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Abstract In this paper, we report on the formalization of a synthetic proof of Pappus' theorem. We provide two versions of the theorem: the first one is proved in neutral geometry (without assuming the parallel postulate), the second (usual) version is proved in Euclidean geometry. The proof that we formalize is the one presented by Hilbert in *The Foundations of Geometry*, which has been described in detail by Schwabhäuser, Szmielew and Tarski in part I of *Metamathematische Methoden in der Geometrie*. We highlight the steps that are still missing in this later version. The proofs are checked formally using the Coq proof assistant. Our proofs are based on Tarski's axiom system for geometry without any continuity axiom. This theorem is an important milestone toward obtaining the arithmetization of geometry, which will allow us to provide a connection between analytic and synthetic geometry.

1 Introduction

Several approaches for the foundations of geometry can be used. Among them we can cite the synthetic approach and the analytic approach. In the synthetic approach, we start with some geometric axioms such as Hilbert's axioms or Tarski's axioms. In the analytic approach, a field is assumed and geometric objects are defined by their coordinates. The two approaches are interesting: the synthetic approach allows to work in any model of the given axioms and it does not require to assume the existence of a field. The analytic approach has the advantage that definitions of geometric objects and transformations are easier, and the existence of coordinates allows to use algebraic approaches for computations and/or automated deduction. One of the main results that can be expected from a geometry is the arithmetization of this geometry: the construction of the field of coordinates. This is our main objective. Pappus's theorem is a very important theorem in geometry, since Pappus's theorem holds for some projective plane if and only if it is a projective plane over a commutative field. It is an important milestone in the arithmetization of geometry.

In this paper, we describe the mechanization of a synthetic proof of Pappus's theorem in the context of Tarski's neutral geometry.

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In our development we formally proved the theorems of the first sixteen chapters of Schwabhäuser, Szmielew and Tarski's book [SST83], using the Coq proof assistant. To formalize these chapters, we had to establish many lemmas that are implicit in Tarski's development. Many of them are trivial but essential in a proof assistant, but some of them are not obvious and are missing. For the formalization of the eleven lemmas of the thirteen chapter of the book we had to introduce more than 200 lemmas; about ten of them are not obvious. For example, to establish the proof of some lemmas, Schwabhäuser, Szmielew and Tarski use implicitly the fact that given a line l, two points not on l, are either on the same side of l or on both sides. We also devoted some chapters to concepts that are not treated in [SST83]: vectors, quadrilaterals, parallelograms, projections, orientation on a line, perpendicular bisector, sum of angles. We base our formalization on the tactics and lemmas already partially described in [Nar07, BN12, BNSB14a, BNSB14b, BNS15a].

Pappus' theorem is proved in the thirteenth chapter of [SST83]. The proof is based on the one presented by Hilbert [Hil60]. This proof is not expressed in the language of first-order logic as it involves second-order definitions, such as the concept of equivalence classes of segments congruent to a given segment. A proof is given in the parallel case and a second one in the non parallel case, which is the only one we will treat in this paper.

After giving an overview of the existing formalizations of Pappus' theorem (Sec. 2), we present the axiom system and main definitions (Sec. 3), in particular the definition of ratio of length using angles. Then, we present the proof of the theorem in neutral geometry (Sec. 4).

2 Related work: other formal proofs related to Pappus' theorem

Pappus's statement can either be considered as an axiom or a theorem depending on the context. Hessenberg's theorem states the Pappus property implies Desargues property. This theorem has already been formalized in Coq by Bezem and Hendriks using coherent logic [BH08], by Magaud, Narboux and Schreck using the concept of rank [MNS12] and by Oryszczyszyn and Prazmowski using the Mizar proof assistant [OP90].

We do not present here the first formal proof of Pappus' theorem. Pappus' theorem has been checked by Narboux using a formalization of the area method in Coq [JNQ12][1] and by Pottier and Théry using Gröbner's bases [GPT11][2]. But these formal proofs can not be used in our context. The proof using the area method is based on an axiom system containing the axioms of a field and axioms about the ratio of segment length, but we want to prove Pappus' theorem in order to construct the field. The proof using Gröbner's bases is based on the algebraization of the statement, which can be justified from a geometric point of view only if we can perform (following Descartes) the arithmetization of geometry and this requires Pappus' theorem.

Most of the proofs we found in books are based directly or indirectly on the arithmetization of geometry. For instance the proofs using Thales' theorem, Ceva's theorem or Menelaüs' theorem rely on the fact that the ratio of distances can be defined and manipulated algebraically. The proofs based on homogeneous coordinates

[1] http://dpt-info.u-strasbg.fr/~narboux/AreaMethod/AreaMethod.examples_4.html
[2] http://www-sop.inria.fr/marelle/CertiGeo/pappus.html

A1	Symmetry	$AB \equiv BA$
A2	Pseudo-Transitivity	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
A3	Cong Identity	$AB \equiv CC \Rightarrow A = B$
A4	Segment construction	$\exists E\ A\text{–}B\text{–}E \wedge BE \equiv CD$
A5	Five-segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
A6	Between Identity	$A\text{–}B\text{–}A \Rightarrow A = B$
A7	Inner Pasch	$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow$ $\exists X\ P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$
A8	Lower Dimension	$\exists ABC\ \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$
A9	Upper Dimension	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q$ $\Rightarrow A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B.$
A10	Parallel postulate	$\exists XY\ (A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$ $A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y)$

Table 1: Tarski's axiom system for neutral geometry

require also to have a field. We are aware of only two synthetic proofs of Pappus' theorem: the one published by Hilbert [Hil60], which we formalized, and a proof using some kind of homothetic transformations by Diller and Boczeck. Indeed, a proof of Pappus' theorem can be derived quite easily using homothetic transformations. But the geometric definition of homothetic transformations without using coordinates, nor distances are non-trivial. Diller and Boczeck described a way to define homethetic transformations geometrically using the concept of half-rotations in the fourth Chapter of [BG74].

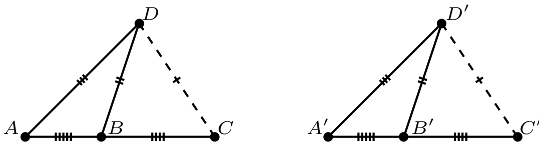
3 Context

In this section we will first present the axiomatic system we used as a basis for our proofs as well as the required definitions.

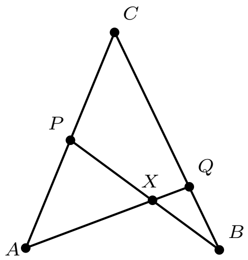
3.1 Tarski's geometry

Let us recall that Tarski's axiom system is based on a single primitive type depicting points and two predicates, namely the betweenness relation, which we write .–.–. and congruence, which we write by $\equiv$. $A\text{–}B\text{–}C$ means that A, B and C are collinear and B is between A and C (and B may be equal to A or C). $AB \equiv CD$ means that the segments AB and CD have the same length. We use neither the continuity nor the Archimedean axiom.

