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# An annotated bibliography on the thickness, outerthickness, and arboricity of a graph

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## 1 Introduction

This bibliography introduces literature on graph thickness, outerthickness and arboricity. In addition to the pointers to the literature we also give some conjectures concerning known open problems on the area.

The bibliography given is most likely incomplete. The authors welcome supplementing information by e-mail (`tp@cs.uta.fi`).

## 2 Thickness

The following conjecture was given by Harary [26]:

> Prove or disprove the following conjecture: For any graph $G$ with 9 points, $G$ or its complementary graph $\overline{G}$ is nonplanar.

The problem is the same as determining whether $K_9$ is *biplanar* or not, that is, a union of two planar graphs. The problem was solved independently by Battle et al. [6] and Tutte [51] by constructing all subgraphs for $K_9$. They showed that $K_9$ is not biplanar. Tutte [52] generalized the problem by defining the concept of the *thickness* of a graph.

**Definition 2.1.** *The graph-theoretical thickness (thickness, for short) of a graph, denoted by $\Theta(G)$, is the minimum number of planar subgraphs into which the graph can be decomposed.*

The thickness of a planar graph is 1 and the thickness of a nonplanar graph is at least 2. Thickness has applications, for example, in VLSI (Very Large Scale Integration) design [1] and network design [48].

It was long an open question whether $\Theta(K_{16}) = 3$ or 4. Harary offered 10 pounds to anyone who could compute $\Theta(K_{16})$. Finally a professor of French literature, Jean Mayer [42], won the prize by showing that $\Theta(K_{16}) = 3$.

The NP-status of thickness was solved by Mansfield.

**Theorem 2.2 ([40]).** *Determining the thickness of a graph is* NP*-complete.*

The only non-trivial graph classes with known thicknesses are the complete graphs, complete bipartite graphs, and hypercubes. The optimal solution for the thickness of complete graphs $K_n$ was given for almost all values of $n$ by Beineke and Harary [12]. A decade later Alekseev and Gonchakov [3], and independently Vasak [53], solved the remaining cases.

**Theorem 2.3 ([3, 12, 53]).** *For complete graphs,* $\Theta(K_n) = \lfloor \frac{n+7}{6} \rfloor$*, except that* $\Theta(K_9) = \Theta(K_{10}) = 3$.

See Figure 1 for a decomposition of $K_9$ into three planar subgraphs.

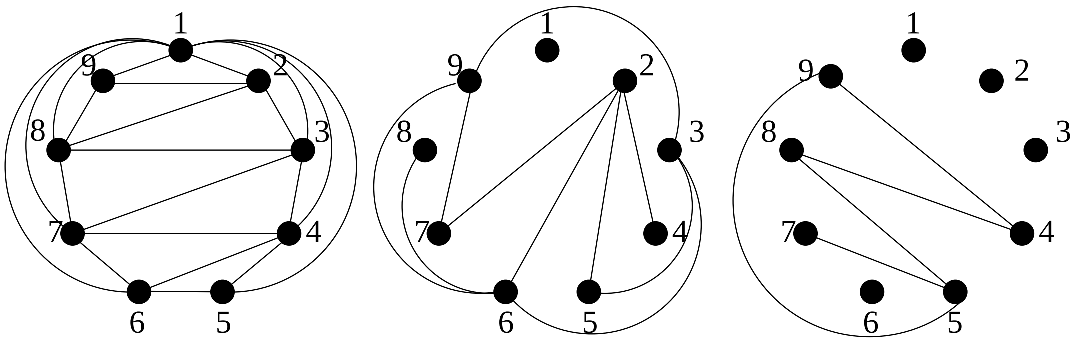


Figure 1: A minimum planar decomposition of $K_9$.

For complete bipartite graphs $K_{m,n}$, thickness is solved for almost all values of $m$ and $n$.

**Theorem 2.4 ([13]).** *For complete bipartite graphs,* $\Theta(K_{m,n}) = \lceil \frac{mn}{2(m+n-2)} \rceil$*, except possibly when $m$ and $n$ are odd, and there exists an integer $k$ satisfying* $n = \lfloor \frac{2k(n-2)}{n-2k} \rfloor$.

If $m = n$, Theorem 2.4 has the following shorter form.

**Corollary 2.5.** $\Theta(K_{n,n}) = \lfloor \frac{n+5}{4} \rfloor$.

The thickness of hypercubes (an $n$-cube is denoted by $Q_n$) was determined by Kleinert [37].

**Theorem 2.6 ([37]).** $\Theta(Q_n) = \lceil \frac{n+1}{4} \rceil$.

Next we give two lower bounds for thickness, see Beineke et al. [13] for references concerning their origin. The first lower bound is a direct application of Euler's polyhedron formula.

