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# On Polynomial General Helices in $n$-Dimensional Euclidean Space $\mathbb{R}^n$

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**Abstract.** In this work, we study the so-called polynomial general helices in an arbitrary dimensional Euclidean space. First, we give a method to construct helices from polynomial curves in $n$-dimensional Euclidean space $\mathbb{R}^n$, and another method to construct polynomial general helices in $\mathbb{R}^n$ from polynomial general helices in $\mathbb{R}^{n+1}$ or $\mathbb{R}^{n+2}$. Then, we proceed with a method to construct rational helices from polynomial general helices.

**Mathematics Subject Classification.** 53A05.

**Keywords.** Harmonic curvature, Polynomial general helix, Rational curve, Rational helix.

## 1. Introduction

Helical structures is a significant workframe in the study of differential geometry. They have been studied to a great extent for a long time and are being studied even today. We can observe these structures in nature, in architecture, in simulation of kinematic motion, or in the design of highways and mechanic tools, etc [2,4,6,15].

A general helix is defined by the property that its tangent vector field makes a constant angle with a fixed direction which is called the axis of the general helix in Euclidean 3-space. This well-known result was stated by Lancret in 1802 [10] and first proved by de Saint Venant in 1845. A necessary and sufficient condition for a curve $\alpha$ to be a general helix is to have the ratio of its curvature to torsion constant. If both curvature and torsion are non-zero constants, then the curve is called a circular helix [10,14].

Harmonic curvature functions were defined earlier by Özdamar and Hacisalihoğlu in [11]. The authors generalized helices from $\mathbb{R}^3$ to $\mathbb{R}^n$ and

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then gave a characterization. Recently, many studies have been published on general helices [3,8,12].

In [5], the authors studied Pythagorean-hodograph (PH) curves in $R^3$ which are also polynomial general helices. Also in [13] authors studied Pythagorean-hodograph (PH) curves in $R^5$ and $R^9$.

The notion of a general helix in $\mathbb{R}^3$ can be generalized to higher dimensions in many ways. In [12], the same definition is proposed but in $\mathbb{R}^n$. In [8] the definition is more restrictive: the fixed direction makes a constant angle with all vectors of the Frenet frame. It is easy to check that this definition only works in odd dimensions. Moreover, in the same paper, it is proven that this definition is equivalent to the fact that the ratios $\frac{k_1}{k_2}, \frac{k_3}{k_4}, \ldots, \frac{k_{n-4}}{k_{n-3}}, \frac{k_{n-2}}{k_{n-1}}$, where curvatures $k_i$ are constants. This statement is related with the Lancret theorem for general helices in $\mathbb{R}^3$.

This paper is organized in the following fashion. We begin in Sect. 2 by recalling some preliminary results about general helices. In Sect. 3, we give a method to construct general helices from polynomial curves in $n$-dimensional Euclidean space $\mathbb{R}^n$, and another method to construct polynomial general helices in $\mathbb{R}^n$ from polynomial general helices in $\mathbb{R}^{n+1}$ or $\mathbb{R}^{n+2}$. Finally in Sect. 4, we show how to build rational helices from polynomial general helices.

## 2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ be an arbitrary curve in the Euclidean $n$-space $\mathbb{R}^n$. Recall that the curve $\alpha$ is said to be of unit speed (or parameterized by arclength function $s$) if $\langle \alpha'(t), \alpha'(t)\rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{R}^n$ given by

$$\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i,$$

for each $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. In particular, the norm of a vector $X \in \mathbb{R}^n$ is given by $||X||^2 = \langle X, X\rangle$. Let $\{V_1, V_2, \ldots, V_n\}$ be the moving Frenet frame along a space curve $\alpha$, where $V_i$ $(i = 1, 2, \ldots, n)$ denote $i$th Frenet vector field. Then the Frenet formulas are given by