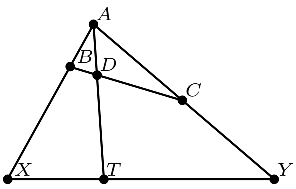
Notice that lines can be represented by pairs of distinct points using the collinearity predicate. Angles can be represented by triple of points and an angle congruence predicate is introduced in Chapter eleven of [SST83].



(a) Five-segment axiom



(b) Inner Pasch's axiom



(c) Parallel postulate

Fig. 1: Illustration for three axioms

The symmetry axiom (A1 on Table 1) for equidistance together with the transitivity axiom (A2) for equidistance imply that the equi-distance relation is an equivalence relation. The identity axiom for equidistance (A3) ensures that only degenerate line segments can be congruent to a degenerate line segment. The axiom of segment construction (A4) allows to extend a line segment by a given length. The five-segments axiom (A5) is similar to the Side-Angle-Side principle, but expressed without mentioning angles, using the betweenness and congruence relations only (Fig. 1a). The lengths of $\overline{AB}$, $\overline{AD}$ and $\overline{BD}$ fix the angle $\widehat{CBD}$. The identity axiom for betweenness expresses that the only possibility to have B between A and A is to have A and B equal. Tarski's relation of betweenness is non-strict, unlike Hilbert's. The inner form of the Pasch's axiom (Fig. 1b) is a variant of the axiom that Pasch introduced in [Pas76] to repair the defects of Euclid. Pasch's axiom intuitively says that if a line meets one side of a triangle and does not pass through the endpoints of that side, then it must meet one of the other sides of the triangle. Inner Pasch is a form of the axiom that holds even in 3-space, i.e. does not assume a dimension axiom. The lower 2-dimensional axiom asserts that the existence of three non-collinear points.The upper 2-dimensional axiom means that all the points are coplanar. The version of the parallel postulate (A10) is a statement which can be expressed easily in the language of Tarski's geometry (Fig. 1c). It is equivalent to the uniqueness of parallels or Euclid's 5th postulate. This equivalence has been formalized in [BNS15b].

3.2 Formalization of Tarksi's geometry in Coq

Contrary to the formalization of Hilbert's axiom system [DDS00, BN12], which leaves room for interpretation of natural language, the formalization in Coq of Tarski's axiom system is straightforward, because the axioms are stated very precisely. We define the axiom system using two type classes [SO08]. Type classes are collections of types, and functions manipulating those types as well as proofs about these functions. Type classes bring modularity: the axioms are not hard coded but are implicit hypotheses for each lemma. The first type class regroups the axioms for neutral geometry in any dimension greater than one. The second one ensures that the space is of dimension two. The formalization is given in Figure 2. We work in intuitionist logic but assuming decidability of equality of points. We do not give details about this in this paper; see [BNSB14a] for further details about decidability issues. Beeson has studied a constructive version of Tarski's geometry [Bee15].

3.3 Main Definitions

Before explaining the proof of Pappus' theorem, we need to introduce some definitions involved in this proof. Throughout the first twelve chapters of [SST83] numerous concepts are introduced and many properties are proved about them. We will explain here only the definitions involved in the proof of Pappus' theorem.

The collinearity of three points A B C, denoted by Col $A\,B\,C$, is defined using betweenness relation:

Definition 1 `Col`

$$\text{Col } A\,B\,C := A\text{—}B\text{—}C \vee B\text{—}A\text{—}C \vee A\text{—}C\text{—}B$$

The `Out` relation asserts that given three collinear points, two of them are on the same side of the third one. It can also be seen as the fact that B belongs to the half-line OA. To assert that A and B are on the same side of O we write: $O{-\!\!\!\!/\!\!\!-}A{\leftrightarrow}B$[3]

Definition 2 `Out`

$$O{-\!\!\!\!/\!\!\!-}A{\leftrightarrow}B := O \neq A \wedge O \neq B \wedge (O\text{—}A\text{—}B \vee O\text{—}B\text{—}A)$$

The midpoint relation can be defined using betweenness and segment congruence. We denote that M is the midpoint of A and B by $A{-\!\!\!+\!\!\!-}M{-\!\!\!+\!\!\!-}B$.

Definition 3 `Midpoint`

$$A{-\!\!\!+\!\!\!-}M{-\!\!\!+\!\!\!-}B := A\text{—}M\text{—}B \wedge AM \equiv BM$$

The midpoint relation is used to define orthogonality. Orthogonality needs three definitions.

[3] Note that we do not use the same notation as in the book [SST83].

```
Class Tarski_neutral_dimensionless := {
  Tpoint : Type;
  Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
  Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
  between_identity : forall A B, Bet A B A -> A=B;
  cong_pseudo_reflexivity : forall A B : Tpoint, Cong A B B A;
  cong_identity : forall A B C : Tpoint, Cong A B C C -> A = B;
  cong_inner_transitivity : forall A B C D E F : Tpoint,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
  inner_pasch : forall A B C P Q : Tpoint,
      Bet A P C -> Bet B Q C ->
      exists X, Bet P X B /\ Bet Q X A;
  five_segment : forall A A' B B' C C' D D' : Tpoint,
    Cong A B A' B' ->
    Cong B C B' C' ->
    Cong A D A' D' ->
    Cong B D B' D' ->
    Bet A B C -> Bet A' B' C' -> A <> B -> Cong C D C' D';
  segment_construction : forall A B C D : Tpoint,
    exists E : Tpoint, Bet A B E /\ Cong B E C D;
  lower_dim : exists A, exists B, exists C, ~ (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

Class Tarski_2D `(Tn : Tarski_neutral_dimensionless) := {
  upper_dim : forall A B C P Q : Tpoint,
    P <> Q -> Cong A P A Q -> Cong B P B Q -> Cong C P C Q ->
    (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

Class Tarski_2D_euclidean `(T2D : Tarski_2D) := {
  euclid : forall A B C D T : Tpoint,
    Bet A D T -> Bet B D C -> A<>D ->
    exists X, exists Y,
    Bet A B X /\ Bet A C Y /\ Bet X T Y
}.

Class EqDecidability U := {
  eq_dec_points : forall A B : U, A=B \/ ~ A=B
}.
```

Fig. 2: Formalization of the axiom system in Coq

The first definition, is called `Per` and noted $\measuredrightangle\, A B C$, it denotes that ABC is a right triangle at B:

Definition 4 `Per`

$$\measuredrightangle\, A B C := \exists C', C\text{-}B\text{-}C' \wedge AC \equiv AC'$$

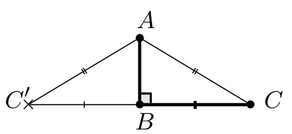


Fig. 3: Definition of `Per`

Note that this definition includes degenerate cases since $A = B$ or $C = B$ conforms to the previous definition[4].

The next definition called `Perp_at` asserts that two lines AB and CD are orthogonal and intercepts in a point P. We denote this by $AB \underset{P}{\perp} CD$.

[4] This definition is called R in [SST83]. We call it `Per` because we want to keep single letter notations for points.

