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Abstract: We consider *Hausdorff discretization* from a metric space E to a discrete subspace D, which associates to a closed subset F of E any subset S of D minimizing the Hausdorff distance between F and S; this minimum distance, called the *Hausdorff radius* of F and written $r_H(F)$, is bounded by the resolution of D. We call a closed set F *separated* if it can be partitioned into two non-empty closed subsets F_1 and F_2 whose mutual distances have a strictly positive lower bound. Assuming some minimal topological properties of E and D (satisfied in $\mathbb{R}^n$ and $\mathbb{Z}^n$), we show that given a non-separated closed subset F of E, for any $r > r_H(F)$, every Hausdorff discretization of F is connected for the graph with edges linking pairs of points of D at distance at most $2r$. When F is connected, this holds for $r = r_H(F)$, and its greatest Hausdorff discretization belongs to the partial connection generated by the traces on D of the balls of radius $r_H(F)$. However, when the closed set F is separated, the Hausdorff discretizations are disconnected whenever the resolution of D is small enough. In the particular case where $E = \mathbb{R}^n$ and $D = \mathbb{Z}^n$ with norm-based distances, we generalize our previous results for $n = 2$. For a norm invariant under changes of signs of coordinates, the greatest Hausdorff discretization of a connected closed set is axially connected. For the so-called *coordinate-homogeneous* norms, which include the L_p norms, we give an adjacency graph for which all Hausdorff discretizations of a connected closed set are connected.

Keywords: metric space, topological connectivity, adjacency graph, partial connection, closed set, Hausdorff discretization

MSC: 05C40, 54E35, 68U10

1 Introduction

Hausdorff discretization is a metric approach to the problem of discretizing Euclidean sets. It was introduced in [12, 13] in the general setting of an arbitrary metric space (E, d) and a subspace D of E which is "discrete" in the sense that every bounded subset of D is finite (we say then that D is *boundedly finite*).

A *Hausdorff discretization* of a non-empty compact subset K of E, is a subset S of the discrete space D that minimizes the Hausdorff distance between K and any subset of D. This minimum Hausdorff distance is given by the *Hausdorff radius* of K: $r_H(K) = \max_{x \in K} d(x, D)$. See Figures 1 and 2.

In our framework, the role played by the grid step h in the common discretization pair $E = \mathbb{R}^n$, $D = (h\mathbb{Z})^n$, is held by the *covering radius* r_c: the least $r > 0$ such that E is covered by the union of all balls $B_r(p) = \{q \in$

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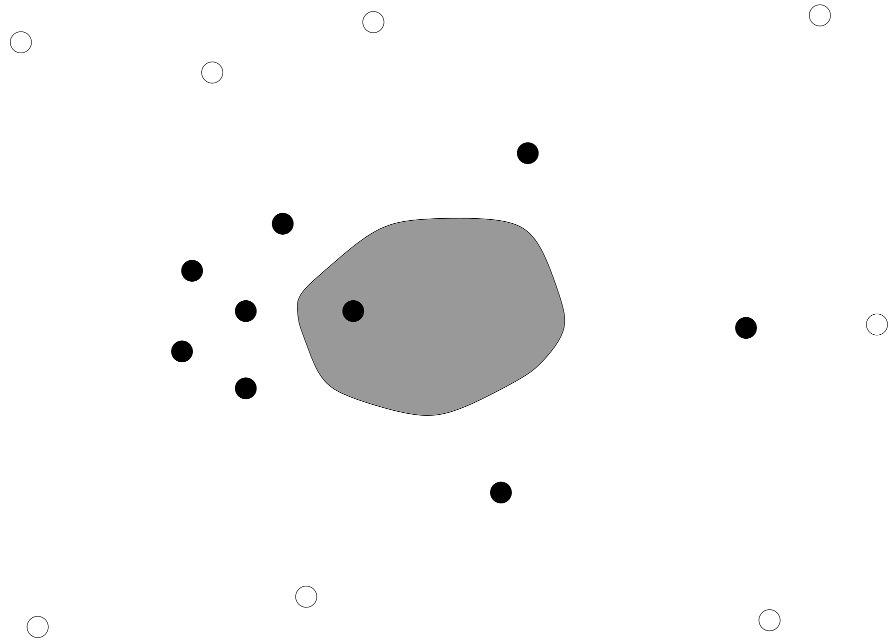

Figure 1: Here $E = \mathbb{R}^2$, D is a discrete set of irregularly spaced points, and we take the Euclidean distance d_2. A compact set K (in grey), and its greatest Hausdorff discretization (shown with filled dots), which consists of all points of D whose distance to K is at most $r_H(K)$.

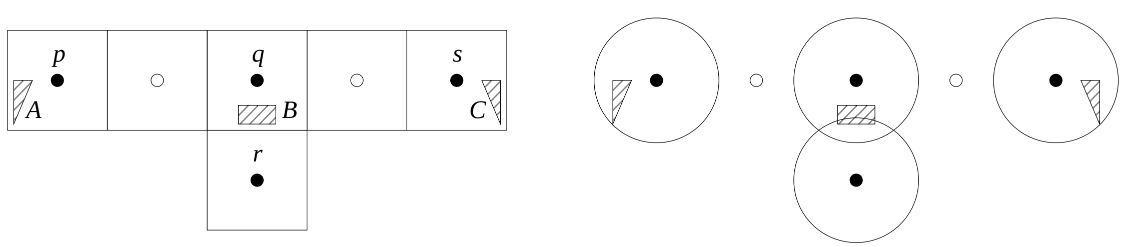

Figure 2: Here $E = \mathbb{R}^2$, $D = \mathbb{Z}^2$, and we take the Euclidean distance d_2. Left: A compact set $K = A \cup B \cup C$ overlayed with discrete points p, q, r, s and their digital cells. Right: We show the balls of Hausdorff radius $r_H(K)$ centered about p, q, r, s; since these balls are all those that intersect K, the greatest Hausdorff discretization of K is $\{p, q, r, s\}$; now since the balls centered about p, q, s cover K, $\{p, q, s\}$ is also a Hausdorff discretization of K, and there is no other one than $\{p, q, r, s\}$ and $\{p, q, s\}$.

$E \mid d(p,q) \leq r\}$ with $p \in D$. Equivalently, $r_c = \sup_{x \in E} d(x, D)$. Then, we have $r_H(K) \leq r_c$, so the Hausdorff distance between K and its Hausdorff discretizations is always bounded by the resolution of D.

The set $\mathcal{M}_H(K)$ of Hausdorff discretizations of K is non-empty, finite and closed under non-empty unions. It has thus a greatest element, called the *greatest Hausdorff discretization* of K and written $\Delta_H(K)$. Actually, $\Delta_H(K)$ consists of all points of D whose distance to K is at most $r_H(K)$, see Figures 1 and 2 again.

Both $\mathcal{M}_H(K)$ and $\Delta_H(K)$ were characterized in [12], and their relation with other types of discretization was analysed in [13]. For instance, we showed that when digital cells constitute the Voronoi diagram of D, every cover discretization is a Hausdorff discretization. On the other hand, in morphological discretization by dilation, provided that the dilation of D is E (every point of E belongs to the structuring element associated to a point of D), the Hausdorff distance between a compact and its discretization is bounded by the radius of the structuring element.

In [14, 15] we extended our theoretical framework to the discretization of non-empty closed subsets of E instead of compact ones.

In Section 5 of [22], we analysed the preservation of connectivity by Hausdorff discretization in the case where $E = \mathbb{R}^2$, $D = \mathbb{Z}^2$ and the metric d is induced by a norm N such that for $(x_1, x_2) \in \mathbb{R}^2$ and $\varepsilon_1, \varepsilon_2 = \pm 1$, $N(\varepsilon_1 x_1, \varepsilon_2 x_2) = N(x_1, x_2) = N(x_2, x_1)$, for example the L_p norm ($1 \leq p \leq \infty$), see (1) next page. We showed then that for a non-empty connected closed subset F of E,

1. for $N \neq L_1$, every Hausdorff discretization of F is 8-connected;
2. the greatest Hausdorff discretization of F is 4-connected.

There was an error in [22]: we overlooked the condition $N \neq L_1$ in item 1; it was later pointed out by D. Wagner (private communication), and indeed for $N = L_1$ we show a counterexample in Section 4 (Figure 8).

Since Section 5 of [22] gives the starting point of the present paper, we can look closely at the proof of the above result: it depends only on the properties of closed balls. Indeed, items 1 and 2 follow from the corresponding two facts:

1. when $N \neq L_1$, for any $x, y \in D$ such that $B_{r_c}(x) \cap B_{r_c}(y) \neq \emptyset$, x is 8-adjacent to y;
2. for any $x \in E$ and $r > 0$, $B_r(x) \cap D$ is 4-connected.

The goal of this paper is to extend these connectivity preservation properties of Hausdorff discretization to the most general situation: we consider the discretizations of any closed subset of a space E into a "discrete" subspace D of "bounded resolution", in other words D is boundedly finite and $0 < r_c < \infty$. The space E will be assumed to be "continuous" in some sense, in other words some results may require additional conditions, such as E being *boundedly compact* (that is, every bounded and closed subset of D is compact) and having the *middle point property* (for any $p, q \in E$, there is some $x \in E$ such that $d(p,x) = d(x,q) = \frac{1}{2} d(p,q)$). Note that all conditions considered here are satisfied by $\mathbb{R}^n$ and $\mathbb{Z}^n$ with a norm-based metric.

This extension of connectivity preservation has been made possible by the development of a very broad framework for the notion of connectivity, namely the concept of *connection* [10, 18] and its recent generalization to that of *partial connection* [11].

We will obtain several general results. Let F be a non-empty closed subset of the boundedly compact space (E, d). For any $s \geq 0$, we say that F is *s-separated* if F can be partitioned into two non-empty closed subsets F_1 and F_2 such that $d(x_1, x_2) > s$ for all $x_1 \in F_1$ and $x_2 \in F_2$. Then we say that F is *separated* if F is s-separated for some $s > 0$ or, equivalently, F can be partitioned into two non-empty closed subsets F_1 and F_2 such that $\inf\{d(x_1, x_2) \mid x_1 \in F_1, x_2 \in F_2\} > 0$. Finally, F is *non-separated* if it is not separated, that is, for every $s > 0$, F is not s-separated. If F is connected, then it is non-separated but the converse is false, see for instance Example 2 (in Subsection 3.1) and Figure 5 (in Subsection 3.2). With this definition, we get the following:

– If F is connected, then (see Theorem 18 in Subsection 3.2):
  1. every Hausdorff discretization of F is connected in the graph on D where we join by an edge any two points whose distance apart is at most $2r_H(F)$ (twice the Hausdorff radius);

2. the greatest Hausdorff discretization $\Delta_H(F)$ of F belongs to the partial connection generated by all $B_{r_H(F)}(x) \cap D$, $x \in F$.

– If F is disconnected but non-separated, then (see Theorem 19 in Subsection 3.2), for any $r > r_H(F)$:

1. every Hausdorff discretization of F is connected in the graph on D where we join by an edge any two points whose distance apart is at most $2r$;
2. the greatest Hausdorff discretization $\Delta_H(F)$ of F belongs to the partial connection generated by all $B_r(x) \cap \Delta_H(F)$, $x \in F$.

– If F is s-separated for some $s > 0$, then (see Proposition 21 in Subsection 3.3), for any $r \leq s/4$, every Hausdorff discretization of F will be disconnected in the graph on D where we join by an edge any two points whose distance apart is at most $2r$. In particular, if $r_c < s/4$, this will be the case for some $r > r_H(F)$.

We can interpret the above results in the framework of discretizations in multiple resolutions: the Hausdorff discretizations of a non empty closed subset F of the boundedly compact space E remain connected when the resolution of D tends to zero iff F is non-separated. See Theorem 23 in Subsection 3.3.

In the second part of our paper, we consider the particular case where $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and d is induced by a norm invariant under any change of sign or permutation of the basis vectors of $\mathbb{R}^n$, for example the L_p norm ($1 \leq p \leq \infty$):

$$\begin{aligned} \|(x_1, \ldots, x_n)\|_p &= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad (p < \infty) \ , \\ \|(x_1, \ldots, x_n)\|_\infty &= \max(|x_1|, \ldots, |x_n|) \ . \end{aligned} \tag{1}$$

We will then see that every Hausdorff discretization of F will be connected according to an adjacency graph depending on the norm, while the greatest Hausdorff discretization of F will be connected for the $2n$-adjacency relation, linking any two points of $\mathbb{Z}^n$ that differ by 1 in exactly one coordinate. For $n = 2$, this will give the result of [22] mentioned above.

The paper is organized as follows. Our notation is summarized in Table 1. Section 2 recalls the theoretical background: first partial connections and connections [11], then some special families of closed sets [14, 15], and finally Hausdorff discretization [12, 13]. It also discusses related works by other authors. Section 3 gives our general results about the connectivity of discretizations of connected, non-separated or separated sets. Then Section 4 considers the particular case of $E = \mathbb{R}^n$ and $D = \mathbb{Z}^n$ with a coordinate-homogeneous norm. Finally Section 5 concludes.

Throughout our paper, the statement of a well-known or published fact will be designated "Property", while the words "Lemma", "Proposition" and "Theorem" will be reserved for new results.

2 Theoretical Background

We will recall some known concepts and results, first about *connections* and *partial connections* [11], then about some families of closed subsets in a metric space [14, 15], and finally in the theory of *Hausdorff discretization* [12, 13]. We also summarize related works.

Let E be any non-empty set. We write $\mathcal{P}(E)$ for the family of all subsets of E. Given three sets X, Y, Z, we say that X is *partitioned into* Y and Z if $\{Y, Z\}$ is a partition of X, that is, $Y \subseteq X$, $Z \subseteq X$, $Y \neq \emptyset$, $Z \neq \emptyset$, $Y \cap Z = \emptyset$ and $X = Y \cup Z$; then we say that X is *partitioned by* Y and Z if X is partitioned into $X \cap Y$ and $X \cap Z$, that is, $X \cap Y \neq \emptyset$, $X \cap Z \neq \emptyset$, $X \cap Y \cap Z = \emptyset$ and $X \subseteq Y \cup Z$.

Table 1: Notation and terminology used in this paper: *Notation* for the mathematical notation, *Name* for the designation, *Reference* for where it is defined (subsection, then property, definition or equation).

Notation	*Name*	*Reference*
$\|x\|_p$	L_p norm of x	§ 1, Eq. (1)
$\mathrm{Con}(\mathcal{B})$ / $\mathrm{Con}^*(\mathcal{B})$	connection /partial connection generated by $\mathcal{B}$	§ 2.1, Prop. 1 and 2
$B_r(x)$ / $B_r^\circ(x)$	closed / open ball of radius r centered about x	§ 2.2, Eq. (3)
$\mathcal{F}_{\mathrm{bc}}(E)$ $\mathcal{F}_{\mathrm{bf}}(E)$ $\mathcal{F}_{\mathrm{p}}(E)$	family of boundedly compact, of boundedly finite and of proximinal subsets of E	§ 2.2, after Def. 3
δ_r / δ_r° / δ_r^+	dilation / open dilation / upper extension of radius r	§ 2.2, Eq. (4), (5) and (6)
h_d	oriented Hausdorff distance	§ 2.3, Eq. (10)
H_d	Hausdorff distance	§ 2.3, Eq. (9)
r_c	covering radius	§ 2.3, Eq. (11)
$r_H(F)$	Hausdorff radius of F	§ 2.3, Eq. (12)
Δ_r / Δ_r^+	discretization / upper discretization of radius r	§ 2.3, Eq. (13) and (14)
$\mathcal{M}_H(F)$	set of all Hausdorff discretizations of F	§ 2.3, after Prop. 8
$\Delta_H(F)$	greatest Hausdorff discretization of F	§ 2.3, after Prop. 9
$G_r(S)$ / $G_r^Z(S)$		§ 3.2, Def. 7
$r_c(\rho)$	covering radius for ρ	§ 3.3, Eq. (15)

2.1 Connections and Partial Connections

Connections were defined by Serra [18] and further analysed by Ronse [10], who then generalized them to partial connections [11]:

Definition 1. *A* partial connection on $\mathcal{P}(E)$ *is a family* $\mathcal{C} \subseteq \mathcal{P}(E)$ *such that*

1. $\emptyset \in \mathcal{C}$*, and*
2. *for any* $\mathcal{B} \subseteq \mathcal{C}$ *such that* $\bigcap \mathcal{B} \neq \emptyset$*, we have* $\bigcup \mathcal{B} \in \mathcal{C}$*.*

We call the partial connection $\mathcal{C}$ *a* connection on $\mathcal{P}(E)$ *if it satisfies the following third condition:*

3. *for all* $p \in E$*,* $\{p\} \in \mathcal{C}$*.*

Note that condition 2 remains valid for $\mathcal{B} = \emptyset$. Indeed, $\bigcap \mathcal{B} = E \neq \emptyset$ and $\bigcup \mathcal{B} = \emptyset \in \mathcal{C}$ thanks to condition 1. Elements of a partial connection $\mathcal{C}$ are said to be *connected*; those of $\mathcal{P}(E) \setminus \mathcal{C}$ are said to be *disconnected*. For $X \in \mathcal{P}(E)$, a *connected component* of X is any element of $\mathcal{P}(X) \cap \mathcal{C} \setminus \{\emptyset\}$ that is maximal for inclusion. When $\mathcal{C}$ is a connection, the connected components of X constitute a partition of X; when $\mathcal{C}$ is a partial connection, they constitute a partition of the subset $\bigcup(\mathcal{P}(X) \cap \mathcal{C})$, the union of all connected subsets of X.