$$\begin{cases} V_1^{'}(t) = \nu(t)k_1(t)V_2(t) \\ V_i^{'}(t) = \nu(t)(-k_{i-1}(t)V_{i-1}(t) + k_i(t)V_{i+1}(t)), \quad i = 2, 3, \ldots, n-1 \\ V_n^{'}(t) = -\nu(t)k_{n-1}(t)V_{n-1}(t)s \end{cases} \tag{2.1}$$

where $\nu(t) = ||\alpha'(t)||$ and $k_i$ $(i = 1, 2, \ldots, n-1)$ denote the $i$th curvature function of the curve [3,9]. We call $\alpha$ a regular curve of order $m$ (where $m \leqslant n$), if and only if for any $t \in I$,

$$\{\alpha'(t), \alpha''(t), \ldots, \alpha^{(m)}(t)\}$$

is a linearly independent subset of $\mathbb{R}^n$.

In this paper, we assume that Frenet frame of the curve are given by Gram–Schmidt method [7]. If the curve lies in a hyperplane of $\mathbb{R}^n$, then it is said that $\alpha$ is a $(n-1)$-flat curve [12]. It is well known that $\alpha$ is $(n-1)$-flat curve in $\mathbb{R}^n$ if and only if $k_{n-1}(t) = 0$ [12].

**Proposition 2.1.** *A curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ is a general helix if and only if the function* $\det(\alpha^{''}(t), \alpha^{'''}(t), \ldots, \alpha^{(n+1)}(t))$ *is identically zero, where $\alpha^{(i)}$ represents the $i$th derivative of $\alpha$ with respect to $t$). Equivalently, $\alpha$ is general helix if and only if $\alpha_T$ is a $n$-flat curve, where $\alpha_T : I \subset \mathbb{R} \to \mathbb{S}^n$ is tangent indicatrix of the curve* [11].

**Definition 2.1.** Let $\alpha$ be a curve in $\mathbb{R}^n$. Harmonic curvatures of the curve $\alpha$ are defined by

$$H_i : I \subset \mathbb{R} \to \mathbb{R}\,, i = 0, 1, 2, \ldots, n-2,$$

$$H_i = \begin{cases} 0, & i = 0 \\ \frac{k_1}{k_2} & i = 1 \\ \frac{1}{k_{i+1}}\left[\frac{1}{\nu}H'_{i-1} + k_i H_{i-2}\right], & i = 2, 3, \ldots, n-2 \end{cases}$$

[11].

**Theorem 2.1.** *Let $\alpha$ be a general helix in $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{H_1, H_2, \ldots, H_{n-2}\}$ be denote the Frenet frame and the higher ordered harmonic curvatures of the curve, respectively. Then, the following equation holds*

$$\langle V_{i+2}, X\rangle = H_i \langle V_1, X\rangle, \quad 1 \leqslant i \leqslant n-1, \tag{2.2}$$

*where $X$ is the axis of the helix $\alpha$* [3].

**Corollary 2.1.** *If $X$ is the axis of the helix $\alpha$, then we can write*

$$X = \lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_n V_n.$$

*From the Theorem 2.1, we get*

$$\lambda_j = \langle V_j, X\rangle = H_{j-2}\langle V_1, X\rangle,\ 1 < j \leqslant n$$

*where $\langle V_1, X\rangle = \cos\theta = constant$.*

*By the definition of the harmonic curvature, we obtain*

$$X = \cos\theta(V_1 + H_1 V_3 + \cdots + H_{n-2}V_n).$$

*Also,*

$$D = V_1 + H_1 V_3 + \cdots + H_{n-2}V_n$$

*is a axis of the helix $\alpha$* [3].

**Definition 2.2.** A curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ with $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$ is called a polynomial curve in $\mathbb{R}^n$ if $\alpha_i(t)$, $1 \leqslant i \leqslant n$ is a polynomial function whose coefficients are real numbers.

