An Exploration of the Confusion Between Concept and Formalization Amongst the Community of Teachers in Physics

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Abstract
This study focuses on the links between concepts in physics and the mathematical formalisms that translate them. A physics concept ought to be explored from an epistemological disciplinary perspective, one that shouldn't be confused with the formalization process that aims at translating it. The notion of divergence of a vector field can be used to highlight the confusions that might exist between concept and formalization. Using an internet survey, an important proportion of French professors of higher education were asked to give the definition of the divergence of a vector field. 80% of the answers defined that term as the sum of the partial derivatives of the components of the field in relation to the corresponding coordinates. The paper shows how Maxwell and Heaviside have clarified this concept and how they have shown that an intrinsic definition based on vector analysis leads to the correct articulation between former concepts and new ones. By defining divergence as the limit of the electric flux per unit volume through a closed surface when the volume tends towards zero, the introduced concept takes root in previous knowledge whose limits were highlighted; it helps in pursuing the initial reflection and hence in making more sense. The poll showed surprisingly that this definition rarely appears. This article shows that much work on Science teaching combined with History of Science may improve teaching efficiency despite the great amount of results that the discipline has already achieved.

1. Introduction

The ties between physics and mathematics, its main writing system, has been an important research issue for a long time, at the level of high school as well as for introductory physics in higher education and also in upper levels.
Bagno, Berger and Eylon [1] have described student's attitude towards the activity focused on

the interpretation of formulae. Bing and Redish [2] have analyzed how intermediate level students connect mathematical skills with physical concepts and situations and propose a classification of the so-called "warrants" that is capable of identifying student's epistemological framings. Bollen, Van Kampen and De Cock [3] have shown that if students are quite skilled at doing calculations in the field of electrodynamics, they struggle with interpreting graphical representations of vector fields and applying vector calculus to physical situations. As they write, "We have found strong indications that traditional instruction is not sufficient for our students to fully understand the meaning and power of Maxwell's equations in electrodynamics". Chasteen, Pollock, Pepper and Perkins [4] have shown that using student-centered methods at the upper-division may improve outcomes. Hudson [5] has shown that though good mathematical skills are not a guarantee of success in physics, the performance in the physics will be poor unless the student reaches good mathematical skills. Karam [6] has investigated the subtle structural role of mathematics in physics teaching. A couple of studies have shown the inherent difficulties associated at the concept of electric and magnetic field like for instance Guisasola, Almudí, Salinas, Zuza and Ceberio [7], Guisasola, Zuza and Almudi [8] or Kesonen, Asikainen and Hirvonen [9]. Among other results and papers, one may note that there is often some confusions in the student's mind between the physics concepts and their mathematical formulations (Yeatts, [10], Pepper, Chasteen, Pollock & Perkins, [11]). McMillan and Swadener, [12] have shown in the context of electrostatics, that though students may calculate properly, they exhibit major misconceptions about the problem situation. Savelsbergh, De Jong and Ferguson-Hessler [13] show also in the context of electromagnetism that expertise, the so-called situational knowledge, comes along with time and experience and though teaching is also the art of accelerating the process, teachers have to remain modest in what they can expect from their students. Last but not least, Kuo, Hull, Gupta and Elby [14] have shown that once the difficulties are overcome and once students have developed their own mental representation of how maths and physics are bound together, they obtain good results in problem solving.

The French tendency to over-represent mathematical formalisms in the teaching of physics has already been highlighted. In a previous article, the Authors of the present article, [15] showed that the didactic contract between students and teachers implicitly lies on the symbolic manipulation of formulae which are, however, emptied of their meaning. In that previous article, the authors showed that the students respond to a question in physics using a mathematical formula – formula that is often wrong and sometimes even absurd. The physical significance of the concept which is at the core of the question being asked, is clearly altered. That same article also showed that this didactic contract is probably correlated to the proportion of French writings in physics used, which makes one think that the epistemology of physics is intimately related to the symbolic manipulation of formulae.