**Theorem 2.7.** *Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Then $\Theta(G) \geq \lceil \frac{m}{3n-6} \rceil$.*

If a graph does not contain any triangles, as it is for bipartite graphs, a tighter lower bound can be derived.

**Theorem 2.8.** *Let $G = (V, E)$ be a graph with $|V| = n$, $|E| = m$ and with no triangles. Then $\Theta(G) \geq \lceil \frac{m}{2n-4} \rceil$.*

The lower bounds of Theorems 2.7 and 2.8 are also the exact values for the thickness of almost all complete and complete bipartite graphs.

Wessel [55] gave lower and upper bounds for the thickness of a graph as a function of the minimum and maximum degree. The upper bound was independently given also by Halton [25].

**Theorem 2.9 ([25, 55]).** *Let $G$ be a graph with minimum degree $\delta$ and maximum degree $\Delta$. Then it holds that $\lceil \frac{\delta+1}{4} \rceil \leq \Theta(G) \leq \lceil \frac{\Delta}{2} \rceil$.*

Halton conjectured a stronger upper bound $\Theta(G) \leq \lceil \frac{\Delta+2}{4} \rceil$. Sýkora et al. [50] gave a counterexample by constructing a class of regular graphs of degree $d$ with thickness $\lceil d/2 \rceil$. The construction shows that the upper bound of Theorem 2.9 is tight.

Dean et al. [22] gave an upper bound as a function of the number of edges.

**Theorem 2.10 ([22]).** *Let $G$ be a graph with $m$ edges, then it holds that $\Theta(G) \leq \lfloor \sqrt{m/3} + 3/2 \rfloor$.*

Czabarka et al. [19] have presented a bound for the thickness of a graph by using the crossing number of the graph in question.

The thickness of degree-constrained graphs is studied by Bose and Prabhu [14], and results for the thickness of random graphs are given by Cooper [18]. Mutzel et al. [33] have shown that the thickness of the class of graphs without $K_5$-minors is at most two.

The *genus* of a graph is the minimum number of handles that must be added to the plane to embed the graph without any crossings. Asano has studied the thickness of graphs with genus at most 2 [4, 5]. Thickness results for other surfaces are reported by White and Beineke [56] and Ringel [49].

# 3 Outerthickness

Instead of decomposing the graph into planar subgraphs, outerthickness seeks a decomposition into outerplanar subgraphs.

**Definition 3.1.** *The outerthickness of a graph, denoted by $\Theta_o(G)$, is the minimum number of outerplanar subgraphs into which the graph can be decomposed.*

Outerthickness seems to be studied first in Geller's unpublished manuscript (see [27], pp. 108 and 245), where it was shown that $\Theta_o(K_7)$ is 3 by similar exhaustive search as in the case of the thickness of $K_9$. See Figure 2 for a decomposition of $K_7$ into three outerplanar subgraphs.

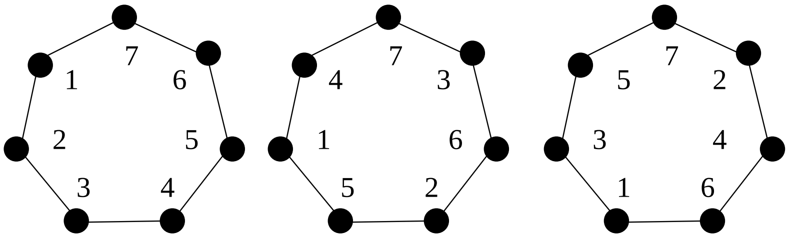


Figure 2: A minimum outerplanar decomposition of $K_7$.

The outerthickness of complete graphs was solved by Guy and Nowakowski.

**Theorem 3.2 ([59]).** *For complete graphs, $\Theta_o(K_n) = \lceil \frac{n+1}{4} \rceil$, except that $\Theta_o(K_7) = 3$.*

The same authors also gave optimal solutions for the outerthickness of complete bipartite graphs and hypercubes.

**Theorem 3.3 ([60]).** *For complete bipartite graphs with $m \leq n$, $\Theta_o(K_{m,n}) = \lceil \frac{mn}{2m+n-2} \rceil$.*

**Theorem 3.4 ([59]).** $\Theta_o(Q_n) = \lceil \frac{n+1}{3} \rceil$.

It is possible to apply Euler's polyhedron formula to derive lower bounds for outherthickness similarly as for the graph thickness.

**Theorem 3.5 ([58]).** *Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Then $\Theta_o(G) \geq \lceil \frac{m}{2n-3} \rceil$.*

**Theorem 3.6 ([58]).** *Let $G = (V, E)$ be a graph with $|V| = n$, $|E| = m$ and with no triangles. Then* $\Theta_o(G) \geq \lceil \frac{m}{3n/2-2} \rceil$.