In the following theorem, Camci et al. [3] gave the explicit characterization for a non-degenerate curve to be a general helix by using the harmonic curvatures of the curve:

**Theorem 2.2.** *Let $\alpha$ be a non-degenerate curve in $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{H_1, H_2, \ldots, H_{n-2}\}$ denote the Frenet frame and harmonic curvatures of the curve, respectively. Then, $\alpha$ is a general helix if and only if $H^{'}_{n-2} + \nu k_{n-1} H_{n-3} = 0$.*

## 3. Polynomial General Helices

As we know for Euclidean space $\mathbb{R}^3$, there are characterizations and a lot of examples about helices. For Euclidean spaces $\mathbb{R}^n$, $n \geq 4$ there is only one example when $n = 5$ [1]. Especially, there is no example when n is even. In this section, we examine polynomial general helices in $\mathbb{R}^n$ depending on whether $n$ is even or odd.

### 3.1. Polynomial General Helices in $\mathbb{R}^n$ When n is Even

Consider the curve $\alpha$ given by,

$$\alpha(t) = \left(a_1 t, \frac{a_2}{2}\, t^2, \frac{a_3}{3}\, t^3, \ldots, \frac{a_{n-1}}{n-1}\, t^{n-1}, \frac{a_n}{n-1}\, t^{n-1} + \frac{a_{n+1}}{n+1}\, t^{n+1}\right).$$

So,

$$\det(\alpha'(t), \alpha''(t), \ldots, \alpha^{(n)}(t)) = 1!2!3! \cdots (n-2)!n! \left(\prod_{i=1}^{n-1} a_i\right) a_{n+1} t.$$

Therefore; if $a_{n+1} \prod_{i=1}^{n-1} a_i \neq 0$, then $\alpha$ is a regular polynomial curve of order $n$.

**Theorem 3.1.** *Let*

$$\alpha(t) = \left(a_1 t, \frac{a_2}{2}\, t^2, \frac{a_3}{3}\, t^3, \frac{a_4}{3}\, t^3 + \frac{a_5}{5}\, t^5\right)$$

*be a curve in* $\mathbb{R}^4$ *with* $1 \leqslant j \leqslant 3$, $b_j \in \mathbb{R}^+$ *and*

$$a_1 = b_1, a_2^2 = 2b_1 b_2, a_3^2 = 2b_1 b_3, a_4 = b_2, a_5 = b_3$$

*then* $\alpha$ *is a polynomial general helix which makes a constant angle*

$$\theta = \arccos\left(\frac{1}{\|D\|}\right)$$

*with the fixed direction*

$$\frac{D}{\|D\|}$$

*where*

$$D = (1, 0, 0, 1).$$

*Proof.* By making calculations, we have

$$V_1(t) = \left(\frac{b_1}{b_1+b_2 t^2+b_3 t^4}, \frac{\sqrt{2b_1 b_2}\, t}{b_1+b_2 t^2+b_3 t^4}, \frac{\sqrt{2b_1 b_3}\, t^2}{b_1 + b_2 t^2 + b_3 t^4}, \frac{b_2\, t^2 + b_3\, t^4}{b_1 + b_2 t^2 + b_3 t^4}\right),$$

$$k_3(t) = \frac{12\, t \sqrt{2b_1 b_2\, (b_2 + 4b_3\, t^2)} {b_3}^{3/2}}{f^2(t)}$$

$$H_1(t) = \frac{\sqrt{b_1} \left(b_2 + 4b_3\, t^2\right)^{3/2}}{f(t)},$$

$$H_2(t) = \frac{\sqrt{2b_2 b_3} \left(b_1 - 3b_3\, t^4\right)^2}{f(t)},$$

where

$$f(t)=\sqrt{2b_1^2b_2b_3+18b_2b_3^3\,t^8+b_1\left(b_2^3+12b_2^2b_3\,t^2+36b_2b3^2t^4+64b_3^3\,t^6\right)}.$$

Then,

$$H_2^{'}+\nu k_3H_1=0.$$

From Theorem 2.2, $\alpha$ is a polynomial general helix. By using Corollary 2.1 we have

$$D=(1,0,0,1).$$

Therefore

$$\left\langle V_1(t),\frac{D}{\|D\|}\right\rangle=\frac{1}{\sqrt{2}}.$$

This completes the proof. □

*Example* 3.1. If we take $b_1=1, b_2=b_3=2$ in Theorem 3.1 then we have,

$$\beta(t)=\left(t,t^2,\frac{2}{3}\,t^3,\frac{2}{3}\,t^3+\frac{2}{5}\,t^5\right).$$

