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GENERALIZED LOCAL MORREY SPACES AND FRACTIONAL INTEGRAL OPERATORS WITH ROUGH KERNEL

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Let $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ be the fractional maximal and integral operators with rough kernels, where $0<\alpha<n$. We study the continuity properties of $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ on the generalized local Morrey spaces $LM^{\{x_0\}}_{p,\varphi}$. We prove that the commutators of these operators with local Campanato functions are bounded. Bibliography: 34 titles.

1 Introduction

For $x\in\mathbb{R}^n$ and $r>0$ we denote by $B(x,r)$ the open ball centered at x with radius r and by $|B(x,r)|$ the Lebesgue measure of $B(x,r)$. Let $\Omega\in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ ($n\geqslant 2$) equipped with the normalized Lebesgue measure $d\sigma$ and $s>1$. For any $0<\alpha<n$ the *fractional integral operator with rough kernel* $I_{\Omega,\alpha}$ is defined by

$$I_{\Omega,\alpha}f(x)=\int\limits_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy$$

and the *related fractional maximal operator with rough kernel* $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha}f(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int\limits_{B(x,t)}|\Omega(x-y)|\,|f(y)|dy.$$

If $\alpha=0$, then $M_\Omega\equiv M_{\Omega,0}$ is the Hardy–Littlewood maximal operator with rough kernel. If $\Omega\equiv 1$, it is obvious that $I_{\Omega,\alpha}$ is the Riesz potential I_α and $M_{\Omega,\alpha}$ is the maximal operator M_α.

Theorem A. *Suppose that $\Omega\in L_s(S^{n-1})$, $1<s\leqslant\infty$, is homogeneous of degree zero, $0<\alpha<n$, $1\leqslant p<n/\alpha$, and $1/q=1/p-\alpha/n$. If $s'\leqslant p$ or $q<s$, then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p>1$. If $q<s$, then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p=1$.*

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Let b be a locally integrable function on $\mathbb{R}^n$. For $0 < \alpha < n$ we define the commutators generated by fractional maximal and integral operators with rough kernels and b as follows:

$$M_{\Omega,b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |b(x) - b(y)||f(y)||\Omega(x-y)|dy,$$

$$[b, I_{\Omega,\alpha}]f(x) = b(x) I_{\Omega,\alpha} f_1(x) - I_{\Omega,\alpha}(bf)(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} [b(x) - b(y)] f(y) dy.$$

Theorem B. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $b \in BMO(\mathbb{R}^n)$. If $s' \leqslant p$ or $q < s$, then the operators $M_{\Omega,b,\alpha}$ and $[b, I_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

The classical Morrey spaces $M_{p,\lambda}$ were first introduced by Morrey in [1] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator, and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [2]–[4]. Properties and applications of classical Morrey spaces can be found in [5]–[8] (cf. also the references therein).

In this paper, we establish the boundedness of the operators $I_{\Omega,\alpha}$ from generalized local Morrey spaces $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$, where $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, and from the space $LM^{\{x_0\}}_{1,\varphi_1}$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2}$, $1 < q < \infty$, $1 - 1/q = \alpha/n$. In the case $b \in CBMO_{p_2}$, we find sufficient conditions on the pair (φ_1, φ_2) that ensure the boundedness of the commutator operators $[b, I_{\Omega,\alpha}]$ from $LM^{\{x_0\}}_{p_1,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$, where $1 < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, $1/q_1 = 1/p_1 - \alpha/n$.

We write $A \lesssim B$ if $A \leqslant CB$, where C is a positive constant independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are *equivalent*.

2 Generalized Local Morrey Spaces

It is convenient to define a generalized Morrey space as follows.

Definition 2.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, and let $1 \leqslant p < \infty$. The space of all functions $f \in L^{\rm loc}_p(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-1/p} \|f\|_{L_p(B(x,r))}$$

is called the *generalized Morrey space* and is denoted by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$. The *weak generalized Morrey space* $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ is introduced as the space of all functions $f \in WL^{\rm loc}_p(\mathbb{R}^n)$ such that

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-1/p} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to Definition 2.1, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{(\lambda-n)/p}$:

$$M_{p,\lambda} = M_{p,\varphi}\big|_{\varphi(x,r)=r^{(\lambda-n)/p}}, \quad WM_{p,\lambda} = WM_{p,\varphi}\big|_{\varphi(x,r)=r^{(\lambda-n)/p}}.$$