The above expressions “connected set” and “connected component” must always be understood in the context of a given (partial) connection. Let us give three well-known examples of connections. In a topological space E, a subset X is connected if it cannot be partitioned by two open sets G_1 and G_2; equivalently, it cannot be partitioned by two closed sets F_1 and F_2. Then the set of connected subsets of E constitutes a connection [18]. It is easily seen that the closure of a connected set is connected, hence the connected components of a closed set are closed. Another connection is made of path-connected sets, that is, sets X such that for any $x, y \in X$, there is a path joining x to y, i.e., a continuous map $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. This second connection contains the first one, since any path-connected set is connected. A third example is given by connectivity in a graph; we will describe it precisely after Property 2. Other examples of connections and partial connections can be found in [10, 11, 20].

Property 1. *An intersection of connections on $\mathcal{P}(E)$ is a connection on $\mathcal{P}(E)$, an intersection of partial connections on $\mathcal{P}(E)$ is a partial connection on $\mathcal{P}(E)$, and $\mathcal{P}(E)$ is the greatest (partial) connection on $\mathcal{P}(E)$. Thus for any family $\mathcal{B}$ of subsets of E, there is a least partial connection including $\mathcal{B}$ and a least connection including $\mathcal{B}$.*

The least partial connection (resp., connection) including $\mathcal{B}$ is called the *partial connection* (resp., *connection*) *generated by* $\mathcal{B}$ and it is written $\mathsf{Con}^\star(\mathcal{B})$ (resp., $\mathsf{Con}(\mathcal{B})$).

Definition 2. *Let $\mathcal{B}$ be a family of non-empty subsets of E.*

1. *For $p, q \in E$, p and q are said to be* chained by $\mathcal{B}$ *if there are $B_0, \dots, B_n \in \mathcal{B}$ ($n \geq 0$) such that $p \in B_0$, $q \in B_n$ and $B_{t-1} \cap B_t \neq \emptyset$ for all $t = 1, \dots, n$. Such a sequence $B_0, \dots, B_n$ is called a* chain, *and its length is $n + 1$.*
2. *For $A \in \mathcal{P}(E)$, we say that A is* chained by $\mathcal{B}$ *if $\mathcal{B} \subseteq \mathcal{P}(A)$ and any two points of A are chained by $\mathcal{B}$.*

Obviously $\emptyset$ (as set of points of E) is chained by $\emptyset$ (as set of non-empty subsets of E). In a non-empty subset A of E, a chain between two points has always a strictly positive length. A point x is chained to itself if $x \in B$ for some $B \in \mathcal{B}$; thus, the binary relation on E linking two points when they are chained by $\mathcal{B}$, is generally not reflexive. However, it is symmetrical and transitive; hence this relation induces an equivalence relation on $\bigcup \mathcal{B}$ (not on E).

Property 2. *Given a family $\mathcal{B}$ of non-empty subsets of E, $\mathsf{Con}^\star(\mathcal{B})$ is the set of all $X \in \mathcal{P}(E)$ chained by $\mathcal{P}(X) \cap \mathcal{B}$; thus $X \in \mathsf{Con}^\star(\mathcal{B})$ if and only if for any $y, z \in X$, there are $B_0, \dots, B_n \in \mathcal{B}$ ($n \geq 0$) such that $y \in B_0$, $z \in B_n$, $B_i \subseteq X$ for $0 \leq i \leq n$ and $B_i \cap B_{i+1} \neq \emptyset$ for $0 \leq i < n$. Now, $\mathsf{Con}(\mathcal{B})$ is obtained by adding to $\mathsf{Con}^\star(\mathcal{B})$ all singletons $\{x\}$, $x \in E$.*

For example, in a graph with vertex set V, let $\mathcal{B}$ be the set of pairs of distinct vertices that are linked by an edge. Then $\mathsf{Con}(\mathcal{B})$ is the set of all parts X of V such that the graph induced on X is connected [10]. On the other hand, $\mathsf{Con}^\star(\mathcal{B})$ consists of such parts X of size at least 2, or singletons $X = \{p\}$ with a loop on p; in other words, isolated vertices are excluded from $\mathsf{Con}^\star(\mathcal{B})$. Indeed, graph connectivity is based on chains of edges, which can possibly be of length 0 (isolated vertices), while $\mathsf{Con}^\star(\mathcal{B})$ is based on chains of length at least 1.

When $X \notin \mathsf{Con}^\star(\mathcal{B})$, let $Y = \bigcup\big(\mathcal{P}(X) \cap \mathcal{B}\big)$; in other words, Y is the set of all points in X that are covered by some $B \in \mathcal{B}$ such that $B \subseteq X$. Then $Y \subseteq X$, and the binary relation on Y linking $x, y \in Y$ iff x and y are chained by $\mathcal{P}(X) \cap \mathcal{B}$, is an equivalence relation, whose equivalence classes are the connected components of X according to the partial connection $\mathsf{Con}^\star(\mathcal{B})$; thus for $x \in Y$, the connected component of X containing x is the set of all $y \in Y$ such that x and y are chained by $\mathcal{P}(X) \cap \mathcal{B}$, while for $x \in X \setminus Y$, x belongs to no connected component; in particular, when $Y = \emptyset$, X has no connected component. Now, the connected components of X according to $\mathsf{Con}(\mathcal{B})$ are those according to $\mathsf{Con}^\star(\mathcal{B})$, plus the singletons of $X \setminus Y$.

Let A and B be two non-empty sets. For any map $\psi : \mathcal{P}(A) \to \mathcal{P}(B)$ and $x \in A$, write $\psi(x)$ for $\psi(\{x\})$. When $B = A$, the map ψ is said “on $\mathcal{P}(A)$”. A map $\delta : \mathcal{P}(A) \to \mathcal{P}(B)$ such that for any $Z \in \mathcal{P}(A)$ we have $\delta(Z) = \bigcup_{z \in Z} \delta(z)$ is called a *dilation*. We finally recall what is called a *connection by dilation* [10, 11, 18]:

Property 3. *Let $\mathcal{C}$ be a partial connection on $\mathcal{P}(E)$ and let δ be a dilation on $\mathcal{P}(E)$ such that for any $x \in E$, $\delta(x) \neq \emptyset$. Then $\mathcal{C}^{\delta} = \{X \in \mathcal{P}(E) \mid \delta(X) \in \mathcal{C}\}$ is a partial connection on $\mathcal{P}(E)$. Furthermore, if for any $x \in E$, $x \in \delta(x)$ and $\delta(x) \in \mathcal{C}$, then $\mathcal{C}^{\delta}$ is a connection and $\mathcal{C} \subseteq \mathcal{C}^{\delta}$.*

2.2 Some families of closed sets in a metric space

The reader is assumed to be familiar with basic topological and metric notions, such as a metric (or distance function), a metric space, a bounded set, an open set, a closed set, the relative topology on a subset, a compact space and a compact set, see [6]. We recall some more advanced definitions and results [14, 15].

Let (E, d) be a metric space, where $E \neq \emptyset$ and d is a metric on E. For $X \subseteq E$ and $p \in E$ we define the distance between p and X as

$$d(p, X) = \inf\{d(p, x) \mid x \in X\} \ . \tag{2}$$

Note that $|d(p, X) - d(q, X)| \leq d(p, q)$, thus the function $E \to \mathbb{R} : p \mapsto d(p, X)$ is continuous.

For $r \geq 0$ and $x \in E$ we define the *closed ball* $B_r(x)$ and the *open ball* $B^{\circ}_r(x)$ *of radius r centered about x*, by

$$B_r(x) = \{y \in E \mid d(x, y) \leq r\} \quad \text{and} \quad B^{\circ}_r(x) = \{y \in E \mid d(x, y) < r\} \ . \tag{3}$$

Note that for $r = 0$ we have $B_0(x) = \{x\}$ and $B^{\circ}_0(x) = \emptyset$. The open balls of radius > 0 constitute the basis of the *metric topology* on E.

Given $X \subseteq E$, (X, d) is a metric space, and the metric topology of (X, d) coincides with the relative topology on X induced by the metric topology of (E, d). We will consider several topological or metric properties on a metric space (E, d) (compact, boundedly compact, boundedly finite, ...), and we will speak of such a property for a subset X of E to mean that the metric space (X, d) has that property.

Given a family $\mathcal{S}(E)$ (defined from some property) of subsets of the space E, we will write $\mathcal{S}'(E)$ for $\mathcal{S}(E)\setminus\{\emptyset\}$, the family of non-empty elements of $\mathcal{S}(E)$. For any $X \subseteq E$, we will write $\mathcal{S}(X)$ (resp., $\mathcal{S}'(X)$) for the corresponding family in the relative topology or metric of X.

Let us write: $\mathcal{F}(E)$ for the family of closed subsets of E, $\mathcal{K}(E)$ for that of compact subsets of E, and $\mathsf{Fin}(E)$ for that of finite subsets of E; then $\mathcal{P}'(E)$, $\mathcal{F}'(E)$, $\mathcal{K}'(E)$ and $\mathsf{Fin}'(E)$ will designate the restrictions of these families to non-empty subsets of E.

Note that for $X \subseteq E$, $\mathcal{K}(X) = \mathcal{K}(E) \cap \mathcal{P}(X)$ and for $X \in \mathcal{K}(E)$, $\mathcal{K}(X) = \mathcal{F}(X)$. For $X \in \mathcal{F}(E)$, $\mathcal{F}(X) = \mathcal{F}(E) \cap \mathcal{P}(X)$.

The following definition and property were given in [14, 15]:

Definition 3. *A metric space (E, d) is called*

- boundedly compact *if every bounded and closed subset of E is compact, equivalently, for every $p \in E$ and $r > 0$, $B_r(p)$ is compact;*
- boundedly finite *if every bounded subset of E is finite, equivalently, for every $p \in E$ and $r > 0$, $B_r(p)$ is finite.*

A subset X of E is called proximinal *if either $X = \emptyset$, or for every $y \notin X$, there is some $x \in X$ minimizing the distance to y, that is, $d(y, x) = d(y, X)$.*

For instance, for the metrics induced by L_p norms, $\mathbb{R}^n$ is boundedly compact and $\mathbb{Z}^n$ is boundedly finite. Write $\mathcal{F}_{\mathrm{bc}}(E)$, $\mathcal{F}_{\mathrm{bf}}(E)$ and $\mathcal{F}_{\mathrm{p}}(E)$ respectively, for the family of boundedly compact subsets of E, of boundedly finite subsets of E, and proximinal subsets of E.

Property 4. *In any metric space (E, d):*

1. *$\mathcal{K}(E) \cup \mathcal{F}_{\mathrm{bf}}(E) \subseteq \mathcal{F}_{\mathrm{bc}}(E) \subseteq \mathcal{F}_{\mathrm{p}}(E) \subseteq \mathcal{F}(E)$ and $\mathcal{K}(E) \cap \mathcal{F}_{\mathrm{bf}}(E) = \mathsf{Fin}(E)$.*
2. *If E is boundedly compact, then every closed subset of E is boundedly compact, that is $\mathcal{F}_{\mathrm{bc}}(E) = \mathcal{F}_{\mathrm{p}}(E) = \mathcal{F}(E)$.*
3. *If E is boundedly finite, then every subset of E is boundedly finite, that is $\mathcal{F}_{\mathrm{bf}}(E) = \mathcal{F}_{\mathrm{bc}}(E) = \mathcal{F}_{\mathrm{p}}(E) = \mathcal{F}(E) = \mathcal{P}(E)$.*

We will now see the relation of these properties with dilations by balls. For $r \geq 0$ we define three operators $\mathcal{P}(E) \to \mathcal{P}(E)$, two of which are dilations: first δ_r, the *dilation of radius* r, given by

$$\delta_r(X) = \bigcup_{x \in X} B_r(x) \quad , \tag{4}$$

then δ_r°, the *open dilation of radius* r, given by

$$\delta_r^\circ(X) = \bigcup_{x \in X} B_r^\circ(x) \quad , \tag{5}$$

and finally δ_r^+, the *upper extension of radius* r (which is generally not a dilation), given by

$$\delta_r^+(X) = \bigcap_{s>r} \delta_s^\circ(X) = \bigcap_{s>r} \delta_s(X) \quad . \tag{6}$$

We have

$$\begin{aligned} \delta_r(X) &= \{p \in E \mid B_r(p) \cap X \neq \emptyset\} \quad , \\ \delta_r^\circ(X) &= \{p \in E \mid d(p, X) < r\} \quad , \\ \delta_r^+(X) &= \{p \in E \mid d(p, X) \leq r\} \quad . \end{aligned} \tag{7}$$

Note that $\delta_r^\circ(X)$, being a union of open balls, is open. On the other hand, $\delta_r^+(X)$, being the inverse image of the closed interval $[0, r]$ by the continuous function $x \mapsto d(x, X)$, is closed.

For $r = 0$ we have $\delta_0(X) = X$, $\delta_0^\circ(X) = \emptyset$ and $\delta_0^+(X) = \overline{X}$. Now, $\delta_r(X)$, $\delta_r^\circ(X)$ and $\delta_r^+(X)$ increase with r, and they satisfy the inclusions

$$\delta_r^\circ(X) \subseteq \delta_r(X) \subseteq \delta_r^+(X) \subseteq \delta_s^\circ(X) \quad \text{for } s > r \quad .$$

In [14] we characterized proximinal sets and boundedly compact spaces in terms of properties of δ_r:

Property 5. *In a metric space (E, d), a set X is proximinal if and only if for every $r \geq 0$ we have $\delta_r(X) = \delta_r^+(X)$. In particular, when X is proximinal, $\delta_r(X)$ is closed.*

Property 6. *The following properties are equivalent in a metric space (E, d):*

1. *E is boundedly compact.*
2. *For every non-empty compact K and $r > 0$, $\delta_r(K)$ is compact.*
3. *For every non-empty closed F and $r > 0$, $\delta_r(F)$ is boundedly compact.*

2.3 Hausdorff Discretization

Let us first recall from [12, 14, 15] some basic concepts and facts about the Hausdorff metric. We start with non-empty compact sets. We remind first that for any $K \in \mathcal{K}'(E)$, every continuous function $K \to \mathbb{R}$ has a compact image, in particular it attains a maximum and a minimum.

Given $K \in \mathcal{K}'(E)$ and $p \in E$, there exists $y_p \in K$ such that $d(p, y_p) = d(p, K)$, so $d(p, K) = \min\{d(p, x) \mid x \in K\}$. For $X, Y \in \mathcal{K}'(E)$, the set of $d(x, Y)$ for $x \in X$ attains a maximum, so we define

$$h_d(X, Y) = \max\{d(x, Y) \mid x \in X\} \quad , \tag{8}$$

which we call the *oriented Hausdorff distance from X to Y*; thus there exist $x^\star \in X$ and $y^\star \in Y$ such that $d(x^\star, y^\star) = d(x^\star, Y) = h_d(X, Y)$. We define then the *Hausdorff distance between X and Y* as

$$H_d(X, Y) = \max\bigl(h_d(X, Y), h_d(Y, X)\bigr) \quad . \tag{9}$$

Now, H_d is a metric on the space $\mathcal{K}'(E)$, it is thus called the *Hausdorff metric*. For $X, Y \in \mathcal{K}'(E)$ and $r \geq 0$, we have $h_d(X, Y) \leq r$ iff $X \subseteq \delta_r(Y)$, while $H_d(X, Y) \leq r$ iff both $X \subseteq \delta_r(Y)$ and $Y \subseteq \delta_r(X)$. Thus $h_d(X, Y)$ is the least $r \geq 0$ such that $X \subseteq \delta_r(Y)$, while $H_d(X, Y)$ is the least $r \geq 0$ such that both $X \subseteq \delta_r(Y)$ and $Y \subseteq \delta_r(X)$.

One can extend the Hausdorff metric from $\mathcal{K}'(E)$ to $\mathcal{F}'(E)$. Given two non-empty closed sets X and Y, we set

$$h_d(X, Y) = \sup\{d(x, Y) \mid x \in X\} \quad ; \tag{10}$$

now we define the Hausdorff distance $H_d(X, Y)$ as in (9). Then H_d is a *generalized metric* on $\mathcal{F}$; by this we mean that H_d satisfies the axioms of a metric, with the only difference that it can take infinite values.

