Approximate Implicitization using Chebyshev Polynomials

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[1] joint work with Tor Dokken (SINTEF)

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Contents

Introduction

Representation of Curves and Surfaces

- Parametric representation: Rational surface given by

$$\mathbf{p}(s,t) = (p_1(s,t), p_2(s,t), p_3(s,t), h(s,t)) \quad \text{for} \quad (s,t) \in \Omega$$

  and bivariate polynomials $p1, p2, p3, h$ (homogeneous form).
- Implicit (algebraic) representation: Surface given by

$$\{(x,y,z,w) : q(x,y,z,w) = 0\}.$$

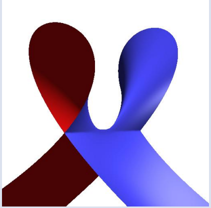
  where q is a polynomial in homogeneous form.
- For *intersection algorithms* it is useful to have both representations available...

Introduction

Motivation - Intersection Algorithms

(a) Surface-surface intersection

(b) Surface self-intersection



(c) Surface raytracing

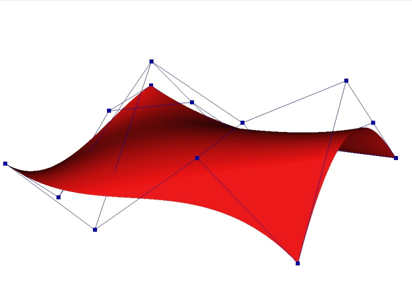
Implicitization

Exact methods

- Traditional methods give exact results:
  - Gröbner bases,
  - Resultants and moving curves/syzygies [Sederberg, 1995],
  - *Linear algebra*.
- Often performed using symbolic computation.
- Surface implicitization can result in very high degrees.
- Algorithms are often slow (especially Gröbner bases).

Implicitization

Implicit degree of parametric surfaces



- Tensor-product bicubic patch
- 16 control points
- Total implicit degree 18
- Defined implicitly by 1330 coefficients!
- Approximation is desirable

Implicitization

Approximate methods

- *Approximate* methods where the degree m can be chosen are desirable:
  - keep the degree low,
  - better stability for floating pt. implementation,
  - faster algorithms.
- Approximation should be good within a region of the parametric curve/surface.
- Algorithms give exact results if the degree is high enough.

Approximate Implicitization

Preliminaries

- First, describe implicit polynomial q in a basis $(q_k)_{k=1}^{M}$, of degree m :
$$q(\mathbf{x}) = \sum_{k=1}^{M} b_k q_k(\mathbf{x})$$
with unknown coefficients $\mathbf{b}$.
- A good error measure is given by algebraic distance $q(\mathbf{p}(s))$.

Approximate Implicitization

Original method (singular value decomposition)

- Original method [Dokken, 1997], gives general framework:
- Form matrix $\mathbf{D} = (d_{jk})_{jk=1}^{L,M}$ such that

$$q(\mathbf{p}(s)) = \sum_{k=1}^{M} b_k q_k(\mathbf{p}(s)) = \sum_{k=1}^{M} b_k \sum_{j=1}^{L} \alpha_j(s) d_{jk}.$$

  where $(\alpha_j)_{j=1}^{L}$ is a polynomial basis in s.
- An approximation is given by right singular vector $\mathbf{v}_{\min}$ corresponding to smallest singular value of $\mathbf{D}$.

Approximate Implicitization

Original method

- Choosing different polynomial bases solves different approximation problems:
- Orthogonal bases solve continuous least squares problems

$$\min_{\|\mathbf{b}\|_2=1} \int_{\Omega} q(\mathbf{p}(s))^2 w(s)\ \mathrm{d}s.$$

- Bernstein/Lagrange bases solve problems which approximate the least squares problem.

