| F(\cdot) \right\|_{\ell^q} \right\|_{L^{\Phi}(B(x,t))} \Phi^{-1}\left(r^{-n}\right) \frac{dt}{t}
\end{aligned}$$

thanks to inequality (2.5). Since $\| \|MF(\cdot)\|_{\ell^q} \|_{L^{\Phi}(B(x,r))} \le \mathrm{I} + \mathrm{II}$, we arrive at (8.12) by Lemma 8.8. Finally, when $\Phi \in \Delta_2$ inequality (8.13) directly follows from (8.8). □

Theorem 8.10 *Let $\Phi \in \Delta_2$ be a Young function and $1 < q \le \infty$. Suppose that we are given a Young function Φ and $\phi_1, \phi_2 \in \mathcal{G}_\Phi$ such that an estimate uniform over $x \in \mathbb{R}^n$ and $r > 0$;*

$$\int_r^\infty \Phi^{-1}\left(t^{-n}\right) \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\phi_1(x,s)}{\Phi^{-1}(s^{-n})} \right) \frac{dt}{t} \lesssim \phi_2(x,r) \tag{8.14}$$

holds. Then;

1. *if $\Phi \in \Delta_2 \cap \nabla_2$, then the maximal operator M is bounded from $\mathcal{M}_{\phi_1,\Phi}(\ell^q, \mathbb{R}^n)$ to $\mathcal{M}_{\phi_2,\Phi}(\ell^q, \mathbb{R}^n)$, that is,*

$$\|MF\|_{\mathcal{M}_{\phi_2,\Phi}(\ell^q)} \lesssim \|F\|_{\mathcal{M}_{\phi_1,\Phi}(\ell^q)} \tag{8.15}$$

holds for all $F \in \mathcal{M}_{\phi_1,\Phi}(\ell^q, \mathbb{R}^n)$;

2. *if* $\Phi \in \Delta$, *then the maximal operator* M *is bounded from* $\mathcal{M}_{\phi_1,\Phi}(\ell^q,\mathbb{R}^n)$ *to* $\mathrm{W}\mathcal{M}_{\phi_2,\Phi}(\ell^q,\mathbb{R}^n)$, *that is*,

$$\|MF\|_{\mathrm{W}\mathcal{M}_{\phi_2,\Phi}(\ell^q)} \lesssim \|F\|_{\mathcal{M}_{\phi_1,\Phi}(\ell^q)} \tag{8.16}$$

holds for all $F \in \mathcal{M}_{\phi_1,\Phi}(\ell^q,\mathbb{R}^n)$.

Proof We use (8.14) as follows: Fix $x \in \mathbb{R}^n$. Define

$$v_1(t) \equiv \frac{\Phi^{-1}(t^{-n})}{\phi_1(x,t)}, \qquad v_2(t) \equiv \frac{1}{\phi_2(x,t)}, \qquad w(t) \equiv \frac{1}{t}$$

and consider the weighted Hardy operator:

$$H_w^* g(t) = \int_t^\infty g(s)\frac{ds}{s} \quad (t>0)$$

where $g \in \mathfrak{M}^+(0,\infty)$. Note that we are in the position of applying (8.14).

As a consequence applying Theorem 8.6, we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\phi_2(x,r)} \int_r^\infty \Phi^{-1}\big(t^{-n}\big) \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,t))} \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^n} \sup_{r>0} \frac{1}{\phi_2(x,r)} \int_r^\infty \Phi^{-1}\big(t^{-n}\big) \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,t))} \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^n} \Big\| H_w^*\Big(\big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,\cdot))}\Big) \Big\|_{L^{\infty,v_2}(0,\infty)} \\
&\lesssim \sup_{x\in\mathbb{R}^n} \Big\| \Big(\big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,\cdot))}\Big) \Big\|_{L^{\infty,v_1}(0,\infty)} \\
&= \sup_{x\in\mathbb{R}^n} \sup_{r>0} \frac{1}{\phi_1(x,r)} \Phi^{-1}\big(r^{-n}\big) \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))} \\
&= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\phi_1(x,r)} \Phi^{-1}\big(r^{-n}\big) \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,r))} \\
&\le \|F\|_{\mathcal{M}_{\phi_1,\Phi}(\ell^q)}.
\end{aligned} \tag{8.17}$$