In this current article, our aim is to attempt to show that this didactic contract also draws its negative strength from the confusions that exist between the conceptualization and the formalization of physics but, this time, within the very mind of teachers. This assertion, if proven, shouldn't be too surprising to us – since teachers are, to a large extent, the very authors of the textbooks that tend to over-emphasize the symbolic manipulations of the terms being taught at the detriment of their meaning in physics.

This article presents the results of a survey carried out in France using some five to six hundred physics higher education teachers. Before giving its results, we'll present the concept of physics on which it focused, its mathematical formalization and the subtle links that can lead one to not detect the hidden pedagogic agenda that could generate the negative effects of over-representation of mathematics.

2.Theoretical background

In order to define the concept of divergence of a vector field, one needs to position oneself in the case of the Gauss theorem for an electric field. It is generally in this context that students first encounter the concept of integral of flux of a field through a surface. It might not be the best context in which to reflect, since there is no transport of matter (mass or electric charge) which would somehow reassure the student – who is intuitively used to relating the notion of flux to a phenomenon of transport. It would be more helpful to present the concept of flux of a field through a surface in the context of a microscopic model of electric current or of mechanics of fluids.

Whichever way, let us first remind ourselves of the Gauss theorem. The flux of an electric field through any type of closed surface equals the total electric charge in the volume enclosed by this surface. Classically, one can write this as follows:

$$\oiint_S \vec{E}.\overrightarrow{dS} = \frac{Q_{int}}{\varepsilon_0} \qquad (1)$$

Expressed as an integral, the Gauss theorem deals with a macroscopic surface and volume in that one can make them as big as one wishes. However, this result doesn't tell us anything concerning the way in which the total charge closed by the surface is being distributed within the inside volume. If one wishes to know more about the distribution of charges, one needs, for instance, to divide the volume corresponding to the Gauss surface by two and to apply the theorem on two new objects. This operation allows one to somehow refine the understanding of the distribution of charges within the initial volume. If one wants to refine the results further, one can, in absolute terms, divide the volume ad infinitum. One can thus define the divergence of the electric field as the limit of the electric flux per unit volume leaving the closed surface when this volume tends towards zero. The definition that emerges from this reasoning therefore makes more sense and leads to important and classic results. We consider this definition, equation 2, as the first one in the rest of this article.

$$\mathrm{div}\,\vec{E} = \lim_{V\to 0} \frac{\oiint_S \vec{E}.\overrightarrow{dS}}{V} \qquad (2)$$

When one re-writes the classic Gauss theorem (in its integral form) and when one applies to the two members of the equation the division by the volume and then the passage to the limit, when this volume tends towards zero, one clearly obtains the divergence of the electric field in the left member of the equation whilst the total electric charge divided by the volume tends towards the local density of charge – hence the first equation of Maxwell, called the Maxwell-Gauss equation which is nothing more than the differential reformulation of its integral form:

$$\mathrm{div}\,\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (3)$$

To a large extent, this new concept of divergence is quite close to the elementary concept of instantaneous speed. If one defines the total time covered in one journey, one cannot say much about how the journey was effectively travelled. One starts by defining the average speed by dividing the distance covered by the time spent travelling, and then one makes the travelling time tend towards zero in order to define the instant speed. Thus, the divergence of a vector field can be defined as an operation of spatial differentiation. It allows one to obtain the local flux per unit volume, defined in each spatial point in the same way as the instantaneous speed at each temporal point. Using the definition of the divergence as the limit of flux per unit volume when the volume tends towards zero, one falls back onto the integral definition which leads to the famous Green-Ostrogradsky formula, that we will consider as our second definition in the rest of this article:

$$\oiint_S \vec{E}.\overrightarrow{dS} = \iiint_V \mathrm{div}\,\vec{E}.dV \qquad (4)$$