Approximate Implicitization

Least squares / weak approximation

- Introduced in [Dokken, 2001], [Corless et al., 2001]:

$$\min_{\|\mathbf{b}\|_2=1} \int_\Omega q(\mathbf{p}(s))^2 w(s)\, \mathrm{d}s.$$

- Method: Form matrix $\mathbf{M} = (m_{kl})_{k,l=1}^{M}$,

$$m_{kl} = \int_\Omega q_k(\mathbf{p}(s)) q_l(\mathbf{p}(s)) w(s)\, \mathrm{d}s$$

- The eigenvector corresponding to the smallest eigenvalue as the solution.

Approximate Implicitization

Orthogonal basis method

The original method using orthogonal polynomials can be used instead:

- Choose a basis $(T_j)_{j=1}^{L}$ that is orthonormal w.r.t. w :

$$
\begin{aligned}
(\mathbf{M})_{kl} &= \int_\Omega q_k(\mathbf{p}(s))q_l(\mathbf{p}(s))w(s)\ \mathrm{ds} \\
&= \int_\Omega \left(\sum_{j=1}^{L} T_j(s)d_{jk}\right)\left(\sum_{i=1}^{L} T_i(s)d_{ik}\right) w(s)\ \mathrm{ds} \\
&= \sum_{i=1}^{L}\sum_{j=1}^{L} d_{jk}d_{ik}\int_\Omega T_j(s)T_i(s)w(s)\ \mathrm{ds} \\
&= \sum_{j=1}^{L} d_{jk}d_{jl} \\
&= (\mathbf{D}^T\mathbf{D})_{kl}
\end{aligned}
$$

Approximate Implicitization

Comparison of methods

- The two methods are mathematically equivalent.
- Singular values of $\mathbf{D}$ are square roots of eigenvalues of $\mathbf{D}^T\mathbf{D} = \mathbf{M}$, thus smallesr condition numbers for $\mathbf{D}$.
- Original method is more numerically stable.
- Original method avoids costly integration of high degree polynomials.

Approximate Implicitization

Why Chebyshev polynomials?

- Near equioscillating behaviour in algebraic error function.
- Number of roots appears to correspond to convergence rates.
- Fast algorithm - based on point sampling, fast Fourier transform (FFT).
- Solves a least squares problem.
- Directly generalizable to tensor-product surfaces.

Approximate Implicitization

Convergence rates of approximate implicitization

Implicit degree	1	2	3	4	5	6
Convergence rate	2	5	9	14	20	27

Curves in $\mathbb{R}^2$

Implicit degree	1	2	3	4	5	6
Convergence rate	2	3	5	7	10	12

Surfaces in $\mathbb{R}^3$

- Convergence as we approximate smaller regions of the curve or surface.

Approximate Implicitization

Algorithm - Chebyshev method

- Generate parametric samples $\mathbf{p}_j = \mathbf{p}(t_j)$ at Chebyshev nodes $t_j = (\cos((j-1)\pi/(L-1)) + 1)/2$, for $j = 1, \ldots, L$.
- Compute a matrix $\mathbf{D}_0 = (q_k(\mathbf{p}_j))_{j=1,k=1}^{L,M}$.
- Compute $\mathbf{D}$ by applying Discrete Cosine Transform to columns of $\mathbf{D}_0$ (using fast Fourier transform methods).
- Perform SVD of $\mathbf{D}$ $(= \mathbf{U}\Sigma\mathbf{V}^T)$.

Examples

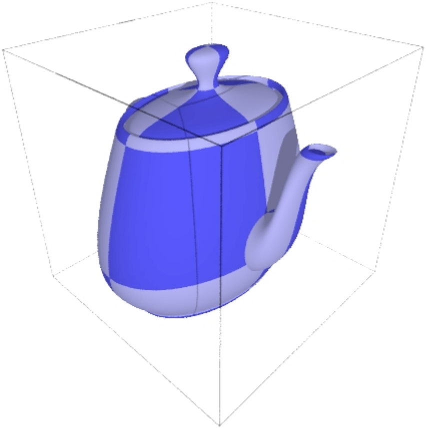
Numerical stability of weak method

- Exact implicitization of degree 5 curve using double precision:

