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Hermite-Hadamard-type inequalities for new different kinds of convex dominated functions

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Abstract. In this paper, several new convex dominated functions are established. Also, some new Hadamard type inequalities are obtained.

Key words. m-convex dominated function, Hermite-Hadamard inequality, (α, m)-convex function, r-convex function.

Farklı türden yeni baskın konveks fonksiyonlar için Hermite-Hadamard tipli eşitsizlikler

Özet. Bu makalede, birkaç yeni baskın fonksiyon sunulmuştur. Ayrıca, bazı yeni Hadamard tipli eşitsizlikler elde edilmiştir.

Anahtar kelimeler. m-konveks baskın fonksiyon, Hermite-Hadamard eşitsizliği, (α, m)-konveks fonksiyon, r-konveks fonksiyon.

1 Introduction

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard inequality.

In [7], Toader defined m-convexity as follows:

Definition 1.1 *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $(-f)$ is $m-$convex.

In [3], Dragomir proved the following theorem.

Theorem 1.1 *Let $f:[0,\infty)\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$ and $0\leq a<b$. If $f\in L_1[a,b]$, then the following inequality hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ &\leq \frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right]. \end{aligned} \tag{1.2}$$

In [4, 5], the authors connect together some disparate threads through a Hermite-Hadamard motive. The first of these threads is the unifying concept of a g-convex dominated function. Similarly, in [8], Kavurmacı et al. introduced the following class of functions and then proved a theorem for this class of functions related to (1.2).

Definition 1.2 *Let $g:[0,b]\to\mathbb{R}$ be a given m-convex function on the interval $[0,b]$. The real function $f:[0,b]\to\mathbb{R}$ is called $(g,m)-$convex dominated on $[0,b]$ if the following condition is satisfied:*

$$\begin{aligned} |\lambda f(x)+m(1-\lambda)f(y)-f(\lambda x+m(1-\lambda)y)| &\leq \lambda g(x)+m(1-\lambda)g(y) \\ &\quad -g(\lambda x+m(1-\lambda)y) \end{aligned}$$

for all $x,y\in[0,b]$, $\lambda\in[0,1]$ and $m\in[0,1]$.

Theorem 1.2 *Let $g:[0,\infty)\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$ and $0\leq a<b$. Then, $f:[0,\infty)\to\mathbb{R}$ is a (g,m)-convex dominated mapping. If $f\in L_1[a,b]$, then one has the inequalities:*

$$\left|\frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{b-a}\int_a^b \frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx-g\left(\frac{a+b}{2}\right),$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\right|$$

$$\leq \frac{1}{2}\left[\frac{g(a)+mg\left(\frac{a}{m}\right)}{2}+m\frac{g\left(\frac{b}{m}\right)+mg\left(\frac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx.$$

In [9], definition of (α,m)-convexity was introduced by Miheşan as follows:

Definition 1.3 *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $(\alpha,m)-$convex, where $(\alpha,m)\in[0,1]^2$, if we have*

$$f(tx+m(1-t)y)\leq t^\alpha f(x)+m(1-t^\alpha)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

Denote by $K_m^\alpha(b)$ the class of all (α,m)-convex functions on $[0,b]$ for which $f(0)\leq 0$. If we take $(\alpha,m)=\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$, it can be easily seen that (α,m)-convexity reduces to increasing: α-starshaped, starshaped, m-convex, convex and α-convex, respectively.

In [14], Set et al. proved the following Hadamard-type inequalities for (α,m)-convex functions.

Theorem 1.3 *Let $f:[0,\infty)\to\mathbb{R}$ be an (α,m)-convex function with $(\alpha,m)\in(0,1]^2$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]\cap L_1\left[\frac{a}{m},\frac{b}{m}\right]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\frac{f(x)+m(2^\alpha-1)f\left(\frac{x}{m}\right)}{2^\alpha}dx. \tag{1.3}$$

Theorem 1.4 *Let $f:[0,\infty)\to\mathbb{R}$ be an (α,m)-convex function with $(\alpha,m)\in(0,1]^2$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{1}{2}\left[\frac{f(a)+f(b)+m\alpha f\left(\frac{a}{m}\right)+m\alpha f\left(\frac{b}{m}\right)}{\alpha+1}\right]. \tag{1.4}$$

For the recent results based on the definition above, see papers [1,2,10,11,13].