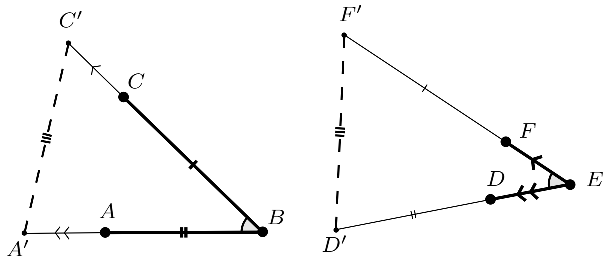


Fig. 4: Definition of `CongA`

Definition 5 `Perp_at`

$$AB \underset{P}{\perp} CD := A \neq B \;\wedge\; C \neq D \;\wedge\; \text{Col}\ P\,A\,B \;\wedge\; \text{Col}\ P\,C\,D \;\wedge$$
$$(\forall U\,V, \text{Col}\ U\,A\,B \Rightarrow \text{Col}\ V\,C\,D \Rightarrow \llcorner U\,P\,V)$$

The third definition allows to assert that two lines AB and CD are orthogonal if there exists a point P such as $AB \underset{P}{\perp} CD$.

Definition 6 `Perp`

$$AB \perp CD := \exists P, AB \underset{P}{\perp} CD$$

Tarski, Schwabhäuser and Szmielew introduce the double orthogonality $\perp\!\!\!\perp$ in order to prove Pappus' theorem. This definition asserts that there exists the lines AB and CD have a common perpendicular passing though P. We write it $AB \underset{P}{\perp\!\!\!\perp} CD$. In Euclidean geometry, this definition is equivalent to the fact the lines AB and CD are parallel but it is not true in neutral geometry.

Definition 7 `Perp2`

$$AB \underset{P}{\perp\!\!\!\perp} CD := \exists X, \exists Y, \text{Col}\ P\,X\,Y \;\wedge\; XY \perp AB \;\wedge\; XY \perp CD$$

The angle congruence relation called `CongA` is denoted by $ABC \mathrel{\widehat{=}} DEF$ and defined as follows (Fig. 4).

Definition 8 `CongA`

$$\begin{aligned} ABC \mathrel{\widehat{=}} DEF := A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge & \\ \exists A', \exists C', \exists D', \exists F', \quad & B\text{—}A\text{—}A' \;\wedge\; AA' \equiv ED \\ & \wedge\; B\text{—}C\text{—}C' \;\wedge\; CC' \equiv EF \\ & \wedge\; E\text{—}D\text{—}D' \;\wedge\; DD' \equiv BA \\ & \wedge\; E\text{—}F\text{—}F' \;\wedge\; FF' \equiv BC \\ & \qquad\qquad \wedge\; A'C' \equiv D'F' \end{aligned}$$

It can be proved that two angles are equal if and only if it is possible to extend them to obtain two congruent triangles.

The InAngle relation asserts that a point P is inside an angle ABC. It is denoted by $P \mathbin{\widehat{\in}} A\,B\,C$.

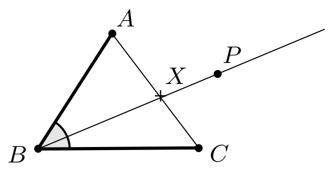


Fig. 5: Definition of `InAngle`

Definition 9 `InAngle`

$$P \mathbin{\widehat{\in}} ABC := A \neq B \wedge C \neq B \wedge P \neq B \wedge \exists X, A\text{—}X\text{—}C \wedge (X = B \vee B \not\leftrightarrow X \leftrightarrow P)$$

Note that the case $X = B$ occurs if ABC is a flat angle when B is between A and C.

Using the $\widehat{\in}$ relation we can define an order relation over angles called `Lea` and denoted by $\widehat{\leq}$ and a strict version `Lta` denoted by $\widehat{<}$.

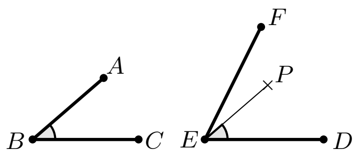


Fig. 6: Definition of `Lea`

Definition 10 `LeA`

$ABC \mathrel{\widehat{\leq}} DEF := \exists P, P \mathbin{\widehat{\in}} D\,E\,F \wedge ABC \mathrel{\widehat{=}} DEP$

Definition 11 `LtA`

$$ABC \mathrel{\widehat{<}} DEF := ABC \mathrel{\widehat{\leq}} DEF \wedge \neg ABC \mathrel{\widehat{=}} DEF$$

We can now define `Acute` angles as angles that are less than a right angle. We denote the fact that ABC is acute by $\measuredangle ABC$.

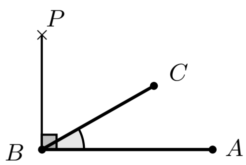


Fig. 7: Definition of `Acute`

Definition 12 `Acute`

$$\measuredangle ABC := \exists P, \llcorner A\,B\,P \wedge ABC \mathrel{\widehat{<}} ABP$$

To end this section, we provide the Table 2 which summarizes all our definitions and notations.

3.4 Lengths, angles and cosine

Up to now we have dealt only with congruence relations over segment lengths ($\equiv$) and angle measures ($\widehat{=}$). To prove Pappus' theorem, it is necessary to introduce the notion of length and angle as equivalence classes over these congruence relations. This is possible since $\equiv$ and $\widehat{=}$ are equivalence relations. Note that we can not use the concept of angle measure nor distance measure, because their definition would require a continuity axiom and a field.

The length of segments is defined as an equivalence class over $\equiv$ relation.

Definition 13 `Q_Cong`

$$Q\_Cong(l) := \exists A, \exists B, \forall X\ Y,\ XY \equiv AB \Leftrightarrow l(X, Y)$$

Coq	Notation
`Bet A B C`	$A–B–C$
`Cong A B C D`	$AB \equiv CD$
`Col A B C`	Col $A\,B\,C$
`Out O A B`	$O \text{-//-} A \leftrightarrow B$
`Midpoint M A B`	$A \text{-+-} M \text{-+-} B$
`Per A B C`	◺ $A\,B\,C$
`Perp_at P A B C D`	$AB \underset{P}{\perp} CD$
`Perp A B C D`	$AB \perp CD$
`Perp2 A B C D P`	$AB \underset{P}{\perp\!\!\perp} CD$
`CongA A B C D E F`	$ABC \stackrel{\frown}{=} DEF$
`InAngle P A B C`	$P \mathbin{\widehat{\in}} A\,B\,C$
`LeA A B C D E F`	$ABC \mathbin{\widehat{\leq}} DEF$
`LtA A B C D E F`	$ABC \mathbin{\widehat{<}} DEF$
`Acute A B C`	$\measuredangle ABC$

Table 2: Summary of notations

If l is a length (`Q_Cong(l)`), then l is a predicate such as $l(X, Y)$ is true if and only if $XY \equiv AB$. AB is a representative of the class l.

We define a predicate `EqL` asserting that two lengths are equal:

Definition 14 `EqL`

$$EqL(l_1, l_2) := \forall XY,\ l_1(X, Y) \Leftrightarrow l_2(X, Y)$$

Since we proved that the binary relation `EqL` is reflexive, symmetric and transitive we can denote $EqL(l_1, l_2)$ by $l_1 = l_2$
In Coq, we use the setoid rewriting mechanism, we therefore declare the equivalence using:

```
Global Instance eqL_equivalence : Equivalence EqL.
```

The null length is defined as the class of segments that are congruent to a degenerated one:

Definition 15 Q_Cong_Null

$$Q\_Cong\_Null(l) := Q\_Cong(l) \wedge \exists A, l(A, A)$$

Similarly, we can define angle measure.