$$
sing(\mathbf{D}) = \begin{pmatrix} \vdots \\ 2.45 \times 10^{-6} \\ 6.05 \times 10^{-7} \\ 3.59 \times 10^{-7} \\ 4.58 \times 10^{-8} \\ 1.24 \times 10^{-8} \\ 6.15 \times 10^{-18} \end{pmatrix}, \quad eig(\mathbf{M}) = \begin{pmatrix} \vdots \\ 6.02 \times 10^{-12} \\ 3.65 \times 10^{-13} \\ 1.29 \times 10^{-13} \\ 2.09 \times 10^{-15} \\ 1.50 \times 10^{-16} \\ 6.84 \times 10^{-19} \end{pmatrix}
$$

Examples

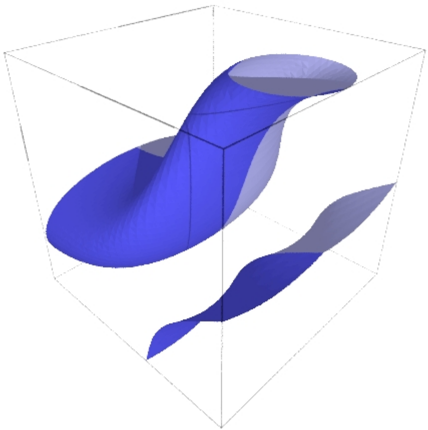
Newell's 32 teapot patches:



- 32 parametric patches.
- All patches are bicubic.

Examples

Implicitization of teapot spout patches:



- Exact implicit degree 18.
- Approximated by degree 6 surfaces.
- Extra branches present.
- Can combine with other approximations to remove branches.

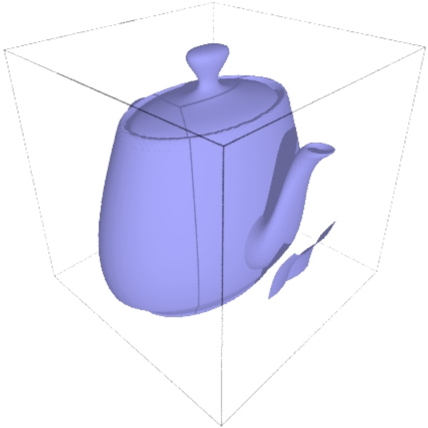
Examples

Implicitization degrees of Newells' teapot

	Exact m	Approximate m 32 patches
rim	9	4
upper body	9	3
lower body	9	3
upper handle	18	4
lower handle	18	4
upper spout	18	5
lower spout	18	6
upper lid	13	3
lower lid	9	4
bottom	15	3

Examples

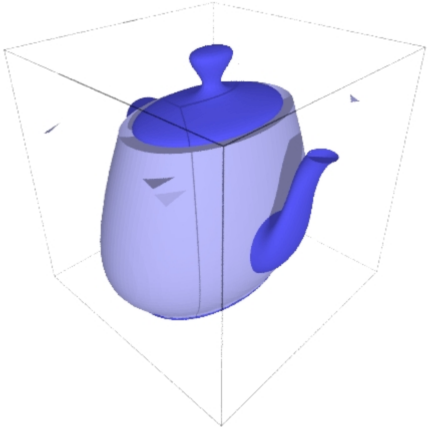
Implicitization of 32 teapot patches:



- 32 approximately implicitized bicubic patches.
- All patches of degree ≤ 6.
- Extra branches present.
- No continuity conditions used.

Examples

Implicit teapot with fewer patches:



- 26 parametric patches.
- 5 approximately implicitized patches.
- All patches of degree ≤ 6.

Thank you!

References:

- T. Dokken, *Aspects of intersection algorithms and approximations*, Ph.D. thesis, Univ. of Oslo, (1997).
- R.M. Corless et al., Numerical implicitization of parametric hypersurfaces with linear algebra, *Artificial Intelligence and Symbolic Computation*, Springer, (2001).