Definition 2.2. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, and let $1 \leqslant p < \infty$. The space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1}\, |B(0,r)|^{-1/p}\, \|f\|_{L_p(B(0,r))}$$

is called the *generalized local Morrey space* and is denoted by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$. The *weak generalized Morrey space* $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ is introduced as the space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1}\, |B(0,r)|^{-1/p}\, \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 2.3. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, and let $1 \leqslant p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ the *generalized local Morrey space* $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}$$

and the *weak generalized Morrey space* $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is the space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty.$$

According to Definition 2.3, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{(\lambda-n)/p}$:

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{(\lambda-n)/p}}, \quad WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{(\lambda-n)/p}}.$$

Wiener [9, 10] looked for a way to describe the behavior of a function at infinity. The conditions he considered were related to appropriate weighted L_q spaces. Beurling [11] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to convolution expressed as the union of certain weighted L_q space and $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [12] observed that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\geqslant 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}, \tag{2.1}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \leqslant 1\}$ and χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leqslant 2^k\}$, $k = 1, 2, \ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called the *Beurling algebra*, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{2.2}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (2.1) and (2.2) instead of $k \geqslant 0$ there. If $\lambda < 0$ or $\lambda > n$, then $LM_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$, $\dot{B}_{p,\mu} = LM_{p,\varphi}\big|_{\varphi(0,r)=r^{\mu n}}$, and $W\dot{B}_{p,\mu} = WLM_{p,\varphi}\big|_{\varphi(0,r)=r^{\mu n}}$.

To study relationships between central BMO spaces and Morrey spaces, Alvarez, Guzman-Partida, and Lakey [13] introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu \in [-1/p, 0]$. If $\mu < -1/p$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-1/p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. The weak central Morrey spaces is defined by $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

Inspired by the aforesaid, we consider the boundedness of fractional integral operators with rough kernels on generalized local Morrey spaces and give the central bounded mean oscillation estimates for their commutators.

3 Fractional Integral Operators with Rough Kernels in $LM_{p,\varphi}^{\{x_0\}}$

In this section, we use the following assertion about the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \ 0 < t < \infty,$$

where w is a fixed function nonnegative and measurable on $(0,\infty)$. In the case $w = 1$, it was proved in [14].

Theorem 3.1. *Let v_1, v_2, and w be positive almost everywhere and measurable functions on $(0,\infty)$. Then*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) \leqslant C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{3.1}$$

for some $C > 0$ and all nonnegative nondecreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{3.2}$$

Moreover, if C^* is the minimal value of C in (3.1), then $C^* = B$.

Remark 3.2. In (3.1) and (3.2) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Proof of Theorem 3.1. *Sufficiency.* Assume that (3.2) holds. If F, G are nonnegative functions on $(0,\infty)$ and F is nondecreasing, then

$$\operatorname*{ess\,sup}_{t>0} F(t)G(t) = \operatorname*{ess\,sup}_{t>0} F(t) \operatorname*{ess\,sup}_{s>t} G(s), \quad t > 0. \tag{3.3}$$

By (3.3), we have

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) = \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty g(s)w(s) \frac{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} ds$$

$$\leqslant \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} \ \operatorname*{ess\,sup}_{t>0} g(t) \ \operatorname*{ess\,sup}_{t<\tau<\infty} v_1(\tau)$$

$$= \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} \operatorname*{ess\,sup}_{t>0} g(t)v_1(t) \leqslant B \operatorname*{ess\,sup}_{t>0} g(t)v_1(t).$$

Necessity. Assume that the inequality (3.1) holds. The function

$$g(t) = \frac{1}{\operatorname{ess\,sup}_{t<\tau<\infty} v_1(\tau)}, \quad t > 0,$$

is nonnegative and nondecreasing on $(0, \infty)$. Thus,

$$B = \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} \leqslant C \operatorname*{ess\,sup}_{t>0} \frac{v_1(t)}{\operatorname{ess\,sup}_{t<\tau<\infty} v_1(\tau)} \leqslant C.$$

Hence $C^* = B$. □

In [15], the following assertion was proved by using the fractional integral operator with rough kernel $I_{\Omega,\alpha}$. It contains the result of [16, 17].

Theorem 3.3. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero. Suppose that $0 < \alpha < n$, $1 \leqslant s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\varphi(x, r)$ satisfies the conditions*

$$c^{-1}\varphi(x, r) \leqslant \varphi(x, t) \leqslant c\,\varphi(x, r) \tag{3.4}$$

if $r \leqslant t \leqslant 2r$, where c ($\geqslant 1$) is independent of t, r, $x \in \mathbb{R}^n$ and

$$\int_r^\infty t^{\alpha p}\varphi(x, t)^p \frac{dt}{t} \leqslant C\, r^{\alpha p}\varphi(x, r)^p, \tag{3.5}$$

where C is independent of x and r. Then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.