Using again the precious analogy with the instant speed, what is useful to understand is the fact that the Ostrogradsky formula is nothing but an integral re-formulation of the definition of the divergence in the same way than the two following equations are strictly equivalent (in the case of a cinematic on the axis x):

$$v(t) = \frac{dx}{dt} \quad \text{and} \quad \Delta x = x_2 - x_1 = \int_1^2 v(t).\,dt$$

There is neither more nor less information in the integral formulation than there is in the differential formulation. Thus, equations (2) and (4) are rigorously equivalent and one can hardly legitimately talk about an "Ostrogradsky theorem" since one can shift from the differential to the integral formulation in a quasi-immediate way. From a pedagogical perspective, however, which one is best to use? It seems to us that the differential formulation is quite clearly more meaningful. In the same way that it would be strange to define an instantaneous speed using an integral relation, it seems obvious that it is best to define the divergence of a vector field using a differential formulation. When the vector field is described using Cartesian co-ordinates, a simple calculus can help express its divergence in an operational way in the following way. The method consists in defining a rectangle box that is infinitely small, defined by small variations dx, dy and dz of the three co-ordinates. The calculus is described in the Physics lesson of Berkeley (Purcell, 2011). The final result is as follows:

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (5)$$

Is it relevant to present this result as the original definition of the divergence of a vector field? We do not think so. This definition was derived from our first definition (differential formulation). It will be easy, from this first definition, to come up with the expression of the divergence into other systems of (cylindrical or spherical) co-ordinates.

To conclude on this first part, we would like to reaffirm that the three outcomes are derived from a didactical hierarchy which stops us from considering them as definitions that are equally meaningful.

The first definition derives from the need to go beyond the integral formulation of the Gauss law which does not allow for an identification of a precise distribution of charges within the macroscopic volume under consideration. The second definition derives from the first one and comes from its integral reformulation, in view of highlighting the link between the flux integral and the volumic integral of the divergence. The didactical hierarchy relates to the students' culture – that is: their ease with differentiating rather than integrating. Finally, the third result should not be, in our view, interpreted as a definition per se – one could even say that it is difficult to attach a meaning to it. It is, rather, an operational formula which helps in expressing in a concrete way, the divergence of a field when one knows the actual co-ordinates of that field.

It is important to understand the pedagogical or didactical dilemmas that one can derive from the values of these definitions or formulations of one similar concept in physics. The first definition is necessary in order to go beyond the limits of the formulation of the Gauss theorem in its integral form; it extends it, refines it, and is naturally articulated around what comes before it. Thus, the new concept is truly rooted in a continuity of ideas. It does not fall from the sky, but it is strongly related to what the student is supposed to have already understood – provided that he/she reasonably 'digested' the concept of 'ascending ideas'. Provided that the student has understood the necessity of the concept, he is progressively in a position to associate to that concept a profound meaning that can help him/her structure his/her understanding of it and his or her capacity to implement it. One must also insist on the fact that the concept of divergence corresponds to a spatial differentiation. We have already seen that, to a large extent, it is quite close to the definition of instant speed which is a founding concept in physics, one that most students understand and use well and, most of all, one that becomes, in higher education, an integral part of learning outcomes and culture. When one has the opportunity to discover a new concept that has integrated the learning culture of the students, it becomes quickly very clear that, as a teacher, one has to grab that opportunity.

Teaching with analogies has been investigated a lot. Research shows that analogies may be a powerful tool provided a few conditions are fulfilled. Harrison and Treagust [16] have shown that it is 'essential that the analogy be familiar to as many students as possible, that shared attributes be precisely identified by the teacher and/or students, and that the unshared attributes should be explicitly identified'. In this case, instantaneous speed is for sure a known concept at this level, the main shared attribute is differentiation or the limit in one point and the main unshared is that the differentiation is temporal in the case of speed whereas it is spatial in the case of divergence. Haglund and Jeppsson [17] have proved that self-generated analogies may help provided that some precautions are taken. It seems possible to conduct students to discover themselves the need to divide the macroscopic volume to overcome the question of how the total inner charge is distributed and hence, to find out by themselves the analogy with instantaneous speed.