If $\Phi \in \Delta_2 \cap \nabla_2$, then Theorem 8.6, Lemma 8.9 and (8.17) yield

$$\begin{aligned}
\|MF\|_{\mathcal{M}_{\phi_2,\Phi}(\ell^q)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\phi_2(x,r)} \int_r^\infty \Phi^{-1}\big(t^{-n}\big) \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,t))} \frac{dt}{t} \\
&\lesssim \|F\|_{\mathcal{M}_{\phi_1,\Phi}(\ell^q)},
\end{aligned}$$

which proves (8.15). Likewise, if $\Phi \in \Delta_2$, then we have

$$\begin{aligned}
\|MF\|_{\mathrm{W}\mathcal{M}_{\phi_2,\Phi}(\ell^q)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\phi_2(x,r)} \int_r^\infty \Phi^{-1}\big(t^{-n}\big) \big\| \|F(\cdot)\|_{\ell^q} \big\|_{L^\Phi(B(x,t))} \frac{dt}{t} \\
&\lesssim \|F\|_{\mathcal{M}_{\phi_1,\Phi}(\ell^q)},
\end{aligned}$$

which proves (8.16). □

As a corollary of the vector-valued inequality, we have;

Corollary 8.11 *Let Φ be a Young function and $1 < q \le \infty$. Suppose that $\phi \in \mathcal{G}_\Phi$ satisfies*

$$\int_r^\infty \phi(x,t)\frac{dt}{t} \lesssim \phi(x,r) \quad \bigl(x \in \mathbb{R}^n, r > 0\bigr),$$

where the implicit constant does not depend on x and r. Then;

1. *whenever $\Phi \in \Delta_2 \cap \nabla_2$ and $F \in \mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$,*

$$\|MF\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)} \lesssim \|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)}$$

with the implicit constant independent of F;

2. *whenever $\Phi \in \Delta_2$ and $F \in \mathcal{M}_{\phi,\Phi}(\ell^q, \mathbb{R}^n)$,*

$$\|MF\|_{\mathrm{w}\mathcal{M}_{\phi,\Phi}(\ell^q)} \lesssim \|F\|_{\mathcal{M}_{\phi,\Phi}(\ell^q)}$$

with the implicit constant independent of F.

In [69, Lemma 2.13] one can find also a result parallel to Corollary 8.11. Note that [69, Lemma 2.13] is called the modular inequality, while Corollary 8.11 is the vector-valued norm inequality. If we apply these results, then our results carry over to the case when Φ depends on x.

9 Concluding Remarks

9.1 Comparison of Many Generalized Orlicz–Morrey Spaces

To the best knowledge of the authors, there exist three generalized Orlicz–Morrey spaces.

Definition 9.1 Let $\Phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ and $\phi : \mathcal{Q} \to (0,\infty)$ be suitable functions. Let f be a measurable function.

1. For a cube $Q \in \mathcal{Q}$ define the (ϕ, Φ)-*average* over Q of f by:

$$\|f\|_{(\phi,\Phi);Q} \equiv \inf\left\{\lambda > 0 : \frac{\phi(Q)}{|Q|}\int_Q \Phi\left(x, \frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

Define the *generalized Orlicz–Morrey space $\mathcal{L}_{\phi,\Phi}(\mathbb{R}^n)$ of the first kind* to be the Banach space equipped with the norm: $\|f\|_{\mathcal{L}_{\phi,\Phi}} \equiv \sup\{\|f\|_{(\phi,\Phi);Q} : Q \in \mathcal{Q}\}$.