The following assertion, containing the results of [16, 17], was proved in [18, 19] (cf. also [14] and [20]–[24]).

Theorem 3.4. *Suppose that $0 < \alpha < n$, $1 \leqslant p < n/\alpha$, $1/q = 1/p - \alpha/n$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty t^{\alpha-1}\varphi_1(0, t)dt \leqslant C\, \varphi_2(0, r), \tag{3.6}$$

where C is independent of r. Then the operators M_α and I_α are bounded from LM_{p,φ_1} to LM_{q,φ_2} for $p > 1$ and from LM_{1,φ_1} to WLM_{q,φ_2} for $p = 1$.

Lemma 3.5. *Suppose that $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero. Suppose that $0 < \alpha < n$, $1 \leqslant p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If $p > 1$ and $s' \leqslant p$ or $q < s$, then*

$$\|I_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

for any ball $B(x_0,r)$ and all $f \in L_p^{\rm loc}(\mathbb{R}^n)$. Moreover, if $p=1<q<s$, then

$$\|I_{\Omega,\alpha} f\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_1(B(x_0,t))} dt \tag{3.7}$$

for any ball $B(x_0,r)$ and all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Suppose that $0<\alpha<n$, $1 \leqslant s' \leqslant p < n/\alpha$, and $1/q = 1/p - \alpha/n$. We denote by $B=B(x_0,r)$ the ball centered at x_0 with radius r. Representing f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \qquad r>0, \tag{3.8}$$

we have

$$\|I_{\Omega,\alpha} f\|_{L_q(B)} \leqslant \|I_{\Omega,\alpha} f_1\|_{L_q(B)} + \|I_{\Omega,\alpha} f_2\|_{L_q(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $I_{\Omega,\alpha} f_1 \in L_q(\mathbb{R}^n)$, and $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, it follows that

$$\|I_{\Omega,\alpha} f_1\|_{L_q(B)} \leqslant \|I_{\Omega,\alpha} f_1\|_{L_q(\mathbb{R}^n)} \leqslant C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where the constant $C>0$ is independent of f.

It is clear that $x \in B$ and $y \in \complement(2B)$ imply

$$\frac{1}{2}|x_0 - y| \leqslant |x-y| \leqslant \frac{3}{2}|x_0-y|.$$

We get

$$|I_{\Omega,\alpha} f_2(x)| \leqslant 2^{n-\alpha} c_1 \int\limits_{\complement_{(2B)}} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^{n-\alpha}} dy.$$

By the Fubini theorem,

$$\int\limits_{\complement_{(2B)}} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^{n-\alpha}} dy \approx \int\limits_{\complement_{(2B)}} |f(y)||\Omega(x-y)| \int\limits_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy$$

$$\approx \int\limits_{2r}^{\infty} \int\limits_{2r \leqslant |x_0-y| \leqslant t} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}} \lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}}.$$

Applying the Hölder inequality, we get

$$\int\limits_{\complement_{(2B)}} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^{n-\alpha}} dy \lesssim \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \|\Omega(\cdot - y)\|_{L_s(B(x_0,r))} \, |B(x_0,t)|^{1-1/p-\frac{1}{s}} \, \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{3.9}$$

Moreover, for all $p \in [1, \infty)$

$$\|I_{\Omega,\alpha} f_2\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{3.10}$$

Thus,

$$\|I_{\Omega,\alpha} f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2B)} \approx r^{\frac{n}{q}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \leqslant r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Thus,

$$\|I_{\Omega,\alpha} f\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

For $1 < q < s$ the Fubini theorem and Minkowski inequality yield

$$\begin{aligned}
\|I_{\Omega,\alpha} f_2\|_{L_q(B)} &\leqslant \Big(\int_B \Big| \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}} \Big|^q \Big)^{1/q} \\
&\leqslant \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_q(B)} dy \frac{dt}{t^{n+1-\alpha}} \\
&\leqslant r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_s(B)} dy \, \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.
\end{aligned} \tag{3.11}$$

For $p = 1 < q < s \leqslant \infty$ from the weak $(1, q)$ boundedness of $I_{\Omega,\alpha}$ and (3.11) it follows that

$$\begin{aligned}
\|I_{\Omega,\alpha} f_1\|_{WL_q(B)} &\leqslant \|I_{\Omega,\alpha} f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\
&= \|f\|_{L_1(2B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.
\end{aligned} \tag{3.12}$$