Using (6,7), and the convention that the infimum (in $\mathbb{R}^+$) of an empty set is equal to ∞, we get:

Property 7. *For every $X, Y \in \mathcal{F}'(E)$ and for every $r \geq 0$, $h_d(X, Y) \leq r$ if and only if $X \subseteq \delta_r^+(Y)$, while $H_d(X, Y) \leq r$ if and only if both $X \subseteq \delta_r^+(Y)$ and $Y \subseteq \delta_r^+(X)$. In particular:*

- $h_d(X, Y) = \inf\{r > 0 \mid X \subseteq \delta_r(Y)\}$.
- *If $h_d(X, Y) < \infty$, then it is the least $r \geq 0$ such that $X \subseteq \delta_r^+(Y)$.*
- $H_d(X, Y) = \inf\{r > 0 \mid X \subseteq \delta_r(Y)$ *and* $Y \subseteq \delta_r(X)\}$.
- *If $H_d(X, Y) < \infty$, then it is the least $r \geq 0$ such that both $X \subseteq \delta_r^+(Y)$ and $Y \subseteq \delta_r^+(X)$.*

As in $\mathbb{R}^+$ the supremum and infimum of an empty set are equal to 0 and ∞ respectively, we can extend h_d and H_d to the empty set. We get then

$$\begin{aligned} h_d(\emptyset, \emptyset) = H_d(\emptyset, \emptyset) = 0 \quad ; \\ \forall F \in \mathcal{F}', \quad h_d(\emptyset, F) = 0 \text{ and } h_d(F, \emptyset) = H_d(F, \emptyset) = \infty \quad . \end{aligned}$$

We can now recall the theory of Hausdorff discretization [12, 14, 15]. We have a metric space (E, d), and let $D \subset E$, $D \neq \emptyset$. Here E will be the "continuous" space and D will be the "discrete" space. We define the *covering radius* (of D for the metric d) as the positive number

$$r_c = \sup_{x \in E} d(x, D) = h_d(E, D) \quad . \tag{11}$$

The covering radius r_c is in some way a measure of the resolution of D. We assume the following:

Axiom 1. *[12] $\emptyset \subset D \subset E$, D is boundedly finite and $r_c < \infty$.*

By Property 4, for every $x \in E$, there exists $y_x \in D$ such that $d(x, D) = d(x, y_x)$, thus for $r \geq 0$ we have $d(x, D) \leq r$ iff $x \in \delta_r(D)$, cf. Property 5 and (7). Hence $r_c > 0$ and r_c is the least $r > 0$ such that $\delta_r(D) = E$. Note that every subset of D is closed and that a subset of D is compact iff it is finite; thus, by analogy with the corresponding subsets in E, we will write $\mathcal{F}'(D)$ for the family of all non-empty subsets of D, and $\mathcal{K}'(D)$ for the family of all non-empty finite subsets of D.

Example 1. *With Axiom 1, E is not necessarily boundedly compact. Take $E = \mathbb{R} \times [-1/2, +1/2]^{\mathbb{N}}$ and $D = \mathbb{Z} \times \{0\}^{\mathbb{N}}$. Thus E consists of all infinite sequences of reals $(r, s_n)_{n \in \mathbb{N}} = (r, s_0, \ldots, s_n, \ldots)$, where $r \in \mathbb{R}$ and $-1/2 \leq s_n \leq 1/2$ for all $n \in \mathbb{N}$, while D consists of all integer sequences in E, namely $(m, 0, \ldots, 0, \ldots)$ with $m \in \mathbb{N}$. We take the distance induced by the L_∞ norm: for all $(r, s_n)_{n \in \mathbb{N}}, (r', s_n')_{n \in \mathbb{N}} \in E$,*

$$d\big((r, s_n)_{n \in \mathbb{N}}, (r', s_n')_{n \in \mathbb{N}}\big) = \sup\big\{|r - r'|, |s_n - s_n'| \bigm| n \in \mathbb{N}\big\} \quad .$$

Then D is boundedly finite, $r_c = 1/2$, but E is not boundedly compact, because it has infinite dimension: covering a ball of radius $1/2$ requires an infinity of balls of radius $1/4$.

For any $F \in \mathcal{F}'(E)$, define the *Hausdorff radius* of F:

$$r_H(F) = \sup_{x \in F} d(x, D) = h_d(F, D) \quad . \tag{12}$$

Then $r_H(F) \leq r_c$, and $r_H(F)$ is the least $r \geq 0$ such that $F \subseteq \delta_r(D)$. The particular case $F = E$ gives $r_H(E) = r_c$. For $K \in \mathcal{K}'(E)$, the above supremum is attained, thus $r_H(K) = \max_{x \in K} d(x, D)$.

For $r \geq 0$, the *discretization of radius r* is the map $\Delta_r : \mathcal{P}(E) \to \mathcal{P}(D)$ defined by setting $\forall X \subseteq E$:

$$\Delta_r(X) = \delta_r(X) \cap D = \{p \in D \mid B_r(p) \cap X \neq \emptyset\} \ . \tag{13}$$

Then Δ_r is a dilation $\mathcal{P}(E) \to \mathcal{P}(D)$, and $\Delta_r(X)$ is finite for every bounded X. For $x \in E$, we write $\Delta_r(x)$ for $\Delta_r(\{x\})$. Next, the *upper discretization of radius r* is the map $\Delta_r^+ : \mathcal{P}(E) \to \mathcal{P}(D)$ given by setting $\forall X \subseteq E$:

$$\Delta_r^+(X) = \bigcap_{s>r} \Delta_s(X) = \delta_r^+(X) \cap D = \{p \in D \mid d(p, X) \leq r\} \ . \tag{14}$$

The last two equalities follow from (6) and (7). In general, Δ_r^+ is not a dilation. By Property 5, when F is proximinal, we have $\Delta_r^+(F) = \Delta_r(F)$.

We will now examine the Hausdorff distance between non-empty closed subsets of E and non-empty subsets of D, and between non-empty compacts subsets of E and non-empty finite subsets of D. Indeed, for $K \in \mathcal{K}'(E)$ and $S \in \mathcal{F}'(D)$, since K is bounded, if $H_d(K, S)$ is finite, then S will be bounded, hence finite: $S \in \mathcal{K}'(D)$.

Property 8. *For any $F \in \mathcal{F}'(E)$ and $S \in \mathcal{F}'(D)$, $H_d(F, S) \geq r_H(F)$. For any $r \geq r_H(F)$, $H_d(F, S) \leq r$ if and only if both $S \subseteq \Delta_r^+(F)$ and $F \subseteq \delta_r(S)$. In particular $F \subseteq \delta_r(\Delta_r^+(F))$, thus $\Delta_r^+(F)$ is the greatest $S \in \mathcal{F}'(D)$ with $H_d(K, S) \leq r$. When $F \in \mathcal{F}_\mathrm{p}'(E)$, $\Delta_r^+(F) = \Delta_r(F)$. When $F \in \mathcal{K}'(E)$, we have $S \in \mathcal{K}'(D)$.*

Given $F \in \mathcal{F}'(E)$, we call a *Hausdorff discretization of F* any $S \in \mathcal{F}'(D)$ which minimizes the Hausdorff distance to F:

$$\forall\, T \in \mathcal{F}'(D), \quad H_d(F, S) \leq H_d(F, T) \ .$$

In [12, 13], we said a *Hausdorff discretizing set* of K. The family of Hausdorff discretizations of F is written $\mathcal{M}_H(F)$.

Property 9. *For $F \in \mathcal{F}'(E)$, $r_H(F)$ minimizes the Hausdorff distance between F and elements of $\mathcal{F}'(D)$:*

$$r_H(F) = \min\{H_d(F, T) \mid T \in \mathcal{F}'(D)\} \ .$$

Thus $\mathcal{M}_H(F)$ is non-empty and $H_d(F, S) = r_H(F)$ for every $S \in \mathcal{M}_H(F)$. For any $S \in \mathcal{F}'(D)$, $S \in \mathcal{M}_H(F)$ if and only if both $S \subseteq \Delta_{r_H(F)}^+(F)$ and $F \subseteq \delta_{r_H(F)}(S)$. Moreover, $F \subseteq \delta_{r_H(F)}\left(\Delta_{r_H(F)}^+(F)\right)$, so $\Delta_{r_H(F)}^+(F)$ is the greatest element of $\mathcal{M}_H(F)$.

Again, when $F \in \mathcal{F}_\mathrm{p}'(E)$, $\Delta_{r_H(F)}^+(F) = \Delta_{r_H(F)}(F)$, and when $F \in \mathcal{K}'(E)$, we have $\mathcal{M}_H(F) \subseteq \mathcal{K}'(D)$: every Hausdorff discretization of a compact set is finite. It is easily seen that $\mathcal{M}_H(F)$ is closed under *non-empty* unions, and in [14] we showed that it is *down-continuous*: for a decreasing sequence $(S_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{M}_H(K)$, $\bigcap_{n\in\mathbb{N}} S_n \in \mathcal{M}_H(K)$.

The greatest element of $\mathcal{M}_H(F)$ is called the *greatest Hausdorff discretization of F*, and we write it $\Delta_H(F)$; thus $\Delta_H(F) = \Delta_{r_H(F)}^+(F)$. In [12, 13], we called it the *maximal Hausdorff discretization* of K.

Following the above remark that $H_d(\emptyset, \emptyset) = 0$ and $H_d(F, \emptyset) = \infty$ for $F \in \mathcal{F}'(E)$, we deduce that the only Hausdorff discretization of the empty set (in E) is the empty set (in D).

2.4 Related Works

Though Hausdorff discretization is a broad discretization scheme [12, 13, 22], several other approaches are described in the literature, primarily for the space $E = \mathbb{R}^n$ and the discrete subspace $D = \mathbb{Z}^n$. Many of them were investigated from the topological point of view, specially about the connectedness preservation property. Write o for the origin of $\mathbb{R}^n$ and $\mathbb{Z}^n$, and for any $p \in \mathbb{Z}^n$, let $C(p) = B_{1/2}^\infty(p)$, the ball of radius $1/2$ for the L_∞ norm (see Figure 3 (c) for $n = 2$); then, $C(p)$ is the digital cell around p.

Let us first clarify the terminology concerning digital adjacency and connectivity. There are essentially two notations. On the one hand, from a combinatorial point of view, k-adjacency designates the one where each digital point has k neighbours, for instance the 4- and 8-adjacencies in $\mathbb{Z}^2$, then the 6-, 18- and 26-adjacencies in $\mathbb{Z}^3$. On the other hand, in a coordinate-based approach, for $0 \leq k \leq n-1$, two points of $\mathbb{Z}^n$ are said to be *k-adjacent* if they differ by 1 in at least 1 and at most $n-k$ coordinates, the other coordinates being equal; to avoid confusion with the combinatorial notation, we will say that they are *k/n-adjacent*. Two particular adjacencies stand out, the *axial* and *diametral* ones. Two digital points p and q are *axially adjacent* if $\|p-q\|_1 = 1$, that is, p and q are $(n-1)/n$-adjacent (or $2n$-adjacent in the combinatorial notation). Now p and q are *diametrally adjacent* if $\|p-q\|_\infty = 1$, that is, p and q are $0/n$-adjacent (or (2^n-1)-adjacent in the combinatorial notation). The graph-theoretical connectivity corresponding to the k-adjacency is called k-connectivity.

In *discretization by dilation* [12], one chooses a structuring element $A \subset \mathbb{R}^n$, then the discretization of a subset X of $\mathbb{R}^n$ is $\Delta^A_\oplus(X) = \{p \in \mathbb{Z}^n \mid A_p \cap X \neq \emptyset\}$, where A_p is the translate of A positioned on p. Taking $A = C(o) = B^\infty_{1/2}(o)$, one obtains the *supercover* discretization, made of all digital points whose cell intersects the Euclidean set. It is well-known that the supercover discretization of a connected subset of $\mathbb{R}^n$ is axially connected.

For $n = 2$, another example is the Freeman/Bresenham *grid-intersect* discretization, which uses the unit cross structuring element of Figure 3 (a). Then the discretization of a connected set will be diametrally connected. Sekiya and Sugimoto [16] considered the discretization by dilation of connected curves in $\mathbb{R}^2$, using the two structuring elements of Figure 3 (b) and (c); the latter gives the supercover, which is axially connected. On the other hand the former (b) gives a discretization which is intermediate between the grid-intersect and the supercover; they showed that the discretization of a path-connected curve will be diametrally connected, and it is easy to check that the result also holds for any connected subset of $\mathbb{R}^2$. Note that both structuring elements (a) and (b) are not covering [12] (i.e., the union of translates of the structuring element by points of D does not cover E), so a non-empty subset of $\mathbb{R}^2$ can have an empty discretization.

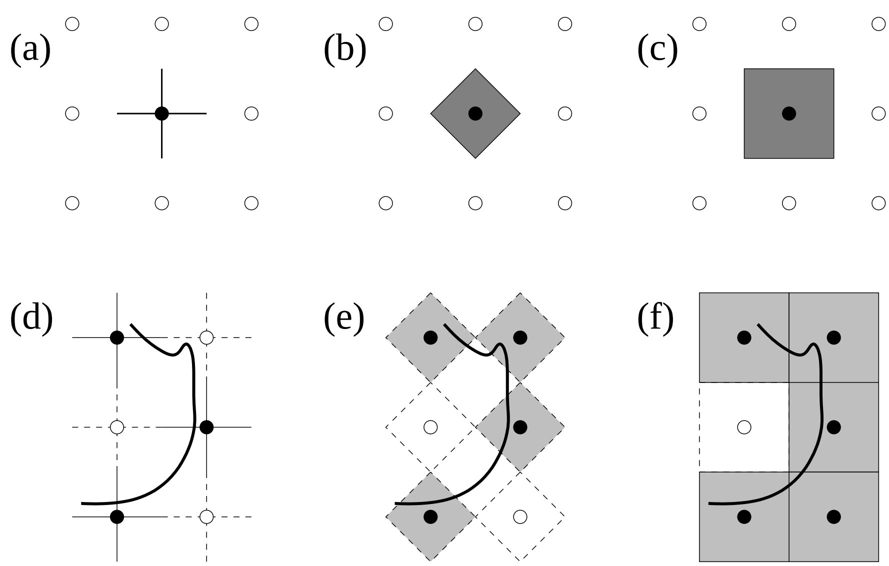

Figure 3: Here $n = 2$. Top: 3 structuring elements used in morphological discretization: (a) the unit cross for the grid-intersect discretization; (b) $B^1_{1/2}(o)$, the ball of radius 1/2 for the L_1 norm [16]; (c) $C(o) = B^\infty_{1/2}(o)$, the unit cell or ball of radius 1/2 for the L_∞ norm. Bottom: the discretization by dilation of a curve, for each of the 3 structuring elements, will be diametrally connected; for the supercover discretization (f), it will be axially connected.

Given the high computational cost of discretization by dilation, Sekiya and Sugimoto suggest an approximation that leads to a slightly smaller discretization. Assuming that a curve C is given by an analytical equation $f(x) = 0$ ($x \in \mathbb{R}^2$), where f is a continuous function $\mathbb{R}^2 \to \mathbb{R}$, the approximated discretization will consist of all digital points p such that A_p has two corners $x, y \in \mathbb{R}^2$ with $f(x) > 0$ and $f(y) < 0$; by continuity, there will be some $z \in A_p$ with $f(z) = 0$, so this discretization is included in the one by dilation by A. They showed that with the structuring elements of Figure 3 (b) and (c), the approximated discretization will still be diametrally connected for (b) and axially connected for (c).

A particular case of discretization by dilation is the *Gauss discretization*, where the structuring element is reduced to the singleton made of the origin: $A = \{o\}$; in other words, a set $X \subset \mathbb{R}^n$ is discretized as $X \cap \mathbb{Z}^n$. Pavlidis [9] and Serra [17] give local geometric conditions (see Figure 4) on the boundary points of $X \subset \mathbb{R}^2$ that ensure a homeomorphic reconstruction of X from its discretization (the set X is reconstructed as the union of the unit squares centerered in $X \cap \mathbb{Z}^2$). In particular, under these conditions, the discretization of a subset X of $\mathbb{R}^2$ yields an axially connected graph iff X is connected. This result was later extended by Gross *et al.* in [5, 8] to threshold area discretizations where an integer point p of $\mathbb{Z}^2$ belongs to the dicretization set if the area of the intersection between X and the unit square centered in p is above some fixed threshold.

