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Dynamic Inventory Routing Problem with Profit Maximization

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Preface

The title of this Master's thesis is "Dynamic Inventory Routing Problem with profit maximization".

It was written at Molde University College – Specialized University in Logistics, according to the requirements for the Master of Science in Logistics degree. The work was supervised by Professor of Quantitative Logistics of Molde University College (Norway) Lars Magnus Hvattum.

Special thanks goes to Professor Lars Magnus for his patience and gentle corrections, and to Mohamed Ben Ahmed who co-supervised the thesis. We are grateful for your guidance throughout the work. We also want to thank our children Ethan and Tatum for their understanding and encouragement. Your warm hugs, kisses and sweet words cheered us on during this program.

In this thesis, we review the literature pertaining to the inventory routing models. We present a formal problem description. Furthermore, we provide a detailed description of two IRP models with profit maximization along with their linearization technique (adopted from Zaitseva 2017).

The proposed models correspond to two market types: monopoly and perfect competition. Computational experiments were conducted on a set of benchmark instances and concluding remarks and interpretations were provided in a later stage.

Summary

With the increasing need for competitive advantage in supply chains today, more and more businesses are addressing inventory routing decisions, in an integrated manner particularly through the Vendor Managed Inventory (VMI) approach. This combined approach of making decisions about routing, inventory and delivery strategies, is opposed to traditional strategies where these problems are solved separately. Clearly, this strategy offers significant advantage of lowered logistical costs but at the cost of producing a complex combinatorial optimization problem termed inventory routing problem (IRP).

Typical IRP aims at minimizing the total costs incurred when these decisions are concurrently made while ensuring that customers do not experience stock-out in the process. Notwithstanding, most supply chains measure their success through the maximization of the overall profit in the chain. Therefore, we focus this thesis on developing IRP models with profit maximization objective.

More precisely, we investigate the dynamic and deterministic versions of the IRP, both of which are derived through the extension of a static deterministic variant of the IRP by applying the rolling horizon technique on 18 scenarios composed from 3 customer sets and 6 planning horizon lengths. We generated 72 instances by testing the scenarios on 4 model types – cost minimization, profit maximization, monopoly and perfect competitive models. The last two models are market structures (monopoly and perfect competition) suppliers can assume and the thesis focuses on them.

The characteristics of these market types and the way they maximize profits made a huge impact on our results. This is because where the monopolist maximizes profit by setting prices at values where it forfeits satisfying customer demand because of increased price, the perfectly competitive market acts differently. It maximizes profit by deriving an optimal production quantity as it is not designed to alter prices but take the price derived from the equilibrium of demand and supply.

From our computation experiment, we found trends, trade-offs and behaviors of the dynamic models in the instances which we tested them, one of which is we were unable to avoid stock-outs at the customers, as we recorded costs and penalties related to shortages and lost sales.

One general trend was that profits increased with larger customer sets and longer planning horizon lengths, and so did the computational time. We also discovered that when static variants of the IRP was used with an objective to maximize profits, values were underestimated, while some outputs were overestimated.

Contents

1. INTRODUCTION

Due to the competitive nature of businesses in the logistics and transport industry, firms are now forced to change their goals from optimizing their own business units to focusing on optimizing the whole supply chain, which is why it has been stated that competition is now between supply chains instead of between each product line or organization (Andersson et al. 2010). The overall goal of supply chain management is to integrate organizational units and coordinate flows of material, information, and money so that the competitiveness of the supply chain is improved (Stadtler 2008). To achieve this integration requires the coordination of logistics activities such as procurement, material management and transportation to achieve optimum, or near optimum performance regarding cost, efficiency and service level.
As logistics cost forms a significant part of a nation's GDP (In Norway, for example, this Logistics cost comprises 14% of the GDP) (Hansen 2010). There is a major opportunity for improvement, mainly through coordination. The need for coordination has resulted in the trend towards the centralization of the decision-making process responsible for the management of distribution and replenishment.

Historically, inventory management and routing have been managed separately in industries. But an interrelation exists between these logistical decisions because in order to determine which customers to serve and the quantity to deliver to the selected customers the routing cost information is needed so that the marginal profit, which is calculated as the difference between revenue and delivery cost for each customer can be computed with accuracy. This interrelationship between inventory allocation and vehicle routing has motivated the modelling of these two logistical activities simultaneously as the integrated Inventory Routing Problem (IRP). The IRP can be seen as an extension of the Vehicle Routing Problem (VRP), but unlike the VRP were the customers specify the order they want to receive and the supplier aims to satisfy this specified demand and simultaneously minimize its total distribution cost, the supplier determines the order quantity (through some input from the customer) and the delivery time.

More and more companies are becoming aware of their supply chain performance and the benefits of coordination and integration of the various components in the management of their supply chain. They are aware of competitive advantage that can be gained through elimination of redundancies and increase in capacity utilization, through elimination of inefficiencies that arise from high distribution costs in their supply chain/distribution networks. Technology, and advancement in communication systems have made available abundant data and reliable information systems, which have eased coordination within the supply chain, encouraging businesses to further embrace the practice of IRP. Many industries are therefore increasingly applying the concept of IRP to their routing and inventory problems. Early applications deal with products like gases, chemicals and automobile. Later applications include routing and inventory management for ammonia, groceries, industrial gases, bitumen, calcium carbonate slurry, frozen products, frozen products, auto parts, blood and petrochemical products (Andersson et al. 2010). IRP has also been applied in the maritime industry, different from other IRP applications because of much longer transit times, (days instead of hours), with destinations often international (Moin and Salhi 2007).

The main objective of every supply chains should be to maximize the overall profitability of the chain, with profitability defined as the difference between revenue generated from satisfying customers and the overall cost across the supply chain (Chopra and Meindl 2016). So, for an IRP, which typically has an objective to minimize costs, this does not translate to the profit maximization objective of a supply chain, as reducing the total cost does not guarantee maximum profit, even though cost minimization is a necessary condition for profit maximization. (Zaitseva 2017) worked on inventory routing problems with profit maximization, where she examined the market structures the supplier can take, the mechanisms controlling the prices and demand and how they affect the IRP. She examined a monopolistic and perfectively competitive market structure for the supplier in a Vendor Managed Inventory setting, constructed from static IRP models, from which she derived interesting trends, trade-offs and behaviours of the different market types in the IRP model.

This thesis plans on extending the static models of (Zaitseva 2017) to dynamic models, in order to examine the results for the purpose of answering the following research questions:

- Can we observe the same trends, trade-offs and behaviours when the model becomes dynamic?
- What are the differences in trends, trade-offs and behaviours of the model when it becomes dynamic?
- Were some results gotten from the static model overestimated, with underestimated variables?

- Can we determine an optimal combination of planning horizon length, market type and customer set from all our test instances that best maximises profit?
- Does the rolling horizon approach yield better results considering the consequence of longer computational time?

The rest of this thesis is organized in the following way. Chapter 2 provides a literature review, which highlights what has been done before related to the IRP. In Chapter 3, problem description and models formulations for dynamic IRP models with explanations are provided. Chapter 4 presents the methodology, chapter 5 discusses the computational results and analyses them. The concluding remarks of the research are provided in Chapter 6.

2. LITERATURE REVIEW

The IRP combines inventory management, vehicle routing and schedules for the delivery of materials. Bell is considered the pioneer of IRP (Bell et al. 1983). IRP arises as a consequence of vendor managed inventory (VMI), where inventory management, vehicle routing and scheduling decisions are integrated and made by the supplier simultaneously, in order to determine when to serve a given customer, how much to replenish when this customer is served and how to combine customers into vehicle routes. The IRP creates an opportunity for the reduction of total routing, inventory and delivery costs through combined, in place of separate optimization. The IRP is a difficult combinatorial optimization problem characterized by the integration of inventory management and vehicle routing decisions (Coelho, Cordeau, and Laporte 2014). The value added from logistics is accomplished via product availability, accuracy in inventory, demand management and ease of placing orders.

Traditionally, inventory management and routing have been managed as different entities in the industry, however, an increasing number of supply chains players are becoming aware of the possibility of synchronizing production and inventory related decisions at the supplier and customer locations. This is evidenced by the adoption of the VMI, a policy under which the vendor is responsible for inventory decisions at the customer location. This policy gives vendors the freedom to choose the size and time of deliveries of products to customers, while the customer is protected against stock-outs occurrences. An ideal scenario would be that under VMI, there will be an integration of inventory management and transportation planning, however, the currently available ERP systems and planning systems do not have such capability as there exists no commercially available system that provides decision support for combined inventory management and routing (Andersson et al. 2010). A typical IRP is concerned with the distributing of a single product type to a geographically dispersed set of customers, using a homogeneous vehicle fleet. The supplier has unlimited product quantity at the factory and the customers have their own storage capacity and rate of consumption. The objective is to minimize total transportation cost over a given planning horizon, with a commitment to prevent stock-outs at all the customers (Song and Furman 2013). These assumptions and simplifications in the definition of the IRP limits the application of its models to real word problems. Therefore, there exists variations of the basic IRP which try to infuse as many practical features of real world scenarios as possible into their models (Song and Furman 2013).

The rest of this literature review focuses on discussing the variety of extensions of the inventory routing problems. In section 2.1, we review the various possible combinations of assumptions that the IRP problems can take using combinations of classification criteria from (Andersson et al. 2010). In Section 2.2, we review literature on dynamic inventory routing problems, then in section 2.3, we discuss methods used to determine feasible solutions in inventory routing problems and finally, in 2.4, we discuss literature on inventory routing problems with profit maximization with emphasis on (Zaitseva 2017), whose IRP models, which capture profit maximization in IRP and the effect of the nature of the type of market an organization is situated in (monopolistic or perfectly competitive markets) has on the generated profits. We will extend these models to enable us reach the goals of this thesis.

2.1 Extensions of the Inventory Routing Problem

Several assumptions can be made when combining inventory management and routing decisions, and almost every possible variant of these assumptions have been made, so many that every reviewed paper in literature treats a new version of the IRP (Andersson et al. 2010). Assumptions and aspects in IRP can be grouped according to the following criteria, which are: time horizon, structure, routing, inventory policy, inventory decisions, fleet composition and fleet size. (Andersson et al. 2010).

Three different modes are used when classifying time horizons, which reflects planning periods in IRP problems. These are instant, finite or infinite. Instant time horizon is used to describe a planning horizon of a problem, which needs only one visit per customer because of the very short length of time. When the planning period requires more than one visit at the customer, the IRP problem has a time horizon which is finite. Finite planning periods are further subdivided into fixed or rolling horizons, fixed when the planning period finite and ends naturally at the end to the horizon, with no link between the time before and after the horizon, therefore long term effects do not need to be handled. A fixed single-day approach simplified IRP problems greatly, which made them popular initially. In single-day models, the IRP is optimized in single-day slices. This approach to the IRP did not consider future deliveries and was deemed myopic, as it postponed all deliveries for the future, resulting in infeasibilities and does not utilize good opportunities in the present time (Campbell and Savelsbergh 2004). Multi-day models have become more prevalent. Although more computationally demanding, they tend to proffer solutions with better quality as they model long-term effects of short term decisions. The rolling horizon approach applied by Baird et al, 2002 as seen in (Campbell et al. 1998) involved, scheduling customers to routes for a two-week period, but executing only the schedules for the first week. The rolling horizon principle entails revising schedules regularly thereby applying more up to date data as they become known (Jaillet et al. 2002). This way, the events of the first week is influenced by the future, which in this instance, is the second week. For infinite time horizons the decisions being made are centred on distribution strategies rather than scheduling (Andersson et al. 2010). An example is the permanent routing or periodic routing, which involved creating a p-day schedule and repeating it for an unlimited time (Campbell and Savelsbergh 2004).

In recent times, literature on IRP with very short (instant) planning horizons have been scarce. This is because they do not have industrial relevance, as from an industrial perspective, combined inventory management and routing problems are determined on a tactical and operational level and therefore finite time horizons, which give solutions that can be implemented in day to day planning and also gives ideas about operational decisions are naturally adopted (Andersson et al. 2010). (Federgruen and Zipkin 1984) modelled a single period IRP problem. Their work aimed at obtaining optimal replenishment quantities and vehicle routes for the customers that, minimized the inventory holding, transportation and shortage costs in one period (Moin and Salhi 2007). Even though single period models do not consider planning on a long term, these models are relevant because they sometimes provide the foundation for studying multi-period models (Moin and Salhi 2007).