Then from (3.10) and (3.12) we get the inequality (3.7). □

Theorem 3.6. *Suppose that $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero. Suppose that $0 < \alpha < n$, $1 \leqslant p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $s' \leqslant p$ or $q < s$. If (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leqslant C \, \varphi_2(x_0, r), \tag{3.13}$$

where C is independent of r, then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$ for $p>1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi_2}$ for $p=1$. Moreover, for $p>1$

$$\|M_{\Omega,\alpha}f\|_{LM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|I_{\Omega,\alpha}f\|_{LM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}$$

and for $p=1$

$$\|M_{\Omega,\alpha}f\|_{WLM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|I_{\Omega,\alpha}f\|_{WLM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{LM^{\{x_0\}}_{1,\varphi_1}}.$$

Proof. By Lemma 3.5 and Theorem 3.1 with $v_2(r)=\varphi_2(x_0,r)^{-1}$, $v_1(r)=\varphi_1(x_0,r)^{-1}r^{-\frac{n}{p}}$, and $w(r)=r^{-\frac{n}{q}}$, we have

$$\begin{aligned}\|I_{\Omega,\alpha}f\|_{LM^{\{x_0\}}_{q,\varphi_2}} &\lesssim \sup_{r>0}\varphi_2(x_0,r)^{-1}\int_r^\infty \|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\\ &\lesssim \sup_{r>0}\varphi_1(x_0,r)^{-1}\,r^{-\frac{n}{p}}\,\|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}\end{aligned}$$

for $p>1$ and for $p=1$

$$\begin{aligned}\|I_{\Omega,\alpha}f\|_{WLM^{\{x_0\}}_{q,\varphi_2}} &\lesssim \sup_{r>0}\varphi_2(x_0,r)^{-1}\int_r^\infty \|f\|_{L_1(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\\ &\lesssim \sup_{r>0}\varphi_1(x_0,r)^{-1}\,r^{-n}\,\|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{1,\varphi_1}}.\end{aligned}$$

□

Corollary 3.7. *Suppose that $\Omega\in L_s(S^{n-1})$, $1<s\leqslant\infty$, is homogeneous of degree zero. Suppose that $0<\alpha<n$, $1\leqslant p<n/\alpha$, $1/q=1/p-\alpha/n$, and $s'\leqslant p$ or $q<s$. If (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}dt \leqslant C\,\varphi_2(x,r),$$

where C is independent of x and r, then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from M_{p,φ_1} to M_{q,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p=1$. Moreover, for $p>1$

$$\|M_{\Omega,\alpha}f\|_{M_{q,\varphi_2}} \lesssim \|I_{\Omega,\alpha}f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}$$

and for $p=1$

$$\|M_{\Omega,\alpha}f\|_{WM_{q,\varphi_2}} \lesssim \|I_{\Omega,\alpha}f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Corollary 3.8. *Suppose that $1\leqslant p<\infty$, $0<\alpha<n/p$, $1/q=1/p-\alpha/n$, and (φ_1,φ_2) satisfies the condition (3.13). Then the operators M_α and I_α are bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$ for $p>1$ and from $M^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi_2}$ for $p=1$.*

Remark 3.9. Corollary 3.7 was proved in [24] in the case $s=\infty$. The condition (3.13) in Theorem 3.6 is weaker than the condition (3.6) in Theorem 3.4 (cf. [24]).

4 Commutators of Fractional Integral Operators with Rough Kernels in $LM_{p,\varphi}^{\{x_0\}}$

Let T be a linear operator. For a function b we define the commutator $[b,T]$ by

$$[b,T]f(x) = b(x)\,Tf(x) - T(bf)(x)$$

for any suitable function f. If $\widetilde{T}$ is a Calderón-Zygmund singular integral operator, a well-known result of Coifman, Rochberg, and Weiss [25] states that the commutator $[b,\widetilde{T}]f = b\,\widetilde{T}f - \widetilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón–Zygmund operators plays an important role for studying the regularity of solutions of elliptic partial differential equations of second order (cf., for example, [5]–[7]). Chanillo [26] proved that the commutator $[b, I_\alpha]f = b\,I_\alpha f - I_\alpha(bf)$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ ($1 < p < q < \infty$, $1/q = 1/p - \alpha/n$) if and only if $b \in BMO(\mathbb{R}^n)$.

