ISMO KORKEE

On Meet and Join Matrices Associated with Incidence Functions

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Contents

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Ismo Korkee

Abstract

This thesis consists of six articles and their summaries. The main theme of the thesis is to demonstrate the usefulness of lattice theory in developing another branch of mathematics, namely number theory.

In this thesis we present formulae for the properties of meet and join matrices, which are lattice-theoretic concepts defined as follows. Let $(P, \leq, \wedge, \vee)$ be a lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and let $f : P \to \mathbb{C}$ be a function. Then the meet matrix $(S)_f$ and the join matrix $[S]_f$ are defined by $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$.

The concepts of meet and join matrices have their origins in number theory. It is well known that $(\mathbb{Z}_+, |, \text{gcd}, \text{lcm})$ is a lattice, where $|$ is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. In this lattice S is a finite set of positive integers, $f : \mathbb{Z}_+ \to \mathbb{C}$ is an arithmetical function and the meet and join matrices are called the GCD and LCM matrices, which are defined as $((S)_f)_{ij} = f(\text{gcd}(x_i, x_j))$ and $([S]_f)_{ij} = f(\text{lcm}(x_i, x_j))$.

In the first article some new bounds for the determinant of meet matrices and formulae for the inverse of meet matrices are given.

In the second article we treat join matrices by interpreting them as meet matrices in the dual lattice. We present a method on how to translate all the results found for meet matrices into join matrices. Thus some new bounds for the determinant of join matrices and formulae for the inverse of join matrices are given. Further, by assuming that f is a semi-multiplicative function, we obtain formulae for meet matrices and join matrices in more complicated cases.

In the third article we present a solution for the following drawback. So far, in applicable formulae for the determinant and the inverse of meet matrices the set S is usually assumed to be meet-closed (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$). We extend these formulae to hold for all sets S by providing

calculation formulae for $\det(S)_f$ and $(S)_f^{-1}$ in terms of meet-closed subsets and supersets of S. Applications and the effectivity of the method are also considered. Naturally, by using the ideas of the second article the results can also be translated into join matrices.

In the fourth article we show that the following structure, which appears frequently in number theory, generalizes the concept of meet matrix. If every principal order ideal of P is finite, then we define the function Ψ on $P \times P$ by $\Psi(x, y) = \sum_{0 \leq z \leq x \wedge y} f(0, z)g(z, x)h(z, y)$, where f, g, h are incidence functions of P and $0 = \min P$. Thus we present formulae for the determinant and the inverse of the generalized meet matrix $[\Psi(x_i, x_j)]$. From these formulae we obtain some new formulae for meet matrices. Similar generalizations can also be found for join matrices.

In the fifth article we present new formulae for the Möbius function μ_S of S in terms of the Möbius function μ of P and thus obtain new inverse formulae for meet and join matrices. The new formulae for μ_S also explain lattice-theoretically their counterparts in number theory. In the fifth article we also propose a lattice-theoretic solution to the following drawback in number theory. Define the unitary divisibility relation $||$ on $\mathbb{Z}_+$ by $x \,||\, y \Leftrightarrow (x \mid y \text{ and } \gcd(x, y/x) = 1)$. Then the greatest common unitary divisor (gcud) of integers always exists, but defining the least common unitary multiple (lcum) conventionally fails. We introduce a new lattice $(\mathbb{Z}_+^\infty, ||, \text{gcud}, \text{lcum})$ in which we may define properly the GCUD matrix $(S)_f^{**}$ and the LCUM matrix $[S]_f^{**}$ as $((S)_f^{**})_{ij} = f(\text{gcud}(x_i, x_j))$ and $([S]_f^{**})_{ij} = f(\text{lcum}(x_i, x_j))$ respectively.

In the sixth article we characterize the matrix divisibility of the join matrix by the meet matrix in the ring $\mathbb{Z}^{n \times n}$ (i.e. when $[S]_f = M(S)_f$ for some $M \in \mathbb{Z}^{n \times n}$) in terms of the usual divisibility in $\mathbb{Z}$. We show that it is the lattice-theoretic structure of S that mainly determines whether or not $(S)_f$ divides $[S]_f$. To demonstrate this we present two algorithms for constructing sets S such that $(S)_f$ divides $[S]_f$. We also classify all sets with at most 5 elements possessing this property.