Some contributions to literature capture the long term effect of tactical decisions in IRP. These include (Burns et al. 1985) and (Anily and Federgruen 1990) whose objective was to determine a long-term integrated replenishment strategy, which combines inventory rules and routing patterns that enable retailers to meet their demands, while minimizing long-run average system-wide transportation and inventory costs. (Chan and Simchi-Levi 1998) showed that long-run average cost can be minimized in a multi-echelon distribution system with an effective inventory control policy and vehicle routing strategy. (Adelman 2004) also worked on a paper to determine feasible replenishment strategies that minimize average transportation and inventory cost in an infinite horizon. (Kleywegt, Nori, and Savelsbergh 2002, 2004) attempted to coordinate inventory replenishment and transportation in a way that minimized costs over an infinite horizon. (Hvattum and Løkketangen 2009) modelled the IRP problem as a discounted infinite horizon Markov Decision Problem aimed at an optimal policy with a replenishment strategy that maximized the long term discounted total profit. Papers that modelled the IRP within a multi-period finite planning horizon include (Solyalı, Cordeau, and Laporte 2012), (Coelho, Laporte, and Cordeau 2012a), (Bertazzi et al. 2013).

Demand is another classification criteria for IRP problems. There exist several variants of the IRP depending on the nature of the demand at the customers (stochastic or deterministic). Combined inventory management and routing problems are seen as practical, rather than theoretical constructs and are therefore stochastic in nature. Stochastic Inventory Routing Problem (SIRP) acknowledges that the demand of customers can be probabilistic in nature and that the best policy for replenishment will take into consideration the probability distribution of future demands (Hvattum and Løkketangen 2009). Literature that includes stochasticity in their IRP models include (Kleywegt, Nori, and Savelsbergh 2002, 2004), (Hvattum and Løkketangen 2009) (Adelman 2004), (Coelho, Laporte, and Cordeau 2012a), (Bertazzi et al. 2013).

Combining the length of the planning horizons (finite versus infinite) and nature of demand at customers (stochastic versus deterministic) literature in IRP can be distinguished into those that studied infinite horizon IRP with stochastic demands; (Kleywegt, Nori, and Savelsbergh 2002, 2004), (Adelman 2004), (Hvattum and Løkketangen 2009), IRP with constant deterministic demands and infinite horizons (Anily and Federgruen 1990), (Burns et al. 1985), (Chan and Simchi-Levi 1998), IRP with constant deterministic demand and finite horizons (Solyalı, Cordeau, and Laporte 2012), IRP with finite planning horizon and stochastic demand (Federgruen and Zipkin 1984), (Coelho, Laporte, and Cordeau 2012a), (Bertazzi et al. 2013).

Topology is another classification criteria for IRP models. Three modes have been identified: one-to-one, one-to-many and many-to-many (Andersson et al. 2010). The one-to-many topology is the dominant mode for road based inventory routing problem, where a single facility serves a set of customers using a fleet of vehicles. The central facility is a depot, where the vehicles begin and end their routes and where the goods are stored before they are delivered to the customers. This is not the case for maritime transportation, which is characterized by the absence of centralized depot and the possibility of loading and unloading vessels at different ports. Many-to-many is the prevalent mode in such a setting (Andersson et al. 2010). A review of literature showing studies with a one-to-many topology includes (Hvattum and Løkketangen 2009), (Anily and Federgruen 1990) (Burns et al. 1985), (Chan and Simchi-Levi 1998), (Coelho, Laporte, and Cordeau 2012a). Some papers where the mode is many-to-many, and of course maritime based with multiple products transported include IRP studies by (Al-Khayyal and Hwang 2007), (Christiansen et al. 2011),)(Hemmati et al. 2016), (Song and Furman 2013), (Christiansen et al. 2013). Studies by (Anily and Federgruen 1990), (Burns et al. 1985), (Chan and Simchi-Levi 1998)used a fixed partition policy to decide which customers to visit first in their single warehouse multi-retailer inventory routing problem. This involves partitioning the set of retailers into a number of sets, such that each retailer is uniquely assigned to a single set and each set is served separately, i.e., whenever a retailer in a set is served, all other retailers in the set are served as well.

Routing: Two types of routing characterize IRP. These are the vehicle routing problem (VRP) and the pick-up delivery problem (PDP). In the VRP setting all routes originate from and end from a central warehouse which serves also as a depot. This type of setting is prevalent for road-based VRP. Meanwhile, two types of deliveries are known for the VRP mode. These are direct deliveries where goods picked up by a vehicle from a central warehouse are delivered only to a single facility before the vehicle returns to the warehouse. A multiple deliveries type is one in which goods picked up from the warehouse are delivered to multiple facilities before the vehicle returns to the warehouse. (Burns et al. 1985) analysed the trade-off between inventory, transportation and setup costs both in the case of direct deliveries and peddling (dispatching trucks that deliver items to more than one customer per load) and concluded that for each delivery strategy, the trade-off depended on the shipment size. The optimal shipment size in Peddling (multiple deliveries) is a full truckload, while the optimal shipment size for direct deliveries is given by the economic order quantity (EOQ) (Burns et al. 1985). (Coelho, Cordeau, and Laporte 2012b) allowed direct deliveries to take place from the supplier to any customers in their IRP model with stochastic demand and finite planning horizon by subcontracting direct deliveries to carriers. (Kleywegt, Nori, and Savelsbergh 2002) also studied direct deliveries in their stochastic model with an infinite planning horizon and concluded that they have higher effectiveness when the economic order quantities of all customers are large compared to the vehicle capacity. (Gallego and Simchi-Levi 1990) as seen in (Solyalı, Cordeau, and Laporte 2012) concluded that the long term effect of direct shipping is at least 94% effective overall IRP strategies whenever minimal economic lot size is at least 71% of truck size and this effectiveness deteriorates as economic lot size gets smaller. Making exclusive use of direct deliveries simplifies the problem because it removes the routing dimension from it. Direct deliveries from the supplier and lateral transhipment between customers have also been used in conjunction with multi-customer routes to increase the flexibility of the system (Coelho, Cordeau, and Laporte 2014). (Coelho, Laporte, and Cordeau

2012b) used lateral transhipment as a means of mitigating stock-outs when demand exceeded the available inventory. Emergency transhipments proved to be a valuable option for decreasing average stock-outs, while reducing distribution costs significantly. The type of delivery that characterizes the PDP (Pick-up and Delivery Problem) is known as continuous, since there is no start or end warehouse for pickup and delivery of goods. It is more common in maritime applications (Andersson et al. 2010) and is studied in (Al-Khayyal and Hwang 2007), (Christiansen et al. 2011), (Hemmati et al. 2016), (Song and Furman 2013, Christiansen et al. 2013).

In terms of inventory decisions, there are four cases: fixed, stock-out, lost sales and back-order. In a fixed case, inventory is not allowed to be negative. In a stock-out case, inventory is allowed to be negative, however, an emergency delivery to the customer takes place, while in a case of a lost sale, the sale is lost when stock-out occurs. In the case of back-order, the demand is postponed until later (Andersson et al. 2010). Stock-out situation can be observed in the supermarket industry, when the consumption of a specific product is quite high so that the regular resupplying policy is not able to satisfy all the customer requirements in the same period during the time horizon, also possible when the demands are stochastic and the capacity of the vehicle is limited relative to the volume of products required by the customer. (Bertazzi et al. 2013) focuses on an IRP with stochastic demand, where stock-outs may occur during the time horizon. The paper assumes that when the inventory level is negative, the excess demand is not backlogged and a penalty cost is incurred. The objective of the paper is to devise a shipping strategy that minimizes total cost, which is given as the sum of the expected inventory costs, routing cost, plus the penalty cost for stock-out at the customer.

The vehicle fleet can be characterized in terms of composition and size. In terms of composition, a vehicle fleet can be considered as homogeneous or heterogeneous. A fleet is said to be homogeneous if it has the same characteristics such as speed, fixed cost, variable cost, equipment, and size. A fleet is considered as heterogeneous when one or all of the characteristics are different. In terms of size, a vehicle fleet can be categorized as single, multiple or unconstrained. A single fleet consists of one single vehicle. In a multiple fleet variant, there are a number of vehicles, which might be a constraining factor. For an unconstrained fleet, there are no restrictions on the number of vehicles that can be used (Andersson et al. 2010).

2.2 Dynamic Inventory Routing Problem

Dynamic inventory routing problem is a logistical problem characterised by the simultaneous consideration of three decisions: Routing, which involves organising the physical movement of goods between different geographic sites like depots, warehouses, production and retail points; Inventory, which involves quantities and values of the goods being moved and dynamism which involves taking repeated decisions at different times with some time horizon, with earlier decision affecting later ones. (Baita et al. 1998) Dynamic inventory routing problem is highly prevalent in everyday experiences, however, a huge amount of literature available covers mostly two aspects of the problem considered in pairs – Inventory and routing.

Dynamic IRP is characterized by the gradual revelation of customers' demand over time. It involves continuous re-optimization of the problem based on the newly received information. Meanwhile, the received information can be deterministic (known with certainty) or stochastic. Real life inventory routing problems are obviously stochastic as no customer will use the product the same way every single day (Campbell et al. 1998). Usage is pretty predictable and customers generally use about the same amount each day if their total usage for several days in a roll is observed. (Campbell and Savelsbergh 2004), in their research inspired by Praxair, an international industrial gas company, their basic model assumed that usage by their customers of the gas they delivered was deterministic. In a dynamic and stochastic inventory routing problem, the customer demand that is revealed over time is characterized by a probabilistic distribution pattern. In order to solve a dynamic problem, it is necessary to propose a solution policy such as the optimization of a static instance in the event of the availability of new information. Another policy is to make use of forecasts, or the probabilistic knowledge of future information (Coelho, Cordeau, and Laporte 2014). Dynamic and Stochastic IRP can be solved by means of a proactive or reactive policy. On one hand, reactive policy involves the observation of the state of the system prior to making decisions regarding routing and delivery. On the other hand, proactive policy entails combining both the observation of the current state and the use of forecasting of future demand in the planning process (Coelho, Cordeau, and Laporte 2014). With regards to reactive policy, the replenishment decision takes place at the end of the period after demand has occurred. The problem also involves the selection of customers to serve with supplier's vehicles and through direct deliveries, which is an NP-hard problem, which, however, may be solved exactly using mixed-integer linear program based on the size of the instance and the fact that the problem is solved once for a given period (Coelho, Cordeau, and Laporte 2014). Regarding proactive policy, the forecast of future demand is being used to make current decisions.

Three decisions that affect the performance of the algorithm are: the choice of the forecasting method such as the use of exponential smoothening, which is capable of identifying changes in the mean, trend or seasonality, the length of the forecasting and rolling horizon and the method of incorporation of future demand forecasts in an IRP heuristic. The IRP being used is the adaptive large neighbourhood search (ALSN) (Coelho, Cordeau, and Laporte 2014). There is a positive relationship between the inventory holding cost and the solution cost, however, the proactive policy is shown to perform better than the reactive policy under situations of both increase and decrease in the inventory holding cost (Coelho, Cordeau, and Laporte 2014). The main features are the following: The use of demand forecast, the use of transhipment reduces stock-outs and does not make the problem more difficult to solve since it can be incorporated into the min-cost network flow problem that is used to solve the delivery subproblem. The first alterative solves the problem as if all information was available from the beginning (in hindsight). The myopic dynamic heuristic uses only information that is known with certainty to solve problems for each stage (Hvattum and Løkketangen 2009).

2.3 Feasible solution methods to Inventory Routing Problems

IRP are among the most important and most challenging extension of optimizing vehicle routing problems in which inventory control, routing decisions and delivery schedules have to be made simultaneously. The IRP represents a non-deterministic, polynomial-time hard (NP-hard) problem. The routing component, vehicle routing problem, makes the problem difficult. Even when only one customer is considered, some variants of the IRP remain computationally hard.

Several exact, metaheuristic and hybrid methods have been used to find feasible solutions for inventory routing problems and its variants.

Exact algorithms relying on branch-and-cut was developed by (Archetti et al. 2007) capable of solving instances for single – products and single vehicle versions of the IRP. (Coelho and Laporte 2013b) increased the scope of an exact approach based on the branch-and-cut algorithm put forward by Archetti et al to include multiples product and multiple vehicle variants of the IRP.