In [12], the power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined by

$$M_r(x,y;\lambda)=\begin{cases}(\lambda x^r+(1-\lambda)y^r)^{\frac{1}{r}}, & r\neq 0,\\ x^\lambda y^{1-\lambda}, & r=0.\end{cases}$$

A positive function f is r-convex on $[a,b]$

$$f(\lambda x+(1-\lambda)y)\leq M_r(f(x),f(y);\lambda), \tag{1.5}$$

if for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.

The generalized logarithmic mean of order r of positive numbers x,y is defined by

$$L_r(x,y)=\begin{cases}\frac{r}{r+1}\frac{x^{r+1}-y^{r+1}}{x^r-y^r}, & r\neq 0,1, x\neq y,\\ \frac{x-y}{\ln x-\ln y}, & r=0, x\neq y,\\ xy\frac{\ln x-\ln y}{x-y}, & r=-1, x\neq y,\\ x, & x=y.\end{cases} \tag{1.6}$$

In [6], the following theorem was proved by Gill et al. for $r-$convex functions.

Theorem 1.5 *Suppose that f is a positive $r-$convex function on $[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq L_r(f(a),f(b)). \tag{1.7}$$

If f is a positive r-concave function, then the inequality is reversed.

In the following sections our main results are given: We establish several new convex dominated functions and then we obtain new Hadamard-type inequalities.

2 $(g-(\alpha,m))$-convex dominated functions

Definition 2.1 *Let $g:[0,b]\to\mathbb{R}$, $b>0$ be a given (α,m)-convex function on the interval $[0,b]$. The real function $f:[0,b]\to\mathbb{R}$ is called $(g-(\alpha,m))$-convex dominated on $[0,b]$ if the following condition is satisfied:*

$$\begin{aligned}|\lambda^\alpha f(x)+m(1-\lambda^\alpha)f(y)-f(\lambda x+m(1-\lambda)y)| &\leq \lambda^\alpha g(x)+m(1-\lambda^\alpha)g(y)\\ &\quad -g(\lambda x+m(1-\lambda)y)\end{aligned} \tag{2.1}$$

for all $x,y\in[0,b]$, $\lambda\in[0,1]$ and $(\alpha,m)\in[0,1]^2$.

The next simple characterization of (α,m)-convex dominated functions holds.

Lemma 2.1 *Let $g:[0,b]\to\mathbb{R}$ be an (α,m)-convex function on the interval $[0,b]$ and let the function $f:[0,b]\to\mathbb{R}$. The following statements are equivalent:*

(i) *f is $(g-(\alpha,m))$-convex dominated on $[0,b]$.*

(ii) *The mappings $g-f$ and $g+f$ are (α,m)-convex functions on $[0,b]$.*

(iii) *There exist two (α,m)-convex mappings h,k defined on $[0,b]$ such that*

$$f=\tfrac{1}{2}(h-k)\quad \textit{and}\quad g=\tfrac{1}{2}(h+k)\ .$$

Proof. Condition (2.1) is equivalent to

$$\begin{aligned} & g(\lambda x+m(1-\lambda)y)-\lambda^{\alpha}g(x)-m(1-\lambda^{\alpha})g(y)\\ \leq\ & \lambda^{\alpha}f(x)+m(1-\lambda^{\alpha})f(y)-f(\lambda x+m(1-\lambda)y)\\ \leq\ & \lambda^{\alpha}g(x)+m(1-\lambda^{\alpha})g(y)-g(\lambda x+m(1-\lambda)y)\end{aligned}$$

for all $x,y\in I$, $\lambda\in[0,1]$ and $(\alpha,m)\in[0,1]^2$. The two inequalities may be rearranged as

$$\begin{aligned}(g+f)(\lambda x+m(1-\lambda)y) &\leq \lambda^{\alpha}(g+f)(x)+m(1-\lambda^{\alpha})(g+f)(y),\\ (g-f)(\lambda x+m(1-\lambda)y) &\leq \lambda^{\alpha}(g-f)(x)+m(1-\lambda^{\alpha})(g-f)(y)\end{aligned}$$

which are equivalent to the (α,m)-convexity of $g+f$ and $g-f$, respectively.