Clearly the results in these six articles obtained for meet and join matrices are lattice-theoretic generalizations for the results of GCD and LCM matrices and GCUD and LCUM matrices in number theory.

Key Words and Phrases: Meet matrix, meet-semilattice, join matrix, join-semilattice, lattice, incidence function, semi-multiplicative function, determinant, inverse matrix, Gram-Schmidt orthogonalization process, partitioned matrix, divisibility of matrices, GCD matrix, LCM matrix, GCUD matrix, LCUM matrix, Smith's determinant.

Preface

I want to express my sincere thanks to Professor Pentti Haukkanen for introducing me to the subject and being of great help and support throughout my study. He has done exemplary recearch within the subject. I am also grateful to Mrs. Virginia Mattila for checking the English of the thesis. Pentti Haukkanen has also assisted me in English.

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Tampere, May 2006
Ismo Korkee

List of original articles

The thesis comprises the following six original articles:

[A] I. Korkee and P. Haukkanen, Bounds for determinants of meet matrices associated with incidence functions, *Linear Algebra and its Applications* 329: 77–88 (2001).

[B] I. Korkee and P. Haukkanen, On meet and join matrices associated with incidence functions, *Linear Algebra and its Applications* 372: 127–153 (2003).

[C] I. Korkee and P. Haukkanen, On meet matrices with respect to reduced, extended and exchanged sets, *JP Journal of Algebra, Number Theory and Applications* 4: 559–575 (2004).

[D] I. Korkee and P. Haukkanen, On a general form of meet matrices associated with incidence functions, *Linear and Multilinear Algebra* 53: 309–321 (2005).

[E] I. Korkee, A note on meet and join matrices and their special cases GCUD and LCUM matrices, *International Journal of Pure and Applied Mathematics*, 1–11, in print.

[F] I. Korkee and P. Haukkanen, On the divisibility of meet and join matrices, manuscript submitted, 1–16 (2005).

1 Introduction

1.1 On roots of the subject in number theory

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n positive integers with $x_1 < x_2 < \cdots < x_n$ and let f be an arithmetical function (i.e., a complex-valued function on $\mathbb{Z}_+$). Let (x_i, x_j) denote the greatest common divisor (gcd) of x_i and x_j and define the $n \times n$ matrix $(S)_f$ by $((S)_f)_{ij} = f((x_i, x_j))$. We refer to $(S)_f$ as the GCD matrix on S with respect to f. The set S is said to be gcd-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$. The set S is said to be factor-closed if it contains every positive divisor of each $x_i \in S$. Clearly, a factor-closed set is always gcd-closed but the converse does not hold.

Let $[x_i, x_j]$ denote the least common multiple (lcm) of x_i and x_j and define the $n \times n$ matrix $[S]_f$ by $([S]_f)_{ij} = f([x_i, x_j])$. We refer to $[S]_f$ as the LCM matrix on S with respect to f. The set S is said to be lcm-closed if $[x_i, x_j] \in S$ whenever $x_i, x_j \in S$. The set S is said to be multiple-closed if it is lcm-closed and $x_i \mid d \mid x_n \Rightarrow d \in S$ holds for any $x_i \in S$. Here $\mid$ stands for the usual divisibility relation of integers.

In 1876 H.J.S. Smith [34] stated in his famous determinant formula that if S is factor-closed, then

$$\det(S)_f = \prod_{k=1}^{n} (f * \mu)(x_k), \tag{1.1}$$

where μ is the number-theoretic Möbius function and $*$ is the Dirichlet convolution of arithmetical functions, see [4, 31]. Smith also calculated $\det[S]_f$ in a more special case [34, (3.)].

More than a hundred years after Smith, Bourque and Ligh [9] calculated $\det(S)_f$ and $(S)_f^{-1}$ on gcd-closed sets. They [11] also calculated $\det[S]_f$ and $[S]_f^{-1}$ on factor-closed sets. Haukkanen and Sillanpää [16] calculated $\det(S)_f$ and $\det[S]_f$ on gcd-closed and lcm-closed sets.