The difficulty of the problem increases as the number of nodes (customers) and/or vehicles increase. It is a well-known fact that mixed integer problems of such sizes are relatively hard to solve to optimality using branch-and-cut methods, which is why the Lagrangean-based approach has been used to efficiently generate lower and upper bounds (Liu and Chen 2011). The Lagrangean-based approach enables the removal of "complicated" constraints and their incorporation into the objective function with the help of Lagrangean multipliers. This results in obtaining relaxed problems that can be solved efficiently (Liu and Chen 2011). It has been observed that most of the hard problems can be viewed as easy problems, complicated by a relatively small set of side constraints. If the side constraints are dualized, a Lagrangean problem is produced that is easy to solve and whose optimal value is a lower bound (for minimization problems) on the optimal value of the original problem. Therefore, the linear programming relaxation can be replaced by a Lagrangean problem for the provision of bounds in a branch and bound algorithm (Fisher 1985). For all applications, the Lagrangean problem has been solvable in polynomial and pseudo-polynomial time (Fisher 1985).
(Simić and Simić 2013) discussed biologically inspired computing called evolutionary algorithm, which develops an algorithm inspired by nature to solve highly complex IRPs, particularly IRPs that cannot be addressed in a satisfactory way by the traditional approach. It models natural processes, such as selection, recombination, mutation, migration, locality and neighbourhood. These metaheuristics are modern techniques for searching complex space for an optimum. Evolutionary Algorithm has become the method of choice for optimization problems that are too complex to be solved using deterministic techniques like linear programming. Most real-world problems involve simultaneous optimization of several mutually concurrent objectives. multi objective evolutionary algorithms are able to find optimal trade-offs in order to get a solution that is overall optimal (Simić and Simić 2013). Some of these algorithms include genetic algorithm, Tabu search, simulated annealing, all of which can be successfully applied. Genetic Algorithm is a stochastic search technique that maintains a population of individuals which represent a set of potential solutions in the search space. It attempts to combine the good features found in each individual using a structured, yet randomised information exchange to construct individuals who are better suited to their environment than the individuals that they were created from. Genetic Algorithm believes that through the evolution of better and better individuals, the desired solution would be found. (Moin, Salhi, and Aziz 2011) applied Genetic Algorithm to their multi-period, multi-supplier, single warehouse with capacitated vehicle inventory routing problem model. (Park, Yoo, and

Park 2016) also applied genetic algorithm to their IRP with lost sales under a vendor managed inventory strategy in a two-echelon supply chain comprised of a single manufacturer and multiple retailers (one – to - many) model, with multiple objectives, one of which is to maximize profit. (Christiansen et al. 2011) applied Genetic Algorithm to their multi product, multi sourced, multi objective, heterogeneous fleet maritime IRP model, which depicts a real life problem faced by a Norwegian Cement production company. (Javid and Azad 2010), Qin, Miao and Zhang, 2014) as seen in (Roldán, Basagoiti, and Coelho 2016) applied local search operators to the IRP models. (Sajjadi and Cheraghi, 2011), (Liu and Lin, 2005) and (Li et al, 2013), as seen in (Roldán, Basagoiti, and Coelho 2016) used simulation annealing to integrate location decisions in the IRP model (Coelho, Cordeau, and Laporte 2012a).

The hybridization of techniques has become prevalent because of the growing awareness that they outperform individual computational intelligence techniques. It is a synergic combination of multiple techniques used to build an efficient solution. It combines various algorithmic ideas and does not rely on a single search strategy. (Archetti et al. 2012) explored a heuristic for the solution of its IRP that combines a Tabu search scheme with mixed integer programming models.

2.4 Inventory Routing Problems with Profit Maximization

The aim of most of the papers reviewed so far is to determine for each delivery time instant, the set of customers to visit, the quantity of each product to ship to each customer and the route of each vehicle that minimizes the overall cost consisting of transportation, inventory holding and storage costs (Moin and Salhi 2007). A few papers however, have an objective to maximize total profit. (Chien, Balakrishnan, and Wong 1989) had a profit maximizing objective for their IRP model; the inventory allocation and vehicle routing decisions seek to maximize the total revenue less the transportation and penalty costs from the supplier. The interrelationship between the inventory allocation and vehicle routing decision is such that, in order to determine which customers must be served and the amount to supply each selected customer, information about the routing costs needs to be known so that the marginal profit (revenue minus delivery cost) for each customer can be accurately computed. (Chien, Balakrishnan, and Wong 1989) worked on maximizing total profit for their model that was characterised by a one – to – many topology, with deterministic demand and fixed capacities for the supplier and customers. The entire demand of the customers need not be satisfied, but there is a penalty cost imposed per unit of unsatisfied demand. The model was based on a single period approach that passed some information from one period to the next through the inter-period inventory flow and could be seen to simulate a multi-period planning model. The problem required a joint consideration of the demand selection decision (when the available inventory is less than the total demand and / or the revenues of serving some customers could not cover the routing costs incurred) and the routing of vehicles to deliver the allocated inventory to the selected customers so that the total profit is maximized. Demand selection decision was integrated into the model and was determined by the profit margins, vehicle capacities and the amount of inventory available at the supplier. The problem employed a Lagrangean relaxation approach to generate upper and lower bounds, and a heuristic method to obtain feasible solutions that give lower bounds for the integrated problem (Chien, Balakrishnan, and Wong 1989). Fisher et al, (1982) and Bell et al (1983) studied the inventory routing problem at Air Products, a producer of industrial gases. Their objective was also to maximize profit from product distribution over several days. Rather than considering a totally random set of demands or deterministic demands, demand is given by upper and lower bounds on the amount to be distributed to each customer for every period of the planning horizon. They then formulated an integer program that captured delivery volumes, assignment of customers to routes, assignment of vehicles to routes and assignment of start time for routes. The integer program was then solved using Lagrangean dual ascent approach.

(Zaitseva 2017) worked on a static, deterministic, one – to – many, multi-period Inventory Routing Problem with an objective to maximize profit. The author developed two models based on the assumption that the company was operating in a monopolistic and then a perfectly competitive market. The model assuming a monopolistic condition was used to determine the optimal trade-off between volume and margin, according to the adopted demand function. The model assuming a perfectly competitive market was used to determine the appropriate quantity with which profit can be maximised using the adopted cost function. With the monopolistic market model,(Zaitseva 2017) determined an optimal combination of price and demand for each discrete time period, which could increase profit, and also created a possibility to adjust price and demand to increase profit. The objective function generates a non-linear programming model which was linearized.

This thesis extends the work done by (Zaitseva 2017). We present a finite time period using a rolling horizon approach, with which we explore schedules for 1, 2, 3, 4, 5 and 6 day planning horizons respectively, but only implement the first day.

The table 2.1 below shows literature reviewed for this work and the characteristics of the IRP problems they worked on.

Reference	Time Horizon		Structure			Routing		Inventory Policy		Inventory Decisions			Fleet Composition		Fleet Size			Demand	
			One-to	One-to	Many-to			Maximal	Order-up										
	Finite	Infinite	one	Many	Many	Direct	Multiple	Level (ML)	to level (OU)	Lost sales	Backlogging	Fixed	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained	deterministic	stochastic
(Adelman 2004)		✓		✓			✓			✓			✓			✓			✓
(Al-Khayyal and Hwang 2007)	✓				✓		✓					✓		✓		✓		✓	
(Al-Ameri, Shah, and Papageorgiou 2008)	✓				✓		✓	✓		✓				✓		✓			✓
(Anily and Federgruen 1990)		✓		✓			✓	✓		✓			✓				✓	✓	
(Archetti et al. 2012)	✓			✓			✓	✓					✓		✓			✓	
(Archetti et al. 2007)	✓			✓			✓	✓					✓		✓			✓	
(Bell et al. 1983)	✓			✓		✓		✓		✓		✓		✓			✓		✓
(Burns et al. 1985)	✓			✓		✓	✓	✓		✓			✓			✓		✓	
(Campbell et al. 1998)		✓		✓			✓	✓		✓			✓			✓			
(Campbell and Savelsbergh 2004)	✓			✓			✓	✓				✓	✓			✓		✓	
(Chan and Simchi-Levi 1998)		✓		✓			✓					✓	✓				✓	✓	
(Chien, Balakrishnan, and Wong 1989)	✓			✓			✓	✓			✓			✓		✓		✓	
(Christiansen et al. 2011)	✓				✓		✓	✓						✓		✓		✓	
Coelho, Cordeau, Laporte, 2012a	✓			✓			✓	✓				✓	✓			✓			✓
Coelho, Cordeau, Laporte, 2012b	✓			✓		✓	✓		✓			✓	✓			✓			✓
(Coelho, Cordeau, and Laporte 2013)	✓			✓			✓	✓				✓	✓			✓			
(Federgruen and Zipkin 1984)	✓			✓			✓	✓		✓				✓		✓			✓
(Gallego and Simchi-Levi 1990)		✓		✓		✓	✓			✓			✓				✓	✓	
(Hvattum and Løkketangen 2009)		✓		✓			✓	✓		✓			✓			✓			✓
(Jaillet et al. 2002)	✓				✓		✓	✓		✓			✓						✓
(Kleywegt, Nori, and Savelsbergh 2002)		✓		✓		✓		✓		✓			✓			✓			✓
(Kleywegt, Nori, and Savelsbergh 2004)		✓		✓			✓	✓		✓			✓			✓			✓
(Moin, Salhi, and Aziz 2011)	✓			✓			✓	✓		✓			✓					✓	
(Roldán, Basagoiti, and Coelho 2016)	✓			✓			✓		✓	✓			✓		✓				✓

Table 2.1: IRP problems from literature and their characteristics adopted from (Andersson et al. 2010),(Baita et al. 1998), (Coelho,

Reference	Time Horizon		Structure			Routing		Inventory Policy		Inventory Decisions			Fleet Composition		Fleet Size			Demand	
	Finite	Infinite	One-to one	One-to Many	Many-to Many	Direct	Multiple	Maximal Level (ML)	Order-up to level (OU)	Lost sales	Backlogging	Fixed	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained	deterministic	stochastic
(Solyalı, Cordeau, and Laporte 2012)	✓			✓			✓	✓		✓	✓		✓		✓				✓
(Song and Furman 2013)	✓				✓			✓						✓		✓			
(Trudeau and Dror 1992)	✓		✓				✓	✓		✓			✓			✓			✓
(Zaitseva 2017)	✓			✓			✓	✓		✓			✓			✓		✓	

Table 2.1: IRP problems from literature and their characteristics. Adopted from (Andersson et al. 2010),(Baita et al. 1998), (Coelho, Cordeau, and Laporte 2013)

3. PROBLEM DESCRIPTION

Inventory routing problems (IRPs) are complex combinatorial optimisation logistic problems that involve managing inventory and vehicle routing decisions simultaneously. It involves vendor managed inventory (VMI) where the resupplying policy of several retailers over a short or long term planning period is organised by a single production plant, single warehouse or simply a single supplier. The supplier plans the deliveries, deciding the time, quantity and route of the delivery vehicles. This thesis focuses on the dynamic variation of the IRP, where the quantities demanded by the customers are gradually revealed over time, but at the beginning of each planning horizon, where the inventory related decisions are being made, we lack full knowledge of the future demands. We have made basic assumptions about this inventory routing problem, we assume that we are dealing with a single product type, from a single supplier to a set of customers having varying demand over a finite planning horizon. The objective of the planning is to determine an optimal assignment of vehicles to customers and the sequence of the vehicle visits to the assigned customers.

The problem considers only one mode of transportation, which is a truck with a given capacity, we assume that the trucks are a homogeneous fleet, and the route of the truck must begin and end at the supplier's facility. The problem considers that the vehicles are able to perform one route per time period, from the supplier to a subset of customers and the total demand on each route must be less than or equal to the vehicle's capacity. A predefined visit scenario is available for each customer and we disallow the use of lateral transhipments between customers as a means of avoiding stock out in instances where actual demand is high, instead, we allow the incurrence of a penalty due to lost sales and the excess demand is not backlogged. We assume that the manufacturer has enough inventory to meet all the demand during the planning horizon, but the inventory at the customers is limited. Each customer has a maximum inventory level, hence the quantity sent to each of them raises their inventory level to its maximum, in an order-up-to policy. No vehicle loading and unloading cost is considered. As this is a dynamic problem, it is assumed that the set of routes is dynamic and change from one time period to the other; the problem does not consider any bounds (upper or lower) to the length of the individual routes. We assume the transportation cost is measured as a Euclidian distance and so the cost matrix satisfies the triangle inequality. The nodes are considered as customers (and node 0 is considered as the supplier/depot) and the edges are used to travel from one node to the other. As distance is the cost measure, we will assume a symmetric cost matrix.

We assume that for every new planning horizon, the routes change to ensure the cheapest possible routes are selected. We also assume that our problem is a pure delivery type and no time windows are requested by the customer for the delivery of the products. In addition, our problem can be characterized by the following: finite planning horizon with a rolling horizon approach; inventory holding costs are considered at both the supplier and customer locations; a deterministic consumption rate is to be considered.