Although the second definition (the Green-Ostrogradsky) is, mathematically speaking, strictly equivalent to the first one, it is certainly less relevant due to the way in which it cumulates a new idea and its integral reformulation. If the additioning of the concept to its reformulation does not present a particular problem to a physicist, it might to a student who will tend to prefer, as much as possible, to isolate the concept in its 'purest' form, from its subtle mathematical reformulation - which tends to add some difficulty to the already challenging experience of being confronted to a new idea.

Finally, the operational formula given by equation (5), if presented as a definition of the divergence of a vector field, deprives the student from the rational chain of ideas from which it derives, from the Gauss theorem, and thus implicitly infers that the concept simply appears, ex nihilo, and insidiously communicates the idea that physics emerges from the revealed truth, from magical thoughts. One then sees physics as 'hocus pocus', a modern form of alchemy, in which the construction of formulae derives from divine art forms; the resolution of problems is reached thanks to a wizard chanting the right spell – privilege which, of course, only belongs to the best few initiated to that sort of mystery.

One will note that the elements mentioned above have been presented in the order in which they had been written in the famous 'Berkeley lecture in physics' [18].

To conclude with this paragraph, Huang, Wang, Chen and Zhang, [19] in a distinguished paper have shown that this teaching approach obtains good results. This article demonstrates the soundness of the didactical hierarchy exposed in the former lines.

3. A brief historical study

The history of science, and in particular that of vector analysis, is also likely to shed some light on these questions.

Research on science teaching has been carried out for a long time. Pocoví [20] has proven how history may help in the context of conceptual change. Karam and Krey [21] have investigated the subtle connections between physics concepts and their writing system, mathematics, on a historical and philosophical point of view. The following lines are written with the very same point of view.

These historical events have mainly been compiled in Michael Crowe's book [22] , of which the elements explored below are derived. One may also refer to Stolze [23].

Since Descartes, the manipulation of vectors had been reduced to that of triplets of coordinates. However, the state of knowledge at that time was such that it was impossible to multiply or divide those triplets since those operations hadn't been defined. The evolution of physics and the mathematicians' will to identify a vector analysis at the third dimension first led people to explore vectors through complex numbers. It was in that context that Sir William Rowan Hamilton invented quaternions, defined as an extension of complex numbers at the third dimension. The Hamilton quaternions theory was well received and, to a large extent, allowed vectors of the third dimension to be formalized in the context of that tool. James Clerk Maxwell was one of the main researchers to appreciate its relevance in the context of his studies in electromagnetism.

Thus, from the development of field theory in physics emerged the need for vector analysis, in particular in the area of electromagnetism. It is therefore not surprising to notice that whilst, in the 1860s, Maxwell initially presented his equations using Cartesian coordinates, he reformed them in 1873 in his "Treatise on Electricity and Magnetism", jointly presenting them using coordinates and quaternions notations. In that reference, Maxwell starts with a chapter covering mathematical preliminaries. After mentioning Descartes's discovery of his system of coordinates, he writes in [24], pages 8 and 9:

"But for many purposes of physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other, although the ideas connected with it did not receive their full development till Hamilton made the next great step in dealing with space, by the invention of his Calculus of Quaternions.

As the methods of Descartes are still the most familiar to students of science, and as they are really the most useful for purposes of calculation, we shall express all our results in the Cartesian form. I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use to us in the study of all parts of our subject, and especially in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few expressions of Hamilton's, than by the ordinary equations."

Here, Maxwell clearly demonstrates how concepts and their intrinsic definitions must be understood and exist independently from their re-formulations in a system of coordinates and this, whilst recognizing that, for practical reasons, it is also necessary to generally re-formulate them in a system of Cartesian coordinates.