Hong [18] provided a lower bound for $\det(S)_f$. He defined the class $\mathcal{C}_S$ of arithmetical functions by $\mathcal{C}_S = \{f : (x \in S, d \mid x) \Rightarrow (f * \mu)(d) > 0\}$ and showed that if $f \in \mathcal{C}_S$, then

$$\det(S)_f \geq \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ t < k}} (f * \mu)(d) \tag{1.2}$$

and the equality holds if and only if S is gcd-closed. Hong [18] also obtained an upper bound for $\det(S)_f$ and bounds for $\det[S]_f$.

Bourque and Ligh [10] generalized the concept of GCD matrix by defining the arithmetical function Ψ of two variables as

$$\Psi(t,r) = \sum_{d|(t,r)} f(d)g(t/d)h(r/d), \tag{1.3}$$

where f, g, h are usual arithmetical functions. They calculated the determinant and the inverse of the generalized GCD matrix $[\Psi(x_i, x_j)]$ when S is a factor-closed set. Hong [19] presented a formula for $\det[\Psi(x_i, x_j)]$ on gcd-closed set S, where either $g \in \mathcal{L}_S$ or $h \in \mathcal{L}_S$. Here the class $\mathcal{L}_S$ of arithmetical functions is defined by $\mathcal{L}_S = \{g : (x_i, x_j \in S,\ x \mid x_i \mid x_j) \Rightarrow g(x_i/x) = g(x_j/x)\}$.

There is a large number of generalizations and analogues of GCD and LCM matrices in the literature. For a general account, see [17]. In this thesis we also consider briefly GCUD and LCUM matrices, which are called the unitary analogues of GCD and LCM matrices, see [12, 16]. Since we will use lattice-theoretic methods to solve a problem concerning GCUD and LCUM matrices, we return to the subject again in Subsection 1.4.

1.2 On lattice-theoretic preliminaries of the earlier study

Let $(P, \leq)$ be a partially ordered set. We call P a meet-semilattice if for any $x, y \in P$ there exists a unique $z \in P$ such that

(i) $z \leq x$ and $z \leq y$,

(ii) $(w \in P,\ w \leq x$ and $w \leq y) \ \Rightarrow\ w \leq z$.

In such a case z is called the meet of x and y and is denoted by $x \wedge y$. For each $x \in P$ the principal order ideal $\downarrow x$ is defined by $\downarrow x = \{y \in P \mid y \leq x\}$. See [7].

Let $(P, \leq, \wedge)$ be a meet-semilattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P such that $x_i < x_j \Rightarrow i < j$ and let f be a complex-valued function on P. Then the $n \times n$ matrix $(S)_f$, where $((S)_f)_{ij} = f(x_i \wedge x_j)$, is called the meet matrix on S with respect to f. We say that S is lower-closed if $(x_i \in S,\ y \in P,\ y \leq x_i) \Rightarrow y \in S$. We say that S is meet-closed if $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$. It is clear that a lower-closed set is always meet-closed but the converse does not hold. We say that S is an a-set if $x_i \wedge x_j = a$ for all $i \neq j$.

Haukkanen [13] introduced meet matrices and obtained a formula for $\det(S)_f$ on meet-closed sets and a formula for $(S)_f^{-1}$ on lower-closed sets, see also [29, 36]. Since $(\mathbb{Z}_+, |, \gcd)$ is a meet-semilattice, the results obtained for meet matrices are lattice-theoretic generalizations for the results of GCD matrices.

1.3 On lattice-theoretic preliminaries of the current study

In this subsection we write preliminaries only to such an extent that are needed to understand the summaries of the articles in Subsections 2.1–2.6. Detailed assumptions, e.g., concerning P, are presented in the articles.

Let $(P, \leq)$ be a partially ordered set. We call P a join-semilattice if for any $x, y \in P$ there exists a unique $z \in P$ such that

(i) $x \leq z$ and $y \leq z$,

(ii) $(w \in P,\ x \leq w \text{ and } y \leq w) \ \Rightarrow\ z \leq w.$

In such a case z is called the join of x and y and is denoted by $x \vee y$. A join-semilattice, which is also a meet-semilattice, is called a lattice.

Let $(P, \leq, \wedge)$ be a meet-semilattice and define the partial order $\sqsubseteq$ on P by $x \sqsubseteq y \Leftrightarrow y \leq x$. Then for any $x, y \in P$ there exists a unique $z = x \sqcup y = x \wedge y$ that satisfies (i) and (ii) above for $\sqsubseteq$. Thus $(P, \sqsubseteq, \sqcup)$ is a join-semilattice and it is said to be the dual of $(P, \leq, \wedge)$, see [7].