Local Campanato spaces are defined is as follows.

Definition 4.1. Let $1 \leqslant q < \infty$, and let $0 \leqslant \lambda < 1/n$. A function $f \in L_q^{\rm loc}(\mathbb{R}^n)$ belong to the *central Campanato space* $CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ if

$$\|f\|_{CBMO_{q,\lambda}^{\{x_0\}}} = \sup_{r>0} \left(\frac{1}{|B(x_0,r)|^{1+\lambda q}} \int\limits_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|^q dy \right)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} f(y)dy.$$

We define

$$CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \big\{ f \in L_q^{\rm loc}(\mathbb{R}^n) : \|f\|_{CBMO_{q,\lambda}^{\{x_0\}}} < \infty \big\}.$$

Lu and Yang [27] introduced the central BMO space $CBMO_q(\mathbb{R}^n) = CBMO_{q,0}^{\{0\}}(\mathbb{R}^n)$. Note that $BMO(\mathbb{R}^n) \subset CBMO_q^{\{x_0\}}(\mathbb{R}^n)$, $1 \leqslant q < \infty$. The space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ can be regarded as a local version of the space $BMO(\mathbb{R}^n)$ of bounded mean oscillation at the origin. However, these spaces have quite different properties. The classical John–Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that for any $1 \leqslant q < \infty$ functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition

$$\sup_{r>0} \Big(\frac{1}{|B|} \int\limits_B |f(y) - f_B|^q dy \Big)^{1/q} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ depends on q. If $q_1 < q_2$, then $CBMO_{q_2}^{\{x_0\}}(\mathbb{R}^n) \subsetneqq CBMO_{q_1}^{\{x_0\}}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John — Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Lemma 4.2. *Suppose that b is a function in $CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1 \leqslant q < \infty$, $0 \leqslant \lambda < 1/n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}} \int\limits_{B(x_0,r_1)} |b(y) - b_{B(x_0,r_2)}|^q dy \right)^{\frac{1}{q}} \leqslant C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big) \|b\|_{CBMO_{q,\lambda}^{\{x_0\}}},$$

where $C > 0$ is independent of b, r_1, and r_2.

In [15], the following assertion was proved for the commutators of fractional integral operators with rough kernels. It contains the result of [16, 17].

Theorem 4.3. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero, and $b \in BMO(\mathbb{R}^n)$. Suppose that $0 < \alpha < n$, $1 \leqslant s' < p < n/p$, $1/q = 1/p - \alpha/n$, and $\varphi(x,r)$ satisfies (3.4) and (3.5). Then the operator $[b, I_{\Omega,\alpha}]$ is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

Lemma 4.4. *Suppose that $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero. Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $b \in CBMO^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $0 \leqslant \lambda < 1/n$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, $1/q_1 = 1/p_1 - \alpha/n$. Then for $s' \leqslant p$ or $q_1 < s$*

$$\|[b, I_{\Omega,\alpha}]f\|_{L_q(B(x_0,r))} \lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt$$

for any ball $B(x_0, r)$ and all $f \in L^{loc}_{p_1}(\mathbb{R}^n)$.

Proof. Suppose that $1 < p < \infty$, $0 < \alpha < n/p$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, and $1/q_1 = 1/p_1 - \alpha/n$. As in the proof of Lemma 3.5, we represent f in the form (3.8) and have

$$\begin{aligned}[b, I_{\Omega,\alpha}]f(x) &= \big(b(x) - b_B\big) I_{\Omega,\alpha} f_1(x) - I_{\Omega,\alpha}\Big(\big(b(\cdot) - b_B\big) f_1\Big)(x) \\ &\quad + \big(b(x) - b_B\big) I_{\Omega,\alpha} f_2(x) - I_{\Omega,\alpha}\Big(\big(b(\cdot) - b_B\big) f_2\Big)(x) \equiv J_1 + J_2 + J_3 + J_4.\end{aligned}$$

Hence

$$\|[b, I_{\Omega,\alpha}]f\|_{L_q(B)} \leqslant \|J_1\|_{L_q(B)} + \|J_2\|_{L_q(B)} + \|J_3\|_{L_q(B)} + \|J_4\|_{L_q(B)}.$$