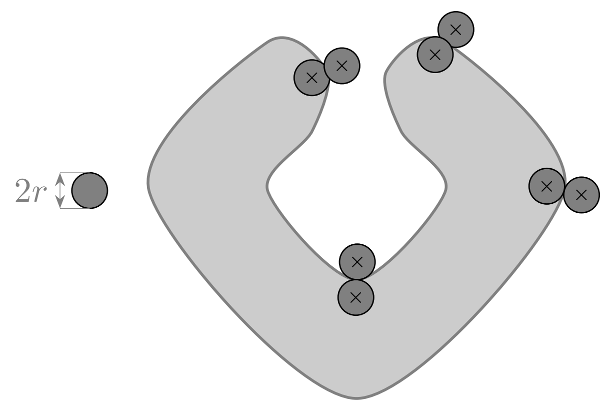

Figure 4: A geometric local property of the boundary that ensures connectedness preservation at a given scale: at each point p of the boundary B, there exist two balls on both sides whose intersection with B is exactly $\{p\}$.

The *minimal cover* of a set $X \subset \mathbb{R}^n$ was studied by Brimkov *et al.* in [3]. It is the smallest union of unit cells of $\mathbb{R}^n$, centered on integer points, that includes X. The integer points of the minimal cover of a simply connected n-dimensional manifold of $\mathbb{R}^n$ form an axially connected graph [3]. When X is an h-dimensional surface (the image of $\mathbb{R}^h$ into $\mathbb{R}^n$ by a continous function), another kind of minimal cover, the *k-discretization* (where $0 \leq k \leq n-1$), is described in the same article. It is a subset of the supercover that is minimal in the sense that the deletion of any of its unit cells will change the topology of the k/n-adjacency graph. Moreover, it is required that this cover includes some h-dimensional surface, not necessarily X itself. In the paper, there is another requirement but it not used in the proof of the following theorem: this k-discretization is $(h-1)/n$-connected.

Later, Brimkov *et al.* [4] showed that the discretization by dilation by an Euclidean ball of radius r (there called *r-offset discretization*) of a bounded path-connected subset of $\mathbb{R}^n$ is axially connected if $r \geq \sqrt{n}/2$ and is diametrally connected if $r \geq \sqrt{n-1}/2$. Furthermore, the values $\sqrt{n}/2$ and $\sqrt{n-1}/2$ are minimal for such a statement. This result is first extended to connected sets in [2] provided, for the first claim, that r is greater than or equal to the Hausdorff radius of the set. Then, in [1] it is extended to disconnected sets whose closure is connected provided the conditions on the radius r are strict inequalities.

Incidentally, Brimkov *et al.* [2] obtain the following result about Hausdorff discretizations (for the Euclidean distance): for any connected subset of $\mathbb{R}^n$, its greatest Hausdorff discretization is axially connected, while any other Hausdorff discretization is diametrally connected if the Hausdorff radius is less than 1 and may be diametrally disconnected otherwise.

In [21], the authors consider the Euclidean space $E = \mathbb{R}^2$, the discrete space $D(\rho) = \rho\mathbb{Z}^2$ of grid step $\rho > 0$, and a metric based on a norm N such that $N(\pm x_1, \pm x_2) = N(\pm x_2, \pm x_1) = N(x_1, x_2)$ and the only point at distance $N(1/2, 1/2)$ from both $(0, 0)$ and $(1, 1)$ is $(1/2, 1/2)$ (this excludes in particular the L_1 norm). For

$X \subseteq D(\rho)$, let $R_\rho(X)$ be its Euclidean reconstruction given by the union of closed digital cells (relatively to $D(\rho)$) of points of X: $R_\rho(X) = \bigcup_{p\in X} B^\infty_{\rho/2}(p)$. For $r > 0$, we say that F is *r-convex* if for any point $x \notin F$ at distance at most r from F, this distance is attained by a unique point in F, and for any $r' \leq r$, the closed ball of radius r' centered about x has a connected intersection with F.

Given a non-empty closed subset F of E and a Hausdorff discretization M^ρ of F in $D(\rho)$, several topological relations between F and the reconstruction $R_\rho(M^\rho)$ are announced (without proof), which hold when ρ is sufficiently small. First, any r-convex non-empty closed subset F of E is homotopically equivalent to $R_\rho(M^\rho)$ for any Hausdorff discretization M^ρ of F. Next, when both F and the closure of its complement are r-convex and form a union of closed balls of radius r, a Hausdorff discretization M^ρ of F will not contain a *singular configuration* [7] (a 2×2 digital square with one diagonal in the figure and the other in the complement), and $R_\rho(M^\rho)$ will be a bordered 2D manifold (that is, locally homeomorphic to a closed half-plane); in particular, if F is a compact bordered 2D manifold, then F and $R_\rho(M^\rho)$ will be homeomorphic.

3 General Results

Our goal is to show that every Hausdorff discretization of a non-empty “connected” closed subset of E is “connected”, and that conversely every Hausdorff discretization of a non-empty “disconnected” set will be “disconnected” when the resolution of the discrete space D is small enough. Here the connectivity in D will be in the graph with edges linking points distant by at most twice the Hausdorff radius. The “connectedness” of a closed subset F of E is more complicated. When F is compact, it is ordinary topological connectivity. However, a disconnected non-compact closed set can have a connected discretization when its connected components are asymptotic to each other (cf. Example 2). We will thus introduce the notion of a *s-separated* closed set, which is partitioned into two closed sets with mutual distances always $> s$. Then a non-empty closed set F will have its Hausdorff discretizations connected at all resolutions if and only if F is not s-separated for any $s > 0$. Such sets, called *non-separated*, constitute a connection on $\mathcal{F}(E)$, which contains topologically connected sets.

Our method of proof is similar to the one used in Section 5 of [22] to show the reciprocal link between discrete and continuous connectivities for compact sets, in the particular case where $E = \mathbb{R}^2$, $D = \mathbb{Z}^2$ and the distance is based on a homogeneous norm.

Subsection 3.1 introduces some needed notions: the middle point and interval properties in a metric space, and s-separated closed sets. Subsection 3.2 shows how, under some general conditions, the Hausdorff discretization of a non-empty non-separated closed set F gives a discrete set which is connected in the graph with edges linking points of D that are “close enough” (see Definition 7). Conversely, Subsection 3.3 shows how this fails for s-separated closed sets; in particular it considers discretization in a sequence of discrete spaces whose resolution tends to zero, and conditions under which the discretizations are connected at all resolutions.

3.1 Further Continuity Properties in a Metric Space

Let (E, d) be a metric space. For $r, s \geq 0$, the triangular inequality implies that for any $X \in \mathcal{P}(E)$, $\delta_r(\delta_s(X)) \subseteq \delta_{r+s}(X)$. The equality holds if for any $p, q \in E$ such that $d(p, q) = r + s$, there exists $x \in E$ such that $d(p, x) = r$ and $d(x, q) = s$. We will in fact consider two conditions, a first one weaker and a second one stronger; they will be equivalent in a boundedly compact space.

Definition 4. *We say that the metric space (E, d):*

1. *has the* middle point property *if for any $p, q \in E$, there is some $x \in E$ such that $d(p, x) = d(x, q) = \frac{1}{2} d(p, q)$.*
2. *has the* interval property *if for any $p, q \in E$, there is a map $f : [0, 1] \to E$ such that $f(0) = p$, $f(1) = q$ and for $0 \leq \alpha < \beta \leq 1$ we have $d\big(f(\alpha), f(\beta)\big) = (\beta - \alpha) d(p, q)$.*

Every metric induced by a norm on a vector space satisfies the interval property. When d has the interval property, we obtain $\delta_r(\delta_s(X)) = \delta_{r+s}(X)$ for any $r, s \geq 0$ and $X \in \mathcal{P}(E)$.

Lemma 10. *1. If (E, d) has the middle point property and for every $p, q \in E$ there is a compact subset $K_{p,q}$ of E containing all $x \in E$ with $d(p, x) + d(x, q) = d(p, q)$, then d has the interval property.*

2. If (E, d) has the interval property, then it has the middle point property and for any $p \in E$ and $r \geq 0$, $B_r(p)$ is path-connected.

Proof. 1. Assume the middle point property. Let $U = \{\frac{k}{2^n} \mid k, n \in \mathbb{N},\ 0 \leq k \leq 2^n\}$. We construct $f : U \to E$ by induction on the exponent n. For $n = 0$, $f(0) = p$ and $f(1) = q$; for $n \geq 0$ and $0 \leq k \leq 2^n - 1$, given $x = f\left(\frac{k}{2^n}\right)$ and $y = f\left(\frac{k+1}{2^n}\right)$, we choose for $f\left(\frac{2k+1}{2^{n+1}}\right)$ any $z \in E$ such that $d(x, z) = d(z, y) = \frac{1}{2} d(x, y)$. By induction, we get that for $0 \leq k \leq 2^n - 1$, $d\left(f\left(\frac{k}{2^n}\right), f\left(\frac{k+1}{2^n}\right)\right) = \frac{1}{2^n} d(p, q)$. Let $\alpha, \beta \in U$ such that $\alpha < \beta$; then there are $a, b, n \in \mathbb{N}$ with $0 \leq a < b \leq 2^n$ such that $\alpha = \frac{a}{2^n}$ and $\beta = \frac{b}{2^n}$. We get then:

$$d(f(\alpha), f(\beta)) = d\left(f\left(\frac{a}{2^n}\right), f\left(\frac{b}{2^n}\right)\right)$$
$$\leq \sum_{k=a}^{b-1} d\left(f\left(\frac{k}{2^n}\right), f\left(\frac{k+1}{2^n}\right)\right) = \frac{b-a}{2^n} d(p, q) = (\beta - \alpha) d(p, q) \ .$$

Similarly, $d\big(f(0), f(\alpha)\big) \leq \alpha d(p, q)$ and $d\big(f(\beta), f(1)\big) \leq (1 - \beta) d(p, q)$. Hence

$$\begin{aligned} d(p, q) = d\big(f(0), f(1)\big) &\leq d\big(f(0), f(\alpha)\big) + d\big(f(\alpha), f(\beta)\big) + d\big(f(\beta), f(1)\big) \\ &\leq \alpha d(p, q) + d\big(f(\alpha), f(\beta)\big) + (1 - \beta) d(p, q) \ , \end{aligned}$$

thus $(\beta - \alpha) d(p, q) \leq d\big(f(\alpha), f(\beta)\big)$. From the double inequality, the equality $d\big(f(\alpha), f(\beta)\big) = (\beta - \alpha) d(p, q)$ follows. In particular, $f(\alpha) \in K_{p,q}$ for all $\alpha \in U$. Since f is a continuous function $U \to K_{p,q}$ and the compact metric space $K_{p,q}$ is complete, there is a continuous extension of f to a function $[0, 1] \to K_{p,q}$. More precisely, given $\alpha \in [0, 1]$, for any $n \in \mathbb{N}$ we set $\alpha_n = \lfloor 2^n \alpha \rfloor / 2^n$; then $\alpha_n \leq \alpha < \alpha_n + \frac{1}{2^n}$, so $\lim_{n\to\infty} \alpha_n = \alpha$; then the $f(\alpha_n)$, $n \in \mathbb{N}$ constitute a Cauchy sequence in $K_{p,q}$, which converges in $K_{p,q}$ to some point that we define as $f(\alpha)$. For $0 \leq \alpha < \beta \leq 1$, by continuity we have

$$d\big(f(\alpha), f(\beta)\big) = \lim_{n\to\infty} d\big(f(\alpha_n), f(\beta_n)\big) = \lim_{n\to\infty} (\beta_n - \alpha_n) d(p, q) = (\beta - \alpha) d(p, q) \ .$$

Therefore (E, d) has the interval property.

2. Assume the interval property. The middle point property follows from taking $x = f\left(\frac{1}{2}\right)$. Given $r \geq 0$, for any $q \in B_r(p)$, we have a continuous map $f : [0, 1] \to E$ with $f(0) = p$, $f(1) = q$ and for $\alpha \in [0, 1]$, $d\big(p, f(\alpha)\big) = \alpha d(p, q) \leq r$, that is, $f(\alpha) \in B_r(p)$; then f is a path $[0, 1] \to B_r(p)$ joining p to q; hence $B_r(p)$ is path-connected. □

Corollary 11. *Let E be boundedly compact. Then (E, d) has the middle point property if and only if it has the interval property; then for any $p \in E$ and $r \geq 0$, $B_r(p)$ is path-connected.*

Proof. For $p, q \in E$, let $r = d(p, q)$; then for any $x \in E$ with $d(p, x) + d(x, q) = d(p, q) = r$, we have $x \in B_r(p)$, which is compact; we can thus apply Lemma 10 with $K_{p,q} = B_r(p)$. □

The connectedness of closed balls is useful in relation to Property 3: if all $B_r(p)$ are connected, then for a connected set X, $\delta_r(X)$ will be connected.

Definition 5. *Let $F \in \mathcal{F}'(E)$ and $s \geq 0$. We says that F is* s-separated *if F can be partitioned into $F_1, F_2 \in \mathcal{F}'(E)$ such that for any $x_1 \in F_1$ and $x_2 \in F_2$, $d(x_1, x_2) > s$. We say that F is* separated *if for some $s > 0$, F is s-separated, and that F is* non-separated *if for every $s > 0$, F is not s-separated.*

Note that F is separated iff it can be partitioned into two non-empty closed subsets F_1 and F_2 such that $\inf\{d(x_1, x_2) \mid x_1 \in F_1, x_2 \in F_2\} > 0$. For $s > s' \geq 0$, if F is s-separated, then it is s'-separated. Moreover, F is

0-separated iff it is disconnected, that is, partitioned into two non-empty closed sets. Thus a connected set is not 0-separated, hence it is non-separated.

Lemma 12. *A non-empty compact set K is non-separated if and only if it is connected.*

Proof. If K is connected, it is not 0-separated, thus it is not s-separated for any $s > 0$. If K is disconnected, then it is partitioned by two closed sets F_1 and F_2, so it is partitioned into two compact sets K_1 and K_2. By the compactness of K_1, K_2 and the continuity of the distance function, the set of all $d(x_1, x_2)$ for $x_1 \in K_1$ and $x_2 \in K_2$ reaches a minimum a; since K_1 and K_2 are disjoint, $a > 0$, so K is s-separated for any $s > 0$ such that $s < a$. □

Example 2. *A non-compact closed set can be both disconnected (0-separated) and non-separated. For instance (cf. Section 5 of [22]), in $\mathbb{R}^2$ with the Euclidean distance, the closed set $F = \{(\pm x, 1/x) \mid x \in]0, 1]\}$ is partitioned into the two closed sets $F_1 = \{(-x, 1/x) \mid x \in]0, 1]\}$ and $F_2 = \{(+x, 1/x) \mid x \in]0, 1]\}$ (which are its connected components), with $d\big((-x, 1/x), (+x, 1/x)\big) = 2x$, tending to 0 for $x \to 0$. Note that $\delta_r(F)$ is connected for any $r > 0$.*

Proposition 13. *Let (E, d) be boundedly compact and satisfying the middle point property. Then for any $F \in \mathcal{F}'(E)$ and $s > 0$, F is s-separated if and only if $\delta_{s/2}(F)$ is disconnected. In particular, F is non-separated if and only if for any $r > 0$, $\delta_r(F)$ is connected.*

Proof. Every closed set X is proximinal by Property 4, so $\delta_{s/2}(X)$ is closed by Property 5.

If F is s-separated, then it is partitioned into two closed sets F_1, F_2 such that for any $x_1 \in F_1$ and $x_2 \in F_2$, $d(x_1, x_2) > s$. If we had $p \in B_{s/2}(x_1) \cap B_{s/2}(x_2)$, then we would get $d(x_1, x_2) \leq d(x_1, p) + d(p, x_2) \leq s$, a contradiction. Thus $B_{s/2}(x_1) \cap B_{s/2}(x_2) = \emptyset$ for all $x_1 \in F_1$ and $x_2 \in F_2$, hence $\delta_{s/2}(F_1)$ and $\delta_{s/2}(F_2)$ are disjoint. As for $i = 1, 2$, $\delta_{s/2}(F_i)$ is closed and $\delta_{s/2}(F_i) \supseteq F_i \neq \emptyset$, and $\delta_{s/2}(F_1) \cup \delta_{s/2}(F_2) = \delta_{s/2}(F_1 \cup F_2) = \delta_{s/2}(F)$, $\delta_{s/2}(F)$ is disconnected.

If $\delta_{s/2}(F)$ is disconnected, it is partitioned into $A_1, A_2 \in \mathcal{F}'(E)$. As $F \subseteq \delta_{s/2}(F)$, F is the disjoint union of the two closed sets $F_1 = A_1 \cap F$ and $F_2 = A_1 \cap F$. For $i = 1, 2$ and $x_i \in F_i$, $B_{s/2}(x_i)$ is connected by Corollary 11, so it cannot be partitioned by A_1 and A_2; we deduce that $B_{s/2}(x_i) \subseteq A_i$. If we had $F = F_i$, we would get $B_{s/2}(x) \subseteq A_i$ for all $x \in F$, so $\delta_{s/2}(F) \subseteq A_i$, a contradiction; therefore $\{F_1, F_2\}$ is a partition of F. If we had $x_1 \in F_1$ and $x_2 \in F_2$ with $d(x_1, x_2) \leq s$, then by the middle point property there would be $p \in E$ such that $d(x_1, p) = d(x_2, p) = d(x_1, x_2)/2 \leq s/2$, so $p \in B_{s/2}(x_1) \cap B_{s/2}(x_2)$; but $B_{s/2}(x_i) \subseteq A_i$ $(i = 1, 2)$, so we would have $p \in A_1 \cap A_2$, a contradiction. Therefore for all $x_1 \in F_1$ and $x_2 \in F_2$ we have $d(x_1, x_2) > s$, thus F is s-separated.