We define the mappings f,g as $f=\frac{1}{2}(h-k)$ and $g=\frac{1}{2}(h+k)$. Then, if we sum and subtract f,g, respectively, we have $g+f=h$ and $g-f=k$. By condition *(ii)* of Lemma 2.1, the mappings $g-f$ and $g+f$ are (α,m)-convex on $[0,b]$, so h,k are (α,m)-convex mappings, too. ■

Theorem 2.2 *Let $g:[0,\infty)\to\mathbb{R}$ be an (α,m)-convex function with $(\alpha,m)\in[0,1]\times(0,1]$ and $0\leq a<b$. Then, $f:[0,\infty)\to\mathbb{R}$ is $(g-(\alpha,m))$-convex dominated mapping. Moreover, if $f\in L_1[a,b]\cap L_1\left[\frac{a}{m},\frac{b}{m}\right]$, then the inequality*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\frac{f(x)+m(2^{\alpha}-1)f\left(\frac{x}{m}\right)}{2^{\alpha}}dx-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{1}{b-a}\int_a^b\frac{g(x)+m(2^{\alpha}-1)g\left(\frac{x}{m}\right)}{2^{\alpha}}dx-g\left(\frac{a+b}{2}\right),\end{aligned}\tag{2.2}$$

and if $f \in L_1[a,b]$ then the inequality

$$\left|\frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{\alpha+1}+m\alpha\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{\alpha+1}\right]-\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\right|$$
$$\leq\frac{1}{2}\left[\frac{g(a)+mg\left(\frac{a}{m}\right)}{\alpha+1}+m\alpha\frac{g\left(\frac{b}{m}\right)+mg\left(\frac{b}{m^2}\right)}{\alpha+1}\right]-\frac{1}{b-a}\int_a^b\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx \tag{2.3}$$

holds.

Proof. By Definition 2.1 with $\lambda=\frac{1}{2}$, as the mapping f is $(g-(\alpha,m))$-convex dominated function, we have that

$$\left|\frac{f(x)+m(2^\alpha-1)f(y)}{2^\alpha}-f\left(\frac{x+my}{2}\right)\right|\leq\frac{g(x)+m(2^\alpha-1)g(y)}{2^\alpha}-g\left(\frac{x+my}{2}\right)$$

for all $x,y\in[0,\infty)$ and $(\alpha,m)\in(0,1]^2$. If we choose $x=ta+(1-t)b$, $y=(1-t)\frac{a}{m}+t\frac{b}{m}$, and $t\in[0,1]$, then we get

$$\left|\frac{f(ta+(1-t)b)+m(2^\alpha-1)f\left(\frac{(1-t)a+tb}{m}\right)}{2^\alpha}-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{g(ta+(1-t)b)+m(2^\alpha-1)g\left(\frac{(1-t)a+tb}{2}\right)}{2^\alpha}-g\left(\frac{a+b}{2}\right).$$

Integrating over t on $[0,1]$ we deduce that

$$\left|\frac{\int_0^1 f(ta+(1-t)b)\,dt+m(2^\alpha-1)\int_0^1 f\left(\frac{(1-t)a+tb}{m}\right)dt}{2^\alpha}-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{\int_0^1 g(ta+(1-t)b)\,dt+m(2^\alpha-1)\int_0^1 g\left(\frac{(1-t)a+tb}{m}\right)dt}{2^\alpha}-g\left(\frac{a+b}{2}\right)$$

and so inequality (2.2) is proved.

Since f is a $(g-(\alpha,m))$-convex dominated function, we have

$$\begin{aligned}|t^\alpha f(x)+m(1-t^\alpha)f(y)-f(tx+m(1-t)y)| &\leq t^\alpha g(x)+m(1-t^\alpha)g(y)\\ &\quad -g(tx+m(1-t)y)\end{aligned}$$

for all $x, y > 0$ which gives for $x = a$ and $y = \frac{b}{m}$ that

$$\left| t^{\alpha} f(a) + m(1-t^{\alpha}) f\left(\frac{b}{m}\right) - f\left(ta + m(1-t)\frac{b}{m}\right) \right|$$

$$\leq \quad t^{\alpha} g(a) + m(1-t^{\alpha}) g\left(\frac{b}{m}\right) - g\left(ta + m(1-t)\frac{b}{m}\right), \tag{2.4}$$

and for $x = \frac{a}{m}$, $y = \frac{b}{m^2}$ and then multiply with m

$$\left| mtf\left(\frac{a}{m}\right) + m^2(1-t) f\left(\frac{b}{m^2}\right) - mf\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right) \right|$$

$$\leq \quad mtg\left(\frac{a}{m}\right) + m^2(1-t) g\left(\frac{b}{m^2}\right) - mg\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right) \tag{2.5}$$

for all $t \in [0,1]$. By properties of modulus, if we add the inequalities in (2.4) and (2.5), we get

$$\left| t^{\alpha}\left[f(a) + mf\left(\frac{a}{m}\right)\right] + m(1-t^{\alpha})\left[f\left(\frac{b}{m}\right) + mf\left(\frac{b}{m^2}\right)\right] \right.$$

$$\left. - \left[f\left(ta + m(1-t)\frac{b}{m}\right) + mf\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right)\right] \right|$$

$$\leq \quad t^{\alpha}\left[g(a) + mg\left(\frac{a}{m}\right)\right] + m(1-t^{\alpha})\left[g\left(\frac{b}{m}\right) + mg\left(\frac{b}{m^2}\right)\right]$$

$$- \left[g\left(ta + m(1-t)\frac{b}{m}\right) + mg\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right)\right].$$