The curve $\beta$ is a polynomial general helix whose tangent vector field

$$V_1(t)=\left(\frac{1}{1+2\,t^2+2\,t^4},\frac{2\ t}{1+2\,t^2+2\,t^4},\frac{2t^2}{1+2\,t^2+2\,t^4},\frac{2(t^2+t^4)}{1+2\,t^2+2\,t^4}\right).$$

makes a constant angle $\theta=\arccos(\frac{1}{\sqrt{2}})$ with the fixed direction

$$\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right).$$

In the following theorem, we obtain a polynomial general helix in $\mathbb{R}^6$. Since, it can be done similarly to Theorem 3.1, we omit the proof.

**Theorem 3.2.** *Let*

$$\alpha(t)=\left(a_1t,\frac{a_2}{2}\,t^2,\frac{a_3}{3}\,t^3,\frac{a_4}{4}\,t^4,\frac{a_5}{5}\,t^5,\frac{a_6}{5}\,t^5+\frac{a_7}{7}\,t^7\right)$$

*be a curve in* $\mathbb{R}^6$ *with* $1\leqslant j\leqslant 4$, $b_j\in\mathbb{R}^+$ *and*

$$\begin{aligned}&a_1=b_1,a_2^2=2b_1b_2,a_3^2=b_2^2+2b_1b_3,\\&a_4^2=2b_1b_4+2b_2b_3,a_5^2=2b_2b_4,\\&a_6=b_3,a_7=b_4\end{aligned}$$

*then* $\alpha$ *is a polynomial general helix which makes a constant angle*

$$\theta=\arccos\left(\frac{1}{\|D\|}\right)$$

*with the fixed direction*

$$\frac{D}{\|D\|}$$

*where*

$$D=\left(1,0,\frac{b_2}{a_3},0,0,1\right).$$

Now, we give a new theorem for $n\geq 8$.

**Theorem 3.3.** *Let $n \geq 8$ be an even number, $1 \leqslant j \leqslant \frac{n+2}{2}$, $b_j \in \mathbb{R}^+$, $b_{\frac{n+4}{2}} = b_{\frac{n+6}{2}} = \cdots = b_{n-2} = 0$,*

$$a_1 = b_1,\ a_2^2 = 2b_1 b_2,\ a_{n-1}^2 = 2b_{\frac{n-2}{2}} b_{\frac{n+2}{2}},\ a_n = b_{\frac{n}{2}},\ a_{n+1} = b_{\frac{n+2}{2}},$$

$$a_{2k+1}^2 = b_{k+1}^2 + 2\sum_{j=1}^{k} b_j b_{2k-j+2},\ 1 \leqslant k \leqslant \frac{n-4}{2},$$

$$a_{2l}^2 = 2\sum_{j=1}^{l} b_j b_{2l-j+1},\ 2 \leqslant l \leqslant \frac{n-2}{2}.$$

*Then, the curve*

$$\alpha(t) = \left(a_1 t, \frac{a_2}{2}\, t^2, \frac{a_3}{3}\, t^3, \ldots, \frac{a_{n-1}}{n-1}\, t^{n-1}, \frac{a_n}{n-1}\, t^{n-1} + \frac{a_{n+1}}{n+1}\, t^{n+1}\right)$$

*is a polynomial general helix which makes a constant angle*

$$\theta_n = \arccos\left(\frac{1}{\|D_n\|}\right)$$

*with the fixed direction*

$$\frac{D_n}{\|D_n\|}$$

*where,*

$$D_n = \sum_{m=1}^{\frac{n-2}{2}} \frac{b_m}{a_{2m-1}} e_{2m-1} + \frac{b_{\frac{n}{2}}}{a_n} e_n$$