2. For a cube $Q \in \mathcal{Q}$ define the Φ-*average* over Q of f by:

$$\|f\|_{\Phi;Q} \equiv \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(x, \frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

Define the *generalized Orlicz–Morrey space $\tilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)$ of the second kind* to be the Banach space equipped with the norm: $\|f\|_{\tilde{\mathcal{M}}_{\phi,\Phi}} \equiv \sup\{\phi(Q)\|f\|_{\Phi;Q} : Q \in \mathcal{Q}\}$.

The spaces $\mathcal{L}_{\phi,\Phi}(\mathbb{R}^n)$, $\tilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)$ and $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ are defined by Nakai in [37] (with Φ independent of x), by Sawano, Sugano and Tanaka in [54] (with Φ independent of x) and by Deringoz, Guliyev and Samko in [6, Definition 2.3], respectively. According to the examples in [14], we can say that the scales $\mathcal{L}$ and $\tilde{\mathcal{M}}$ are different and that $\tilde{\mathcal{M}}$ and $\mathcal{M}$ are different. However, it is not known that $\mathcal{L}$ and $\mathcal{M}$ are different.

Example 9.2 Let $n = 1$, $\Phi(t) = t^2 + t^3$ and $\phi(t) = \sqrt{t}$. Then $\mathcal{L}_{\phi,\Phi}(\mathbb{R}) = \mathcal{M}^4_2(\mathbb{R}) \cap \mathcal{M}^6_3(\mathbb{R})$. But no matter what function ψ we choose, $\mathcal{L}_{\phi,\Phi}(\mathbb{R})$ and $\mathcal{M}_{\psi,\Phi}(\mathbb{R})$ are not isomorphic. Assume to the contrary that $\mathcal{L}_{\phi,\Phi}(\mathbb{R})$ and $\mathcal{M}_{\psi,\Phi}(\mathbb{R})$ are isomorphic. Then

$$C \max\big(\sqrt[4]{r}, \sqrt[6]{r}\big) \le \|\chi_{[0,r]}\|_{\mathcal{M}_{\psi,\Phi}} = \|\chi_{[0,r]}\|_{\mathcal{L}_{\phi,\Phi}} \le C^{-1} \max\big(\sqrt[4]{r}, \sqrt[6]{r}\big).$$

This implies

$$\|\chi_{[0,r]}\|_{\mathcal{M}_{\psi,\Phi}} > \|\chi_{[0,ar]}\|_{\mathcal{M}_{\psi,\Phi}}$$

for some constant a independent of $r > 0$.

When $0 < r < 1$, we deduce

$$C \le \int_0^r \left(\frac{1}{\psi(r)\sqrt[6]{r}}\right)^3 dt + \int_0^r \left(\frac{1}{\psi(r)\sqrt[6]{r}}\right)^2 dt \le C^{-1}$$

and hence $C\sqrt[6]{r} \le \psi(r) \le C^{-1}\sqrt[6]{r}$. Likewise $C\sqrt[4]{r} \le \psi(r) \le C^{-1}\sqrt[4]{r}$ when $r \ge 1$.

Define a sequence $\{f_j\}_{j=1}^{\infty}$ of Affine mappings by

$$f_j(x) = (2\sqrt{2})^j - (2\sqrt{2})^{j-1} + x.$$

We also define

$$E_0 = [0, 1], \qquad E_{j+1} = E_j \cup f_j(E_j).$$

Then we see that

$$1 \le \|\chi_{E_j}\|_{\mathcal{M}^6_2} \le C$$

for all j; see [52, (4.10)]. Meanwhile, keeping in mind, E_j is made up of 2^j disjoint intervals of length 1

$$\|\chi_{E_j}\|_{\mathcal{M}_{\psi,\Phi}} \le C\|\chi_{E_j}\|_{\mathcal{M}^6_2} \le C < \infty.$$

However,

$$\|f_j\|_{\mathcal{M}^6_3} \ge \frac{1}{\sqrt[6]{(2\sqrt{2})^j}}\sqrt[3]{2^j} = \sqrt[12]{2^j}$$

for any $j \in \mathbb{N}$. This is a contradiction.