Our objective is to maximize profit. Although our problem is an inventory routing problem it differs from the basic inventory routing because unlike the basic IRP which guarantees that the inventory level is at the predetermined level, we aim to maximize profits. Since our problem is that of profit maximization, we do not guarantee that any of the customers will always have the required level of inventory and so may not satisfy all of the customer demand during each planning horizon. Pricing decisions are made with the IRP problem simultaneously because pricing decisions affect the demand decision and then both the inventory and routing decisions. The relationship between pricing, inventory and routing decisions is that higher pricing causes lower demand and then lower quantities are ordered, hence lower inventory. Inversely, lower pricing results in higher demand and then higher order quantity and higher inventory in turn. Since the pricing decision is related to the inventory routing decisions, the profit may decrease when they are made separately.

We assume that the price cannot be zero and a demand function will define the relationship between the price and demand quantity. Iso-elastic and linear demand function are the most commonly used functions for representing a downward sloping price versus demand relationship and for the thesis we assume that the demand function for the customers is linear and the demand lies between a specific range. The assumption of linear demand function holds very well within this range.

3.1 MODEL FORMULATION

Notations used by (Zaitseva 2017) are used as the basis for this work, except for minor modifications that were made to suit our objectives. We also adopt all four mathematical models developed by (Zaitseva 2017); IRP with cost minimization, IRP with profit maximization and the profit maximization models which emphasise two market structure types – monopoly and perfectly competitive market.

To formally describe the problem, we consider a graph $G = (N, E)$ where $N = \{0, \dots, n\}$ is the nodes set or customers and $\{E = (i, j) \in N, i = j\}$ is the set of edges. The depot node is s, it is the common supplier to the customers over a given time horizon T. The horizon over which the problem is defined has a length T and at each time period, $t \in T = \{1,..,T\}$. Each edge (i,j) is associated with travel cost c_{ij}, which is known, and we assume that $c_{ij} = c_{ji}$. Each vehicle has a capacity of Q. During every time period, each customer i consumes an amount of r_i. The inventory holding cost at the customer and supplier are h_i and h_s respectively. The supplier has a maximum inventory level U_s, inventory holding costs h_s, an initial inventory level B_s and a production rate at each time period r_t^s. Unit production costs are defined by a unit costs function $f(r_t^s)$. Each customer defines a maximum inventory level U_i and has an initial inventory level I_i^0 such that $I_i^0 \leq U_i$ are defined for each customer $i \in N$. If the customer i is visited at time t, then the quantity shipped to i at time t is such that the inventory level of the customer reaches its maximum value U_i (an order-up-to level policy is applied). If I_{it} denotes the inventory level of customer i at time t, the shipped quantity is $U_i - I_{it}$ if the shipment is performed at time t, and 0 otherwise. An inventory level at the end of time period t at the supplier and customers is denoted as variables B_t and I_t^i respectively. Parameter n defines a number of available vehicles, which should perform a delivery using a set of routes $K = \{1,2,\dots,k\}$ with costs c_k. A binary parameter a_{ik} equals 1 if customer i is served on route k, 0 otherwise. Each vehicle can perform no more than one route per day. Denoted by Y_{kt}, we introduce a binary variable equal to 1 if route k is used at time t and 0 otherwise. A variable X_{ikt} identifies a quantity of product shipped to customer i at time period t using route k and deliveries take place before the consumption. Note that we are assuming that when the level of inventory at the customer is negative, the excess demand is not backlogged. Therefore in this case, the initial inventory level at the following period is set to be equal to zero for each $t \in T = \{1,2..,T+1\}$. A penalty cost d_i is considered if the inventory level is negative. The decisions to be made are the determination of the following, for each time period and planning horizon:

- The customers to be visited
- The amount to be delivered to each customer and
- The route to be followed in order to maximize profit.

The decision variables for the problem will be α_{it}: the inventory level at the customer and supplier at each period (after consumption), γ_{it}: a binary variable equal to 1 if $\alpha_{it} > 0$ and 0 otherwise, $i \in N, t \in T$.

s_{itk}; a non-negative variable representing the quantity of product shipped to customer i at period t using route k and σ_{kt} ,a binary variable equal to 1 if route k is used at period t and 0 otherwise. β_{it} is a non-negative variable representing the level of stock-out at the customer i at time t, δ_{it}; a binary variable equal to 1 if $\beta_{it} > 0$ and 0 otherwise $i \in N, t \in T$.

3.1.1 MODEL 1: INVENTORY ROUTING PROBLEM WITH COST MINIMIZATION

The objective is to minimize the total cost, comprising of total inventory holding costs at the customers and supplier and total transportation cost summed. The total cost of production is fixed in this case , they therefore do not have an effect on the objective function.

$$Min \sum_{t \in T} \sum_{k \in K} c_k * Y_{kt} + \sum_{i \in N} \sum_{t \in T} h_i * I_{it} + \sum_{t \in T} h_s * B_t * r_t^s + \sum_{t \in T} f(r_t^s)\,(r_t^s) \qquad (1.1)$$

CONSTRAINTS

The constraints are as follows:

1. **Inventory definition at the supplier**

$$B_t = B_{t-1} + r_t^s - \sum_{i \in N} \sum_{k \in K} X_{ikt} \qquad t \in T \qquad (1.2)$$

Constraint (1.2) stipulates that the inventory level at the supplier in period t is defined at the end of the period and is given by its previous inventory level period t, plus the quantity r_t^s made available in period t, minus the total quantity shipped to the customers using the supplier's vehicle in period t.

$$B_{t-1} = r_t^s \leq U_s \qquad t \in T \qquad (1.3)$$

Constraint (1.3) limits the inventory level at the supplier to its maximum.

$$B_0 = B^0 \qquad t \in T \qquad (1.4)$$

Constraint (1.4) defines an initial inventory level at the supplier. The inventory level at a customer i in period t is defined at the end of the period.

2. **Inventory definitions at the customers**

$$I_{it} = I_{i,t-1} + \sum_{k\in K} X_{ikt} - r_i \qquad i \in N, t \in T \qquad (1.5)$$

(1.5) stipulates that the inventory level is given by its previous inventory level period t, plus the quantity X_{ikt} delivered to customer i in period t, minus the total quantity consumed by customer i in period t. The inventory level at the end of period t at customer i is then:

$$I_{i0} = I_0^i \qquad i \in N \qquad (1.6)$$

Constraint (1.6) defines the initial inventory level at each customer.

3. **Maximal inventory level at the customers**

These constraints ensure that the inventory level at the customers will not exceed its maximum level.

$$\sum_{k\in K} X_{ikt} \leq U_i - I_{i,t-1} \qquad i \in N, t \in T \qquad (1.7)$$

Constraint (1.7) guarantees that delivery at each time period takes place only if a customer is visited with a route and this route is used at this time period.

4. **Vehicle Capacity**

$$\sum_{i\in N} X_{ikt} \leq Q * Y_{kt} \qquad t \in T, k \in K \qquad (1.8)$$

Constraint (1.8) guarantees that the vehicle's capacity is not exceeded

5. **Routing constraints**

$$X_{ikt} \leq Q * a_{ik} * Y_{kt} \qquad t \in T, i \in N, k \in K \qquad (1.9)$$

Constraint (1.9) guarantee that a delivery at each time period takes place only if a customer is visited with a route and this route is used at this time period:

$$\sum_{k\in K} Y_{kt} \leq n \qquad t \in T \qquad (1.10)$$

Constraint (1.10) limits the number of routes per time period to the number of vehicles.

6. **Integrality and non-negativity constraints**

$$I_{it} \geq 0 \qquad i \in N, t \in T \qquad (1.12)$$

$$B_t \geq 0 \qquad t \in T \qquad (1.13)$$

$$X_{ikt} \geq 0 \qquad i \in N, k \in K, t \in T \qquad (1.14)$$

$$Y_{kt} \in \{0,1\} \qquad k \in K, t \in T \qquad (1.15)$$

3.1.2 MODEL 2: DYNAMIC INVENTORY ROUTING PROBLEM WITH COST MINIMIZATION

In model 1, the consumption rate for the whole planning horizon is known a priori and does not change during the whole planning horizon. Based on the fixed consumption rate for all the customers in all the time periods, routes are generated and delivery plans made for the planning horizon. The decisions do not change from one period to the other. This can therefore be seen as a deterministic inventory routing problem. On the hand, in a dynamic inventory routing problem, the routing and delivery decisions may change from one period to the other during the planning horizon, due to changes in consumption rate in a scenario where the consumption rate for the rest of the planning horizon is to be forecasted based on historical data. Routing and delivery decisions are made at the end of every period for the next period. The same objective function and constraints as in Model 1 shall apply.

3.1.3 MODEL 3: DYNAMIC INVENTORY ROUTING PROBLEM WITH PROFIT MAXIMIZATION

When the objective is to maximize profit, the supplier can earn revenue from sales, valued at P_i per unit of product shipped to customers, which is a unit price. The demands for all the customers do not need to be satisfied, so some of the demand can be partially fulfilled. This unsatisfied demand results in an incurred penalty b_i for each unit of demand not satisfied and that penalty helps account for the customer's unsatisfied needs. Since it is allowed for consumption to be less than demand, we therefore introduce a variable C_{it}, which is the amount of product consumed by the customer i at time period t. The mathematical formulation for this model is:

$$Max \sum_{i\in N}\sum_{k\in K}\sum_{t\in T} P_i X_{ikt} \sum_{t\in T}\sum_{k\in K} c_k Y_{kt} + \sum_{i\in N}\sum_{t\in T} h_i I_{it} + \sum_{t\in T} h_s B_t \; - \sum_{i\in N}\sum_{t\in T} b_i (r_i * C_{it}) - \sum_{t\in T} f(r_t^s)\, r_t^s \tag{3.1}$$

The same constraints as in Model 1 shall apply except the inventory definition at the customers that will be changed as per below:

$$I_{it} = I_{i,t-1} + \sum_{k\in K} X_{ikt} - C_{it} \qquad i \in N, t \in T \tag{3.2}$$

$$I_{i0} = \; I_{i0} \qquad i \; \in N \tag{3.3}$$

3.1.4 MODEL 4: DYNAMIC INVENTORY ROUTING PROBLEM WITH PROFIT MAXIMIZATION FOR MONOPOLY

When the supplier is a monopolist, prices can be adjusted to maximize profit. There is a limit to how high a monopolist can set the price, because there is an inverse relationship between price and demand. When the price is too high, demand is lowered and in this case, we treat the generated revenue P_i as a variable.. The relationship between demand and product unit price is shown by the function $r_i \leq \; f(\, P_i)$ (Zaitseva 2017).

$$Max \sum_{i\in N}\sum_{k\in K}\sum_{t\in T} p_i * X_{ikt} \sum_{t\in T}\sum_{k\in K} c_k Y_{kt} + \sum_{i\in N}\sum_{t\in T} h_i I_{it} + \sum_{t\in T} h_s B_t - \sum_{i\in N}\sum_{t\in T} b_i (f(p_i C_{it}) - \sum_{t\in t} f(r_t^s)(r_t^s) \tag{4.0}$$

The same constraints as in Model 2 shall apply except the constraint below which stipulates that the amount consumed by the customer is a function of price and so a function appears on the right hand side of the constraints:

$$C_{it} \leq \; f(\, P_i) \qquad i \; \in N \,, t \; \in T \tag{4.1}$$

A linear demand curve will be assumed as in (Zaitseva 2017), Besango and Braeutigam 2010. The non-linear and non-separable form of the objective function occurs because profit is derived by multiplying the variables price and quantity, and this non-linearity will be treated as described in (Zaitseva 2017) and Williams 2013 where the non-separable and non-linear objective function is converted into separable functions and then linearized using the piecewise linear method by the following:

1. Transformation to Separable Form:

This is achieved via the following steps

i) A new variable $Z_i = \sum_{k \in K} \sum_{t \in T} X_{ikt}$ is introduced in order to avoid indices for routes and time periods for every customer, which results in a new term in the objective function being$\sum_{i \in N} P_i Z_i$.

ii) Two new variables W_{1i} and W_{2i} are introduced into the model and related to P_i and Z_i such that

$$W_{1i} = \frac{1}{2}(P_i * Z_i)$$

$$W_{2i} = \frac{1}{2}(P_i * Z_i)$$

If $l_p \leq P_i \leq u^p \; and \; l^z \leq Z_i \leq u_i^z, then \; the \; bounds \; on \; W_{1i} \; and \; W_{2i} \; are$

$$\frac{1}{2}(l^p + l^z) \leq W_{1i} \leq \frac{1}{2}(u^p - u_i^z)$$

$$\frac{1}{2}(l^p - u_i^z) \leq W_{2i} \leq \frac{1}{2}(u^p - l^z)$$

The objective function $\sum_{i \in N} P_i Z_i$ is now replaced by the term $\sum_{i \in N}(W_{1i}^2 * W_{2i}^2)$, which is now a separable non-linear function (as it contains non-linear functions of a single variable.