*and*

$$\{e_1, e_2, \ldots, e_n\}$$

*is the standard orthonormal basis in $\mathbb{R}^n$.*

*Proof.* We can write,

$$\begin{aligned} V_1(t) &= \frac{\alpha'(t)}{\|\alpha'(t)\|} \\ &= \frac{1}{\|\alpha'(t)\|}\left(a_1, a_2\, t, a_3\, t^2, \ldots, a_n\, t^{n-2} + a_{n+1} t^n\right) \\ &= \frac{1}{\|\alpha'(t)\|}\left(b_1, a_2\, t, a_3\, t^2, \ldots, b_{\frac{n}{2}}\, t^{n-2} + b_{\frac{n+2}{2}}\, t^n\right) \end{aligned}$$

If we make the necessary calculations, we will have

$$\|\alpha'(t)\|^2 = \left(\sum_{j=1}^{\frac{n+2}{2}} b_j t^{2(j-1)}\right)^2$$

If we take

$$D_n = \sum_{m=1}^{\frac{n-2}{2}} \frac{b_m}{a_{2m-1}} e_{2m-1} + \frac{b_{\frac{n}{2}}}{a_n} e_n$$

we have

$$\langle V_1(t), D_n\rangle = 1.$$

Therefore

$$\left\langle V_1(t), \frac{D_n}{\|D_n\|}\right\rangle = \cos(\theta_n).$$

This completes the proof of the theorem. □

We can give the following corollary as the result of Theorem 3.3.

**Corollary 3.1.** *In Theorem 3.3, if $b_{\frac{n+2}{2}}$ vanishes, then the polynomial general helix is included in the $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$, so we can consider it to be a helix in $(n-1)$-dimensional Euclidean space, where $n-1$ is an odd number.*

### 3.2. Polynomial General Helices in $\mathbb{R}^n$ When n is Odd

Consider the curve $\alpha$ denoted by,

$$\alpha(t) = \left(a_1\, t, \frac{a_2}{2}\, t^2, \frac{a_3}{3}\, t^3, \ldots, \frac{a_n}{n}\, t^n\right). \tag{3.1}$$

So,

$$\det(\alpha'(t), \alpha''(t), \ldots, \alpha^{(n)}(t)) = 1!2!\ldots(n-1)!\prod_{i=1}^{n} a_i.$$

Therefore, if $\prod_{i=1}^{n} a_i \neq 0$ then $\alpha$ is a regular polynomial curve of order $n$.

**Theorem 3.4.** *Let*

$$\alpha(t) = \left(a_1\, t, \frac{a_2}{2}\, t^2, \frac{a_3}{3}\, t^3, \frac{a_4}{4}\, t^4, \frac{a_5}{5}\, t^5\right)$$

*be a curve in $\mathbb{R}^5$ with $1 \leqslant j \leqslant 3$, $b_j \in \mathbb{R}^+$,*

$$a_1 = b_1, a_2^2 = 2b_1b_2, a_3^2 = b_2^2 + 2b_1b_3, a_4^2 = 2b_2b_3, a_5 = b_3$$

*then $\alpha$ is a polynomial general helix which makes a constant angle*

$$\theta = \arccos\left(\frac{1}{\|D\|}\right)$$

*with the fixed direction*

$$\frac{D}{\|D\|}$$

*where*

$$D = \left(1, 0, \frac{b_2}{a_3}, 0, 1\right)$$

*Proof.* The proof of the above theorem is similar to the proof of the theorem 3.1. □

*Example* 3.2. If we take $b_1 = b_2 = b_3 = 1$ in Theorem 3.4, then we have

$$\gamma(t) = \left(t, \frac{\sqrt{2}}{2}\, t^2, \frac{\sqrt{3}}{3}\, t^3, \frac{\sqrt{2}}{4}\, t^4, \frac{1}{5}\, t^5\right).$$

The curve $\gamma$ is a polynomial general helix whose tangent vector field

$$V_1(t) = \left( \frac{1}{1+t^2+t^4}, \frac{\sqrt{2}\,t}{1+t^2+t^4}, \frac{\sqrt{3}\,t^2}{1+t^2+t^4}, \frac{\sqrt{2}\,t^3}{1+t^2+t^4}, \frac{t^4}{1+t^2+t^4} \right).$$

makes a constant angle $\theta = \arccos(\sqrt{3/7})$ with the fixed direction

$$\left( \sqrt{\frac{3}{7}}, 0, \frac{1}{\sqrt{7}}, 0, \sqrt{\frac{3}{7}} \right).$$

Now, we give a new theorem for $n \geq 7$.