However, quaternions became disused and could not survive the criticisms addressed by the inventors of modern vector analysis towards them – mainly Josiah Willard Gibbs and Oliver Heaviside. At the end of the 19th century, independently from each other and yet nearly simultaneously, these two scholars had invented a type of vector analysis that is very similar to that being taught today. Since Oliver Heaviside did so mostly in the context of electromagnetism, it is on his writings that we will focus next.

In [25],Oliver Heaviside emphasized the need for a method based on vector analysis and, yet, discredited the Quaternions method:

" Against the above stated great advantages of Quaternions has to be set the fact that the operations met with are much more difficult than the corresponding ones in the ordinary system, so that the saving of labour is, in a great measure, imaginary. There is much more thinking to be done, for the mind has to do what in scalar algebra is done almost mechanically. At the same time, when working with vectors by the scalar system, there is a great advantage to be found in continually bearing in mind the fundamental ideas of the vector system. Make a compromise; look behind the easily-managed but complex scalar equations, and see the single vector one behind them, expressing the real thing."

In these few lines, Heaviside acknowledges the operational characteristic of the calculations being carried out on the scalar components of a vector but he also highlights the need to reason using the fundamental ideas derived from a vector system.

In [26] (p. 298), Oliver Heaviside reflects on the intricate links that exist between the concept of vector and its re-formulation in the cartesian system :

"And it is a noteworthy fact that the ignorant men have long been in advance of the learned about vectors. Ignorant people, like Faraday, naturally think in vectors. They may know nothing of their formal manipulation, but if they think about vectors, they think of them as vectors, that is, directed magnitudes. No ignorant man could or would think about the three components of a vector separately, and disconnected from one another. That is a device of learned mathematicians, to enable them to evade vectors. The device is often useful, especially for calculation purposes, but for general purposes of reasoning the manipulation of the scalar components instead of the vector itself is entirely wrong."

It is in this context that he defines the divergence of a vector following our first definition of it. In [26] Heaviside writes (p. 50):

"This being general, if we wish to find the distribution of electrification we must break up the region into smaller regions, and in the same manner determine the electrifications in them. Carrying this on down to the infinity small unit volume, we, by the same process of surface-integration, find the volume-density of the electrification. It is then called the divergence of the displacement.

That is, in general, the divergence of any flux is the amount of the flux leaving the unit volume".

Here is thus the justification, fully backed up by the written works of two renowned physicists of the 19th century, of our first paragraph assertion following which the concept of divergence must be understood in the context of vector analysis rather than be introduced in the form of manipulations of its scalar components using a system of Cartesian coordinates which, as Heaviside explains, "is entirely wrong".

4. The survey: results

That is, in general, the divergence of any flux is the amount of the flux leaving the unit volume".

Responses were of two types. Firstly, and most frequently, a quasi-equation (thus designed in the rest of the article) was given in a very explicit way, despite the rather unconventional graphical representation used in it. Secondly, respondents (although much more rarely so) gave a definition in plain letters and words. In that case, and each time it was possible, that type of response was linked to one of the three equations of paragraph 1. It is here useful to note that certain of the responses combined the quasi equation type with its 'translation', expressed in words. Some others were making a vague reference to one of the three equations of paragraph 1 and were therefore classified in the corresponding category. Some responses remained difficult to classify in one of the categories. Thus, the translation of raw responses to the survey presented in the tables below reflect the author's own interpretation – and this, for at least 10% of the responses. In quite a few cases, the quasi equation was sufficiently explicit, for it to present no ambiguity at all.

For the majority, responses were given in the form of an equation or of a quasi equation. Occasionally, they were accompanied by a text but, with the exception of five cases out of the totality of 76 responses, all responses included an equation or a quasi equation.

Table 1 indicates to what extent the responses can be classified between the three equations of paragraph 1. It is important to notice the high rate of responses for equation (5), always given with a quasi equation that is perfectly explicit and easy to read. The rates for equations (2) and (4) are uncertain because often given in the form of a text, somehow interpreted by the authors.