2. Transformation to linear form: The non-linear terms can be eliminated by piecewise linear approximations by using the λ – formulation method described below:

a. Breakpoints denoted by $W1_{is}$are introduced for functions $g(W_{1i})$ and $g(W_{2i})$ by which the curves are divided into pieces that are approximated by straight lines. Any point between two breakpoints is a weighted sum of these two points.
b. Non-negative weights λ_{is}^{W1} and $\lambda_{is}^{W2}$for the functions $g(W_{1i})$ and $g(W_{2i})$ are introduced.
c. The piecewise approximation ensures the maximization of the product of the weight λ_{is}^{W1} and its corresponding function $g(W_{1i})$ less the product of the weight λ_{is}^{W2} and its corresponding function $g(W_{2i})$

iv) Inclusion of a special ordered set of type 2 (SOS2) constraint whereby at most two adjacent λ_{is}^{W1} can be greater than zero. This constraint guarantees that corresponding W_{1i} and $g(W_{1i})$ always lie on one of the straight line segments between breakpoints. A binary variable $S_{\ is}$that represents the interval between two adjacent breakpoints and equal 1 if the interval is chosen and 0 otherwise is included for the function $g(W_{1i})$ or $(W2i)$ where a convex function is maximized.

4. METHODOLOGY

In this chapter we discuss the strategies we plan to adopt towards achieving the profit maximization objective of our problem, while considering the supplier's market structure. Our approach combines a mixed integer linear programming with the rolling horizon approach, by using the rolling horizon approach to decompose the problem into small sub-problems, which are then optimized using the exact method of mixed-integer linear programming. The proposed approach based on a rolling horizon framework has previously been applied to inventory routing problems with satellite facilities (IRPSF) by (Jaillet et al. 2002) and (Bard et al. 1998) but with a cost minimization objective. (Zaitseva 2017) also worked on ascertaining profit maximization in inventory routing problem with consideration for the type of market the organization operates. We extend the work done by Zaitseva 2017 by introducing dynamic settings implemented under a rolling horizon approach. To the best of our knowledge, no paper exists where the rolling horizon approach has been incorporated into a dynamic inventory routing problem with an objective to maximize profit, while considering the structure of the market the supplier operates in.

4.1 Scenario Generation

Scenarios from our context are time blocks generated from the problem's simulation length, composed of fixed planning horizons and fixed customer sets, over which the performance of the different possible market structures the supplier can take is tested, to determine their maximum profit, revenue, and costs post demand satisfaction etc. For the purpose of this thesis, we have generated 72 different scenarios with simulation length of 30. The scenarios are generated based on 3 fixed customer sizes: 5, 10 and 15 customers and 6 fixed planning horizon lengths of 1, 2, 3, 4, 5 and 6 days, which will be tested over 4 different models. The list of scenarios are shown in table 4.1:

Scenario	Number of Customers	Length of Planning Horizon, day
T1N5	5	1
T2N5	5	2
T3N5	5	3
T4N5	5	4
T5N5	5	5
T6N5	5	6
T1N10	10	1
T2N10	10	2
T3N10	10	3
T4N10	10	4
T5N10	10	5
T6N10	10	6
T1N15	15	1
T2N15	15	2
T3N15	15	3
T4N15	15	4
T5N15	15	5
T6N15	15	6

Table 4.1: Generated Scenarios

- T1N5: 1-day planning horizon with 5 customers
- T1N10: 1-day planning horizon with 10 customers
- T1N15: 1-day planning horizon with 15 customers
- T2N5: 2-day planning horizon with 5 customers
- T2N10: 2-day planning horizon with 10 customers
- T2N15:2--day planning horizon with 15 customers
- T3N5: 3-day planning horizon with 5 customers
- T3N10: 3-day planning horizon with 10 customers
- T3N15: 3-day planning horizon with 15 customers
- T4N5: 4-day planning horizon with 5 customers
- T4N10: 4-day planning horizon with 10 customers
- T4N15: 4-day planning horizon with 15 customers
- T5N5: 5-day planning horizon with 5 customers
- T5N10: 5-day planning horizon with 10 customers
- T5N15: 5-day planning horizon with 15 customers

- T6N5: 6-day planning horizon with 5 customers
- T6N10: 6-day planning horizon with 10 customers
- T6N15: 6- day planning horizon with 15 customers

4.1.1 The Rolling Horizon Approach

Following the rolling horizon approach flowchart in Figure 4.1, the rolling horizon approach operates by dividing a large problem with a long time horizon into smaller sub-problems with shorter and more manageable time horizons. In each of these sub-problems, only part of the scheduling problem is solved in detail, while the remaining part of the time horizon is aggregated.

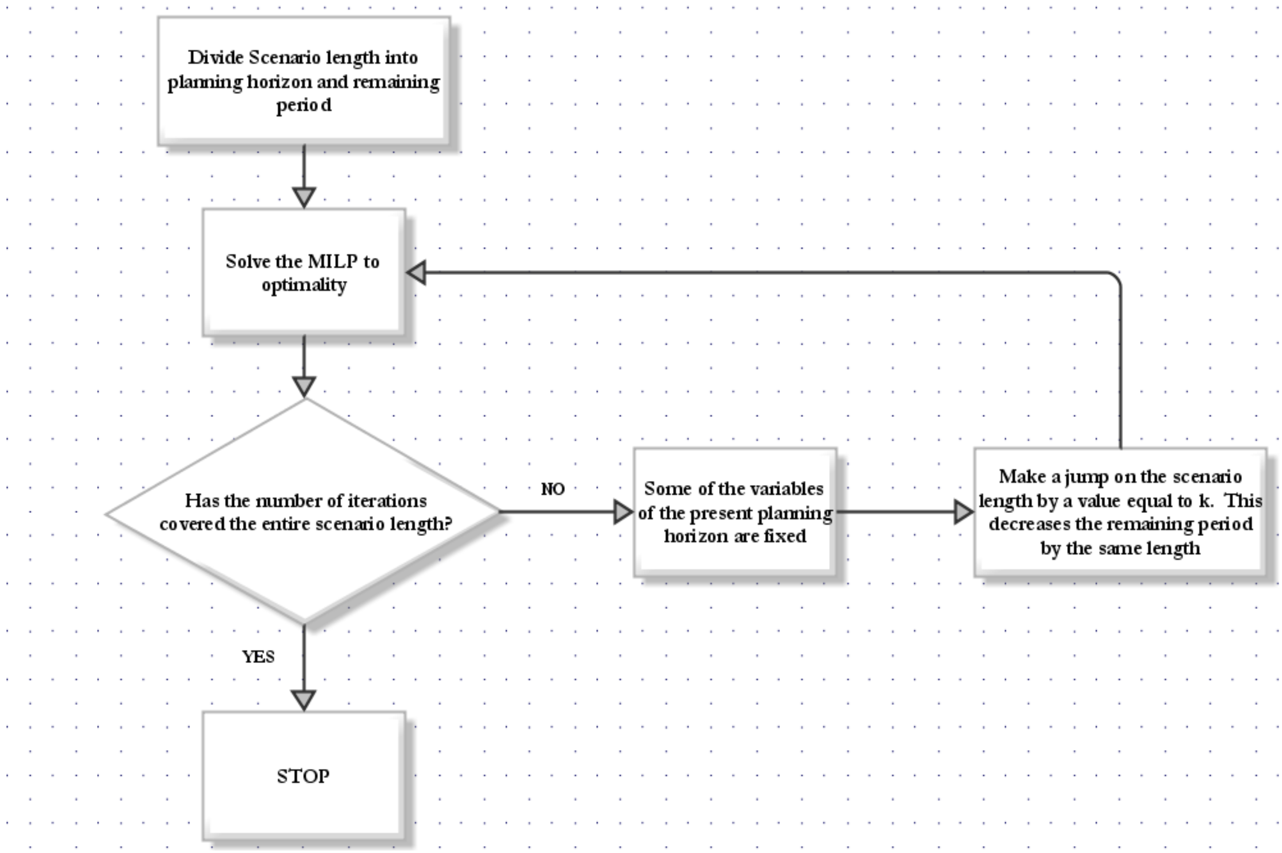

Figure 4.1: Rolling Horizon approach flowchart.

The rolling horizon principle helps to reduce computing times of larger problem because computing time increases as the number of variables or time horizon increases. The main idea behind the rolling horizon is that solving the MILP in detail, to optimality, for a small part of the time horizon is relatively simple compared to solving the MILP in detail for the entire horizon, and this approach substantially reduces computational time (Al-Ameri, Shah, and Papageorgiou 2008). As seen in figure 4.1, the next step is that the sub problems are solved to optimality. The rolling horizon approach ties the model in a loop, and for each iteration, the planning horizon jumps by a value k, which is 1 day in our problem, such that the remaining period on the simulation length reduces by the value of k until the planning horizon covers the entire simulation length and this is done for every scenario that is to be observed.

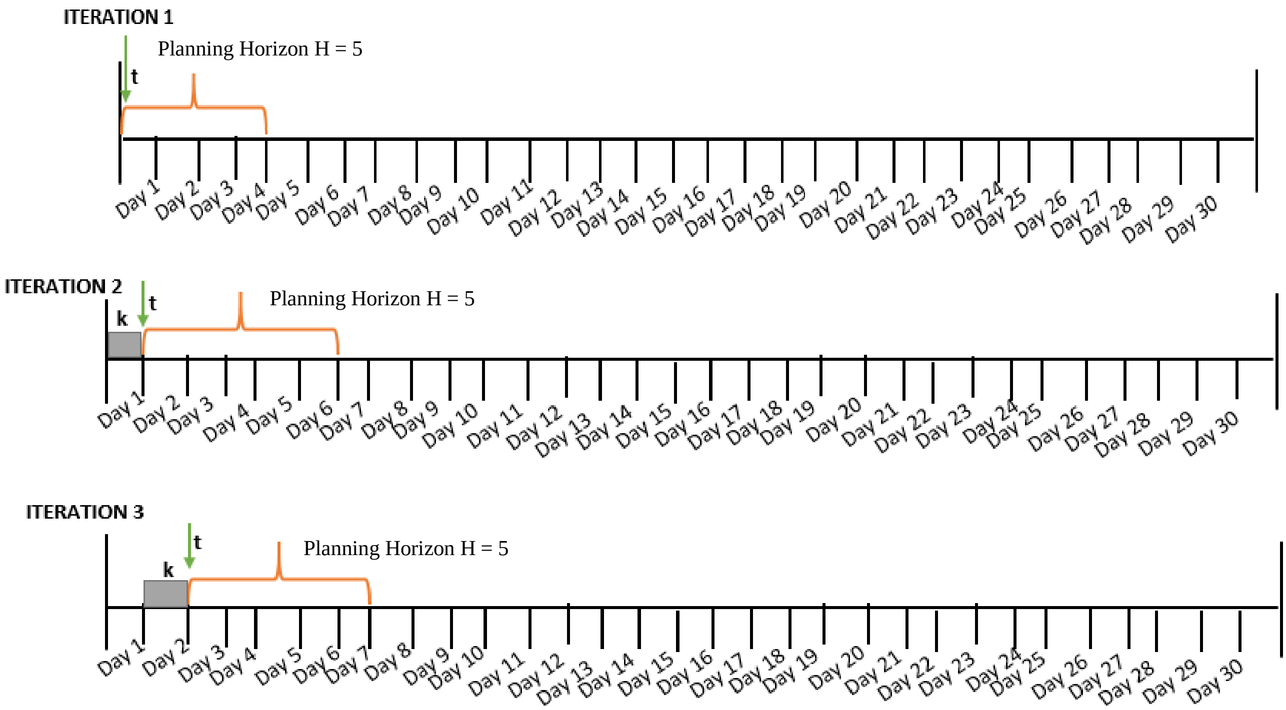

Figure 4.2: Rolling Horizon diagram for five-day planning horizons, adopted from (Fagerholt et al. 2010)

The figure above illustrates the rolling horizon principle where the planning horizon H has a length of 5 days and the shifting of the planning horizon between three consecutive iterations, with a length of the jump in time simulated between iteration of k = 1 (The methodology makes a length of jump in time $K \leq H$). In our work, the length of the simulation covers a fixed 30 day period and (t) represents the present time in the simulation. The planning horizon is fixed at the beginning of the iteration, as seen in figure 1. The present time (t) in the simulation is updated after each jump in time of length k and what was planned up to the new present time is executed. The solver then recalculates a new routing decision using available demand information and this procedure is continued until the total fixed length of the simulation is covered. Therefore, a rolling scheduling horizon method has been adopted to enable the model iteratively works its way through the simulation length of 30 days. A length of the planning horizon should match values that are experienced by the planner and determined based on contractual agreements if any exists. Only demands experienced within the scheduling horizon are presented to the solver.