By the boundedness of $[b, I_{\Omega,\alpha}]$ from $L_{p_1}(\mathbb{R}^n)$ to $L_{q_1}(\mathbb{R}^n)$, it follows that

$$\begin{aligned}\|J_1\|_{L_q(B)} &\leqslant \|\big(b(\cdot) - b_B\big)[b, I_{\Omega,\alpha}]f_1(\cdot)\|_{L_q(\mathbb{R}^n)} \\ &\leqslant \|\big(b(\cdot) - b_B\big)\|_{L_{p_2}(\mathbb{R}^n)}[b, I_{\Omega,\alpha}]f_1(\cdot)\|_{L_{q_1}(\mathbb{R}^n)} \leqslant C\|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+n\lambda} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \\ &= C\|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+\frac{n}{q_1}+n\lambda} \|f\|_{L_{p_1}(2B)} \int\limits_{2r}^{\infty} t^{-1-\frac{n}{q_1}} dt \\ &\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{q}+n\lambda} \int\limits_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt.\end{aligned}$$

For J_2 we have

$$\begin{aligned}\|J_2\|_{L_q(B)} &\leqslant \|[b, I_{\Omega,\alpha}]\big(b(\cdot) - b_B\big) f_1\|_{L_q(\mathbb{R}^n)} \lesssim \|(b(\cdot) - b_B) f_1|\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|b(\cdot) - b_B\|_{L_{p_2}(\mathbb{R}^n)} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+\frac{n}{q_1}+n\lambda} \|f\|_{L_{p_1}(2B)} \int\limits_{2r}^{\infty} t^{-1-\frac{n}{q_1}} dt \\ &\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}+n\lambda} \int\limits_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt.\end{aligned}$$

It is known that $x \in B$ and $y \in {}^{\complement}(2B)$ for J_3, which implies $\frac{1}{2}|x_0 - y| \leqslant |x - y| \leqslant \frac{3}{2}|x_0 - y|$.

For $s' \leqslant p$ the Fubini theorem and Hölder inequality yield

$$\begin{aligned}
|I_{\Omega,\alpha} f_2(x)| &\leqslant c_0 \int\limits_{{}^{\complement}(2B)} |\Omega(x-y)| \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy \\
&\approx \int\limits_{2r}^{\infty} \int\limits_{2r<|x_0-y|<t} |\Omega(x-y)||f(y)| dy \, t^{-1-n-\alpha} dt \\
&\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |\Omega(x-y)||f(y)| dy \, t^{-1-n-\alpha} dt \\
&\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}} t^{-1-\frac{n}{p_1}-\alpha} dt \\
&\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt.
\end{aligned}$$

Hence

$$\begin{aligned}
\|J_3\|_{L_q(B)} &= \|\big(b(\cdot) - b_B\big) I_{\Omega,\alpha} f_2(\cdot)\|_{L_q(\mathbb{R}^n)} \\
&\leqslant \|\big(b(\cdot) - b_B\big)\|_{L_q(\mathbb{R}^n)} \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt \\
&\leqslant \|\big(b(\cdot) - b_B\big)\|_{L_{p_2}(\mathbb{R}^n)} r^{\frac{n}{q_1}} \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt \\
&\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{q}+n\lambda} \int\limits_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt.
\end{aligned}$$

For $q_1 < s$ the Fubini theorem and Minkowski inequality yield

$$\begin{aligned}
\|J_3\|_{L_q(B)} &\leqslant \Bigg(\int\limits_B \Big| \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)||b(x) - b_B||\Omega(x-y)| dy \frac{dt}{t^{n-\alpha+1}} \Big|^q \Bigg)^{\frac{1}{q}} \\
&\leqslant \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|(b(\cdot) - b_B)\Omega(\cdot - y)\|_{L_q(B)} dy \frac{dt}{t^{n-\alpha+1}} \\
&\leqslant \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|b(\cdot) - b_B\|_{L_{p_2}(B)} \, \|\Omega(\cdot - y)\|_{L_{q_1}(B)} dy \, \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+n\lambda} |B|^{\frac{1}{q_1}-\frac{1}{s}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_s(B)} dy \, \frac{dt}{t^{n-\alpha+1}}
\end{aligned}$$

$$\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{q}+n\lambda} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{q}+n\lambda} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} \frac{dt}{t^{\frac{n}{q_1}+1}}. \tag{4.1}$$