Now F is non-separated iff for all $r > 0$ it is not $2r$-separated, that is, $\delta_r(F)$ is connected for any $r > 0$. □

Serra [19] generalized connections from $\mathcal{P}(E)$ to complete lattices. Now $\mathcal{F}(E)$ is a complete lattice, where the infimum and supremum of a family are respectively its intersection and the closure of its union, and it comprises all singletons (as closed balls of radius 0); thus Serra's definition takes here the following form:

Definition 6. *A* partial connection on $\mathcal{F}(E)$ *is a family $\mathcal{C} \subseteq \mathcal{F}(E)$ such that*

1. *$\emptyset \in \mathcal{C}$, and*
2. *for any $\mathcal{B} \subseteq \mathcal{C}$ such that $\bigcap \mathcal{B} \neq \emptyset$, we have $\overline{\bigcup \mathcal{B}} \in \mathcal{C}$.*

The partial connection $\mathcal{C}$ is a connection on $\mathcal{F}(E)$ *if it satisfies the following third condition:*

3. *for all $p \in E$, $\{p\} \in \mathcal{C}$.*

Then Property 1 is also valid with $\mathcal{F}(E)$ instead of $\mathcal{P}(E)$: an intersection of connections (resp., partial connections) on $\mathcal{F}(E)$ is a connection (resp., partial connection) on $\mathcal{F}(E)$.

Proposition 14. *For any $s \geq 0$, the family of closed sets that are not s-separated constitutes a connection on $\mathcal{F}(E)$. The family of non-separated closed sets is also a connection on $\mathcal{F}(E)$.*

Proof. Obviously the empty set and the singletons are closed and cannot be partitioned by two sets, so they are not s-separated. Let $\mathcal{B}$ be a family of closed sets that are not s-separated, such that $\bigcap \mathcal{B} \neq \emptyset$; let $F = \overline{\bigcup \mathcal{B}}$ and let $z \in \bigcap \mathcal{B}$. Suppose that F is s-separated: F is partitioned into $F_1, F_2 \in \mathcal{F}'(E)$ such that for any $x_1 \in F_1$ and $x_2 \in F_2$, $d(x_1, x_2) > s$. As $z \in F$, for some $i = 1, 2$ we have $z \in F_i$. For any $B \in \mathcal{B}$, as $z \in B$, $B \cap F_i \neq \emptyset$; as B is not s-separated, it cannot be partitioned by F_1 and F_2, so we must have $B \subseteq F_i$. We deduce that $\bigcup \mathcal{B} \subseteq F_i$, and as F_i is closed, $F = \overline{\bigcup \mathcal{B}} \subseteq F_i$, which contradicts the partitioning of F. Therefore F is not s-separated.

The family of non-separated closed sets is the intersection, for all $s > 0$, of the families of closed sets that are not s-separated. It is thus an intersection of connections, hence it is a connection on $\mathcal{F}(E)$. □

3.2 Discretizing Non-Separated Sets

Let (E, d) be a metric space and $D \subset E$, $D \neq \emptyset$, satisfying Axiom 1.

Definition 7. *Let $r > 0$, $S \in \mathcal{F}'(D)$ and $Z \in \mathcal{P}'(E)$. Then write $G_r(S)$ and $G_r^Z(S)$ for the two graphs both with vertex set S, and with an edge joining any two* distinct *$p, q \in S$ such that respectively:*

- *$B_r(p) \cap B_r(q) \neq \emptyset$ for $G_r(S)$;*
- *$B_r(p) \cap B_r(q) \cap Z \neq \emptyset$ for $G_r^Z(S)$.*

Note that the all edges of $G_r^Z(S)$ are edges of $G_r(S)$, and that when r grows, the sets of edges of $G_r(S)$ and $G_r^Z(S)$ will be growing. Thus connectedness extends from $G_r^Z(S)$ to $G_r(S)$, and for $r' > r$, from $G_r(S)$ to $G_{r'}(S)$ and from $G_r^Z(S)$ to $G_{r'}^Z(S)$.

For $p, q \in E$, $B_r(p) \cap B_r(q) \neq \emptyset$ implies that $d(p, q) \leq 2r$. If (E, d) has the middle point property, then for $p, q \in E$, we have $B_r(p) \cap B_r(q) \neq \emptyset \iff d(p, q) \leq 2r$, which simplifies the definition of $G_r(S)$. This fact intervenes in the following result, which will be used below.

Lemma 15. *Consider the following three conditions:*

(A) E is a vector space, D is an additive subgroup of E and the metric d is based on a norm N on E.
(B) (E, d) satisfies the middle point property and the set $\{d(p, q) \mid p, q \in D\}$ is boundedly finite.
(C) For any $R > 0$, there exists $r > R$ such that for all $S \subseteq D$, $G_r(S) = G_R(S)$.

Then, (A) implies (B) and (B) implies (C).

Proof. Let $X = \{d(p, q) \mid p, q \in D\}$. If (A) holds, then for $p, q \in D$, $d(p, q) = N(p - q)$, so $X = \{N(x) \mid x \in D\}$. Given the origin o in E, for any $r > 0$, $B_r(o) = \{x \in D \mid N(x) \leq r\}$ is finite by Axiom 1; thus $X \cap [0, r]$ is finite. Hence X is boundedly finite. Now for any $p, q \in D$, $x = \frac{1}{2}(p + q)$ satisfies

$$d(p, x) = d(x, q) = N\left(\frac{p - q}{2}\right) = \frac{1}{2}N(p - q) = \frac{1}{2}d(p, q) \ ,$$

so (E, d) has the middle point property. Therefore (B) follows.

If (B) holds, then for any $R > 0$, $X \cap]2R, 2R + 1]$ is finite; thus there is some $r > R$ such that $X \cap]2R, 2r] = \emptyset$. Thus for any $p, q \in D$, $d(p, q) \leq 2R \iff d(p, q) \leq 2r$; as (E, d) has the middle point property, we get $B_R(p) \cap B_R(q) \neq \emptyset \iff B_r(p) \cap B_r(q) \neq \emptyset$. Hence $G_r(S)$ has the same edges as $G_R(S)$, so the two graphs are equal. Therefore (C) holds. □

We give below a relation between graph connectivity and chaining by sets, which will be used in our main results:

Proposition 16. *Let A be a non-empty subset of E and let $\mathcal{B}$ be a non-empty family of non-empty subsets of A. Let G be the undirected graph with vertex set A and with edges joining all pairs $\{x, y\}$ of distinct elements of A such that there is $B \in \mathcal{B}$ with $x, y \in B$. Then G is connected if and only if A is chained by $\mathcal{B}$.*

Proof. If $|A| = 1$, we have $A = \{p\}$, G is connected, and we must have $\mathcal{B} = \{A\}$, so A is chained by $\mathcal{B}$. Suppose now that $|A| \geq 2$ and consider two distinct $p, q \in A$.

Now, p and q are chained in $\mathcal{B}$ iff there are $n \geq 0$ and $B_0, \dots, B_n \in \mathcal{B}$ such $p \in B_0$, $q \in B_n$ and for $t = 1, \dots, n$, $B_{t-1} \cap B_t \neq \emptyset$. Writing $p = u_0$ and $q = u_{n+1}$, and choosing any $u_t \in B_{t-1} \cap B_t$ ($t = 1, \dots, n$), the statement becomes: there are the two sequences $p = u_0, \dots, u_{n+1} = q$ in A and $B_0, \dots, B_n$ in $\mathcal{B}$ such that for $t = 0, \dots, n$, we have $u_t, u_{t+1} \in B_t$. In other words, we have the sequence $p = u_0, \dots, u_{n+1} = q$ in A such that for $t = 0, \dots, n$, u_t and u_{t+1} are joined by an edge. Hence it is equivalent to: p and q are joined by a chain of edges of G.

We have shown that two *distinct* vertices $p, q \in A$ are joined by a chain of edges in G iff they are chained in $\mathcal{B}$. Assuming this property, p is chained to q and q is chained back to p (both for edges and for $\mathcal{B}$), so p is chained to itself. □

In the above proof, we had to distinguish the case where $|A| = 1$ and the chaining of a point p to itself. Indeed, as said in Subsection 2.1, $\mathtt{Con}^*(\mathcal{B})$, the *partial connection* generated by $\mathcal{B}$, is obtained through chaining by $\mathcal{B}$, with chains of length at least 1, and excluding isolated points (that is, not belonging to any element of $\mathcal{B}$); on the other hand, in a graph we take the *connection* generated by edges, based on chains of edges of length at least 0, including thus isolated vertices (that is, not incident to any edge).

We can now analyse the connectedness of Hausdorff discretizations of non-separated closed sets:

Lemma 17. *Let $r > 0$, $s, t \geq 0$, $X \in \mathcal{P}'(E)$, $Y \in \mathcal{F}'(E)$ and $S \in \mathcal{F}'(D)$, such that Y is not t-separated, $X \subseteq Y \subseteq \delta_s(X)$, $S \subseteq \Delta_{r+s}(X)$ and $X \subseteq \delta_r(S)$. Then $G^Y_{r+s+t}(S)$, $G^X_{r+2s+t}(S)$ and $G_{r+s+t}(S)$ are connected, and S is chained by $\{B_{r+s+t}(y) \cap S \mid y \in Y\}$ and by $\{B_{r+2s+t}(x) \cap S \mid x \in X\}$.*

Proof. As $X \subseteq \delta_r(S)$ and $Y \subseteq \delta_s(X)$, we get $Y \subseteq \delta_s(X) \subseteq \delta_s(\delta_r(S)) \subseteq \delta_{r+s}(S)$; hence for any $y \in Y$ we have $B_{r+s}(y) \cap S \neq \emptyset$. As $S \subseteq \Delta_{r+s}(X)$, for any $p \in S$ we have $B_{r+s}(p) \cap X \neq \emptyset$ by (13).

Consider a partition S into S_1 and S_2. Now $Y \subseteq \delta_{r+s}(S) = \delta_{r+s}(S_1 \cup S_2) = \delta_{r+s}(S_1) \cup \delta_{r+s}(S_2)$. For $i = 1, 2$, every $p \in S_i$ satisfies $B_{r+s}(p) \cap X \neq \emptyset$, hence $\delta_{r+s}(S_i) \cap X \neq \emptyset$; as $X \subseteq Y$, $\delta_{r+s}(S_i) \cap Y \neq \emptyset$. Next, S_i is proximinal by Property 4, so $\delta_{r+s}(S_i)$ is closed by Property 5. Thus Y is the union of the two non-empty closed sets $\delta_{r+s}(S_i) \cap Y$ for $i = 1, 2$. If $d(y_1, y_2) > t$ for all $y_i \in \delta_{r+s}(S_i) \cap Y$ ($i = 1, 2$), then Y is t-separated, a contradiction. Hence there are $y_i \in \delta_{r+s}(S_i) \cap Y$ ($i = 1, 2$) such that $d(y_1, y_2) \leq t$; now $y_i \in B_{r+s}(p_i)$ for some $p_i \in S_i$; then $y_2 \in B_{r+s+t}(p_1)$, so $y_2 \in B_{r+s+t}(p_1) \cap B_{r+s+t}(p_2) \cap Y$; in particular, $y_2 \in B_{r+s+t}(p_1) \cap B_{r+s+t}(p_2)$. As $y_2 \in Y \subseteq \delta_s(X)$, there is some $x \in X$ with $d(x, y_2) \leq s$, so $x \in B_{r+2s+t}(p_1) \cap B_{r+2s+t}(p_2) \cap X$. Thus for any partition $\{S_1, S_2\}$ of S, there are $p_1 \in S_1$ and $p_2 \in S_2$ joined by an edge in each of the graphs $G^Y_{r+s+t}(S)$, $G^X_{r+2s+t}(S)$ and $G_{r+s+t}(S)$, therefore these three graphs are connected.

For two distinct $p, q \in S$, p and q are joined by an edge of $G^Y_{r+s+t}(S)$ iff $B_{r+s+t}(p) \cap B_{r+s+t}(q) \cap Y \neq \emptyset$, that is, there is some $y \in Y$ such that $y \in B_{r+s+t}(p) \cap B_{r+s+t}(q)$, in other words $p, q \in B_{r+s+t}(y)$. Hence: two distinct $p, q \in S$ are joined by an edge of $G^Y_{r+s+t}(S)$ iff there is some $y \in Y$ such that $p, q \in B_{r+s+t}(y) \cap S$. As $B_{r+s+t}(y) \cap S \neq \emptyset$ for all $y \in Y$, we can apply Proposition 16: S is chained by $\{B_{r+s+t}(y) \cap S \mid y \in Y\}$. The same argument with $G^X_{r+2s+t}(S)$ gives that S is chained by $\{B_{r+2s+t}(x) \cap S \mid x \in X\}$. □

Our first application of this result is for the Hausdorff discretization of non-empty connected proximinal sets:

Theorem 18. *Let $F \in \mathcal{F}'_{\mathrm{p}}(E)$ be connected and let $r \geq r_H(F)$. Then:*

1. *For any $S \subseteq \Delta_r(F)$ such that $F \subseteq \delta_r(S)$, the two graphs $G^F_r(S)$ and $G_r(S)$ are connected and S is chained by $\{B_r(x) \cap S \mid x \in F\}$.*
2. *$\Delta_r(F)$ is chained by $\{\Delta_r(x) \mid x \in F\}$.*

In particular, for any $S \in \mathcal{M}_H(F)$, $G_{r_H(F)}(S)$ and $G^F_{r_H(F)}(S)$ are both connected, and $\Delta_H(F)$ is chained by $\{\Delta_{r_H(F)}(x) \mid x \in F\}$.

Proof. We apply Lemma 17 with $s = t = 0$ and $X = Y = F$, and combine it with Property 8, where $\Delta^+_r(F) = \Delta_r(F)$ because F is proximinal. Then $G^F_r(S)$ and $G_r(S)$ are connected and S is chained by $\{B_r(x) \cap S \mid x \in F\}$. Thus item 1 holds.

Now $F \subseteq \delta_r(\Delta_r(F))$, so we apply item 1 with $S = \Delta_r(F)$. For $x \in F$, $B_r(x) \subseteq \delta_r(F)$, so (13) gives

$$B_r(x) \cap \Delta_r(F) = B_r(x) \cap \delta_r(F) \cap D = B_r(x) \cap D = \Delta_r(x) \ .$$

We get thus item 2.

By Property 9, for any $S \in \mathcal{M}_H(F)$, $S \subseteq \Delta^+_{r_H(F)}(F)$ and $F \subseteq \delta_{r_H(F)}(S)$; since $\Delta_H(F) = \Delta^+_{r_H(F)}(F) = \Delta_{r_H(F)}(F)$, the last sentence of the above statement follows. □

In view of Property 2, we can write:

$$\Delta_H(F) \in \mathsf{Con}^*\left(\{\Delta_{r_H(F)}(x) \mid x \in F\}\right) \subseteq \mathsf{Con}^*\left(\{\Delta_{r_H(F)}(x) \mid x \in E\}\right) \ .$$

For any $r > 0$, the set of all $S \subseteq D$ such that $G_r(S)$ is connected, that is, such that S spans a connected subgraph of $G_r(D)$, is a connection. Thus, all Hausdorff discretizations of all connected proximinal subsets of E (including $\emptyset$), belong to the connection of all connected subsets of the graph $G_{r_c}(D)$.

Remark 1. *For an arbitrary $S \in \mathcal{M}_H(F)$, the above result is optimal: in general we cannot get connectivity for a graph $G^F_r(S)$ with $r < r_H(F)$. As we will see in Section 4 (see in particular Property 24), for $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and a distance induced by a coordinate-symmetrical norm, such as the L_p norms (1), for $p = (\frac{1}{2}, \ldots, \frac{1}{2})$, we have $r_c = r_H(\{p\}) = N(p)$. Thus for $o = (0, \ldots, 0)$ and $q = (1, \ldots, 1)$, we have $d(o, p) = d(q, p) = r_H(\{p\})$, so $\{o, q\} \in \mathcal{M}_H(\{p\})$, with $d(o, q) = 2r_H(\{p\})$.*

We now consider the discretization of non-separated sets. The result will be slightly weaker than for connected ones:

Theorem 19. *Let $F \in \mathcal{F}'(E)$ such that one of the following holds:*

(a) F is proximinal and for any $s > 0$, $\delta_s(F)$ is connected.
(b) F is non-separated.