Thus, integrating over t on $[0,1]$, we obtain inequality (2.3). The proof is completed. ■

Remark 2.3 *If we choose $\alpha = 1$ in Theorem 2.2, we get two inequalities of Hermite-Hadamard-type for functions that are (g,m)-convex dominated in Theorem 1.2.*

Theorem 2.4 *Let $g : [0,\infty) \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in [0,1] \times (0,1]$ and $0 \leq a < b$. Then, $f : [0,\infty) \to \mathbb{R}$ is $(g - (\alpha, m))-$convex dominated mapping. Moreover, if $f \in L_1[a,b]$, then the following inequality holds:*

$$\left| \frac{1}{2}\left[\frac{f(a) + f(b) + m\alpha f\left(\frac{a}{m}\right) + m\alpha f\left(\frac{b}{m}\right)}{\alpha+1}\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\leq \quad \frac{1}{2}\left[\frac{g(a) + g(b) + m\alpha g\left(\frac{a}{m}\right) + m\alpha g\left(\frac{b}{m}\right)}{\alpha+1}\right] - \frac{1}{b-a}\int_a^b g(x)\,dx. \tag{2.6}$$

Proof. Since f is a $(g-(\alpha,m))$-convex dominated function, we have

$$\begin{aligned}\left|t^{\alpha}f(a)+m(1-t^{\alpha})f\left(\frac{b}{m}\right)-f\left(ta+m(1-t)\frac{b}{m}\right)\right| &\leq t^{\alpha}g(a)+m(1-t^{\alpha})g\left(\frac{b}{m}\right)\\ &\quad -g\left(ta+m(1-t)\frac{b}{m}\right),\end{aligned}$$

and

$$\begin{aligned}\left|t^{\alpha}f(b)+m(1-t^{\alpha})f\left(\frac{a}{m}\right)-f\left(tb+m(1-t)\frac{a}{m}\right)\right| &\leq t^{\alpha}g(b)+m(1-t^{\alpha})g\left(\frac{a}{m}\right)\\ &\quad -g\left(tb+m(1-t)\frac{a}{m}\right)\end{aligned}$$

for all $t\in[0,1]$. Adding the inequalities above, we get

$$\begin{aligned}&\left|t^{\alpha}[f(a)+f(b)]+m(1-t^{\alpha})\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\right.\\ &\left.-f\left(ta+m(1-t)\frac{b}{m}\right)-f\left(tb+m(1-t)\frac{a}{m}\right)\right|\\ \leq\ & t^{\alpha}[g(a)+g(b)]+m(1-t^{\alpha})\left[g\left(\frac{a}{m}\right)+g\left(\frac{b}{m}\right)\right]\\ &-g\left(ta+m(1-t)\frac{b}{m}\right)-g\left(tb+m(1-t)\frac{a}{m}\right).\end{aligned}$$

Integrating over $t\in[0,1]$ and then by dividing the resulting inequality with 2, we get the desired result. The proof is completed. ■

3 (g,r)-convex dominated functions

Definition 3.1 *Let the positive function $g:[a,b]\to\mathbb{R}$ be a given r-convex function on $[a,b]$. The real function $f:[a,b]\to\mathbb{R}$ is called (g,r)-convex dominated on $[a,b]$ if the following condition is satisfied:*

$$|M_r(f(x),f(y);\lambda)-f(\lambda x+(1-\lambda)y)|\leq M_r(g(x),g(y);\lambda)-g(\lambda x+(1-\lambda)y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.