For $x \in B$ the Fubini theorem and Hölder inequality yield

$$|I_{\Omega,\alpha}\Big(\big(b(\cdot)-b_B\big)f_2\Big)(x)| \lesssim \int_{\complement_{(2B)}} |b(y)-b_B|\,|\Omega(x-y)|\,\frac{|f(y)|}{|x-y|^{n-\alpha}}dy$$

$$\lesssim \int_{\complement_{(2B)}} |b(y)-b_B|\,|\Omega(x-y)|\,\frac{|f(y)|}{|x_0-y|^{n-\alpha}}dy$$

$$\approx \int_{2r}^{\infty} \int_{2r<|x_0-y|<t} |b(y)-b_B|\,|\Omega(x-y)|\,|f(y)|dy\, t^{\alpha-n-1}dt$$

$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}||\Omega(x-y)|\,|f(y)|dy \frac{dt}{t^{n-\alpha+1}}$$

$$+ \int_{2r}^{\infty} |b_{B(x_0,r)}-b_{B(x_0,t)}| \int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)|dy \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \int_{2r}^{\infty} \|(b(\cdot)-b_{B(x_0,t)})f\|_{L_p(B(x_0,t))}\,\|\Omega(\cdot-y)\|_{L_s(B(x_0,t))}\,|B(x_0,t)|^{1-1/p-\frac{1}{s}}\,\frac{dt}{t^{n-\alpha+1}}$$

$$+ \int_{2r}^{\infty} |b_{B(x_0,r)}-b_{B(x_0,t)}|\|f\|_{L_{p_1}(B(x_0,t))}\,\|\Omega(\cdot-y)\|_{L_s(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}}\,t^{\alpha-n-1}dt$$

$$\lesssim \int_{2r}^{\infty} \|b(\cdot)-b_{B(x_0,t)}\|_{L_{p_2}(B(x_0,t))}\|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}}\,dt$$

$$+ \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p_1}(B(x_0,t))}\, t^{n\lambda-1-\frac{n}{q_1}}\,dt$$

$$\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p_1}(B(x_0,t))}\; t^{n\lambda-1-\frac{n}{q_1}}\,dt.$$

Then for J_4 we have

$$\|J_4\|_{L_q(B)} \leqslant \|I_{\Omega,\alpha}\big(b(\cdot)-b_B\big)f_2\|_{L_q(\mathbb{R}^n)}$$

$$\lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p_1}(B(x_0,t))} t^{n\lambda-1-\frac{n}{q_1}}\,dt.$$

For $q_1 < s$ the Fubini theorem and Minkowski inequality yield

$$\|I_{\Omega,\alpha} f_2\|_{L_q(B)} \leqslant \Bigg(\int\limits_B \Big| \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n-a+1}} \Big|^q \Bigg)^{\frac{1}{q}}$$

$$\leqslant \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_q(B)} dy \frac{dt}{t^{n-a+1}}$$

$$\leqslant |B|^{\frac{1}{q}-\frac{1}{s}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_s(B)} dy \, \frac{dt}{t^{n-a+1}}$$

$$\lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n-a+1}} \lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} \frac{dt}{t^{\frac{n}{q_1}+1}}.$$

Then we combine the above estimates and complete the proof of the lemma. □

Theorem 4.5. *Suppose that $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero. Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $b \in CBMO^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $0 \leqslant \lambda < 1/n$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, $1/q_1 = 1/p_1 - \alpha/n$. If for $s' \leqslant p$ or $q_1 < s$ the pair (φ_1, φ_2) satisfies the condition*

$$\int\limits_r^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}-n\lambda+1}} dt \leqslant C \, \varphi_2(x_0,r), \tag{4.2}$$

where C is independent of r, then the operators $M_{\Omega,b,\alpha}$ and $[b, I_{\Omega,\alpha}]$ are bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$. Moreover

$$\|M_{\Omega,b,\alpha} f\|_{LM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|[b, I_{\Omega,\alpha}]f\|_{LM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|b\|_{CBMO^{\{x_0\}}_{p_2,\lambda}} \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.$$

Proof. The statement of Theorem 4.5 follows from Lemma 4.4 and Theorem 3.1 in the same manner as Theorem 3.6. □

For the sublinear commutator of the fractional maximal operator $M_{b,\alpha}$ and the linear commutator of the Riesz potential $[b, I_\alpha]$ from Theorem 4.5 we obtain the following new results.

Corollary 4.6. *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $b \in CBMO^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $0 \leqslant \lambda < 1/n$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, $1/q_1 = 1/p_1 - \alpha/n$, and (φ_1, φ_2) satisfies the condition (4.2). Then the operators $M_{b,\alpha}$ and $[b, I_\alpha]$ are bounded from $LM^{\{x_0\}}_{p_1,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$.*

5 Some Applications

In this section, we apply Theorems 3.6 and 4.5 to some particular operators such as the Marcinkiewicz operator and fractional powers of some analytic semigroups.