Rate of responses (in percent. within the category)	Limit of the volumic flux (equation (2))	Green-Ostrogradsky formulae (equation (4))	Sum of the partial derivatives expressed in Cartesian coordinates (equation (5))
Student	0%	0%	100%
Non-specialist	0%	0%	100%
Person who knows about this subject	9%	12%	79%
Specialist in this subject	12%	13%	75%

Table 1 – allocation of responses between equations (2), (4) and (5) of paragraph 1. In the category 'specialists', 12% responded with the limit of the flux per unit volume (equ.2), 13% responded using the Green-Ostrogradsky formula (equ.4) and 75% responded using the sum of the partial derivatives (equ. (5)).

5. Discussion and conclusion

For the majority, responses were given in the form of an equation or of a quasi equation.

In a first instance, we will comment on the number responses received in total. If compared with the potential total of the target (five hundred to six hundred), this result is disappointing (only seventy six responses). One could object to the validity of our research results by highlighting the fact that this sample is too small to allow any conclusion to be derived from the study. It is regretful that more responses could not be received. Unfortunately, electronic mail is so demanding nowadays that it is understandable that the efficiency of using it for such survey is rather low. Nevertheless, the fact that the rate of responses corresponding to equation (5) reached 80% is noticeable (all categories put together). The rate reaches 75% of the so-called specialists who are susceptible to give an answer that represents well their conception of the concept within three minutes since they are supposed to have thought about it for quite a long time. It is important to emphasize the fact that that rate, already reached after 10 responses, remained stable after that. It is therefore safe to assume that that rate is representative of the surveyed community, despite the low number of answers.

To the extent that the study carried out presents some weaknesses – those of the media relied on to collect the responses; the level of knowledge of the people who responded, and the proportion of interpretation on a rather limited number of responses – the author wishes to focus the discussion on what seems to be mostly based on facts and on what seems most certain in the whole set of results. As it happens, this constitutes, anyway, the main lesson of the study. We are here talking about the importance taken by the rate of equation (5), equation which encompasses the weakest value of definition of the concept, potentially that to which one could assign the least value of definition. One will add that the few responses that make reference to the local flux do so in a very indirect way, whilst the definition specifying that "the limit of the flux per unit volume of A through a closed surface when the closed volume tends towards zero" is, in fact, a very explicit definition, not that complex, containing similarities with the concept of instantaneous speed. So, why is this definition lacking so much from the sample of collected responses? Why does the overwhelming majority of lecturers, including those who consider themselves as specialists in that field, give an equation, though not wrong, is not the best one as an intrinsic definition for the concept?

The answer is probably complex. Certainly, for a confirmed physicist who has understood deeply the concept, the three equations (2), (4) and (5) are strictly equivalent but that is only because as soon as he sees the mathematic equation he instantly thinks about the definition behind the formula itself. Somehow he thinks in “physics language” but talks in “mathematical language” just like one can think and speak in foreign languages once he becomes fluent enough to do so. Yet, what is possible for a confirmed physicist is not as easily done for an undergraduate student. Just because a physicist can instantly identify and think of the formula in a physical sense does not mean it is as easily done for a beginning student who will simply see the formula and use it without ever thinking about the actual meaning behind it.

To conclude, this study asks some questions concerning the epistemological relationship that a scientist has with his / her own discipline. For didactical reasons, it may be helpful to consider what has already been stressed by, among others, Maxwell and Heaviside, two major pioneers of Electrodynamics theory. We have seen the importance of History of Science since the question of the best way to introduce a concept has already been an issue in the 19th century when electrodynamics was still an active field of research. In that way, History of Science sheds some light on didactics. Last but not least, we have also seen the importance of epistemology or Nature of Science since the whole thing is about the epistemology of physics with respect to its main writing system, mathematics.

We believe that these issues are fundamental in the education of future Science teachers to prevent misleading confusions between concept and their mathematical formalization.

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