Let $(P, \leq, \wedge, \vee)$ be a finite lattice. Then P has the least and the greatest element, which we denote by 0 and 1 respectively. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P such that $x_i < x_j \Rightarrow i < j$ and let f be a complex-valued function on P. Then the $n \times n$ matrix $[S]_f$, where $([S]_f)_{ij} = f(x_i \vee x_j)$, is called the join matrix on S with respect to f. We say that S is upper-closed if $(x_i \in S,\, y \in P,\, x_i \leq y) \ \Rightarrow\ y \in S$. We say that S is join-closed if $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$. An upper-closed set is always join-closed but the converse does not hold.

We say that f is a semi-multiplicative function on P if $f(x \wedge y)f(x \vee y) = f(x)f(y)$ for all $x, y \in P$. We adapt this concept from number theory, see [30, p. 49], and many important arithmetical functions possess this property. For example, multiplicative and completely multiplicative functions are all semi-multiplicative, see [28, 32].

Let g be a complex-valued function on $P \times P$ such that $g(x,y) = 0$ whenever $x \not\leq y$. Then we say that g is an incidence function of P. If g and h are incidence functions of P, their sum $g+h$ is defined by $(g+h)(x,y) = g(x,y) + h(x,y)$ and their convolution $g * h$ is defined by $(g*h)(x,y) = \sum_{x \leq z \leq y} g(x,z)h(z,y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the unity δ is defined by $\delta(x,y) = 1$ if $x = y$, and $\delta(x,y) = 0$ otherwise. The zeta function ζ of P is defined by $\zeta(x,y) = 1$ if $x \leq y$, and $\zeta(x,y) = 0$ otherwise. The Möbius function μ of P is the inverse of ζ under convolution. The inverse of f (if it exists) is denoted by f^{-1}. We denote the restriction of an incidence function f on $S \times S$ by f_S and we write $f_S^{-1} = (f_S)^{-1}$ if it exists. We denote the zeta function of S by ζ_S and let $\mu_S = \zeta_S^{-1} = (\zeta_S)^{-1}$. For incidence functions, see [28, p. 294–296].

Since the set S is finite, we can treat a suitable finite sublattice of the divisor lattice $(\mathbb{Z}_+, |, \gcd, \operatorname{lcm})$. Thus all the results for meet and join matrices found in these six articles are lattice-theoretic generalizations for the results of GCD and LCM matrices. This is also true for GCUD and LCUM matrices, and we are now in a position to discuss this subject more precisely.

1.4 On a lattice-theoretic approach to a special number-theoretic concept

Define the unitary divisibility relation $||$ on $\mathbb{Z}_+$ by the formula $x \,||\, y \Leftrightarrow (x \mid y$ and $(x, y/x) = 1)$, where $|$ is the usual divisibility relation on $\mathbb{Z}_+$. The greatest common unitary divisor (gcud) of positive integers x and y always exists and is denoted by $(x,y)^{**}$, see [12]. The $n \times n$ matrix $(S)_f^{**}$, where $((S)_f^{**})_{ij} = f((x_i, x_j)^{**})$, is called the GCUD matrix on S with respect to f. For properties of GCUD matrices, see Haukkanen and Sillanpää [16].

Defining the least common unitary multiple (lcum) conventionally fails. For example, there does not exist any positive integer m such that $2 \,||\, m$ and $4 \,||\, m$. Hansen and Swanson [12] attempted to overcome this difficulty by defining the lcum of x and y as $xy/(x,y)^{**}$. Although this definition is questionable, the study of LCUM matrices (i.e., the $n \times n$ matrices $[S]_f^{**}$, where $([S]_f^{**})_{ij} = f(\operatorname{lcum}(x_i, x_j))$ in the literature has hitherto been based on it.