**Theorem 3.5.** *Let $n \geq 7$ be an odd number, $1 \leqslant j \leqslant \frac{n+1}{2}$, $b_j \in \mathbb{R}^+$, $1 \leqslant j \leqslant \frac{n+1}{2}$, $b_j \in \mathbb{R}^+$, $b_{\frac{n+3}{2}} = b_{\frac{n+5}{2}} = \cdots = b_{n-1} = 0$,*

$$a_1 = b_1, a_2^2 = 2b_1 b_2, a_n = b_{\frac{n+1}{2}},$$

$$a_{2k+1}^2 = b_{k+1}^2 + 2\sum_{j=1}^{k} b_j b_{2k-j+2}, 1 \leqslant k \leqslant \frac{n-3}{2},$$

$$a_{2l}^2 = 2\sum_{j=1}^{l} b_j b_{2l-j+1}, 2 \leqslant l \leqslant \frac{n-1}{2}.$$

*Then, the curve*

$$\alpha(t) = (a_1 t, \frac{a_2}{2}\,t^2, \frac{a_3}{3}\,t^3, \ldots, \frac{a_n}{n}\,t^n)$$

*is a polynomial general helix which makes a constant angle*

$$\theta_n = \arccos\left(\frac{1}{\|D_n\|}\right)$$

*with the fixed direction*

$$\frac{D_n}{\|D_n\|}$$

*where,*

$$D_n = \sum_{m=1}^{\frac{n+1}{2}} \frac{b_m}{a_{2m-1}} e_{2m-1}$$

*and*

$$\{e_1, e_2, \ldots, e_n\}$$

*is the standard orthonormal basis in $\mathbb{R}^n$.*

*Proof.* The proof of the above theorem is similar to the proof of the Theorem 3.3. □

With the help of the Theorem 3.5, one can easily obtain the following important result.

**Corollary 3.2.** *In Theorem 3.5, if $b_{\frac{n+1}{2}}$ vanishes, then the polynomial general helix is included in the $(n-2)$-dimensional hyperplane of $\mathbb{R}^n$, so we can consider it to be a helix in $(n-2)$-dimensional Euclidean space, where $n-2$ is an odd number.*

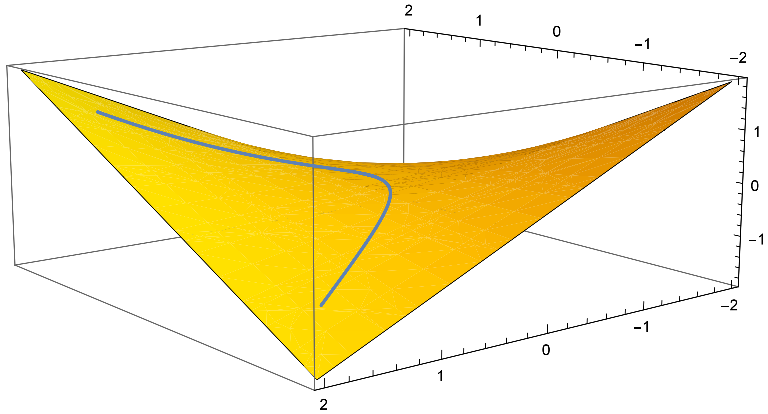


FIGURE 1. Polynomial general helix $\beta(t) = (t, \frac{\sqrt{2}}{2}\, t^2, \frac{1}{3}\, t^3)$ lies on the saddle $z = \frac{\sqrt{2}}{3} xy$

## 4. Constructing Rational Helices from Polynomial General Helices

We can use polynomial general helices to construct rational curves which are general helices. Now, we will give a method of this.