Definition 16 `Q_CongA`

$$Q\_CongA(\alpha) := \exists A, \exists B, \exists C,\ A \neq B\ \wedge\ C \neq B\ \wedge \\ \forall X\ Y\ Z,\ \alpha(X, Y, Z) \Leftrightarrow ABC \stackrel{\frown}{=} XYZ$$

The predicate `EqA` asserts the equality of two angles:

Definition 17 `EqA`

$$EqA(\alpha_1, \alpha_2) := \forall XYZ,\ \alpha_1(X, Y, Z) \Leftrightarrow \alpha_2(X, Y, Z)$$

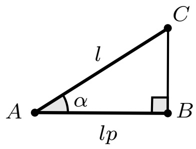


Fig. 8: Definition of `Lcos`

EqA is reflexive, symmetric and transitive, thus we denote $EqA(\alpha_1, \alpha_2)$ by $\alpha_1 = \alpha_2$. The same principle can be applied to define measure of acute angles.

Definition 18 `Q_CongA_Acute`

$Q\_CongA\_Acute(\alpha) := \exists A, \exists B, \exists C, \measuredangle ABC \wedge \forall X\ Y\ Z,\ \alpha(X,Y,Z) \Leftrightarrow ABC \mathrel{\widehat{=}} XYZ$

The proof of Pappus' theorem that we formalize is founded on properties of ratios of lengths and implicitly on the cosine function. The following relation provides a link between two distances and an angle without explicitly building the cosine function. Note that for the traditional construction of the cosine function using series, a continuity axiom is needed. Here, the definition is valid in neutral geometry without any continuity axiom. The relation $Lcos(lp, l, \alpha)$ intuitively means that $lp = l\ \cos(\alpha)$ (Fig. 8).

Definition 19 `Lcos`

$$\begin{aligned} Lcos(lp, l, \alpha) := Q\_Cong(lp)\ \wedge\ Q\_Cong(l)\ \wedge\ Q\_CongA\_Acute(\alpha) \wedge \\ (\exists A, \exists B, \exists C, \llcorner C\,B\,A\ \wedge\ lp(AB)\ \wedge\ l(AC)\ \wedge\ \alpha(BAC)) \end{aligned}$$

We can remark that the definitions `Lcos`, `Q_Cong`, `Q_CongA` and `Q_CongA_Acute` are using higher-order logic as in the original text. Nevertheless, it is possible to give an alternative definition `fo_Lcos` of `Lcos` of arity seven that would allow to prove Pappus's theorem in first-order logic at the cost of using more verbose statements.

$$\begin{aligned} &\text{firstorder\_Lcos}(P, Q, R, S, T, U, V) := \\ &\qquad \exists A, \exists B, \exists C, \llcorner C\,B\,A\ \wedge\ \measuredangle BAC \wedge \\ &\qquad\qquad AB \equiv PQ\ \wedge\ AC \equiv RS\ \wedge\ BAC \mathrel{\widehat{=}} TUV \end{aligned}$$

After the definition of `Lcos`, we can show that length equality and angle equality is compatible with this relation:

Lemma 1 *Lcos_morphism*

$$\forall a, b, c, d, e, f, EqL(a, b) \Rightarrow EqL(c, d) \Rightarrow EqA(e, f) \Rightarrow (Lcos(a, c, e) \Leftrightarrow Lcos(b, d, f))$$

We declare this morphism in Coq's syntax as:

```
Global Instance Lcos_morphism :
 Proper (EqL ==> EqL ==> EqA ==> iff) Lcos.
```

Naturally, the `Lcos` relation is functional:

Lemma 2 *Lcos_existence*

$$\forall \alpha, l, \exists lp, Lcos(lp, l, \alpha)$$

Lemma 3 *Lcos_uniqueness*

$$\forall \alpha, l, l_1, l_2, Lcos(l_1, l, \alpha) \wedge Lcos(l_2, l, \alpha) \Rightarrow EqL(l_1, l_2)$$

Since we have a proof of the existence and the uniqueness of the projected length we can use a functional notation: $\alpha l = lp$ instead of $Lcos(lp, l, \alpha)$.
In the mechanization in Coq of this proof we could use Hilbert's ϵ operator to derive Church's ι operator to mimic this notation [Cas07]. But this would requires adding an axiom such as the `FunctionalRelReification_on` property of the standard library of Coq which states that if we have a functional relation we can obtain the function represented by this relation:

```
Definition FunctionalRelReification_on :=
  forall R:A->B->Prop,
    (forall x : A, exists! y : B, R x y) ->
    (exists f : A->B, forall x : A, R x (f x)).
```

As the proof can be carried without this axiom, we decided to go without it[5].

Now, we define an equality which relates pairs of angles and lengths.

Definition 20 `Lcos_eq`

$$Lcos\_eq(l_1, \alpha_1, l_2, \alpha_2) := \exists\ lp, Lcos(lp, l_1, \alpha_1) \wedge Lcos(lp, l_2, \alpha_2)$$

Since $Lcos\_eq$ is an equivalence relation we will denote $Lcos\_eq(l_1, \alpha_1, l_2, \alpha_2)$ by:

$$\alpha_1 l_1 = \alpha_2 l_2$$

This means intuitively that $l_1 cos(\alpha_1) = l_2 cos(\alpha_2)$ but the cosine function is not explicitly defined.
In the proof of Pappus' theorem we will need to deal with two or three applications of the function of arity two implicitly represented by the ternary `Lcos` predicate. Given two angles we can apply to a length two consecutive orthogonal projections using the predicate `Lcos2`.

Definition 21 `Lcos2`

$$Lcos2(lp, l, \alpha_1, \alpha_2) := \exists\ l_1, Lcos(l_1, l, \alpha_1) \wedge Lcos(lp, l_1, \alpha_2)$$

$Lcos2(lp, l, \alpha_1, \alpha_2)$ can be denoted using a functional notation by $\alpha_2(\alpha_1 l) = lp$.

Given l, α_1, α_2, we proved the existence and the uniqueness of the length lp such that $Lcos2(lp, l, \alpha_1, \alpha_2)$. As previously we can define an equivalence relation `Lcos2_eq`.

[5] Note, however that for arithmetization of geometry we will need to use this axiom to obtain the standard axioms of an ordered field expressed using functions instead of relations [BBN16].