Our approach is as follows. Let $\mathbb{Z}_+^\infty = \mathbb{Z}_+ \cup \{\infty\}$ and define $x \,||\, \infty$ for all $x \in \mathbb{Z}_+^\infty$. Then the greatest common unitary divisor $(x,y)^{**}$ of integers coincides with that given in [12] and in addition $(x,\infty)^{**} = (\infty, x)^{**} = x$ for all $x \in \mathbb{Z}_+^\infty$. Further, then for each $x, y \in \mathbb{Z}_+^\infty$ there exists a unique element $z \in \mathbb{Z}_+^\infty$ such that (i) $x \mid\mid z$, $y \mid\mid z$ and (ii) $(x \mid\mid w, y \mid\mid w) \Rightarrow z \mid\mid w$. The element z may be considered as the natural least common unitary multiple $[x, y]^{**}$ of $\mathbb{Z}_+^\infty$, and we may define the LCUM matrix $[S]_f^{**}$ on S with respect to f as $([S]_f^{**})_{ij} = f([x_i, x_j]^{**})$. Thus we can treat GCUD and LCUM matrices in the lattice $(\mathbb{Z}_+^\infty, ||, \mathrm{gcud}, \mathrm{lcum})$.

Let S be a finite subset of $\mathbb{Z}_+^\infty$. The set S is said to be gcud-closed if $(x_i, x_j)^{**} \in S$ whenever $x_i, x_j \in S$. The set S is said to be ud-closed (unitary divisor closed) if it contains every unitary divisor of x_i for any $x_i \in S$. Analogously, the set S is said to be lcum-closed if $[x_i, x_j]^{**} \in S$ whenever $x_i, x_j \in S$. The set S is said to be um-closed (unitary multiple closed) if it is lcum-closed and $x_i \mid\mid d \mid\mid x_n \Rightarrow d \in S$ holds for any $x_i \in S$. It is clear that ud-closed sets are always gcud-closed but the converse does not hold, and dually, um-closed sets are always lcum-closed but the converse does not hold.

Note that since S is finite, we can treat a suitable finite sublattice of $(\mathbb{Z}_+^\infty, ||, \mathrm{gcud}, \mathrm{lcum})$. Thus the results for meet and join matrices found in these six articles are lattice-theoretic generalizations for the results of GCUD and LCUM matrices.

2 Summaries of articles

2.1 Bounds for determinants of meet matrices associated with incidence functions [A]

Haukkanen [13] previously introduced meet matrices and gave abstract generalizations for the formulae of Bourque and Ligh [9, 11] by providing a formula for $\det(S)_f$ on meet-closed sets and a formula for $(S)_f^{-1}$ on lower-closed sets.

This article continues the work of Haukkanen by generalizing the formulae of Hong [18]. That is, by using the Gram-Schmidt orthogonalization process [25, p. 15] and incidence functions as our methods we obtain a lower bound and an upper bound for $\det(S)_f$ whenever $f \in \mathcal{C}_S$. Here the class $\mathcal{C}_S$ is a lattice-theoretic generalization of the one introduced by Hong in (1.2).

We also provide a new formula for $(S)_f^{-1}$ on meet-closed sets. Note that the sufficient condition of invertibility in the article can easily be written as a necessary and sufficient condition.

2.2 On meet and join matrices associated with incidence functions [B]

In this article we introduce the concept of join matrices, which has not previously been studied in the literature. We consider the dual of the lattice P and interpret $[S]_f$ as the dual of $(S)_f$.

We present a method for how to translate all the results found for meet matrices into join matrices. That is, we provide the dual formulae for all the formulae presented in [A]. Thus we offer a formula for $\det[S]_f$ on join-closed sets, a lower bound and an upper bound for $\det[S]_f$ whenever $f \in \mathcal{D}_S$, and a formula for $[S]_f^{-1}$ on join-closed sets. Here the class $\mathcal{D}_S$ is the dual of $\mathcal{C}_S$.

By assuming that f is a semi-multiplicative function, we obtain formulae similar to those presented in [A] and at the beginning of [B] but in more complicated cases. Thus we present bounds for $\det(S)_f$ and formulae for $(S)_f^{-1}$ on join-closed sets and also bounds for $\det[S]_f$ and formulae for $[S]_f^{-1}$ on meet-closed sets.