### 4.1. Constructing Rational Helix in $\mathbb{R}^4$ from a Polynomial General Helix in $\mathbb{R}^3$

Let $b_1, b_2 \in \mathbb{R}^+$,

$$\begin{aligned} \beta_1(t) &= b_1 t, \\ \beta_2(t) &= \frac{\sqrt{2b_1 b_2}}{2}\, t^2, \\ \beta_3(t) &= \frac{b_2}{3}\, t^3. \end{aligned}$$

Then, $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$ is a polynomial general helix in $\mathbb{R}^3$ which makes angle $\theta = \pi/4$ with the axis $u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ [5].

Now, we want to find a rational curve $\gamma$ in $\mathbb{R}^4$ which makes angle $\theta = \pi/4$ with the axis $v = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. In order to do this; first, we have to write $\gamma$ by using the curve $\beta$ (see Fig. 1).

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a curve in $\mathbb{R}^4$ denoted by

$$\begin{aligned} \gamma_1(t) &= a(t), \\ \gamma_2(t) &= a(t)\,\beta_1(t), \\ \gamma_3(t) &= a(t)\,\beta_2(t), \\ \gamma_4(t) &= a(t)\,\beta_3(t). \end{aligned}$$

where $a(t)$ is a real valued function. Therefore, we have the differential equation

$$\langle V_1(t), v\rangle = \frac{1}{\sqrt{2}}.$$

If we solve this equation, we find

$$a(t) = \frac{c}{-6 + b_1 b_2 t^4}$$

where $c \in \mathbb{R} - \{0\}$. Then, we have

$$\gamma(t) = \frac{c}{-6 + b_1 b_2\, t^4}\left(1, b_1 t, \frac{\sqrt{2 b_1 b_2}}{2}\, t^2, \frac{b_2}{3}\, t^3\right).$$

### 4.2. Constructing Rational Helix in $\mathbb{R}^5$ from a Polynomial General Helix in $\mathbb{R}^4$

Let $b_1, b_2, b_3 \in \mathbb{R}^+$,

$$\begin{aligned}
\beta_1(t) &= b_1 t,\\
\beta_2(t) &= \frac{\sqrt{2 b_1 b_2}}{2}\, t^2,\\
\beta_3(t) &= \frac{\sqrt{2 b_1 b_3}}{3}\, t^3,\\
\beta_4(t) &= \frac{b_2}{3}\, t^3 + \frac{b_3}{5}\, t^5.
\end{aligned}$$

From Theorem 2.2, $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t))$ is a polynomial general helix in $\mathbb{R}^4$ which makes angle $\theta = \pi/4$ with the axis $u = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$.

Now, we want to find a rational curve $\gamma$ in $\mathbb{R}^5$ which makes angle $\theta = \pi/4$ with the axis $v = \left(0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$. In order to do this, first we have to write $\gamma$ by using the curve $\beta$.

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t), \gamma_5(t))$ be a curve in $\mathbb{R}^5$ denoted by

$$\begin{aligned}
\gamma_1(t) &= a(t),\\
\gamma_2(t) &= a(t)\,\beta_1(t),\\
\gamma_3(t) &= a(t)\,\beta_2(t),\\
\gamma_4(t) &= a(t)\,\beta_3(t),\\
\gamma_5(t) &= a(t)\,\beta_4(t),
\end{aligned}$$

where $a(t)$ is a real valued function. Therefore; if we do the necessary calculations, we have

$$\gamma(t) = \frac{c}{-90 + 15 b_1 b_2\, t^4 + 16 b_1 b_3\, t^6}\left(1, b_1\, t, \frac{\sqrt{2 b_1 b_2}}{2}\, t^2, \frac{\sqrt{2 b_1 b_3}}{3}\, t^3, \frac{b_2}{3}\, t^3 + \frac{b_3}{5}\, t^5\right)$$

where $c \in \mathbb{R} - \{0\}$.

We can find rational helices in upper dimensions similarly.

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