Definition 22 `Lcos2_eq`

$$Lcos2\_eq(l_1, \alpha_1, \beta_1, l_2, \alpha_2, \beta_2) := \exists\, lp, Lcos2(lp, l_1, \alpha_1, \beta_1) \wedge Lcos2(lp, l_2, \alpha_2, \beta_2)$$

We proved that `Lcos2_eq` is an equivalence relation, thus we can write the relation $Lcos2\_eq(l_1, \alpha_1, \beta_1, l_2, \alpha_2, \beta_2)$:

$$\beta_1\alpha_1 l_1 = \beta_2\alpha_2 l_2$$

Similarly, given three angles we can apply to a length three consecutive orthogonal projections using the predicate `Lcos3` and that is all we will need for the proof of Pappus' theorem. As previously we can define an equivalence relation `Lcos3_eq` of arity eight that we denote by:

$$\gamma_1\beta_1\alpha_1 l_1 = \gamma_2\beta_2\alpha_2 l_2$$

3.5 Some lemmas involved in the proof of Pappus' theorem

In this section, we describe some lemmas about the pseudo-cosine function that are used in the proof of Pappus's theorem. The first lemma shows that two applications of the pseudo-cosine function commute.

Lemma 4 *(l13_7 in [SST83])*

$$\forall \alpha, \beta, l, la, lb, lab, lba, \\ Lcos(la, l, \alpha) \wedge Lcos(lb, l, \beta) \wedge Lcos(lab, la, \beta) \wedge Lcos(lba, lb, \alpha) \Rightarrow eqL(lab,\ lba)$$

Using the functional notation we have:

$$\forall \alpha, \beta, l, la, lb, lab, lba,\ \alpha l = la \wedge \beta l = lb \wedge \beta la = lab \wedge \alpha lb = lba \Rightarrow lab = lba$$

Using $l13\_7$ we can prove the lemma $Lcos2\_comm$, which is a more convenient version:

Lemma 5 *Lcos2_comm*

$$\forall \alpha, \beta, lp, l, Lcos2(lp, l, \alpha, \beta) \Rightarrow Lcos2(lp, l, \beta, \alpha,)$$

In the original notation using functional symbols we obtain: $\forall \alpha, \beta, l,\ \beta\alpha l = \alpha\beta l$

From the previous lemma $Lcos2\_comm$ we can prove a generalization for the $Lcos3$ predicate.

Lemma 6 *Lcos3_permut1*

$$\forall \alpha, \beta, \gamma, lp, l, Lcos3(lp, l, \alpha, \beta, \gamma) \Rightarrow Lcos3(lp, l, \alpha, \gamma, \beta)$$

Lemma 7 *Lcos3_permut2*

$$\forall \alpha, \beta, \gamma, lp, l, Lcos3(lp, l, \alpha, \beta, \gamma) \Rightarrow Lcos3(lp, l, \gamma, \beta, \alpha)$$

Lemma 8 *Lcos3_permut3*

$$\forall \alpha, \beta, \gamma, lp, l, Lcos3(lp, l, \alpha, \beta, \gamma) \Rightarrow Lcos3(lp, l, \beta, \alpha, \gamma)$$

In a more readable notation we have:

$$\forall \alpha, \beta, \gamma, l,\ \gamma\beta\alpha l = \beta\gamma\alpha l$$

$$\forall \alpha, \beta, \gamma, l,\ \alpha\beta\gamma l = \beta\gamma\alpha l$$

$$\forall \alpha, \beta, \gamma, l,\ \gamma\beta\alpha l = \gamma\alpha\beta l$$

It can be proved that the *Lcos* pseudo function is injective in the sense that:

Lemma 9 *13_6*

$$\alpha l_1 = \alpha l_2 \Rightarrow l_1 = l_2$$

From the previous lemma, we can deduce :

$$\forall\ l_1, \alpha_1, l_2, \alpha_2, \beta, \gamma,\ \gamma\beta\alpha_1 l_1 = \gamma\beta\alpha_2 l_2 \Rightarrow \alpha_1 l_1 = \alpha_2 l_2$$

4 Pappus's theorem

We now have all the required ingredients and we can prove the main theorem. We first provide the statement, then give a brief overview of the proof, we fix the notations before giving the construction and the detailed proof.

4.1 The statement

The traditional formulation of Pappus theorem is the following (Lemma 13.11 in [SST83], Fig.9):

Theorem 1 *Pappus (Euclidean version)*

$$\begin{aligned}\forall O, A, B, C, A', B', C',\ \neg \text{Col } O\,A\,A' \\ \wedge\ \text{Col } O\,A\,B\ \wedge\ \text{Col } O\,B\,C\ \wedge\ B \neq O\ \wedge\ C \neq O \\ \wedge \text{Col } O\,A'\,B' \wedge \text{Col } O\,B'\,C' \wedge B' \neq O \wedge C' \neq O \\ \wedge\, AC' \parallel CA' \wedge BC' \parallel CB' \Rightarrow AB' \parallel BA'\end{aligned}$$

In this paper, we describe the proof of a second version which is valid in neutral geometry (Fig. 10). To express the statement in neutral geometry, we use the predicate $\perp\!\!\perp$ (Definition 7), to add the assumption that the parallel lines have a common perpendicular going through O. This is lemma number 13.10 in [SST83].

Theorem 2 *Pappus (neutral version)*

$$\begin{aligned}\forall O, A, B, C, A', B', C',\ \neg \text{Col } O\,A\,A' \\ \wedge\ \text{Col } O\,A\,B\ \wedge\ \text{Col } O\,B\,C\ \wedge\ B \neq O\ \wedge\ C \neq O \\ \wedge \text{Col } O\,A'\,B' \wedge \text{Col } O\,B'\,C' \wedge B' \neq O \wedge C' \neq O \\ \wedge\ AC' \underset{O}{\perp\!\!\perp} CA'\ \wedge\ BC' \underset{O}{\perp\!\!\perp} CB' \Rightarrow AB' \underset{O}{\perp\!\!\perp} BA'\end{aligned}$$

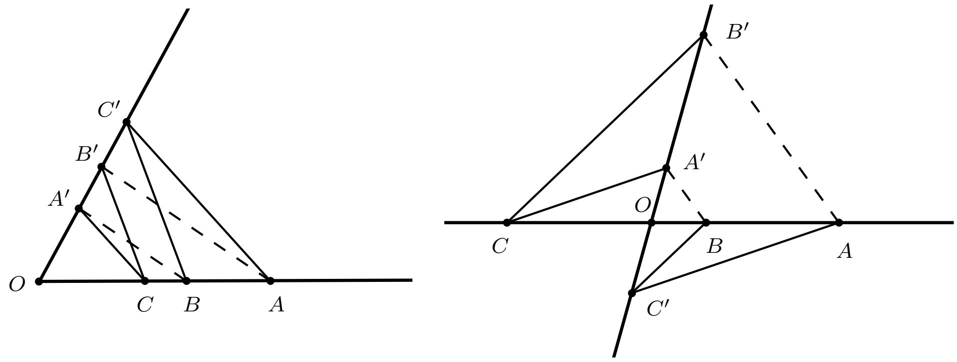


Fig. 9: Two illustrations of Pappus' theorem depending on the configuration of points