Then for all $r > r_H(F)$:

1. *For any $S \in \mathcal{M}_H(F)$, the two graphs $G^F_r(S)$ and $G_r(S)$ are connected.*
2. *$\Delta_H(F)$ is chained by $\{\Delta_r(x) \cap \Delta_H(F) \mid x \in F\}$.*

Proof. We first prove that in both cases (a) and (b), for any $S \in \mathcal{M}_H(F)$ and $r > r_H(F)$, $G^F_r(S)$ and $G_r(S)$ are connected and S is chained by $\{B_r(x) \cap S \mid x \in F\}$. In particular item 1 holds.

(a) Take any $s > 0$. By Property 5, $\delta_s(F)$ is closed and $\Delta^+_{r_H(F)}(F) = \Delta_{r_H(F)}(F)$. Thus Property 9 gives $F \subseteq \delta_{r_H(F)}(S)$ and $S \subseteq \Delta_{r_H(F)}(F) \subseteq \Delta_{r_H(F)+s}(F)$. We apply Lemma 17 with $r = r_H(F)$, $t = 0$, $X = F$ and $Y = \delta_s(F)$: then $G^F_{r_H(F)+2s}(S)$ and $G_{r_H(F)+s}(S)$ are connected, and S is chained by $\{B_{r_H(F)+2s}(x) \cap S \mid x \in F\}$; then $G_{r_H(F)+2s}(S)$ is also connected. The result follows by taking $r = r_H(F) + 2s$.

(b) Take any $s > 0$. By Property 9, $F \subseteq \delta_{r_H(F)}(S)$ and $S \subseteq \Delta^+_{r_H(F)}(F) \subseteq \Delta_{r_H(F)+s}(F)$ (the last inclusion follows from (14)). Now F is not s-separated. We apply Lemma 17 with $r = r_H(F)$, $t = s$, and $X = Y = F$: then with $F = Y$, $G^F_{r_H(F)+2s}(S)$ and $G_{r_H(F)+2s}(S)$ are connected, and S is chained by $\{B_{r_H(F)+2s}(x) \cap S \mid x \in F\}$. The result follows by taking $r = r_H(F) + 2s$.

Now take $S = \Delta_H(F)$. Then $\Delta_H(F)$ is chained by the $\{B_r(x) \cap \Delta_H(F) \mid x \in F\}$; since $\Delta_H(F) \subseteq D$, we have

$$B_r(x) \cap \Delta_H(F) = B_r(x) \cap D \cap \Delta_H(F) = \Delta_r(x) \cap \Delta_H(F) \ .$$

Item 2 follows. □

Thus, for al $r > r_c$, every Hausdorff discretization of any closed subset of E (including $\emptyset$) satisfying condition (a) or (b), belongs to the connection of all connected subsets of the graph $G_r(D)$. In other words, it belongs to the connection which is the intersection of all connections of $G_r(D)$, $r > r_c$.

The fact that $G_r(S)$ is connected for all $r > r_H(F)$ does not necessarily mean that $G_{r_H(F)}(S)$ is connected, as shows the following example:

Example 3. *Let $E = \mathbb{R}^2$, $D = \{(z, \pm(2n + 1 + 2^{-|z|})) \mid z \in \mathbb{Z},\ n \in \mathbb{N}\}$, and $F = \{(y, \pm 2^{-|y|}) \mid y \in \mathbb{R}\}$. We take the distance induced by the L_∞ norm (1). See Figure 5 (a). Each point of F is at distance at most 1 from the closest point in D; for instance in the top right quadrant, for $y \in \mathbb{N}$, $(y, 2^{-y})$ is at distance 1 from $(y, 1 + 2^{-y})$ and $(y+1, 1+2^{-(y+1)})$, while for $y \notin \mathbb{N}$, $y > 0$, $(y, 2^{-y})$ is at distance at most 1 from $(\lceil y \rceil, 1+2^{-\lceil y \rceil})$. Thus $r_H(F) = 1$ and $\Delta_H(F) = \{(z, \pm(1+2^{-|z|})) \mid z \in \mathbb{Z}\}$. Now $G_{r_H(F)}(\Delta_H(F))$ has two connected components, $\{(z, (1+2^{-|z|})) \mid z \in \mathbb{Z}\}$ and $\{(z, -(1 + 2^{-|z|})) \mid z \in \mathbb{Z}\}$. For any $s > 0$ and $z \in \mathbb{N}$ large enough (in absolute value) to have $2^{-|z|} \leq s$, the distance between $(z, 1+2^{-|z|})$ and $(z, -(1+2^{-|z|}))$ is $2(1+2^{-|z|})) \leq 2(r_H(F)+s)$, so $G_{r_H(F)+s}(\Delta_H(F))$ is connected.*

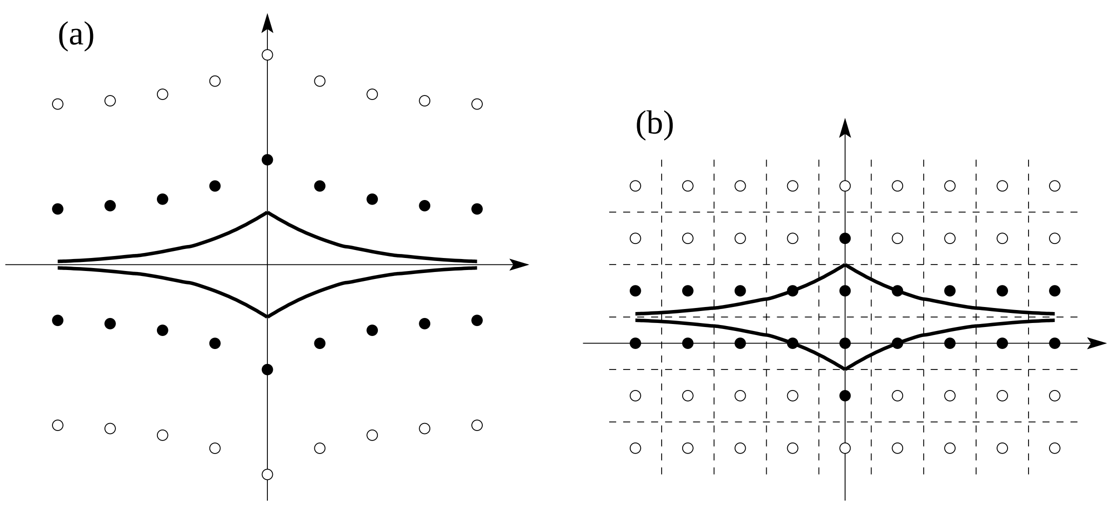

Figure 5: Let $E = \mathbb{R}^2$ with d is induced by the L_∞ norm. In both cases (a,b), the elements of $\Delta_H(F)$ are shown as filled disks, and those of $D \setminus \Delta_H(F)$ as hollow disks. (a) Here $D = \{(z, \pm(2n+1+2^{-|z|})) \mid z \in \mathbb{Z},\ n \in \mathbb{N}\}$, $F = \{(y, \pm 2^{-|y|}) \mid y \in \mathbb{R}\}$, and $r_H(F) = 1$. For all $s > 0$, $G_{r_H(F)+s}(\Delta_H(F))$ is connected, but $G_{r_H(F)}(\Delta_H(F))$ is not connected. (b) Here $D = \mathbb{Z}^2$, $F = \{(y, \frac{1}{2} \pm 2^{-|y|}) \mid y \in \mathbb{R}\}$, and $r_H(F) = r_c = 1/2$. Then $G_{r_H(F)}(\Delta_H(F))$ is connected, but $G^F_{r_H(F)}(\Delta_H(F))$ is not connected, and $\Delta_H(F)$ is not chained by the $\Delta_{r_H(F)}(x) \cap \Delta_H(F)$ for $x \in F$.

However, $G_{r_H(F)}(S)$ will be connected in the "usual" cases where $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and the metric d is based on a norm:

Corollary 20. *Assume the hypothesis of Theorem 19. If (E, d) satisfies one of the conditions (A), (B) or (C) of Lemma 15, then for any $S \in \mathcal{M}_H(F)$, $G_{r_H(F)}(S)$ is connected.*

Indeed, there is then some $r > r_H(F)$ such that $G_r(S) = G_{r_H(F)}(S)$. Such an argument does not apply to $G^F_r(S)$, nor to $\Delta_r(x)$ in the chaining by the $\Delta_r(x) \cap \Delta_H(F)$, so $G^F_{r_H(F)}(S)$ can be disconnected, and $\Delta_H(F)$ is not necessarily chained by the $\Delta_{r_H(F)}(x) \cap \Delta_H(F)$ for $x \in F$, even when $E = \mathbb{R}^n$ and $D = \mathbb{Z}^n$ with a metric based on a norm:

Example 4. *Let $E = \mathbb{R}^2$, $D = \mathbb{Z}^2$, and $F = \{(y, \frac{1}{2} \pm 2^{-|y|}) \mid y \in \mathbb{R}\}$. We take again the distance induced by the L_∞ norm (1). See Figure 5 (b). Each point of F is at distance at most $1/2$ from the closest point in D (to whose digital cell it belongs). Thus $r_H(F) = r_c = 1/2$ and $\Delta_H(F) = (\mathbb{Z} \times \{0, 1\}) \cup \{(0, -1), (0, 2)\}$. Now for $x = (y, \frac{1}{2} + 2^{-|y|})$ ($y \in R$), $\Delta_{r_H(F)}(x) \in (\mathbb{Z} \times \{1\}) \cup \{(0, 2)\} = S_1$, while for $x = (y, \frac{1}{2} - 2^{-|y|})$ ($y \in R$),*

$\Delta_{r_H(F)}(x) \in \big(\mathbb{Z}\times\{0\}\big) \cup \{(0,-1)\} = S_2$. Thus $G^F_{r_H(F)}(\Delta_H(F))$ has two connected components, S_1 and S_2, and the same connected components are obtained in the chaining by the $\Delta_{r_H(F)}(x) \cap \Delta_H(F)$ for $x \in F$. For any $s > 0$ and $y \in \mathbb{N}$ large enough (in absolute value) to have $2^{-|y|} \leq s$, the distance between each $\big(y, \frac{1}{2} \pm 2^{-|y|}\big)$ and each of $(y,0)$ and $(y,1)$ is $\leq \frac{1}{2} + 2^{-|y|} \leq r_H(F)+s$, so $(y,0)$ and $(y,1)$ will be joined together by an edge of $G^F_{r_H(F)+s}(\Delta_H(F))$, and be in a block $\Delta_{r_H(F)+s}(x) \cap \Delta_H(F)$ of the chaining. Hence $G^F_{r_H(F)+s}(\Delta_H(F))$ is connected and $\Delta_H(F)$ is chained by the $\Delta_{r_H(F)+s}(x) \cap \Delta_H(F)$ for $x \in F$.

3.3 Discretizing Separated Sets in Multiple Resolutions

We will now give in some way the converse of the results of the previous subsection, namely that every Hausdorff discretization of a s-separated closed set ($s > 0$) is disconnected (for the graphs in Definition 7) when the Hausdforff radius r_c is small compared to s. Next, we will consider discretization at varying resolutions, and we will see that a non-empty closed set has its Hausdorff discretizations connected at all resolutions if and only if it is non-separated.

Proposition 21. *Consider a s-separated $F \in \mathcal{F}'(E)$, where $s > 0$, and let $S \in \mathcal{M}_H(F)$.*

1. *If $G^F_r(S)$ is connected for some $r > r_H(F)$, then $s < 2r$; if it is for all $r > r_H(F)$, then $s \leq 2r_c$.*
2. *If $G_r(S)$ is connected for some $r > r_H(F)$, then $s < 4r$; if it is for all $r > r_H(F)$, then $s \leq 4r_c$.*

Proof. Take any $r > r_H(F)$. By Properties 8 and 9, $F \subseteq \delta_{r_H(F)}(S) \subseteq \delta_r(S)$ and $S \subseteq \Delta^+_{r_H(F)}(F) \subseteq \Delta_r(F)$. The s-separated set F is partitioned into $F_1, F_2 \in \mathcal{F}'(E)$ such that for all $x_1 \in F_1$ and $x_2 \in F_2$, $d(x_1, x_2) > s$. For $i = 1, 2$, let S_i be the set of all $p \in S$ such that $B_r(p) \cap F_i \neq \emptyset$; as $S \subseteq \Delta_r(F)$, for every $p \in S$ we have $B_r(p) \cap F \neq \emptyset$, hence $S = S_1 \cup S_2$. As $F \subseteq \delta_r(S)$, for every $x \in F$, there is some $p \in S$ such that $x \in B_r(p)$; in particular, taking $x \in F_i$ ($i = 1, 2$), we get $S_i \neq \emptyset$.

If one of the graphs $G^F_r(S)$ or $G_r(S)$ is connected, then $S_1 \cap S_2 \neq \emptyset$ or that connected graph has an edge with extremities $p_1 \in S_1$ and $p_2 \in S_2$.

If $S_1 \cap S_2 \neq \emptyset$, then for $p \in S_1 \cap S_2$ and $i = 1, 2$ we have some $x_i \in B_r(p) \cap F_i$, so $d(p, x_i) \leq r$. Then

$$s < d(x_1, x_2) \leq d(x_1, p) + d(p, x_2) \leq 2r \ .$$

If $G^F_r(S)$ is connected, then it has an edge with extremities $p_i \in S_i$ ($i = 1, 2$), so $B_r(p_1) \cap B_r(p_2) \cap F \neq \emptyset$; now for $x \in B_r(p_1) \cap B_r(p_2) \cap F$, either $x \in F_1$ and then $p_2 \in S_1$, or $x \in F_2$ and then $p_1 \in S_2$. Thus one of p_1, p_2 belongs to both S_1 and S_2, hence $S_1 \cap S_2 \neq \emptyset$, and $s < 2r$. If this inequality holds for any $r > r_H(F)$, we get then $s \leq 2r_H(F) \leq 2r_c$. Thus item 1 holds.

If $G_r(S)$ is connected, then it has an edge with extremities $p_i \in S_i$ ($i = 1, 2$), so $B_r(p_1) \cap B_r(p_2) \neq \emptyset$, thus $d(p_1, p_2) \leq 2r$. Now for $i = 1, 2$, $B_r(p_i) \cap F_i \neq \emptyset$, so we have some $x_i \in B_r(p) \cap \in F_i$, thus $d(p_i, x_i) \leq r$. It follows that

$$s < d(x_1, x_2) \leq d(x_1, p_1) + d(p_1, p_2) + d(p_2, x_2) \leq 4r \ .$$

If this inequality $s < 4r$ holds for any $r > r_H(F)$, we get then $s \leq 4r_H(F) \leq 4r_c$. Thus item 2 holds. □

We will now consider Hausdorff discretization at multiple resolutions. We suppose a set $\mathcal{R}$ of arbitrarily small "grid steps" $\rho > 0$, and for each $\rho \in \mathcal{R}$, a subset $D(\rho)$ of E, where $\emptyset \neq D(\rho) \neq E$, for which we define the *covering radius for* ρ:

$$r_c(\rho) = \sup_{x \in E} d(x, D(\rho)) = h_d(E, D(\rho)) \ . \tag{15}$$

We extend then Axiom 1 as follows:

Axiom 2. *There is a set $\mathcal{R} \subset \{r \in \mathbb{R} \mid r > 0\}$ such that* $\inf \mathcal{R} = 0$ *and:*

1. *for every $\rho \in \mathcal{R}$, there exists $D(\rho)$ such that $\emptyset \subset D(\rho) \subset E$, $D(\rho)$ is boundedly finite and $r_c(\rho) < \infty$;*
2. *$r_c(\rho)$ is an increasing function of ρ and $\lim_{\rho \to 0} r_c(\rho) = 0$.*

Then, given $\rho \in \mathcal{R}$, for every $F \in \mathcal{F}'(E)$, the definitions and notations given for D in Subsection 2.3 extend to $D(\rho)$: the Hausdorff radius $r_H(F, \rho)$, the discretization and upper discretization of radius $r \geq 0$, $\Delta_r(F, \rho)$ and $\Delta_r^+(F, \rho)$, the family $\mathcal{M}_H(F, \rho)$ of Hausdorff discretizations of F, and the greatest Hausdorff discretization of F, $\Delta_H(F, \rho)$.

From now on, we will assume that (E, d) is boundedly compact and it has the middle point property. Then (E, d) has the interval property, every closed ball $B_r(p)$ is connected (Corollary 11), every closed set F is proximinal (Property 4), then for all $r > 0$, $\delta_r(F) = \delta_r^+(F)$, which is closed (Property 5), and for any $s > 0$, F is s-separated if and only if $\delta_{s/2}(F)$ is disconnected (Proposition 13).