4.2 Analysis of results

Important output figures will be obtained for every scenario from the solving process of the short-term routing and scheduling problem. The following figures are to be obtained: total profit, total revenue, total cost, total produced amount, total shipped amount, total consumed amount, total shortage amount and total penalty cost. Figures like total shortage amount and total penalty cost are relevant in situations where customers are not served, which may force the company to break the agreement with the customers, resulting in possible loss of goodwill or excessively higher replenishment cost. In addition to producing near-optimal results, the rolling horizon approach can reduce the computation time.

5 COMPUTATIONAL EXPERIMENT

In this section,we provide some specifications with which the models were generated and run, we describe the proceedure with which the instances were generated and we present results which highlight the performance of our models.

5.1 Specification for Implementation

All formulations were computed with AMPL/CPLEX 12.7.00 through remote desktop access run on a private computer with the following specifications: Intel (R). Core (Stadtler) i5-6400T CPU at 2.2GHz, 2.21GHz with 12.0GB RAM, 64-bit operating system, x64-based processor.
The thesis makes use the AMPL API R interface to access the features of the AMPL interpreter from within the R programming language. The interface was used to directly assign data to AMPL parameters and sets. The benefit of this is that we are able to develop a separate decision support application outside of AMPL and interact with the AMPL solver when required.

5.2 Instance Generation

We generated instances following the standards used for the instance generation in (Zaitseva 2017), except for the horizon, which was modified to fit the problem. Instances were generated according to the following data:

- Number of customers n = 5, 10 and 15.
- Horizon H; equals 1, 2, 3, 4, 5 or 6 periods
- The quantity of product $r_i$consumed by the customers i at time t is generated as a random integer following a discrete uniform distribution in the interval [10, 100].
- The maximum inventory level U_i at the customer equals $r_i g$
- The inventory holding cost h_s at the supplier is 0.3 and inventory holding cost $h_i$at the customers. Both costs are randomly generated in the interval [0.1, 0.5].
- The vehicle capacity Q is $\frac{1.5}{n} \sum_{i \in j} r_i$. n represents the number of available vehicles.
- The coordinates $(X_i - Y_i)$ are the coordinates of the customers i, and are obtained randomly from a discrete uniform distribution in the interval [0, 500.] distance/cost c_{ij} is then calculated as $\sqrt{(X_0 - X_1)^2} + \sqrt{(Y_0 - Y_1)^2}$.
- The maximum number of customers on each route is 2 and 3.

- The number of vehicles is 3
- The demand function is $f(p_i) = -2.5p_i + 113$, where p_i is the unit price.
- The unit price limit for the monopoly is 41, with the corresponding demand 10.5.
- The penalty for unsatisfied demand is $0.2p_i$
- The average cost function $:f(r_t^s) = 0.0005(r_t^s) + 2 + \frac{3}{(r_t^s)}$ where (r_t^s) is the production rate.

5.3 Computational results

We now present the results of our computational experiments.

In table 5.1, we show the results for all 72 instances, the total revenue, total profit, total transportation cost, total inventory holding cost, total production cost, total penalty cost, consumed amount, shortage quantity and computation time for each of the 4 models in each scenario. From this table, we will show trends in total revenues, profits and costs across models and scenarios

Table 5.1: summary of computational results for the dynamic inventory routing problem for the scenarios, across 4 models

Scenario	Model	Total Revenue	Total Profit	Total Cost	Total Transportation Cost	Total Inv Holding Cost Customers	Total Inv Holding Cost Supplier	Total Production Cost	Penalty Cost	Consumed Amount	Shortage Qty	Solve Time
T1N5	mincost	147576	98015	49561	30045	492	6793	12228	0	5790	0	57
T1N5	profit max	147576	85876	61699	42346	2533	4590	12228	0	5790		1
T1N5	monopoly	0	0	0	0	0	0	0	0	0	0	0
T1N5	competition	147576	91407	56169	42346	2533	132	11155	0	5790	0	1
T2N5	mincost	147576	102156	45420	26024	1089	6076	12228	0	5790	0	101
T2N5	profit max	147573	96682	50891	31500	1737	5426	12228	0	5790	0	323
T2N5	monopoly	181029	119308	61721	39630	2519	7344	12228	0	5473	0	130
T2N5	competition	147576	102232	45344	31503	1721	957	11151	0	5790	0	89
T3N5	mincost	147576	99588	47987	28673	1676	5408	12228	0	5790	0	95
T3N5	profit max	147576	105330	42246	22819	1564	5633	12228	0	5790	0	138
T3N5	monopoly	180229	125268	54961	33100	2225	7388	12228	20	5497	3	347
T3N5	competition	147576	111880	35696	22030	1495	1071	11098	0	5790	0	107
T4N5	mincost	147576	104088	43488	24052	1210	5996	12228	0	5790	0	215
T4N5	profit max	147576	104690	42886	23468	1132	6056	12228	0	5790	0	178
T4N5	monopoly	185065	133432	51633	30654	1777	6974	12228	0	5553	0	826
T4N5	competition	136951	101058	35893	22134	1165	1468	11126	0	5790	0	223
T5N5	mincost	147576	103869	43707	24310	1548	5621	12228	0	5790	0	332
T5N5	profit max	147575	106753	40822	21334	1443	5816	12228	0	5790	0	264
T5N5	monopoly	185690	134268	51422	31010	1511	6621	12228	52	5627	8	1241
T5N5	competition	147360	112058	35302	21466	1505	1178	11111	42	5784	6	290
T6N5	mincost	147576	106526	41050	21610	1286	5926	12228	0	5790	0	362
T6N5	profit max	147576	106887	40689	21255	1298	5906	12228	0	5790	0	367
T6N5	monopoly	191496	140541	50955	32764	1531	4421	12228	11	6039	2	3512
T6N5	competition	146472	110295	36177	22236	1353	1321	11047	220	5763	27	336
T1N10	mincost	0	0	0	0	0	0	0	0	0	0	0
T1N10	profit max	317472	191864	125606	57236	6925	16996	44238	211	19050	101	44
T1N10	monopoly	0	0	0	0	0	0	0	0	0	0	0
T1N10	competition	316416	208095	108321	61129	6789	531	39659	211	18948	101	39
T2N10	mincost	317472	191246	126226	59074	3876	19036	44238	0	19050	0	225
T2N10	profit max	317471	196134	121337	54036	4745	18318	44238	0	19050	0	581
T2N10	monopoly	345059	206965	138094	60697	8639	22497	44238	2023	18161	255	1837
T2N10	competition	317462	216260	101202	55288	4542	1762	39605	2	19049	1	251

Table 5.1 continues: summary of computational results for the dynamic inventory routing problem for the scenarios, across 4 models

Scenario	Model	Total Revenue	Total Profit	Total Cost	Total Transportation Cost	Total Inv Holding Cost Customers	Total Inv Holding Cost Supplier	Total Production Cost	Penalty Cost	Consumed Amount	Shortage Qty	Solve Time
T3N10	mincost	317472	187600	129872	62145	6430	17057	44238	0	19050	0	209
T3N10	profit max	317472	190424	127048	59812	4574	18422	44238	0	19050	0	407
T3N10	monopoly	368682	234076	134606	60173	7089	22662	44238	444	19050	61	26171
T3N10	competition	315517	217804	97713	51352	3498	3106	39367	390	18872	178	441
T4N10	mincost	317472	197662	119810	52463	2577	20532	44238	0	19050	0	377
T4N10	profit max	317466	202738	114728	47412	4335	18741	44238	6	19050	3	739
T4N10	monopoly	411837	280018	131819	57799	6845	22467	44238	470	18355	63	34034
T4N10	competition	317082	223996	93086	46731	4719	1980	39579	77	19014	35	670
T5N10	mincost	317472	201656	115816	48590	4182	18804	44238	0	19050	0	830
T5N10	profit max	317452	200993	116459	49184	4291	18157	42763	20	18415	3,5	1072
T5N10	monopoly	405576	284502	121074	52462	6065	20900	41186	459	17023	63	41921
T5N10	competition	316754	224442	92312	46283	4339	2032	39515	143	18997	53	736
T6N10	mincost	317472	204257	113215	45854	4671	18450	44238	0	19050	0	1072
T6N10	profit max	317447	202771	114676	47384	4364	18688	44238	25	19050	11	2247
T6N10	monopoly	436911	313497	123414	50163	5997	21639	44238	1377	17085	189	141990
T6N10	competition	317096	224518	92578	46528	4263	2147	39566	74	19020	30	1565
T1N15	mincost	0	0	0	0	0	0	0	0	0	0	0
T1N15	profit max	0	0	0	0	0	0	0	0	0	0	0
T1N15	monopoly	0	0	0	0	0	0	0	0	0	0	0
T1N15	competition	422895	311692	111203	47759	5500	821	46680	10441	22187	2593	87
T2N15	mincost	0	0	0	0	0	0	0	0	0	0	0
T2N15	profit max	475102	333287	141815	49436	5072	25116	59884	2307	24780	280	157
T2N15	monopoly	0	0	0	0	0	0	0	0	0	0	0
T2N15	competition	461738	352284	109454	48725	4923	1113	52018	2673	23685	488	238
T3N15	mincost	0	0	0	0	0	0	0	0	0	0	0
T3N15	profit max	475102	329371	145731	48903	4857	27362	59884	4725	24780	708	2764
T3N15	monopoly	0	0	0	0	0	0	0	0	0	0	0
T3N15	competition	436229	326773	109455	48395	4148	1319	47818	7775	22773	2007	7978
T4N15	mincost	0	0	0	0	0	0	0	0	0	0	0
T4N15	profit max	473749	332775	140974	49408	3625	25546	59884	2511	24780	364	904
T4N15	monopoly	0	0	0	0	0	0	0	0	0	0	0
T4N15	competition	461135	351846	109287	48457	4988	1027	52022	2793	24313	467	13967

Scenario	Model	Total Revenue	Total Profit	Total Cost	Total Transportation Cost	Total Inv Holding Cost Customers	Total Inv Holding Cost Supplier	Total Production Cost	Penalty Cost	Consumed Amount	Shortage Qty	Solve Time
T4N15	monopoly	0	0	0	0	0	0	0	0	0	0	0
T4N15	competition	461135	351846	109287	48457	4988	1027	52022	2793	24313	467	13967
T5N15	mincost	0	0	0	0	0	0	0	0	0	0	0
T5N15	profit max	462024	323295	138729	48349	5069	25116	57888	2307	24780	280	888
T5N15	monopoly											
T5N15	competition	461562	352550	109009	48349	4918	1106	51928	2708	24272	508	7482
T6N15	mincost	0	0	0	0	0	0	0	0	0	0	0
T6N15	profit max	475102	332530	142572	49805	5005	25445	59884	2433	24780	352	2770
T6N15	monopoly	0	0	0	0	0	0	0	0	0	0	0
T6N15	competition	457639	348199	109437	49292	4744	1145	50764	3492	23808	972	13830

Table 5.1 continues: summary of computational results for the dynamic inventory routing problem for the scenarios, across 4 models

5.3.1 Computation Time

The simulation terminated for each scenario when the iteration reached the fixed 30 days it was simulated to cover. The table below gives us the computation time for each of our 72 instances, which is a combination of planning horizon and number of customers.

Number of Customers	Model	Planning Horizon					
		T1	T2	T3	T4	T5	T6
5	mincost	57	101	95	215	332	362
5	profit_max	1	323	138	178	264	367
5	monopoly	0	130	347	826	1241	3512
5	competition	1	89	107	223	290	336
10	mincost	0	225	209	377	830	1072
10	profit_max	44	581	407	739	1072	2247
10	monopoly	0	1837	26171	34034	41921	141990
10	competition	39	251	441	670	736	1565
15	mincost	0	0	0	0	0	0
15	profit_max	0	157	2764	904	888	2770
15	monopoly	0	0	0	0	0	0
15	competition	87	238	7978	13967	7482	13830

Table 5.2: Computational time in seconds

The table shows that the computational requirement increased progressively with an increase in the number of customers and length of planning horizon.The cost minimization models took the least time to run for all customer sets and all planning horizon lengths. The monopolistic models took the most time to run, for all customer sets and all poanning horizons where feasible results were gotten. Comparing the cost minimization model with 5 customers, it can be seen that as the planning horizon length increases from T1 to T6, the computational requirement increases from 57 secs in the 1 day planning horizon length to 362 seconds in the 6 day planning horizon length for the same customer number of 5. The increase in computation time is reasonable, in accordance with (Al-Ameri, Shah, and Papageorgiou 2008) where it shows that with a rolling horizon approach, as the variables or the time horizon increases, computing times increase greatly. This trend of dramatic increase in computing time can also be seen for the 4 different models, with the model for the monopolistic market condition showing the longest computational time mostly for all scenarios. The scenarios with 15 customers do not follow this trend because the computation time shown in Table 5.2 does not include computational times from the model with the monopolistic market structure, whose computing time was extremely longer than computing times from all other scenarios combined. Adding data from those computing time would have tallied with a dramatically longer computing time as the variables increased.