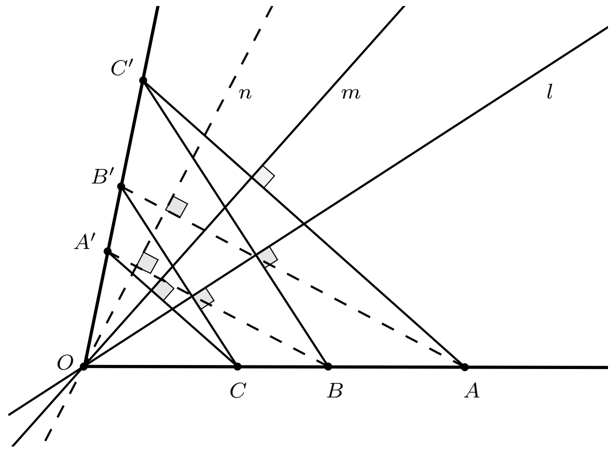


Fig. 10: Main figure for Pappus' theorem in neutral geometry

4.2 Overview of the proof

Before giving a very detailed description of the proof, we provide an overview. First, we construct the two common perpendicular through O of the two pairs of parallel lines $A'C \parallel AC'$ and $BC' \parallel B'C$. Then, we construct the perpendicular to line n to AB' through O. We need to prove that $A'B \perp n$. To reach this goal, we prove that the orthogonal projections N_1 of A' on line n and N_2 of B on n are equal. To prove this equality, it is sufficient to show that the lengths ON_1 and ON_2 are equal and that the two points lie on the same side of O. A difficulty of the formalization is that a rigorous proof needs to deal with the relative positions of the points w.r.t. O. We use the fact that the orthogonal projection preserve betweenness. The equality of lengths is obtained by manipulation of the pseudo-cosine function, a key lemma is the fact that the composition of two pseudo-cosine functions commutes. The main idea of the proof is to use the pseudo-cosine function which allows to express ratios of lengths using congruence class of angles.

4.3 Proof of Pappus' theorem

4.3.1 Notations

To improve readability of the proofs, we will name the different lengths according to Definition 13 (`Q_Cong`).

We will denote the length of OA by $|OA|$ and name it a. That means $Q\_Conq(a) \wedge a(OA)$.

Similarly :

$$\begin{array}{lll} |OA| = a & |OB| = b & |OC| = c \\ |OA'| = a' & |OB'| = b' & |OC'| = c' \end{array}$$

4.3.2 Construction

Since $BC' \underset{O}{\perp\!\!\!\perp} CB'$, there exists a line l perpendicular to BC' and CB' passing through O (Fig. 11a). l intercepts BC' in L and CB' in L'. The acute angle $C'OL \widehat{=} B'OL'$ is called λ.

The acute angle $COL' \widehat{=} BOL$ is called λ'. Using the previously defined notations, we have :

$$\lambda' b = \lambda c' \tag{1}$$

$$\lambda' c = \lambda b' \tag{2}$$

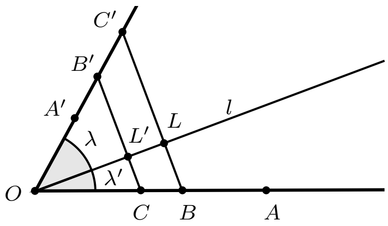
The proof as described in [SST83] and [Hil60] contains a gap here. Indeed it is not trivial to prove that the angles COL' and BOL are congruent. To prove this fact, we need to prove that the points belongs to the same half lines. In order to prove this, one could think of using the fact that parallel projection preserves betweenness. But remember that we are working in neutral geometry, so parallel projection is not a function. Still we can prove the following lemma about $\perp\!\!\!\perp$ which is valid in neutral geometry:

Lemma 10

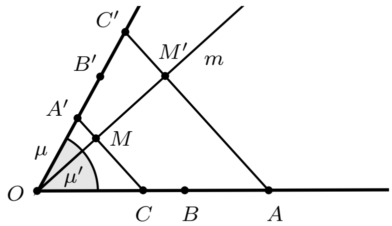
$$\forall OABA'B', O\text{—}A\text{—}B \wedge \text{Col } O\,A'\,B' \wedge \neg\text{Col } O\,A\,A' \wedge AA' \underset{O}{\perp\!\!\!\perp} BB' \;\Rightarrow\; O\text{—}A'\text{—}B'$$

We omit the proof of Lemma 10. Since $AC' \underset{O}{\perp\!\!\!\perp} CA'$, there exists a common perpendicular m to lines AC' and CA' going through O (Fig. 11b). m intercepts AC' in M' and CA' in M. The acute angle $A'OM \widehat{=} C'OM'$ is called μ. The acute angle $COM \widehat{=} AOM'$ is called μ'. To prove these equalities between angles we use lemma 10.

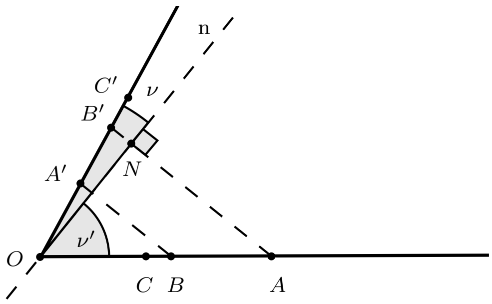
As previously we have :



(a) First notations



(b) Second notations



(c) Third notations

Fig. 11: Notations

$$\mu' a = \mu c' \tag{3}$$

$$\mu' c = \mu a' \tag{4}$$

We call n the orthogonal line to AB' and passing through O (Fig. 11c). n intercepts AB' in N. Similarly acute angle $B'ON$ is called ν and the acute angle AON is called ν'. Translated in terms of lengths, angles and pseudo-cosine it means:

$$\nu b' = \nu' a \tag{5}$$

We will prove that:

$$\nu a' = \nu' b \tag{6}$$

To summarize we have:

$$\lambda' b = \lambda c' \tag{1}$$

$$\lambda' c = \lambda b' \tag{2}$$

$$\mu' a = \mu c' \tag{3}$$

$$\mu' c = \mu a' \tag{4}$$

$$\nu' a = \nu b' \tag{5}$$

and we want to prove that $\nu a' = \nu' b$ (6). We carry out the steps presented in [SST83] page 136.

$$
\begin{aligned}
\lambda'\nu' b &= \nu'\lambda' b && \text{(Lcos2\_comm)}\\
&= \nu'\lambda c' && (1)
\end{aligned}
$$

$$
\begin{aligned}
\mu\lambda'\nu' b &= \mu\nu'\lambda c' && \\
&= \nu'\mu\lambda c' && \text{(Lcos3\_permut)}\\
&= \nu'\lambda\mu c' && \text{(Lcos3\_permut)}\\
&= \nu'\lambda\mu' a && (3)\\
&= \lambda\mu'\nu' a && \text{(Lcos3\_permut)}\\
&= \lambda\mu'\nu b' && (5)\\
&= \mu'\nu\lambda b' && \text{(Lcos3\_permut)}\\
&= \mu'\nu\lambda' c && (2)\\
&= \nu\lambda'\mu' c && \text{(Lcos3\_permut)}\\
&= \nu\lambda'\mu a' && (4)\\
&= \mu\lambda'\nu a' && \text{(Lcos3\_permut)}
\end{aligned}
$$

Thus we have that $\mu\lambda'\nu' b = \mu\lambda'\nu a'$ and as the pseudo-cosine is injective (Lemma 9) we can deduce that $\nu' b = \nu a'$.