9.2 Comparison of the Assumptions of the Theorems

Here we discuss the meaning of the assumptions of Theorems 1.1 and 1.2. In [25], we have proved the following theorems:

Theorem 9.3 [25, Theorem 1] *Suppose that the parameters* p, q, s, t *satisfy*

$$1 < q \le p < \infty, \qquad 1 < t \le s < \infty, \qquad q < t, \qquad p < s.$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}$, $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}_t^s(\mathbb{R}^n)$ *and* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ *fulfill*

$$\|a_j\|_{\mathcal{M}_t^s} \le |Q_j|^{\frac{1}{s}}, \qquad \operatorname{supp}(a_j) \subset Q_j, \qquad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p} < \infty. \tag{9.1}$$

Then $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{M}_q^p} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p}. \tag{9.2}$$

Theorem 9.4 [25, Theorem 2] *Suppose that the real parameters* p, q, L *satisfy*

$$1 < q \le p < \infty, \qquad L \in \mathbb{N}_0.$$

Let $f \in \mathcal{M}_q^p(\mathbb{R}^n)$. *Then there exists a triplet* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}$, $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_L^{\perp}(\mathbb{R}^n)$ *and* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ *such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and that, for all* $v > 0$

$$|a_j| \le \chi_{Q_j}, \qquad \left\| \{\lambda_j \chi_{Q_j}\}_{j=1}^{\infty} \right\|_{\mathcal{M}_q^p(\ell^v)} \lesssim \|f\|_{\mathcal{M}_q^p}. \tag{9.3}$$

Assumption (1.6) and (1.7) correspond to conditions $p < s$ and $q < t$ in Theorem 9.3, respectively. Assumption (1.13) corresponds to $p < \infty$.

According to the counterexample in [52, Proposition 4.1], we know that we can not relax the assumption $q < t$; if this were true for $q = t$, then this would contradict to the counterexample in [52, Proposition 4.1].

However, it is not known that we can relax the assumption $p < s$.

Remark 9.5 According to the best knowledge of the authors, it seems that there are three decompositions for Morrey spaces.

1. In 2005, Kruglyak and Kuznetsov considered the Calderón–Zygmund decomposition [31].
2. In 2007, the "so called" smooth decomposition is obtained [50]. The key idea is to develop the idea obtained in [23, 34, 61, 62]. This decomposition is investigated very intensively in [24, 32, 33, 45, 47, 48]. Later, in [51], by using this atomic decomposition, the above scale turned out to be the one defined by Yang and Yuan [64, 65]. See [66–68, 70] for more for this new scale. We refer to [71] for an exhaustive account of these function spaces as well as the results on this decomposition. In particular, remark that Yang, Yuan and Zhuo obtained the smooth decomposition for Musielak–Orlicz spaces in [69]. Using Corollary 8.11 and the main results in [33], one can obtain the smooth atomic decomposition for the Orlicz–Morrey spaces of the third kind. However, the cost that must be paid is the size of the number N of the moment condition in words of [69, Definition 5.4]. With the results in this paper and the ones in [33], N must be large enough. To overcome this disadvantage, one needs another approach, which is the future work.
3. Probably, [27] is the first work on the non-smooth decomposition of functions Hardy–Morrey space. The paper [25] complements the case when $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ is the classical Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$.