Finally, we turn to consider the usual number-theoretic matrices and give many corollaries of the general case. Since GCD matrices have been examined extensively, many of these results have previously been found. The features of LCM matrices are less known, since – as we note in [B, Section 6] – the Dirichlet convolution of arithmetical functions is not always available. Thus we introduce many new results concerning LCM matrices. One new and remarkable result is a formula for $\det[S]_f$ on an lcm-closed set S without any restrictions on f. Further, a new formula for $[S]_f^{-1}$ on an lcm-closed set S is given.

2.3 On meet matrices with respect to reduced, extended and exchanged sets [C]

In this article we present a solution for the following drawback. So far, in applicable formulae for the determinant and the inverse of meet matrices the set S is usually assumed to be a meet-closed set. In addition, a formula for $\det(S)_f$ on a-sets is given in [13, Corollary of Theorem 3] and an application of the Cauchy-Binet formula [13, Theorem 3] gives $\det(S)_f$ for all S, but the calculation is ineffective. Thus, for the present, row-reduction is usually the only way to calculate $\det(S)_f$ and $(S)_f^{-1}$ for fixed S and f.

In this article we propose calculation formulae for $\det(S)_f$ and $(S)_f^{-1}$ on wider classes of sets than the class of meet-closed sets. By assuming that X is a proper meet-closed subset of S and using partitioned matrices [37] as our method, we can utilize $\det(X)_f$ and $(X)_f^{-1}$ in calculating $\det(S)_f$ and $(S)_f^{-1}$. Analogously, if Y is a proper meet-closed superset of S, we can utilize $\det(Y)_f$ and $(Y)_f^{-1}$ in calculating $\det(S)_f$ and $(S)_f^{-1}$. We also compare the effectiveness of our methods with the effectiveness of row-reduction.

As applications we first combine our two methods for calculating $\det(S)_f$ and $(S)_f^{-1}$ by replacing the "difficult" elements of S with "easy" ones. Second, we obtain known formulae for $\det(S)_f$ and new formulae for $(S)_f^{-1}$, where S is an a-set. Next we give new formulae for the determinant and the inverse of the meet matrix $(X,Y)_f$ on two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, where $((X,Y)_f)_{ij} = f(x_i \wedge y_j)$. The results of this article are new in the divisor lattice. The methods of this article are also appropriate for join matrices.

2.4 On a general form of meet matrices associated with incidence functions [D]

In this article we show that the following structure, which appears frequently in number theory, generalizes the concept of meet matrix. If we assume that every principal order ideal of P is finite, then we define the function Ψ on $P \times P$ by $\Psi(x, y) = \sum_{0 \le z \le x \wedge y} f(0, z)g(z, x)h(z, y)$, where f, g, h are incidence functions of P and 0 is the least element of P. Thus we present formulae for the generalized meet matrix $[\Psi(x_i, x_j)]$. That is, we obtain a formula for $\det[\Psi(x_i, x_j)]$ on meet-closed sets, which generalizes the formula [19, Theorem 1] found by Hong for arithmetical functions. In addition, we derive a formula for the inverse of $[\Psi(x_i, x_j)]$ on meet-closed sets and two inverse formulae on lower-closed set S. These formulae have not previously been presented in the literature for incidence functions. In the setting of arithmetical functions the other inverse formula on lower-closed sets is known in the literature [10, Theorem 1 (ii)].

We also found a similar generalization for join matrices in [27] but we do not include these results in the thesis.

Applying our results to meet matrices we obtain formulae for the determinants of meet matrices, which are the same as those given in [13]. We also obtain a known (presented in [A]) and some new formulae for the inverse of meet matrices. The new results for meet matrices are also new in the setting of GCD matrices.

We also apply our general results to the matrix $[C(x_i, x_j)]$, where $C(m, n)$ is Ramanujan's sum [5], and obtain an inverse formula on gcd-closed set S, which has not hitherto been presented in the literature. As special cases we also obtain some known results.

2.5 A note on meet and join matrices and their special cases GCUD and LCUM matrices [E]

In this article we first discuss an important tool, the Möbius incidence function μ of P, which is a generalization of the usual number-theoretic Möbius function. We present new formulae for the Möbius incidence function μ_S of S in terms of μ. Since μ_S usually occur in the inverse formulae of meet and join matrices, each of such a formula in the literature get a new form. The formulae of μ_S also explain lattice-theoretically their counterparts in number theory, which are rather complicated.