At this stage, Schwabhäuser, Szmielew and Tarski define two points N_1 and N_2, the orthogonal projections of A', respectively B on the line ON. Thus we have $\text{◺}\, O\, N_1\, A'$ and $\text{◺}\, O\, N_2\, B$. Now it is sufficient to prove that $N_1 = N_2$. In the proof given by Hilbert this is not detailed, the theorem is considered to be proved at this stage.

Since O, A, B and C are collinear Schwabhäuser, Szmielew and Tarski distinguish four different cases depending of the relative positions of these points:

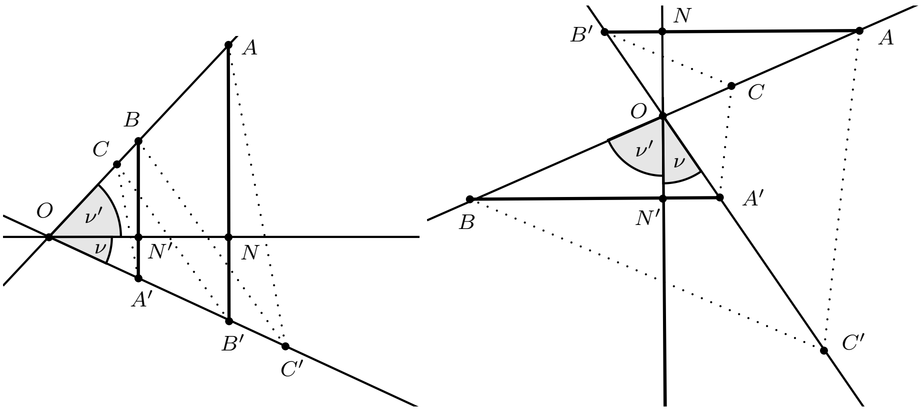
1. $O \not- A \leftrightarrow C$ and $O \not- B \leftrightarrow C$
2. $O \not- A \leftrightarrow C$ and $B — O — C$
3. $A — O — C$ and $O \not- B \leftrightarrow C$
4. $A — O — C$ and $B — O — C$

In our proof, we use a slightly different method. We define the point N' on the line ON such as ON' is of length n'. Two points meet this condition on either side of the point O. We have to distinguish only two cases depending on the relative positions of A, B and O (Fig. 12).

1. $O \not- A \leftrightarrow B$
2. $A — O — B$

Then, we will have to establish that $\text{◺}\, O\, N'\, B$ and $\text{◺}\, O\, N'\, A'$. This will be the subject of sections 4.3.3 and 4.3.4.

Case 1 : $O \not- A \leftrightarrow B$. We build the point N' such as : $|ON'| = n' \wedge O \not- N \leftrightarrow N'$ by using the lemma *ex_point_lg_out* which express that we can build a point on an half line at given distance of the origin:



(a) Case 1 : $O{-}\!\!/\!{-}A\leftrightarrow B$ (b) Case 2 : A—O—B

Fig. 12: Two cases depending on the position of A, B and O

Lemma 11 *ex_point_lg_out*

$$\forall\, l, A, P,\ A \neq P \wedge Q\_Cong(l) \wedge \neg Q\_Cong\_Null(l) \Rightarrow \exists B,\ l(A, B) \wedge A{-}\!\!/\!{-}B\leftrightarrow P$$

Case 2 : A—O—B. The second case can be proved similarly, but we need to build the point N' such as N—O—N' and distance ON' is equal to n'. This can be done using the lemma *ex_point_lg_bet* which express that we can extend a segment by a given length. This is a consequence of the segment construction axiom:

Lemma 12 *ex_point_lg_bet*

$$\forall\, l, A, M,\ Q\_Cong(l) \Rightarrow \exists B,\ l(M, B) \wedge\ A\text{—}M\text{—}B$$

4.3.3 Proof of the fact that $ON'B$ is a right triangle.

The lemma *Lcos_per* helps us to prove $\llcorner O\,N'\,B$. It states that if two lengths and an angle are related by `Lcos` then they form a right triangle, this is consequence of Side-Angle-Side property about congruence of triangles:

Lemma 13 *Lcos_per*

$$\forall\, A, B, C, lp, l, a, Q\_CongA\_Acute(a) \wedge Q\_Cong(l) \wedge Q\_Cong(lp)$$
$$\wedge\ Lcos(lp, l, a) \wedge l(A, C) \wedge lp(A, B) \wedge a(B, A, C) \Rightarrow \llcorner A\,B\,C$$

We apply it in the context :
$Lcos(n', b, \nu') \wedge b(O, B) \wedge n'(O, N') \wedge \nu'(N', O, B) \Rightarrow \llcorner O\,N'\,B$

by assumption we already have:

- $\nu' b = n'$
- $|OB| = b$
- $|ON'| = n'$

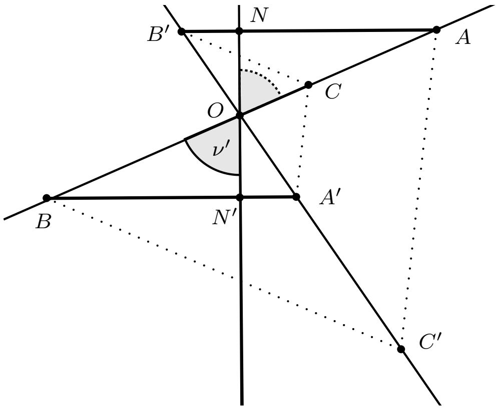


Fig. 13: Case 2

We have only to prove $\nu'(N', O, B)$. This can be done by proving that $N'OB \hat{=} NOA$.

Case 1 $O{\not\!\!-}A{\leftrightarrow}B \wedge O{\not\!\!-}N{\leftrightarrow}N'$. In this case, to prove $N'OB \hat{=} NOA$ we apply the lemma *out_conga*. This lemma is implicit in a traditional proof, it express that $\hat{=}$ is preserved if by prolonging the half-lines that define the angle.

Lemma 14 *out_conga*

$$\forall A, B, C, A', B', C', A_0, C_0, A_1, C_1,$$
$$ABC \hat{=} A'B'C' \ \wedge\ B{\not\!\!-}A{\leftrightarrow}A_0 \ \wedge\ B{\not\!\!-}C{\leftrightarrow}C_0 \ \wedge\ B'{\not\!\!-}A'{\leftrightarrow}A_1 \ \wedge\ B'{\not\!\!-}C'{\leftrightarrow}C_1 \Rightarrow$$
$$A_0BC_0 \hat{=} A_1B'C_1$$

We apply this lemma in the context:

$$NOA \hat{=} NOA \wedge O{\not\!\!-}N{\leftrightarrow}N' \wedge O{\not\!\!-}A{\leftrightarrow}B \wedge O{\not\!\!-}N{\leftrightarrow}N \wedge O{\not\!\!-}A{\leftrightarrow}A \Rightarrow N'OB \hat{=} NOA$$

Formally, the burden is to obtain the $.{\not\!\!-}.{\leftrightarrow}.$ relations.