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References

1. Akbulut, A., Guliyev, V.S., Mustafayev, R.: On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces. Math. Bohem. **137**(1), 27–43 (2012)
2. Andersen, K.F., John, R.T.: Weighted inequalities for vecter-valued maximal functions and singular integrals. Stud. Math. **69**, 19–31 (1980)
3. Burenkov, V.I., Guliyev, H.V.: Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. Stud. Math. **163**(2), 157–176 (2004)
4. Burenkov, V., Gogatishvili, A., Guliyev, V.S., Mustafayev, R.: Boundedness of the fractional maximal operator in local Morrey-type spaces. Complex Var. Elliptic Equ. **55**(8–10), 739–758 (2010)
5. Curbera, P., Garcia-Cuerva, J., Martell, J., Maria, J., Perez, C.: Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals. Adv. Math. **203**(1), 256–318 (2006)
6. Deringoz, F., Guliyev, V.S., Samko, S.: Boundedness of maximal and singular operators on generalized Orlicz–Morrey spaces. In: Operator Theory, Operator Algebras and Applications. Operator Theory: Advances and Applications, vol. 242, pp. 139–158 (2014)
7. Duoandikoetxea, J.: Fourier Analysis. Graduate Studies in Mathematics, vol. 29. Am. Math. Soc., Providence (2001). Translated and revised from the 1995 Spanish original by David Cruz–Uribe
8. Eridani, Gunawan, H., Utoyo, M.I.: A characterization for fractional integral operators on generalized Morrey spaces. Anal. Theory Appl. **28**(3), 263–267 (2012)
9. Eridani, Gunawan, H., Nakai, E., Sawano, Y.: Characterizations for the generalized fractional integral operators on Morrey spaces. Math. Inequal. Appl. **17**(2), 761–777 (2014)
10. Fefferman, C., Stein, E.M.: Some maximal inequalities. Am. J. Math. **93**, 107–115 (1971)
11. Gala, S., Sawano, Y., Tanaka, H.: A new Beale–Kato–Majda criteria for the 3D magneto-micropolar fluid equations in the Orlicz–Morrey space. Math. Methods Appl. Sci. **35**(11), 1321–1334 (2012)
12. Gala, S., Sawano, Y., Tanaka, H.: On the uniqueness of weak solutions of the 3D MHD equations in the Orlicz–Morrey space. Appl. Anal. **92**(4), 776–783 (2013)
13. Gala, S., Ragusa, A.M., Sawano, Y., Tanaka, H.: Uniqueness criterion of weak solutions for the dissipative quasi-geostrophic equations in Orlicz–Morrey spaces. Appl. Anal. **93**(2), 356–368 (2014)
14. Gala, S., Sawano, Y., Tanaka, H.: A remark on two generalized Orlicz–Morrey spaces. J. Approx. Theory **198**, 1–9 (2015)
15. Gogatishvili, A., Mustafayev, R.: New pre-dual space of Morrey space. J. Math. Anal. Appl. **397**(2), 678–692 (2013)
16. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. In: Doctor's Degree Dissertation, Inst. Steklov, Moscow (1994). 329 pp. (in Russian)
17. Guliyev, V.S.: Function spaces, integral operators and two weighted inequalities on homogeneous groups. In: Some Applications, Cashioglu, Baku (1999). 332 pp. (in Russian)
18. Guliyev, V.S.: Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal. Appl. (2009). Article ID 503948, 20 pp.
19. Guliyev, V.S.: Generalized local Morrey spaces and fractional integral operators with rough kernel. J. Math. Sci. (N.Y.) **193**(2), 211–227 (2013)
20. Guliyev, V.S.: Local generalized Morrey spaces and singular integrals with rough kernel. Azerb. J. Math. **3**(2), 79–94 (2013)
21. Gunawan, H., Hakim, D.I., Sawano, Y., Sihwaningrum, I.: Weak type inequalities for some integral operator on generalized nonhomogeneous Morrey spaces. J. Funct. Spaces Appl. **2013** (2013)
22. Hardy, G.H., Littlewood, J.: A maximal theorem with function-theoretic applications. Acta Math. **54**(1), 81–116 (1930)
23. Hedberg, L., Netrusov, Y.: An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation. Mem. Am. Math. Soc. **188** (2007). vi+97 pp.
24. Ho, K.P.: Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces. Anal. Math. **38**(3), 173–185 (2012)