We first show that these assumptions guarantee the existence of $D(\rho)$, $\rho \in \mathcal{R}$, satisfying Axiom 2. We define the *diameter* of E as $\operatorname{diam}(E) = \sup\{d(x, y) \mid x, y \in E\}$.

Proposition 22. *Let (E, d) be boundedly compact and having the middle point property. For any $\rho \in \mathbb{R}$ such that $0 < \rho < \operatorname{diam}(E)$, there exists $D(\rho) \subseteq E$ satisfying Axiom 1 with covering radius $r_c(\rho)$ such that $\rho/2 \leq r_c(\rho) \leq \rho$. Furthermore, there is some $n_0 \in \mathbb{N}$ such that $\mathcal{R} = \{2^{-n} \mid n \in \mathbb{N},\ n \geq n_0\}$ and the family of sets $D(\rho)$, $\rho \in \mathcal{R}$, satisfy Axiom 2.*

Proof. For $0 < \rho < \operatorname{diam}(E)$ there exist $p, q \in E$ such that $d(p, q) > \rho$. By the interval property, there exists $p_0 \in E$ such that $d(p, p_0) = \rho$. Choose $\sigma \in \mathbb{R}$ such that $0 < \sigma < \rho$. For every $n \in \mathbb{N}$ we define the two sets

$$C_n = \{x \in E \mid d(p, x) = \rho + n\sigma\} \quad \text{and}$$
$$R_n = \{x \in E \mid \rho + n\sigma \leq d(p, x) \leq \rho + (n + 1)\sigma\} \ .$$

Then $p_0 \in C_0$ and $C_n \subseteq R_n$ for all $n \in \mathbb{N}$; hence $R_0 \neq \emptyset$. For any $x \in E$, either $d(p, x) \leq \rho$ and $x \in B_\rho(p)$, or $d(p, x) > \rho$ and there is some $n \in \mathbb{N}$ such that $n\sigma < d(p, x) - \rho \leq (n + 1)\sigma$, hence $x \in R_n$. Thus $E = B_\rho(p) \cup \bigcup_{n\in\mathbb{N}} R_n$.

Since R_n is the inverse image by the continuous function $d(p, \cdot)$ of the closed interval $[\rho+n\sigma, \rho+(n+1)\sigma]$, it is closed; now R_n is bounded, and as E is boundedly compact, R_n is compact. For any $x \in R_n$, by the interval property there exists $y \in E$ such that $d(p, y) = \rho + n\sigma$ and $d(y, x) = d(p, x) - d(p, y) \leq \sigma < \rho$; in other words, $y \in C_n$ and $x \in B_\rho^\circ(y)$. Hence the union of open balls $B_\rho^\circ(y)$, $y \in C_n$, covers R_n, and as R_n is compact, there is a finite subset G_n of C_n such that the union of $B_\rho^\circ(y)$, $y \in G_n$, covers R_n; hence $R_n \subseteq \delta_\rho(G_n)$. Since $R_0 \neq \emptyset$, $G_0 \neq \emptyset$.

Let $D(\rho) = \{p\} \cup \bigcup_{n\in\mathbb{N}} G_n$; thus, $D(\rho) \neq \emptyset$. Every bounded subset of $D(\rho)$ is included in some $B_r(p)$, $r > 0$, and we have $r \leq \rho + m\sigma$ for some $m \in \mathbb{N}$, hence it is included in $\{p\} \cup \bigcup_{n=0}^m G_n$, which is finite; thus $D(\rho)$ is boundedly finite. Since $E = B_\rho(p) \cup \bigcup_{n\in\mathbb{N}} R_n$, with $\delta_\rho(p) = B_\rho(p)$ and $R_n \subseteq \delta_\rho(G_n)$ for all n, we have $\delta_\rho(D(\rho)) = E$, hence $r_c(\rho) \leq \rho$. As $d(p, p_0) = \rho$, by the interval property, there exists $p_1 \in E$ such that $d(p, p_1) = \rho/2$; then $p_1 \notin D(\rho)$, so $D(\rho) \neq E$. For $x \in G_n$, $d(p_1, x) \geq d(p, x) - d(p, p_1) = \rho/2 + n\sigma \geq \rho/2$; now, $d(p_1, p) = \rho/2$; as $D(\rho) = \{p\} \cup \bigcup_{n\in\mathbb{N}} G_n$, we deduce that $d(p_1, D(\rho)) = \rho/2$, hence $r_c(\rho) \geq \rho/2$. Therefore Axiom 1 is satisfied and $\rho/2 \leq r_c(\rho) \leq \rho$.

For some $n_0 \in \mathbb{N}$ we have $2^{-n_0} < \operatorname{diam}(E)$. Let $\mathcal{R} = \{2^{-n} \mid n \in \mathbb{N},\ n \geq n_0\}$. For each $n \geq n_0$, we have $2^{-n} < \operatorname{diam}(E)$, and we construct $D(\rho)$ for $\rho = 2^{-n}$, giving $2^{-n-1} \leq r_c(2^{-n}) \leq 2^{-n}$. It follows that for $n < m$, $r_c(2^{-m}) \leq 2^{-m} \leq 2^{-n-1} \leq r_c(2^{-n})$, thus $r_c(\rho)$ is an increasing function of ρ. Obviously $\lim_{\rho\to 0} r_c(\rho) = 0$. Therefore Axiom 2 is satisfied. □

We can now summarize Theorems 18 and 19 and Proposition 21 in the multi-resolution framework of Axiom 2:

Theorem 23. *Let (E, d) be boundedly compact and having the middle point property. Let $\mathcal{R}$ and the family of sets $D(\rho)$, $\rho \in \mathcal{R}$, satisfy Axiom 2. Let $F \in \mathcal{F}'(E)$.*

1. *If F is connected, then for all $\rho \in \mathcal{R}$, $\Delta_H(F, \rho)$ is chained by $\{\Delta_{r_H(F,\rho)}(x) \mid x \in F\}$ and for every $S \in \mathcal{M}_H(F, \rho)$, the two graphs $G_{r_H(F,\rho)}(S)$ and $G^F_{r_H(F,\rho)}(S)$ are connected.*
2. *If F is disconnected but non-separated, then for all $\rho \in \mathcal{R}$, for any $r > r_H(F, \rho)$, $\Delta_H(F, \rho)$ is chained by $\{\Delta_r(x) \cap \Delta_H(F, \rho) \mid x \in F\}$ and for every $S \in \mathcal{M}_H(F, \rho)$, the two graphs $G^F_r(S)$ and $G_r(S)$ are connected.*

3. *If F is s-separated for some $s > 0$, then there is some $\rho_0 \in \mathcal{R}$ such that for every $\rho \in \mathcal{R}$ with $\rho \leq \rho_0$, there is some $r > r_H(F, \rho)$ such that for every $S \in \mathcal{M}_H(F, \rho)$ and every $r' \leq r$, the two graphs $G^F_{r'}(S)$ and $G_{r'}(S)$ are disconnected.*

Let us now briefly consider the convergence to the original closed set of the discretization in resolution ρ when ρ tends to 0. If for every $\rho \in \mathcal{R}$ we choose some $S_\rho \in \mathcal{M}_H(F, \rho)$, as $H_d(F, S_\rho) = r_H(F, \rho) \leq r_c(\rho)$, then we will have $\lim_{\rho\to 0} H_d(F, S_\rho) = 0$, in other words $\lim_{\rho\to 0} S_\rho = F$ for the generalized Hausdorff metric H_d. Thus the Hausdorff discretizations tend to the original closed set when the resolution tends to 0.

Moreover, the argument of Proposition 21 can be extended to show that if we take a constant $c > 1$ and for every $\rho \in \mathcal{R}$ we choose some $S_\rho \subseteq D(\rho)$ (not necessarily a discretization of F) such that $G_{c\cdot r_c(\rho)}(S_\rho)$ is connected, if we have $\lim_{\rho\to 0} H_d(F, S_\rho) = 0$, then F must be non-separated.

4 Coordinate-Homogeneous Norms in $\mathbb{R}^n$ and $\mathbb{Z}^n$

From now on, we assume that $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and the metric d is induced by a norm N: for $x, y \in E$, $d(x, y) = N(x - y)$. Write o for the origin $(0, \dots, 0)$ of $\mathbb{R}^n$. Then $B_1(o)$ is a symmetrical convex compact subset of E with non-empty interior, and for $x \in E$ and $r > 0$, we have $B_r(x) = \{x + ry \mid y \in B_1(o)\}$. In particular, the metric d is topologically equivalent to the Euclidean metric d_2, in other words, there exist $\beta > \alpha > 0$ such that for any $x, y \in E$, $\alpha d_2(x, y) \leq d(x, y) \leq \beta d_2(x, y)$.

Clearly, (E, d) is boundedly compact and satisfies the interval property, see Definition 4: in item 2 we take $f(\alpha) = (1 - \alpha)p + \alpha q$, then we get $f(\beta) - f(\alpha) = (\beta - \alpha)(q - p)$. Moreover, D is boundedly finite. Note also that condition (A) of Lemma 15 is satisfied.

Everything that we will say here can easily be extended to the case where $D = \rho\mathbb{Z}^n$ for a resolution $\rho > 0$; in fact, for $S \in \mathcal{F}'(\rho\mathbb{Z}^n)$ and $F \in \mathcal{F}'(E)$, we have $\rho^{-1}S \in \mathcal{F}'(\mathbb{Z}^n)$, $\rho^{-1}F \in \mathcal{F}'(E)$ and $H_d(F, S) = \rho H_d(\rho^{-1}F, \rho^{-1}S)$.

For $p = (p_1, \dots, p_n) \in D$, let $C(p)$ be the square/cubic/hypercubic unit cell centered about p, that is, is the set of all $x = (x_1, \dots, x_n) \in E$ such that $\|x-p\|_\infty \leq 1/2$, that is, $|x_i - p_i| \leq 1/2$ for each $i = 1, \dots, n$. For any $x \in E$, there is some $p \in D$ such that $x \in C(p)$; we have then $d(x, D) \leq d(x, p) = N(x - p)$, with $x - p \in C(o)$, from which we deduce that $r_c \leq \sup_{x\in C(o)} N(x)$. Therefore Axiom 1 is satisfied.

We say that the norm N is *coordinate-symmetrical* [13] if for any $i = 1, \dots, n$ and $(x_1, \dots, x_n) \in E$ we have

$$N(x_1, \dots, x_i, \dots, x_n) = N(x_1, \dots, -x_i, \dots, x_n) \ .$$

Equivalently, for any $(x_1, \dots, x_n) \in E$ we have

$$N(x_1, \dots, x_n) = N(|x_1|, \dots, |x_n|) \ .$$

For $1 \leq p \leq \infty$, the L_p norm $\| \|_p$, see (1), is coordinate-symmetrical. In the case of the L_p norm, we will write $B^p_r(x)$ for the closed ball of radius r centered about x, cf. (3), and $r_c[p]$ for the covering radius, cf. (11); we have then $r_c[p] = \frac{1}{2}n^{1/p}$ for $p < \infty$, and $r_c[\infty] = \frac{1}{2}$. We will require the following result:

Property 24. *[13] If N is coordinate-symmetrical, then $N(x_1, \dots, x_n)$ is increasing in each of $|x_1|, \dots, |x_n|$: $\forall (x_1, \dots, x_n), (y_1, \dots, y_n) \in E$,*

$$\Big(\forall\, i = 1, \dots, n,\ |y_i| \geq |x_i|\Big) \implies \Big(N(y_1, \dots, y_n) \geq N(x_1, \dots, x_n)\Big) \ .$$

Furthermore, for each $p \in D$ and $x \in C(p)$, $d(x, D) = N(x - p)$, and $r_c = N\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$.

It follows then that a coordinate-symmetrical norm N satisfies

$$\forall\, x \in E, \qquad N(x) \leq N(1, \dots, 1) \cdot \|x\|_\infty \ . \tag{16}$$

We refer the reader to the beginning of Subsection 2.4 for the terminology on digital adjacency and connectivity, in particular the axial adjacency and connectivity.

Proposition 25. *If N is coordinate-symmetrical, then for any $x \in E$ and $r > 0$, $B_r(x) \cap D$ is axially connected.*

Proof. Write $x = (x_1, \ldots, x_n)$. Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ two points of $B_r(x) \cap D$. We show that they are joined in $B_r(x) \cap D$ by a path for the axial adjacency. We use induction on the number k of coordinates on which p and q differ, that is, the number of $i \in \{1, \ldots, n\}$ such that $p_i \neq q_i$. For $k = 0$, $p = q$ and the result is obvious. Suppose now that the result is true for k, and let p and q differ on $k + 1$ coordinates. Take i such that $p_i \neq q_i$; without loss of generality, we can assume that $|p_i - x_i| \leq |q_i - x_i|$, otherwise we exchange p and q in the following argument. Consider the sequence of integers m between p_i and q_i: $m = p_i, p_i+1, \ldots, q_i-1, q_i$ if $p_i < q_i$, while $m = q_i, q_i+1, \ldots, p_i-1, p_i$ if $q_i < p_i$; then $|m-x_i| \leq |q_i-x_i|$. For each such m, let $z(m) = (q_1, \ldots, q_{i-1}, m, q_{i+1}, \ldots, q_n)$ be the point in D whose i-th coordinate is m, and whose j-th coordinate is q_j for $j \neq i$; by Property 24, $N(z(m) - x) \leq N(q - x)$, thus $z(m) \in B_r(x) \cap D$. Hence the $z(m)$ constitute a path for the axial adjacency joining $z(p_i)$ to $z(q_i) = q$ inside $B_r(x) \cap D$; on the other hand, $z(p_i)$ and p differ in k coordinates, so by induction hypothesis they are joined inside $B_r(x) \cap D$ by a path for the axial adjacency. It follows thus that p and q are joined by such a path, and the result is thus satisfied for $k + 1$. □

Combining this result with Theorem 18 (item 2), we deduce the following:

Corollary 26. *Let N be coordinate-symmetrical, let $F \in \mathcal{F}'(E)$ be connected and let $r \geq r_H(F)$. Then $\Delta_r(F)$ is axially connected. In particular, $\Delta_H(F)$ is axially connected.*

Indeed, F will be chained by the axially connected sets $\Delta_r(x) = B_r(x) \cap D$ for $x \in F$. For $n = 2$ ($E = \mathbb{R}^2$ and $D = \mathbb{Z}^2$) we obtain the result of [22]: for a connected $F \in \mathcal{F}'(E)$, the greatest Hausdorff discretization $\Delta_H(F)$ is 4-connected.

We say that the norm N is *coordinate-homogeneous* [13] if it is coordinate-symmetrical and there is a transitive group G of permutations of $\{1, \ldots, n\}$ such that for any $(x_1, \ldots, x_n) \in E$ and $\pi \in G$ we have

$$N(x_1, \ldots, x_n) = N(x_{\pi(1)}, \ldots, x_{\pi(n)}) \ .$$

In other words there is a group of permutations of the coordinates, that acts transitively on them, under which the norm N is invariant. For example a coordinate-symmetrical norm N satisfying $N(x_1, \ldots, x_n) = N(x_2, \ldots, x_n, x_1)$ is coordinate-homogeneous, because the permutation $i \mapsto i + 1$ ($i < n$), $n \mapsto 1$ generates the cyclic permutation group on $\{1, \ldots, n\}$.

In [22] we said that the norm N is *homogeneous* if it is coordinate-symmetrical and invariant under *any* permutation of coordinates; this corresponds to the case where the group G is the symmetrical group consisting of all permutations of $\{1, \ldots, n\}$. For instance, the L_p norm ($1 \leq p \leq \infty$) is homogeneous, so it is coordinate-homogeneous.

In [13] we showed that a coordinate-homogeneous norm N satisfies the following counterpart of (16):

$$\forall\, x \in E, \qquad N(x) \geq \frac{N(1, \ldots, 1) \cdot \|x\|_1}{n} \ . \tag{17}$$

Since $\|(1, \ldots, 1)\|_1 = n$ and $\|(1, \ldots, 1)\|_\infty = 1$, while the covering radius is given by the norm of $(\frac{1}{2}, \ldots, \frac{1}{2})$, we get:

Property 27. *[13] If N is coordinate-homogeneous, then for any $x \in E$,*

$$\frac{\|x\|_1}{\|(1, \ldots, 1)\|_1} = \frac{\|x\|_1}{n} \leq \frac{N(x)}{N(1, \ldots, 1)} \leq \|x\|_\infty = \frac{\|x\|_\infty}{\|(1, \ldots, 1)\|_\infty} \ . \tag{18}$$

Then for any $x \in E$ and $r > 0$,

$$B^\infty_{r/2r_c}(x) \subseteq B_r(x) \subseteq B^1_{nr/2r_c}(x) \ ; \tag{19}$$

in particular for $r = r_c$,

$$B^\infty_{r_c[\infty]}(x) = B^\infty_{1/2}(x) \subseteq B_{r_c}(x) \subseteq B^1_{n/2}(x) = B^1_{r_c[1]}(x) \ . \tag{20}$$

In other words, the closed ball of radius r_c for d is intermediate between the one of radius $r_c[\infty] = \frac{1}{2}$ for the L_∞ norm and the one of radius $r_c[1] = \frac{n}{2}$ for the L_1 norm. We illustrate this in Figure 6 (a) for $n = 2$ and the Euclidean distance (based on the L_2 norm).