In this article [E] we also propose a lattice-theoretic approach to the difficulties concerning LCUM matrices mentioned in Subsection 1.4. Since the systems $(\mathbb{Z}_+, |, \mathrm{gcd})$ and $(\mathbb{Z}_+, ||, \mathrm{gcud})$ are both meet-semilattices, then GCD and GCUD matrices belong to the class of meet matrices. Similarly, $(\mathbb{Z}_+, |, \mathrm{lcm})$ is a join-semilattice and thus LCM matrices belong to the class of join matrices. The basic difficulty concerning LCUM matrices is that $(\mathbb{Z}_+, ||, \mathrm{lcum})$ is not a join-semilattice. Therefore, in this article we solve the problem by considering the extended set $\mathbb{Z}_+^\infty = \mathbb{Z}_+ \cup \{\infty\}$, where ∞ is defined to be a unitary multiple of all positive integers. Then lcum always exists on $\mathbb{Z}_+^\infty$ and $(\mathbb{Z}_+^\infty, ||, \mathrm{gcud}, \mathrm{lcum})$ is a lattice. We present a known determinant formula and a new inverse formula for GCUD matrices on gcud-closed sets S and new determinant and inverse formulae for LCUM matrices on lcum-closed sets S.

Note that our solution makes the theory technically correct, but has also some disadvantages. For example, we may consider the LCUM matrix $[S]_f^{**}$ with $f(m) = m$ only when $[S]_f^{**}$ does not contain ∞.

2.6 On the divisibility of meet and join matrices [F]

Bourque and Ligh [8, 11] were the first to study the divisibility of GCD and LCM matrices in the ring $\mathbb{Z}^{n\times n}$ (i.e. when $[S]_f = M(S)_f$ for some $M \in \mathbb{Z}^{n\times n}$). Hong [20, 21, 22] studied this subject extensively. See also Haukkanen and Korkee [15], which is not included in this thesis.

In this article [F] we study the subject of Bourque, Ligh and Hong on a more general level, in fact, we study the divisibility of meet and join matrices. We present a characterization for the matrix divisibility of the join matrix $[S]_f$ by the meet matrix $(S)_f$ in the ring $\mathbb{Z}^{n\times n}$ in terms of the usual divisibility in $\mathbb{Z}$, where S is a meet-closed set and f is an integer-valued function on P.

We show that it is the lattice-theoretic structure of S that mainly determines whether or not $(S)_f$ divides $[S]_f$. To demonstrate this we present two inductive methods for constructing meet-closed sets S such that $(S)_f$ divides $[S]_f$ under certain conditions on f. For example, all chains and x_1-sets can be constructed using our methods, and thus they possess this divisibility property. We also classify all sets with at most 5 elements possessing this property. Finally, we show what new this study contributes to the divisor lattice.

3 Author's contribution to the articles

The contents of this thesis were accomplished in collaboration between the author of the thesis and the co-author P. Haukkanen. In this section we endeavour to distinguish the respective contributions of the two authors.

Generally, the proportion of the contribution of each co-author is difficult or even impossible to estimate exactly, since the collaboration between author and co-author may appear in many levels. Further, most of the work done for the article is invisible. Surely something can still be said about the matter.

Since P. Haukkanen is the supervisor of the present study, his role has been to guide, advise, conduct, comment the results, provide ideas and references to other works, etc. In all articles he was also the proofreader and has substantially improved and clarified the expression in manuscripts made by the author.

In spite of the significant role of the co-author above, the order of the names of the author and the co-author in the articles still reflects the share of contribution – it may be estimated that more than half of the work was carried out by the main author, I. Korkee. He is mainly responsible for the technical part of the manuscript processes. That is, he explored the subject, achieved new results and carried out their proofs, wrote down and organized the contents of the articles, etc.

In articles [A] and [D] the supervisor proposed that the author should generalize the particular number-theoretic concepts into lattice theory. The main article in this thesis is [B]. Here the supervisor proposed the theme but it was an open question how it should be carried out. In articles [C] and [E] the role of the co-author is smaller than in the others. In [E] his assistance consists mainly on proofreading. In article [F] the collaboration between the author and the co-author was different from that for the earlier articles. In this article the author and the co-author simultaneously explored the same problem and the co-author participated more in the technical part of the manuscript process.

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