25. Iida, T., Sawano, Y., Tanaka, H.: Atomic decomposition for Morrey spaces. Z. Anal. Anwend. **33**(2), 149–170 (2014)
26. Izumi, T., Sawano, Y., Tanaka, H.: Littlewood–Paley theory for Morrey spaces and their preduals. Rev. Mat. Complut. **28**(2), 411–447 (2015)
27. Jia, H., Wang, H.: Decomposition of Hardy–Morrey spaces. J. Math. Anal. Appl. **354**(1), 99–110 (2009)
28. Kalita, E.A.: Dual Morrey spaces. Dokl. Akad. Nauk **361**(4), 447–449 (1998)
29. Kita, H.: Inequalities with weights for maximal functions in Orlicz spaces. Acta Math. Hung. **72**(4), 291–305 (1996)
30. Kokilashvili, V., Krbec, M.M.: Weighted Inequalities in Lorentz and Orlicz Spaces. World Scientific, Singapore (1991)
31. Kruglyak, N., Kuznetsov, E.A.: Smooth and nonsmooth Calderón–Zygmund type decompositions for Morrey spaces. J. Fourier Anal. Appl. **11**(6), 697–714 (2005)
32. Liang, Y., Sawano, Y., Ullrich, T., Yang, D., Yuan, W.: New characterizations of Besov–Triebel–Lizorkin–Hausdorff spaces including coorbits and wavelets. J. Fourier Anal. Appl. **18**(5), 1067–1111 (2012)
33. Liang, Y., Sawano, Y., Ullrich, T., Yang, D., Yuan, W.: A new framework for generalized Besov-type and Triebel–Lizorkin-type spaces. Diss. Math. **489** (2013). 114 pp.
34. Mazzucato, A.: Decomposition of Besov–Morrey spaces. In: Harmonic Analysis at Mount Holyoke, South Hadley, MA, 2001. Contemp. Math., vol. 320, pp. 279–294. Am. Math. Soc., Providence (2003)
35. Nakai, E.: Hardy–Littlewood maximal operator, singular integral operators, and the Riesz potential on generalized Morrey spaces. Math. Nachr. **166**, 95–103 (1994)
36. Nakai, E.: Generalized fractional integral operator on Orlicz–Morrey spaces. In: Banach and Function Spaces, Kitakyushu, 2003, pp. 323–333. Yokohama Publ., Yokohama (2004)
37. Nakai, E.: The Campanato, Morrey and Hölder spaces on spaces of homogeneous type. Stud. Math. **176**(1), 1–19 (2006)
38. Nakai, E.: Orlicz–Morrey spaces and Hardy–Littlewood maximal function. Stud. Math. **188**, 193–221 (2008)
39. Nakai, E.: Orlicz–Morrey spaces and their preduals. In: Banach and Function Spaces, Kitakyushu, 2009, pp. 187–205. Yokohama Publ., Yokohama (2011)
40. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. **262**, 3665–3748 (2012)
41. Nakai, E., Sawano, Y.: Orlicz–Hardy spaces and their duals. Sci. China Math. **57**(5), 903–962 (2014)
42. Noi, T., Nakamura, S., Sawano, Y.: Generalized Morrey spaces and trace operator. Sci. China Math., online
43. Olsen, P.: Fractional integration, Morrey spaces and Schrödinger equation. Commun. Partial Differ. Equ. **20**, 2005–2055 (1995)
44. Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces. Dekker, New York (1991)
45. Rauhut, H., Ullrich, T.: Generalized coorbit space theory and inhomogeneous function spaces of Besov–Lizorkin–Triebel type. J. Funct. Anal. **260**, 3299–3362 (2011)
46. Sawano, Y.: Wavelet characterization of Besov–Morrey and Triebel–Lizorkin–Morrey spaces. Funct. Approx. Comment. Math. **38**, 93–107 (2008). part 1
47. Sawano, Y.: A note on Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces. Acta Math. Sin. **25**, 1223–1242 (2009)
48. Sawano, Y.: Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces on domains in $\mathbb{R}^n$. Math. Nachr. **283**(10), 1–32 (2010)
49. Sawano, Y., Tanaka, H.: Morrey spaces for non-doubling measures. Acta Math. Sin. **21**(6), 1535–1544 (2005)
50. Sawano, Y., Tanaka, H.: Decompositions of Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces. Math. Z. **257**(4), 871–905 (2007)
51. Sawano, Y., Yang, D., Yuan, W.: New applications of Besov-type and Triebel–Lizorkin-type spaces. J. Math. Anal. Appl. **363**, 73–85 (2010)
52. Sawano, Y., Sugano, S., Tanaka, H.: Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces. Trans. Am. Math. Soc. **363**(12), 6481–6503 (2011)
53. Sawano, Y., Sugano, S., Tanaka, H.: Olsen's inequality and its applications to Schrödinger equations. RIMS Kôkyûroku Bessatsu **B26**, 51–80 (2011)
54. Sawano, Y., Sugano, S., Tanaka, H.: Orlicz–Morrey spaces and fractional operators. Potential Anal. **36**(4), 517–556 (2012)
55. Sawano, Y., Hakim, D.I., Gunawan, H.: Non-smooth atomic decomposition for generalized Orlicz–Morrey spaces. Math. Nachr. **288**(14–15), 1741–1775 (2015)
56. Sihwaningrum, I., Suryawan, H.P., Gunawan, H.: Fractional integral operators and Olsen inequalities on non-homogeneous spaces. Aust. J. Math. Anal. Appl. **7**(1) (2010). Art. 14, 6 pp.