5.3.2 Revenue

The table below shows the total revenue for all 72 instances. The revenue represents the total demand satisfied at all customer locations multiplied by the price of each product.

Number of Customers	Planning Horizon						
	Model	T1	T2	T3	T4	T5	T6
5	mincost	147576	147576	147576	147576	147576	147576
5	profit_max	147576	147573	147576	147576	147576	147576
5	monopoly	0	181029	180229	185065	185690	191496
5	competition	147576	147576	147576	136951	147360	146472
10	mincost	0	317472	317472	317472	317472	317472
10	profit_max	317472	317472	317472	317466	317452	317447
10	monopoly	0	345059	368682	411837	405576	436911
10	competition	316416	317462	315517	317082	316754	317096
15	mincost	0	0	0	0	0	0
15	profit_max	0	475102	475102	473749	462024	475102
15	monopoly	0	0	0	0	0	0
15	competition	422895	461738	436229	461135	461562	457639

Table 5.3: Total Revenue across all instances

The revenue in Table 5.3 above is generated for all 4 models across 3 customer sets and 6 planning horizons. It can be seen that the value for the revenue is fairly equal across the instances for each set of customers, this is because the consumed demand is fairly constant across all 4 models, although highest with the monopolistic market across all scenarios.

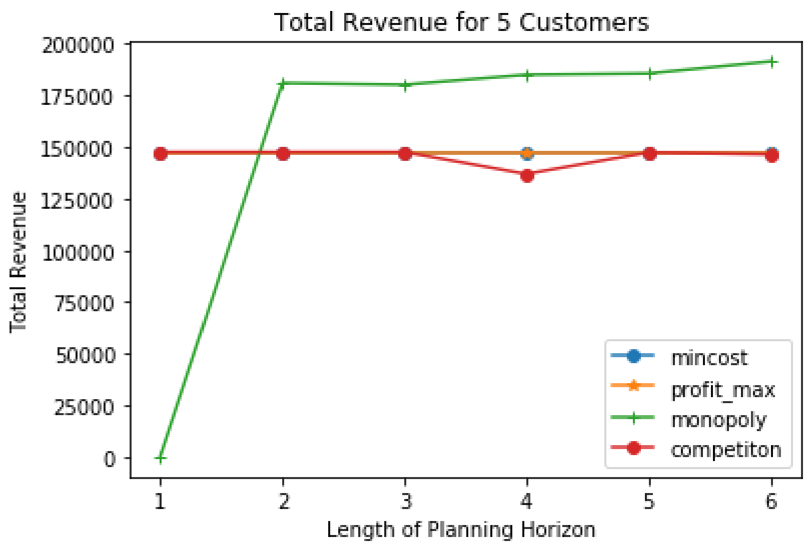

Figure 5.1: Total revenue for customer set of 5 across all 4 models and 6 planning horizons

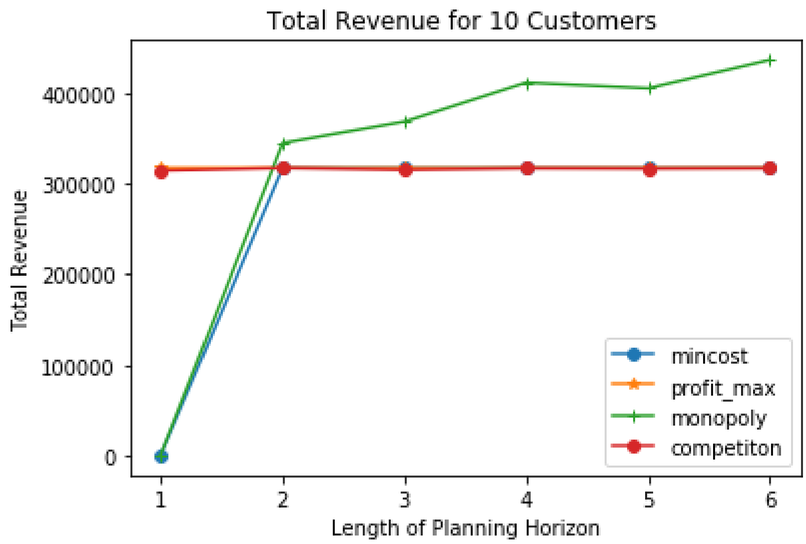

Figure 5.2: Total revenue for customer set of 10 across all 4 models and 6 planning horizons

5.3.3 Profit

Profit is calculated by subtracting the total incurred cost from the total revenue.

Number of Customers	Planning Horizon						
	Model	T1	T2	T3	T4	T5	T6
5	mincost	98015	102156	99588	104088	103869	106526
5	profit_max	85876	96682	105330	104690	106753	106887
5	monopoly	0	119308	125268	133432	134268	140541
5	competition	91407	102232	111880	101058	112058	110295
10	mincost	0	191246	187600	197662	201656	204257
10	profit_max	191864	196134	190424	202738	200993	202771
10	monopoly	0	206965	234076	280018	284502	313497
10	competition	208095	216260	217804	223996	224442	224518
15	mincost	0	0	0	0	0	0
15	profit_max	0	333287	329371	332775	323295	332530
15	monopoly	0	0	0	0	0	0
15	competition	311692	352284	326773	351846	352550	348199

Table 5.4: Total Profit across all instances

Table 5.4 shows the total profit across all instances. The mincost models generated the least profits for all customer sets and all planning horizon lengths. The monopolist's model is seen to have generated the highest profit for every scenario where a feasible result was gotten. This is because the model combines the profit maximizing price-quantity combination on the market demand curve. The monopolistic model could in some instances have increased the price at which products were supplied to the consumer, which will trigger a decrease in demand consumed by the consumer, reducing costs and increasing profits. The perfect competitive market on the other hand, being the price taker, takes prices as given by the market equilibrium of demand and supply, generating profits lower than the monopolist. Had the price of the perfectly competitive model been raised above the given market price, no consumed demand or profit would have been recorded.

Figure 5.3: Total profit for customer set of 5 across all 4 models and 6 planning horizons

Figure 5.3 shows the monopoly with the highest profits when the profits generated with the scenarios with 5 customers were plotted against the planning horizons.

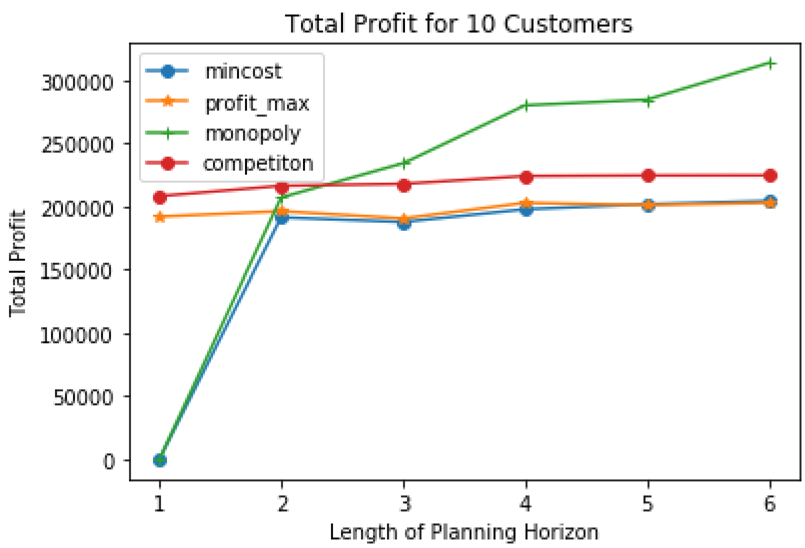

Figure 5.4: Total profit for customer set of 10 across all 4 models and 6 planning horizons

Figure 5.4 shows the monopoly with the highest profits when the profits generated with the scenarios with 10 customers were plotted against the planning horizons. From Table 5.4, it can be seen that as the number of customers and length of the planning horizon increases, the total profit increases across all 4 models.

5.3.3 Cost

Total cost encompasses the cost of production, transportation, inventory holding costs and penalties.

Number of Customers	Model	Planning Horizon					
		T1	T2	T3	T4	T5	T6
5	mincost	49561	45420	47987	43488	43707	41050
5	profit_max	61699	50891	42246	42886	40822	40689
5	monopoly	0	61721	54961	51633	51422	50955
5	competition	56169	45344	35696	35893	35302	36177
10	mincost	0	126226	129872	119810	115816	113215
10	profit_max	125606	121337	127048	114728	116459	114676
10	monopoly	0	138094	134606	131819	121074	123414
10	competition	108321	101202	97713	93086	92312	92578
15	mincost	0	0	0	0	0	0
15	profit_max	0	141815	145731	140974	138729	142572
15	monopoly	0	0	0	0	0	0
15	competition	111203	109454	109455	109287	109009	109437

Table 5.5: Total Cost across all instances

The monopolist is seen to have incurred the most cost of all the models, although for all feasible results obtained for the monopolist, it can be seen that their cost reduces as the length of the planning horizon increases for all customer sets. The cost however doubles as the number of customers increase. This trend is also observed for all models. The high costs of the monopolist can be linked to penalty costs from unsatisfied demands when the model increases the price, causing demand to go unsatisfied. Unsatisfied demand due to increased prices and not lack of product at the supplier means the supplier deals with an added inventory holding and transportation cost, all adding up to high costs for the monopolist. The perfectly competitive market model has the least cost in table 5.5. This means it incurred the least combination of penalty, transportation and inventory holding and production costs. The perfectly competitive model cannot increase its price to maximize profit, but maximizes profit through keeping its costs low, especially its production costs and amount produced, which translates to reduced inventory holding costs.

5.3.4 Production Cost at the Supplier

Table 5.6 shows the cost the supplier incurs from production. The general trend from the table is an increase in production cost as the number of customers increase. The perfectly competitive model generates the lowest production cost across the planning horizon lengths, for all the customer sets.

Number of Customers	Model	Planning Horizon T1	T2	T3	T4	T5	T6
5	mincost	6793	6076	5408	5996	5621	5926
5	profit_max	4590	5426	5633	6056	5816	5906
5	monopoly	0	7344	7388	6974	6621	4421
5	competition	132	957	1071	1468	1178	1321
10	mincost	0	19036	17057	20532	18804	18450
10	profit_max	16996	18318	18422	18741	18157	18688
10	monopoly	0	22497	22662	22467	20900	21639
10	competition	531	1762	3106	1980	2032	2147
15	mincost	0	0	0	0	0	0
15	profit_max	0	25116	27362	25546	25116	25445
15	monopoly	0	0	0	0	0	0
15	competition	821	1113	1319	1027	1106	1145

Table 5.6: Production cost

As stated previously, the perfectly competitive model capitalizes on lowering its production costs to maximize its profits, as it cannot maximize profits through increasing it prices like the monopoly can. For the monopolist, production costs mostly reduces as the length of the planning horizon increases for its feasible results.

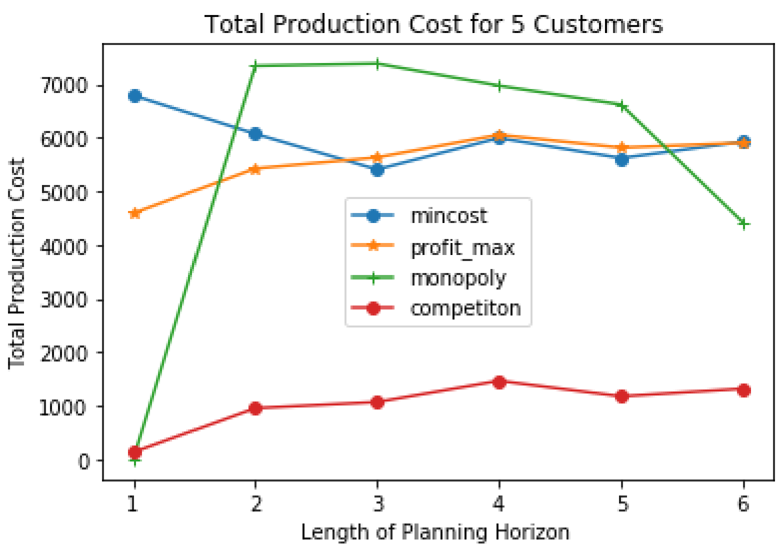

Figure 5.5: Total production cost for customer set of 5 across all 4 models and 6 planning horizons

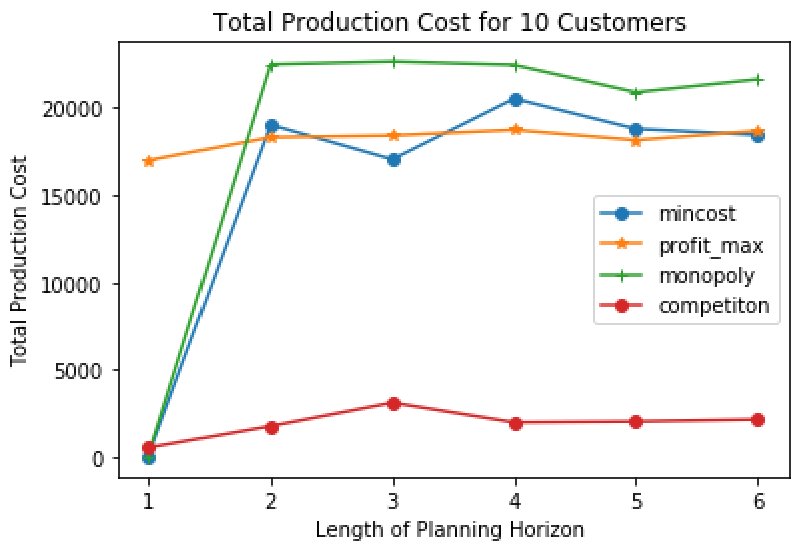

Figure 5.6: Total production cost for customer set of 10 across all 4 models and 6 planning

Figures 5.5 and 5.6 show the trends in total production for 5 and 10 customer sets respectively, across the planning horizon lengths. The reduction in the production cost across planning horizon length is more pronounced in Figure 5.5 with the 5 customer sets. The cost generally increases with increase in customers.