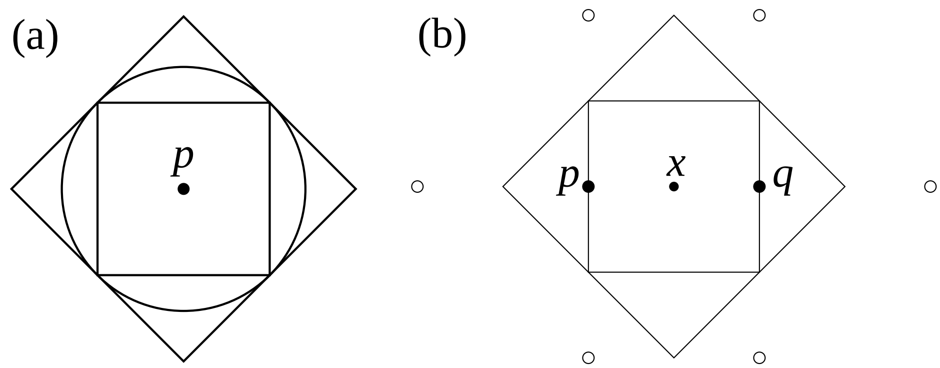

Figure 6: Let $n = 2$ (a) Property 27 for the Euclidean distance: $B^\infty_{r_c[\infty]}(p) \subseteq B^2_{r_c[2]}(p) \subseteq B^1_{r_c[1]}(p)$. (b) Let x be the midpoint between two axially adjacent $p, q \in \mathbb{Z}^2$; for any coordinate-homogeneous norm, as $B^\infty_{r_c[\infty]}(x) \subseteq B_{r_c}(x) \subseteq B^1_{r_c[1]}(x)$ and $B^\infty_{r_c[\infty]}(x) \cap \mathbb{Z}^2 = B^1_{r_c[1]}(x) \cap \mathbb{Z}^2 = \{p, q\}$, we deduce that $\Delta_{r_c}(x) = B_{r_c}(x) \cap \mathbb{Z}^2 = \{p, q\}$.

We see then in Figure 6 (b) that for $n = 2$, the midpoint x between two axially adjacent pixels $p, q \in D$ must satisfy $\Delta_{r_c}(x) = \{p, q\}$. This means that the discrete traces $\Delta_{r_H(F)}(x)$ of balls $B_{r_H(F)}(x)$ will not necessarily be thick, so in view of Corollary 26, for a connected $F \in \mathcal{F}'(E)$, $\Delta_H(F)$ will not necessarily be thick.

In the case of the L_1 norm, since $r_H(F) \leq r_c[1] = \frac{n}{2}$, Corollary 20 gives the following:

Corollary 28. *Let N be the L_1 norm. Let $F \in \mathcal{F}'(E)$ be non-separated and let $r \geq r_H(F)$. Then any Hausdorff discretization of F is connected for the graph with vertex set D and with an edge linking any two distinct $p, q \in D$ such that $\|q-p\|_1 \leq 2r$. In particular, it is connected for the graph with $r = r_c[1] = \frac{n}{2}$, that is, with an edge linking two distinct $p, q \in D$ when $\|q - p\|_1 \leq n$.*

In order to deal with the case when the norm N is not proportional to the L_1 norm, we need the following:

Lemma 29. *Let N be a coordinate-homogeneous norm. For each $i = 1, \dots, n$, let e^i be the i-th canonical basis vector (with i-th coordinate equal to 1, and every other coordinate equal to 0). If for some $j \in \{1, \dots, n\}$ we have $N(e^j) \leq \frac{N(1,\dots,1)}{n}$, then N is proportional to the L_1 norm: $\forall\, x \in E$, $N(x) = \frac{N(1,\dots,1)}{n} \|x\|_1$.*

Proof. For every $i = 1, \dots, n$, e^i can be obtained from e^j by a permutation of coordinates, and as N is coordinate-homogeneous, we get $N(e^i) = N(e^j) \leq \frac{N(1,\dots,1)}{n}$. Let $x = (x_1, \dots, x_n) \in E$. Then $x = \sum_{i=1}^n x_i e^i$ and the norm N gives:

$$N(x) = N\Big(\sum_{i=1}^n x_i e^i\Big) \leq \sum_{i=1}^n |x_i| N(e^i) \leq \Big(\sum_{i=1}^n |x_i|\Big) \frac{N(1, \dots, 1)}{n} \quad ,$$

that is, $N(x) \leq \frac{N(1,\dots,1)}{n} \|x\|_1$. But (17) gives the opposite inequality, therefore the equality $N(x) = \frac{N(1,\dots,1)}{n} \|x\|_1$ holds for any $x \in E$. □

We can now state the counterpart of Corollary 28 in the case of a coordinate-homogeneous norm that is not proportional to the L_1 norm:

Proposition 30. *Let N be a coordinate-homogeneous norm that is not proportional to the L_1 norm. Let $F \in \mathcal{F}'(E)$ be non-separated and let $r \geq r_H(F)$. Then any Hausdorff discretization of F is connected for the graph with vertex set D and with an edge linking any two distinct $p, q \in D$ such that either p and q differ in at least two* *coordinates and* $\|q-p\|_1 \leq \frac{nr}{r_C}$, *or p and q differ in exactly one coordinate and* $\|q-p\|_1 < \frac{nr}{r_C}$. *In particular, it is connected for the graph with* $r = r_C$, *that is, with an edge linking two distinct* $p, q \in D$ *when either p and q differ in at least two coordinates and* $\|q-p\|_1 \leq n$, *or p and q differ in exactly one coordinate and* $\|q-p\|_1 < n$.

Proof. Recall that $N(1,\dots,1) = 2r_C$. Take two distinct $p, q \in D$ such that $B_r(p) \cap B_r(q) \neq \emptyset$; then $N(q-p) \leq 2r$. Suppose first that p and q differ in at least two coordinates; by (18) we have

$$\|q-p\|_1 \leq \frac{nN(q-p)}{N(1,\dots,1)} = \frac{nN(q-p)}{2r_C} \leq \frac{nr}{r_C} ,$$

so p and q are joined by an edge in the graph. Suppose next that p and q differ in exactly one coordinate; thus there is some $j \in \{1,\dots,n\}$ and some $x \in \mathbb{R}$ such that $q - p = xe^j$. By Lemma 29, we have $N(e^j) > \frac{N(1,\dots,1)}{n} = \frac{2r_C}{n}$, hence

$$\|q-p\|_1 = |x| < |x|N(e^j)\frac{n}{2r_C} = N(xe^j)\frac{n}{2r_C} = N(q-p)\frac{n}{2r_C} \leq \frac{nr}{r_C} ;$$

thus $\|q-p\|_1 < \frac{nr}{r_C}$, so p and q are joined by an edge in the graph. Hence any two distinct $p, q \in D$ such that $B_r(p) \cap B_r(q) \neq \emptyset$ are joined by an edge in the graph; thus by Corollary 20 S is connected in that graph. Since $r_H(F) \leq r_C$, we have the result with $r = r_C$. □

Note that for $p, q \in D$ we have $\|q-p\|_\infty < \|q-p\|_1$ when p and q differ in at least two coordinates, but $\|q-p\|_\infty = \|q-p\|_1$ when p and q differ in exactly one coordinate. Thus the condition "either p and q differ in at least two coordinates and $\|q-p\|_1 \leq \frac{nr}{r_C}$, or p and q differ in exactly one coordinate and $\|q-p\|_1 < \frac{nr}{r_C}$" can be written as the conjunction:

$$\|q-p\|_1 \leq \frac{nr}{r_C} \quad \text{and} \quad \|q-p\|_\infty < \frac{nr}{r_C} . \tag{21}$$

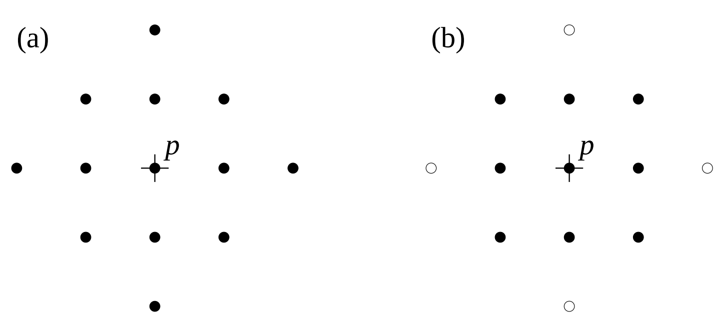

Figure 7: Here $E = \mathbb{R}^2$ and $D = \mathbb{Z}^2$. We show two neighbourhoods of a pixel $p \in D$ identified by a cross. (a) The neighbourhood consists of all $q \in D$ such that $\|q-p\|_1 \leq 2$. (b) Restricting this neighbourhood (a) to $\|q-p\|_\infty < 2$, we remove the 4 endpoints, so we get the 8-neighbourhood made of all $q \in D$ such that $\|q-p\|_\infty \leq 1$.

Let us apply the results of this section to the case where $n = 2$. We assume a coordinate-homogeneous norm N on $\mathbb{R}^2$, which means that for any $(x_1, x_2) \in \mathbb{R}^2$, $N(\pm x_1, \pm x_2) = N(\pm x_2, \pm x_1) = N(x_1, x_2)$. Let F be a non-empty connected closed subset of $\mathbb{R}^2$. By Corollary 26, the greatest Hausdorff discretization $\Delta_H(F)$ will be 4-connected, as shown in [22]. Consider now an arbitrary Hausdorff discretization S of F. We have two cases:

1. N is not proportional to the L_1 norm. By Proposition 30, S will be connected for the graph linking any two distinct $p, q \in D$ such that $\|q-p\|_1 \leq 2$ and $\|q-p\|_\infty < 2$; this is simply the 8-adjacency graph, see Figure 7 (b), thus S will be 8-connected. We obtain thus the result of [22], namely that all Hausdorff discretizations of F are 8-connected.
2. N is the L_1 norm. By Corollary 28, S will be connected for the graph with vertex set D and with an edge linking any two distinct $p, q \in D$ such that $\|q-p\|_1 \leq 2$, see Figure 7 (a). As shown in Figure 8, S will not

always be 8-connected, since we require the adjacency to include the case of two pixels that differ by 2 in exactly one coordinate. This case was overlooked in [22].

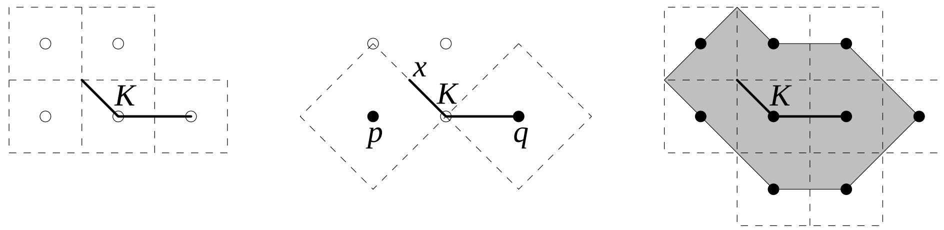

Figure 8: Here $E = \mathbb{R}^2$, $D = \mathbb{Z}^2$ and d is the metric induced by the L_1 norm. Left: the connected compact $K \subseteq E$ and the 5 pixels in D whose square cells intersect K. Middle: the endpoint x of K satisfies $d(x, D) = 1 = r_c$, hence $r_H(K) = 1$; now $H_d(K, \{p, q\}) = 1$, so $\{p, q\}$ is a 8-disconnected Hausdorff discretization of K. Right: the greatest Hausdorff discretization $\Delta_H(K) = \delta_1(K) \cap D$ is 4-connected.

5 Conclusion

This paper is part of a series on *Hausdorff discretization*, the approach to discretization which associates to a closed set F in a "continuous" metric space E any subset S of a "discrete" subspace D which minimizes the Hausdorff distance between S and F [12–15, 22]. Here we investigated the relation between the topological connectivity of that closed set F and the "discrete" connectivity of its discretization S.

As in our previous papers, we have presented our theory in the most general framework possible: E is an arbitrary metric space and D is a *boundedly finite* subset of E with finite covering radius r_c, see (11) and Axiom 1. Then the Hausdorff distance between a closed set F and any of its Hausforff discretizations is equal to its *Hausdorff radius* $r_H(F)$, see (12). We have $r_H(F) \leq r_c$.

For some results we required that E is *boundedly compact* and that it satisfies the *middle point property*, so that it satisfies the *interval property*, see Definition 4 and Corollary 11. Of course, all these properties are indeed satisfied for $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and a distance based on a norm.

For connectivity, we have not restricted ourselves to the topological one (in E) and the graph-theoretical one (in D). We rather based ourselves on the theory of connections and partial connections [10, 11, 18].

We first consider a connected proximinal set F (when E is boundedly compact, every closed set is proximinal). The greatest Hausdorff discretization of F belongs to the partial connection generated by the traces of all balls with radius equal to $r_H(F)$. On the other hand, any Hausdorff discretization S of F will be connected in the graph $G_{r_H(F)}(F)$ where two distinct points of D are joined by an edge if their closed balls of radius $r_H(F)$ intersect. In particular, it will be connected in the graph where two distinct points of D at distance $\leq 2r_c$ are joined by an edge.

We next generalize topological connectivity by considering *non-separated* closed sets, see Definition 5. They include all connected sets, but also non-compact disconnected sets where the connected components are "asymptotic" to each other, see Example 2. The family of non-separated closed sets constitutes a connection on the lattice of closed sets. Given a non-separated closed set F, for all $r > r_H(F)$, any Hausdorff discretization S of F will be connected in the graph $G_r(S)$ where two distinct points of D are joined by an edge if their closed balls of radius r intersect.

When the closed set F is separated, connectivity is not preserved when r_c is small enough. More precisely, if F is s-separated for some $s > 0$, and $r_c < s/4$, then there exists $r > r_c$ such that the graph $G_r(S)$ will be disconnected.

We consider then discretization in multiple subspaces $D(\rho)$ with resolution ρ tending to zero, see Axiom 2; then Hausdorff discretizations of F will be connected at all resolutions if and only F is non-separated.

In a second part of the paper, we consider the case where $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and the metric is induced by a *coordinate-symmetrical* norm (for instance, the L_p norm, $1 \leq p \leq \infty$). Then the greatest Hausdorff discretization of a connected closed subset of E will be axially connected. Note that this result, see Corollary 26, relies on the theory of partial connections and chainings, and it could not be obtained by more conventional methods based on adjacency graphs or Voronoi tesselations.

When the norm is *coordinate-homogeneous* (again, this property holds for the L_p norm) the Hausdorff discretization of a non-separated closed set will be connected for a particular adjacency graph on D. When the norm is proportional to the L_1 norm, two distinct $p, q \in D$ will be adjacent when $\|q - p\|_1 \leq n$. When it is not proportional to the L_1 norm, p and q will be adjacent when either p and q differ in at least two coordinates and $\|q - p\|_1 \leq n$, or p and q differ in exactly one coordinate and $\|q - p\|_1 < n$; an equivalent condition is that $\|q - p\|_1 \leq n$ and $\|q - p\|_\infty < n$.

For $n = 2$, this gives the result of [22]: the greatest Hausdorff discretization of a connected closed subset F of $\mathbb{R}^2$ is 4-connected, and when the norm is coordinate-homogeneous but not proportional to the L_1 norm, every Hausdorff discretization of F will be 8-connected.

For $n > 2$, this graph based on the L_1 norm relies on neighbourhoods that are generally too large, so it is better to use the general theory, giving connectivity for the digital graph with an edge between two digital points p, q such that $d(p, q) \leq 2r_c$.

Our study shows the interest of the recent notion of a partial connection [11]. In particular, for $E = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and a *coordinate-symmetrical* norm, it would be interesting to investigate the partial connection generated by digital traces of balls of a given Hausdorff radius. Indeed, for $n > 2$ it might well be more restricted than the set of all axially connected digital sets, in other words, such sets may have some "thickness" (this is not the case for $n = 2$, see Figure 6 (b)).

It would be interesting to extend our results to related forms of discretization, such as the *discretization by dilation* considered in [12, 13, 15].

Another possible approach would be to associate to every digital set $S \subseteq \mathbb{Z}^n$ its Euclidean representation as the union of cells of its points, $C(S) = \bigcup_{p \in S} C(p)$, and to associate to a closed set $F \subseteq \mathbb{R}^n$ a discretization $S \subseteq \mathbb{Z}^n$ such that F can be homotopically deformed into $C(S)$.

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