Case 2 A–O–$B \wedge N$–O–N'.
In this case, to prove $N'OB \hat{=} NOA$ we have to deal with a pair of vertical angles. This can be done by applying the lemma *l11_13* which states that supplementary angles are congruent if the angles are congruent (Fig. 14):

Lemma 15 *l11_13*

$$\forall A, B, C, D, E, F, A', D',$$
$$ABC \hat{=} DEF \ \wedge\ A\text{–}B\text{–}A' \ \wedge\ A' \neq B \ \wedge\ D\text{–}E\text{–}D' \ \wedge\ D' \neq E \Rightarrow$$
$$A'BC \hat{=} D'EF$$

In the context:

$$N'OB' \hat{=} B'ON' \ \wedge\ N'\text{–}O\text{–}N \ \wedge\ N \neq O \ \wedge\ A\text{–}O\text{–}B \ \wedge\ A' \neq O \Rightarrow NOA \hat{=} BON'$$

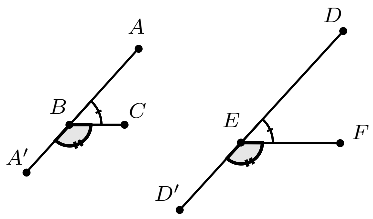


Fig. 14: Congruence of supplementary angles

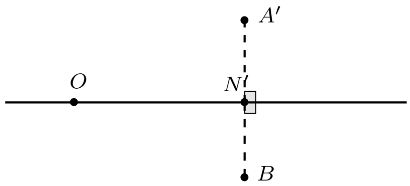


Fig. 15: Case 2: per_per_perp

4.3.4 Proof of the fact that $ON'A'$ is a right triangle.

The proof is similar to the proof of section 4.3.3. But we have before to establish that in the Case 1 we have $O \not\!\!-\!\!- A' \leftrightarrow B'$ and in the Case 2 we have $A' — O — B'$.

This result stems from the fact that projections preserves betweenness. Projection properties have been proved in our developments that are not present in Schwabhäuser, Szmielew and Tarski's work. We deduce two lemmas adapted to the context of the proof which assert that: $A — O — B \Rightarrow A' — O — B'$ and $O \not\!\!-\!\!- A \leftrightarrow B \Rightarrow O \not\!\!-\!\!- A' \leftrightarrow B'$.

4.3.5 Proof of: $ON \perp BA'$

Finally, once we have established $\measuredangle O\,N'\,B$ and $\measuredangle O\,N'\,A'$ we can deduce $ON \perp BA'$ using the lemma *per_per_perp* (Fig. 15):

Lemma 16 *per_per_perp*

$$\begin{aligned}&\forall\, O, N', A', B,\\&\qquad O \neq N' \;\wedge\; A' \neq B \;\wedge\; (A' \neq N' \;\vee\; B \neq N') \;\wedge\; \measuredangle O\,N'\,A' \;\wedge\; \measuredangle O\,N'\,B \Rightarrow\\&\hfill ON' \perp A'B\end{aligned}$$

We have necessarily $A' \neq N' \;\vee\; B \neq N'$ otherwise all the points (O, A, B, C, A', B', C') would be collinear, which is contrary to the hypothesis.
For the same reason we have $A' \neq B$.
On the other hand, $O \neq N'$ since $Lcos(n', a', \nu)$ implies that $\nu = A'ON'$ must be an acute angle because of the definition of $Lcos$.

Since we have the hypothesis $ON \perp B'A$ and we proved $ON \perp BA'$ we deduce from the definition of $\underset{}{\perp\!\!\!\perp}$ that $AB' \underset{O}{\perp\!\!\!\perp} BA'$. QED.

4.4 Some missing lemmas

About lengths

In the proof, Schwabhäuser, Szmielew and Tarski use a notation assigning a name to each congruence class of lengths like $|OA| = a$. In fact such a notation is valid since, given two points A B, there exists a length l such that $l(AB)$.

In Schwabhäuser, Szmielew and Tarski's work no existence lemma is proved, not even mentioned. Such a lemma is of course trivial but necessary in the Coq proof assistant.

Lemma 17 *lg_exists*

$$\forall\ A, B,\ \exists l, Q\_Cong(l) \wedge l(A, B)$$

Conversely, given a length l, we need to prove the existence of two points A and B, such that $l(A, B)$.

Lemma 18 *ex_points_lg*

$$\forall\ l,\ Q\_Cong(l) \Rightarrow \exists A, \exists B, l(A, B)$$

Likewise given a length l and a point A we have a lemma that prove the existence of a point B such that $l(A, B)$

Lemma 19 *ex_point_lg*

$$\forall\ l, A,\ Q\_Cong(l) \Rightarrow \exists B, l(A, B)$$

We also had to derive Lemmas11 and 12.

About angles

Schwabhäuser, Szmielew and Tarski use a notation by assigning a name to each congruence class of angles, for example the class of angles congruent to COL is called λ. As for lengths, such a notation is valid since, given three points A, B, C there exists angle α such as $\alpha(ABC)$.
In Schwabhäuser, Szmielew and Tarski's proof such trivial lemma doesn't appear, but in the Coq proof assistant an angle existence lemma is necessary to assign a name to each angle.

Lemma 20 *ang_exists*

$$\forall\ A, B, C,\ A \neq B \wedge C \neq B \Rightarrow \exists \alpha, Q\_CongA(\alpha) \wedge \alpha(A, B, C)$$

Similarly, the lemma *anga_exists* works for acute angles:

Lemma 21 *anga_exists*

$$\forall\ A, B, C, A \neq B \wedge C \neq B \wedge \measuredangle ABC \Rightarrow \exists \alpha, Q\_ConqA\_Acute(\alpha) \wedge \alpha(A, B, C)$$

For completeness we defined some more existence lemmas, which do not appear in the proof of Pappus' theorem.

- given a point A and an angle α, there exists two points B and C such as $\alpha(A, B, C)$
- given a point B and an angle α, there exists two points A and C such as $\alpha(A, B, C)$
- given two points A, B and an angle α, there exists a point C such as $\alpha(A, B, C)$
- given three points A, B P and an angle α, there exists a point C on the same side of the line AB than P such as $\alpha(A, B, C)$

5 Conclusion

We described a *synthetic proof* of Pappus' theorem for both neutral and euclidean geometry. This is, to our knowledge, the first formal proof of this theorem using a synthetic approach. This is crucial to obtain a coordinate-free version of the proof of this theorem, because this theorem is the main ingredient for building a field and defining a coordinate system. The coordinatization of geometry allows the use of the algebraic approaches for automated deduction in the context of an axiom system for synthetic geometry as shown in [Bee13, BBN16]. The overall proof consists of approximately 10k lines of proof compared to the proof in [Hil60] which is three pages long and the version in [SST83] which is nine pages long. The formalization is tedious because we had to prove many lemmas concerning the relative position of the points and the congruence classes of lengths and angles, which are implicit in the textbooks. The proof we obtained relies on the higher-order logic of Coq, it would be interesting to study how to obtain a first-order proof within Coq or to prove formally that there exists such a first-order proof.

Availability

The full Coq development is available here: `http://geocoq.github.io/GeoCoq/`

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