57. Stein, E.M.: Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)
58. Sugano, S.: Some inequalities for generalized fractional integral operators on generalized Morrey spaces. Math. Inequal. Appl. **14**(4), 849–865 (2011)
59. Sugano, S., Tanaka, H.: Boundedness of fractional integral operators on generalized Morrey spaces. Sci. Math. Jpn. **58**(3), 531–540 (2003). Sci. Math. Jpn. (Online) **8**, 233–242 (2003)
60. Tanaka, H.: Morrey spaces and fractional operators. J. Aust. Math. Soc. **88**(2), 247–259 (2010)
61. Tang, L., Xu, J.: Some properties of Morrey type Besov–Triebel spaces. Math. Nachr. **278**(7–8), 904–917 (2005)
62. Ullrich, T.: Continuous characterizations of Besov–Lizorkin–Triebel spaces and new interpretations as coorbits. J. Funct. Spaces Appl. Article ID 163213, 47 pp.
63. Utoyo, M.I., Nusantara, T., Widodo, B.S.: Fractional integral operator and Olsen inequality in the non-homogeneous classic Morrey space. Int. J. Math. Anal. **6**(29–32), 1501–1511 (2012)
64. Yang, D., Yuan, W.: A new class of function spaces connecting Triebel–Lizorkin spaces and Q spaces. J. Funct. Anal. **255**, 2760–2809 (2008)
65. Yang, D., Yuan, W.: New Besov-type spaces and Triebel–Lizorkin-type spaces including Q spaces. Math. Z. **265**, 451–480 (2010)
66. Yang, D., Yuan, W.: Characterizations of Besov-type and Triebel–Lizorkin-type spaces via maximal functions and local means. Nonlinear Anal. **73**, 3805–3820 (2010)
67. Yang, D., Yuan, W.: Dual properties of Triebel–Lizorkin-type spaces and their applications. Z. Anal. Anwend. **30**, 29–58 (2011)
68. Yang, D., Yuan, W.: Relations among Besov-type spaces, Triebel–Lizorkin-type spaces and generalized Carleson measure spaces. Appl. Anal. **92**(3), 549–561 (2013)
69. Yang, D., Yuan, W., Zhuo, C.: Musielak–Orlicz Besov-type and Triebel–Lizorkin-type spaces. Rev. Mat. Complut. **1**, 93–157 (2014)
70. Yuan, W., Sawano, Y., Yang, D.: Decompositions of Besov–Hausdorff and Triebel–Lizorkin–Hausdorff spaces and their applications. J. Math. Anal. Appl. **369**, 736–757 (2010)
71. Yuan, W., Sickel, W., Yang, D.: Morrey and Campanato Meet Besov, Lizorkin and Triebel. Lecture Notes in Mathematics. Springer, Berlin (2010). xi+281 pp.
72. Zorko, C.: Morrey space. Proc. Am. Math. Soc. **98**(4), 586–592 (1986)