5.1. Marcinkiewicz operator. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, is homogeneous of degree zero and satisfy the cancellation condition. In 1958, Stein [28] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and provides useful tools in harmonic analysis [29]–[32].

The Marcinkiewicz operator is defined by the formula (cf. [33])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

We introduce the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}.$$

It is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$. By the Minkowski inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \leqslant \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \leqslant C I_{\Omega,\alpha}(f)(x).$$

It is known that $\mu_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$ and from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$ (cf. [33]). Hence from Theorems 3.6 and 4.5 we get the following assertions.

Corollary 5.1. *Let $x_0 \in \mathbb{R}^n$, and let $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, be homogeneous of degree zero and satisfy the cancellation condition. Suppose that $0 < \alpha < n$, $1 \leqslant p < n/\alpha$, $1/q = 1/p - \alpha/n$ and for $s' \leqslant p$ or $q_1 < s$ the pair (φ_1, φ_2) satisfies the condition* (3.13). *Then $\mu_{\Omega,\alpha}$ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $M_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$.*

Corollary 5.2. *Let $x_0 \in \mathbb{R}^n$, and let $\Omega \in L_s(S^{n-1})$, $1 < s \leqslant \infty$, be homogeneous of degree zero and satisfy the cancellation condition. Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $b \in CBMO_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leqslant \lambda < \frac{1}{n}$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, $1/q_1 = 1/p_1 - \alpha/n$ and for $s' \leqslant p$ or $q_1 < s$ the pair (φ_1, φ_2) satisfies the condition* (3.13). *Then $[a, \mu_{\Omega,\alpha}]$ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$.*

5.2. Fractional powers of the some analytic semigroups. The theorems of previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 that generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying the Gaussian upper bound, i.e.,

$$|p_t(x,y)| \leqslant \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{5.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where c_1 and $c_2 > 0$ are independent of x, y, and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α (cf., for example, [30, Chapter 5]).

Theorem 5.3. *Let the condition* (5.1) *be satisfied. Suppose that* $1 \leqslant p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, *and the pair* (φ_1, φ_2) *satisfies the condition* (3.13). *Then* $L^{-\alpha/2}$ *is bounded from* $LM^{\{x_0\}}_{p,\varphi_1}$ *to* $LM^{\{x_0\}}_{q,\varphi_2}$ *for* $p > 1$ *and from* $M^{\{x_0\}}_{1,\varphi_1}$ *to* $WLM^{\{x_0\}}_{q,\varphi_2}$ *for* $p = 1$.

Proof. Since the semigroup e^{-tL} has kernel $p_t(x,y)$ satisfying the condition (5.1), it follows that (cf. [34])

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x).$$

By the aforementioned theorems, we have

$$\|L^{-\alpha/2} f\|_{M^{\{x_0\}}_{q,\varphi_2}} \lesssim \|I_\alpha(|f|)\|_{M^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{M^{\{x_0\}}_{p,\varphi_1}}. \qquad \square$$

Let b be a locally integrable function on $\mathbb{R}^n$. The commutator of b and $L^{-\alpha/2}$ is defined by

$$[b, L^{-\alpha/2}] f(x) = b(x) L^{-\alpha/2} f(x) - L^{-\alpha/2}(bf)(x).$$

In [34], the result of [26] was extended from $(-\Delta)$ to the more general operator L as above. More precisely, it was shown in [34] that if $b \in BMO(\mathbb{R}^n)$, then the commutator operator $[b, L^{-\alpha/2}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < q < \infty$ and $1/q = 1/p - \alpha/n$. Then form Theorem 4.5 we get the following assertion.

Theorem 5.4. *Let the condition* (5.1) *be satisfied. Suppose that* $0 < \alpha < n$, $1 < p < n/\alpha$, $b \in CBMO^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $0 \leqslant \lambda < 1/n$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, *and* $1/q_1 = 1/p_1 - \alpha/n$. *If* (φ_1, φ_2) *satisfies the condition* (4.2), *then* $[b, L^{-\alpha/2}]$ *is bounded from* $LM^{\{x_0\}}_{p,\varphi_1}$ *to* $LM^{\{x_0\}}_{q,\varphi_2}$.

The property (5.1) is satisfied for large classes of differential operators (cf., for example, [21]). Other examples of operators estimated from above by Riesz potentials can be found in [21]. In these cases, Theorems 3.6 and 4.5 are also applicable for proving the boundedness of those operators and commutators from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$.

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