Theorem 3.1 *Let the positive function $g:[a,b]\to\mathbb{R}$ be an r-convex function on $[a,b]$ and $0\leq a<b$. Then, $f:[a,b]\to\mathbb{R}$ is (g,r)-convex dominated mapping. Furthermore, if $f\in L_1[a,b]$, then the following inequality holds:*

$$\left|L_r(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq L_r(g(a),g(b))-\frac{1}{b-a}\int_a^b g(x)\,dx$$

for all $x, y \in I$, $\lambda \in [0,1]$ *and* $L_r(f(a), f(b))$ *as in (1.6).*

Proof. By Definition 3.1 with $r = 0$, $f(a) \neq f(b)$, we have

$$\left|f^{\lambda}(a) f^{1-\lambda}(b) - f(\lambda a + (1-\lambda) b)\right| \leq g^{\lambda}(a) g^{1-\lambda}(b) - g(\lambda a + (1-\lambda) b).$$

Integrating the inequality above over λ on $[0,1]$, we get

$$\begin{aligned}\left|f(b)\int_0^1 \left[\frac{f(a)}{f(b)}\right]^{\lambda} d\lambda - \int_0^1 f(\lambda a + (1-\lambda) b)\, d\lambda\right| &\leq g(b)\int_0^1 \left[\frac{g(a)}{g(b)}\right]^{\lambda} d\lambda \\ &\quad - \int_0^1 g(\lambda a + (1-\lambda) b)\, d\lambda.\end{aligned}$$

By a simple calculation, we obtain

$$\left|\frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \leq \frac{g(b) - g(a)}{\ln g(b) - \ln g(a)} - \frac{1}{b-a}\int_a^b g(x)\, dx.$$

The inequality above can be written as

$$\left|L_r(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \leq L_r(g(a), g(b)) - \frac{1}{b-a}\int_a^b g(x)\, dx.$$

For $r = 0$, $f(a) = f(b)$, we have with the same development

$$|f(a) - f(\lambda a + (1-\lambda) b)| \leq g(a) - g(\lambda a + (1-\lambda) b).$$

This inequality can be written as

$$\left|L_r(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \leq L_r(g(a), g(b)) - \frac{1}{b-a}\int_a^b g(x)\, dx.$$

By Definition 3.1 with $r \neq 0, -1$, $f(a) \neq f(b)$, we have

$$\begin{aligned}\left|(\lambda f^r(a) + (1-\lambda) f^r(b))^{\frac{1}{r}} - f(\lambda a + (1-\lambda) b)\right| &\leq (\lambda g^r(a) + (1-\lambda) g^r(b))^{\frac{1}{r}} \\ &\quad - g(\lambda a + (1-\lambda) b).\end{aligned}$$

Integrating the inequality above over λ on $[0,1]$, we have

$$\begin{aligned}\left|\frac{r}{r+1}\frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)} - \frac{1}{b-a}\int_a^b f(x)\, dx\right| &\leq \frac{r}{r+1}\frac{g^{r+1}(a) - g^{r+1}(b)}{g^r(a) - g^r(b)} \\ &\quad - \frac{1}{b-a}\int_a^b g(x)\, dx.\end{aligned}$$

The inequality above can be written as

$$\left|L_r(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \leq L_r(g(a), g(b)) - \frac{1}{b-a}\int_a^b g(x)\, dx.$$

For $r \neq 0$ and $f(a) = f(b)$, similarly

$$\left|(f^r(a))^{\frac{1}{r}} - f(\lambda a + (1-\lambda) b)\right| \leq (g^r(a))^{\frac{1}{r}} - g(\lambda a + (1-\lambda) b).$$

Then integrating the inequality above over λ on $[0,1]$, we have

$$\left|L_r(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq L_r(g(a), g(b)) - \frac{1}{b-a}\int_a^b g(x)\,dx.$$

Finally, let $r = -1$. For $f(a) \neq f(b)$, we have again

$$\begin{aligned}\left|\left(\lambda f^{-1}(a) + (1-\lambda) f^{-1}(b)\right)^{-1} - f(\lambda a + (1-\lambda) b)\right| &\leq \left(\lambda g^{-1}(a) + (1-\lambda) g^{-1}(b)\right)^{-1} \\ &\quad - g(\lambda a + (1-\lambda) b).\end{aligned}$$

Integrating the inequality above over λ on $[0,1]$, we have

$$\begin{aligned}\left|\frac{f(a) f(b)}{f(b) - f(a)}\int_{\frac{1}{f(a)}}^{\frac{1}{f(b)}} \lambda^{-1} d\lambda - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{g(a) g(b)}{g(b) - g(a)}\int_{\frac{1}{f(a)}}^{\frac{1}{f(b)}} \lambda^{-1} d\lambda \\ &\quad - \frac{1}{b-a}\int_a^b g(x)\,dx.\end{aligned}$$

The inequality above can be written as

$$\left|L_{-1}(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq L_{-1}(g(a), g(b)) - \frac{1}{b-a}\int_a^b g(x)\,dx.$$

The proof is completed. ■

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