The lower bounds of Theorems 3.5 and 3.6 are also the exact values for the outerthickness of complete graphs, complete bipartite graphs, and hypercubes.

The following theorem gives lower and upper bounds in the terms of minimum and maximum degree of a graph.

**Theorem 3.7 ([25, 55, 47]).** *For a graph with with minimum degree $\delta$ and maximum degree $\Delta$, it holds that* $\lceil \delta/4 \rceil \leq \Theta_o(G) \leq \lceil \Delta/2 \rceil$.

Since $\Theta_o(G) \geq \Theta(G)$ and the upper bound is tight for thickness [50], it follows that the upper bound is tight also for outerthickness.

Heath [61] has shown that a planar graph can be divided into two outerplanar graphs. Therefore, $\Theta_o(G) \leq 2\Theta(G)$.

# 4 Arboricity

As thickness is defined using planar graphs and outerthickness by using outerplanar graphs, it is natural to continue to tighten the definition by replacing outerplanar graphs by trees. This gives us the concept of arboricity. Hence, the *arboricity* of a graph, denoted by $\Upsilon(G)$, is the minimum number of trees whose union is $G$. Nash-Williams [81] gave the exact solution for arboricity

$$\Upsilon(G) = \max \left\lceil \frac{m_H}{n_H - 1} \right\rceil,$$

where the maximum is taken over all nontrivial subgraphs $H$ of $G$. The number of vertices and edges in $H$ are denoted by $n_H$ and $m_H$, respectively. Applying Nash-Williams' result, Dean et al. [22] showed that $\Upsilon(G) \leq \lceil \sqrt{m/2} \rceil$. This gives also a lower bound for outerthickness.

Trees can be further replaced by stars, caterpillars [74, 79, 77] or linear forests [72, 85]. (The bibliography concerning star, caterpillar, and linear arboricity is by no means complete.)

# 5 Conjectures

Computational experiments [47] have shown that Theorem 2.4 holds for all $m < 30$. For example, it was unknown if $\Theta(K_{17,21})$ is equal to 5 or 6 (the thickness of $K_{13,17}$ is at least 5 due to Euler's polyhedron formula and it cannot be more than $\Theta(K_{18,21}) = 6$ or $\Theta(K_{17,22}) = 6$). In general, the unknown values of $\Theta(K_{m,n})$ are quite rare, for an arbitrary $m$, there are fewer than $m/4$ unsolved cases [10].

**Conjecture 5.1.** *The claim of Theorem 2.4 holds for all complete bipartite graphs.*

Dean et al. [22] have conjectured a tighter upper bound for the thickness as a function of the number of edges in the graph.

**Conjecture 5.2 ([22]).** $\Theta(G) \leq \sqrt{m/16} + O(1)$ *for an arbitrary graph* $G$ *with* $m$ *edges.*

The complexity status of outerthickness is open, but since thickness and maximum planar subgraph problem are *NP*-complete, we conjecture that determining the outerthickness of a graph is also *NP*-complete.

**Conjecture 5.3.** *Determining the outerthickness of a graph is* NP*-complete.*

Dean et al. [22] gave an upper bound for thickness as a function of the number of edges (Theorem 2.10). If their proof technique is applied straightforward to outerplanar graphs, the bound $\lceil \sqrt{m/2} + 1/2 \rceil$ is obtained. The upper bound is of the right order, since the outerthickness of the complete graph with $n$ vertices is $O(n)$. On the other hand, since $\Theta_o(K_n)$ is approximately $\sqrt{m/8}$ and $\Theta_o(K_{n,n})$ is approximately $\sqrt{m/9}$, it seems that the constant is not the best possible. We conjecture the following upper bound for outerthickness.

**Conjecture 5.4.** $\Theta_o(G) \leq \sqrt{m/8} + O(1)$ *for an arbitrary graph* $G$ *with* $m$ *edges.*

Dean et al. [20] proposed an open problem related on bar k-visibility graphs.

**Conjecture 5.5 ([20]).** *Bar k-visibility graphs have thickness no greater than* $k + 1$.

# 6 Related problems

We can also consider other types of subgraphs whose union is the given graph. For an interested reader, we recommend an article by Dujmovic and Wood [23] for further references related to these subgraph classes.

The star arboricity of a graph $G$ is the minimum number of stars whose union is $G$. Similarly, the linear arboricity is the minimum number of linear forests.

In the *book thickness* of a graph, which is sometimes called the *pagenumber* or *stacknumber*, vertices are placed on a line (the spine) and edges are routed without intersections via half-planes (pages) having common boundary with the spine. Book thickness indicates the minimum number of needed pages.

*Geometric thickness* is the smallest number of layers such that the graph can be drawn in the plane with straight line edges and each edge assigned to a layer such that no two edges cross. *Geometric outerthickness*, *geometric arboricity* and *geometric star-aboricity* are defined analogously.

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