5.3.5 Penalty Cost

The penalty cost is incurred by the supplier for not meeting demand requirement of the customer. Table 5.7 shows the penalty costs across all 4 models, 3 customer sets and 6 planning horizons.

Number of Customers	Model	Planning Horizon					
		T1	T2	T3	T4	T5	T6
5	mincost	0	0	0	0	0	0
5	profit_max	0	0	0	0	0	0
5	monopoly	0	0	20	0	52	11
5	competition	0	0	0	0	42	220
10	mincost	0	0	0	0	0	0
10	profit_max	211	0	0	6	20	25
10	monopoly	0	2023	444	470	459	1377
10	competition	211	2	390	77	143	74
15	mincost	0	0	0	0	0	0
15	profit_max	0	2307	4725	2511	2307	2433
15	monopoly	0	0	0	0	0	0
15	competition	10441	2673	7775	2793	2708	3492

Table 5.7: Penalty costs

The perfectly competitive model incurs the most penalty cost of the instances with 5 customers. The model maintains 0 penalty costs between planning horizon lengths of 1 to 3, but begins to accrue penalty costs at the 5th and 6th day planning horizon lengths. This is perhaps due to a trade-off between a combination of production amount and fulfilled demand with which it maximizes the most profit. The penalty costs at the monopoly are also probably due to the price and satisfied demand combination that maximizes the most profits for the model.

5.3.6 Transportation cost

Number of Customers	Planning Horizon						
	Model	T1	T2	T3	T4	T5	T6
5	mincost	30045	26024	28673	24052	24310	21610
5	profit_max	42346	31500	22819	23468	21334	21255
5	monopoly	0	39630	33100	30654	31010	32764
5	competition	42346	31503	22030	22134	21466	22236
10	mincost	0	59074	62145	52463	48590	45854
10	profit_max	57236	54036	59812	47412	49184	47384
10	monopoly	0	60697	60173	57799	52462	50163
10	competition	61129	55288	51352	46731	46283	46528
15	mincost	0	0	0	0	0	0
15	profit_max	0	49436	48903	49408	48349	49805
15	monopoly	0	0	0	0	0	0
15	competition	47759	48725	48395	48457	48349	49292

Table 5.8: Transportation costs

Table 5.8 shows the transportation and production cost generated for all 4 models, across horizon lengths and customer sets. A general trend observed is that transportation cost increases as the number of customers increase. The monopolistic model generates the most transportation cost of all 4 mode across all customer sets and planning horizon lengths.

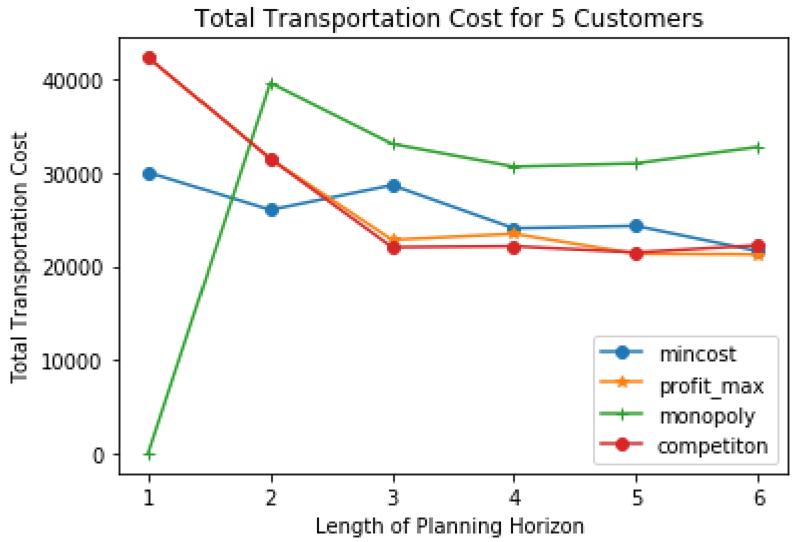

Figure 5.7: Total transportation cost for customer set of 5 across all 4 models and 6 planning horizons

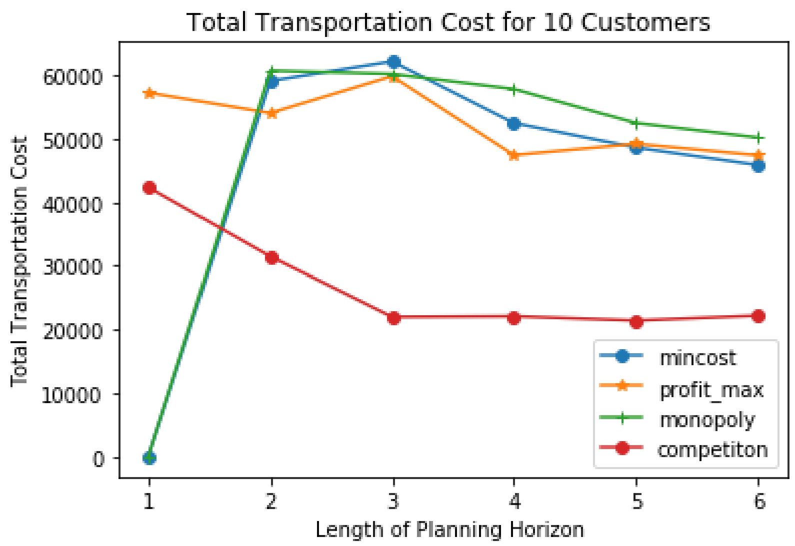

Figure 5.8: Total transportation cost for customer set of 10 across all 4 models and 6 planning horizons

Figures 5.7 and 5.8 show the trends in total transportation costs for 5 and 10 customer sets respectively. Both charts show a generally downward trend in the transportation cost as the planning horizon length increases

5.3.7 Lost sales/Shortages

The total shortages computed in Table 5.9 no particular trend. They show the number of demands the supplier was unable to satisfy and provide the values for which the penalty cost is computed.

Number of Customers	Model	Planning Horizon					
		T1	T2	T3	T4	T5	T6
5	mincost	0	0	0	0	0	0
5	profit_max	0	0	0	0	0	0
5	monopoly	0	0	3	0	8	2
5	competition	0	0	0	0	6	27
10	mincost	0	0	0	0	0	0
10	profit_max	101	0	0	3	3,5	11
10	monopoly	0	255	61	63	63	189
10	competition	101	1	178	35	53	30
15	mincost	0	0	0	0	0	0
15	profit_max	0	280	708	364	280	352
15	monopoly	0	0	0	0	0	0
15	competition	2593	488	2007	467	508	972

Table 5.9: Shortage Cost/Lost sales

5.3.8 Inventory Holding Cost at the suppliers

Table 5.10 shows the inventory holding cost at the supplier for all instances. The costs reduce across all instances as the number of customers increase, however, no trend is notices for most of the instances as the length of the planning horizon increases.

Number of Customers	Planning Horizon						
	Model	T1	T2	T3	T4	T5	T6
5	mincost	6793	6076	5408	5996	5621	5926
5	profit_max	4590	5426	5633	6056	5816	5906
5	monopoly	0	7344	7388	6974	6621	4421
5	competition	132	957	1071	1468	1178	1321
10	mincost	0	19036	17057	20532	18804	18450
10	profit_max	16996	18318	18422	18741	18157	18688
10	monopoly	0	22497	22662	22467	20900	21639
10	competition	531	1762	3106	1980	2032	2147
15	mincost	0	0	0	0	0	0
15	profit_max	0	25116	27362	25546	25116	25445
15	monopoly	0	0	0	0	0	0
15	competition	821	1113	1319	1027	1106	1145

Table 5.10: Inventory holding cost at the supplier

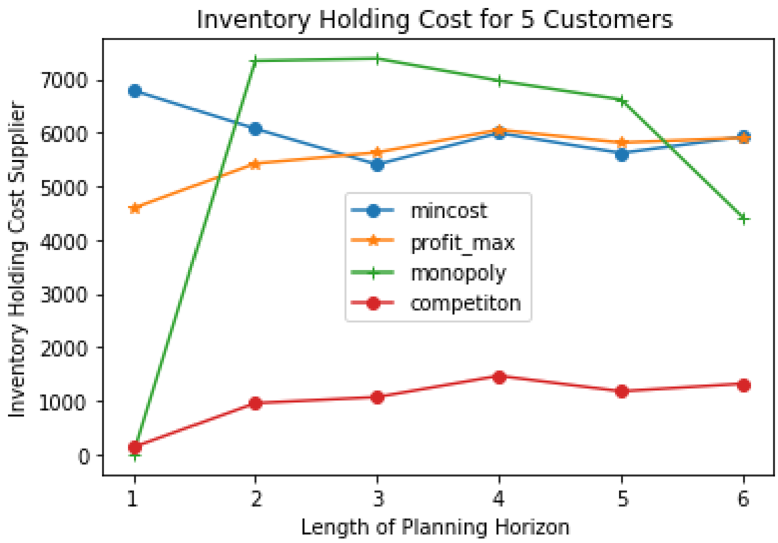

Figure 5.9: Total Inventory holding cost at the supplier cost for customer set of 5 across all 4 models and 6 planning horizons

The monopoly shows a decrease in cost as the length of the planning horizon increases, as seen in figure 5.9. The decrease in cost as the length of the planning increases for the monopoly is not as marked in figure 5.10 below for the 10 customer set.

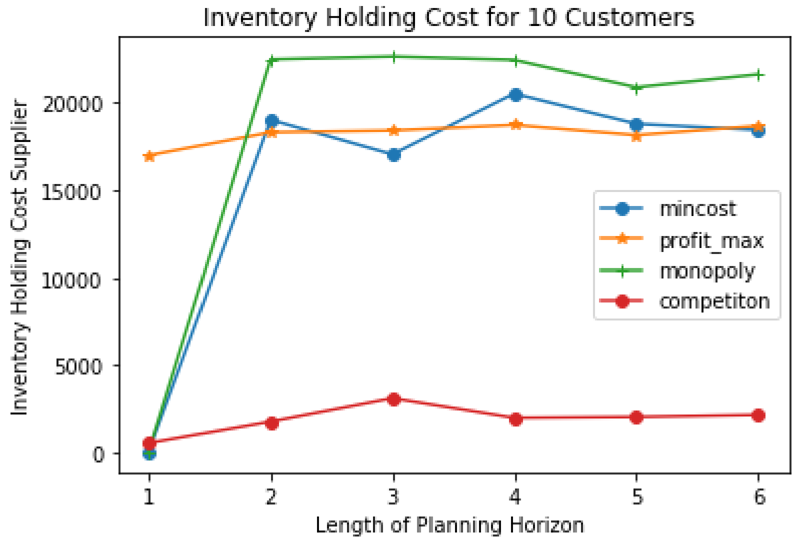

Figure 5.10: Total Inventory holding cost at the supplier cost for customer set of 10 across all 4 models and 6 planning horizons

5.4 Comparison of static and dynamic results

In this section, we compare our results with those from (Zaitseva 2017) in order to highlight overestimated values and underestimated performance results. Table 5.10 gives an overview of all the performance values, which are then compared in the bar charts below. Table 5.11 further states which values are underestimated and which are overestimated.

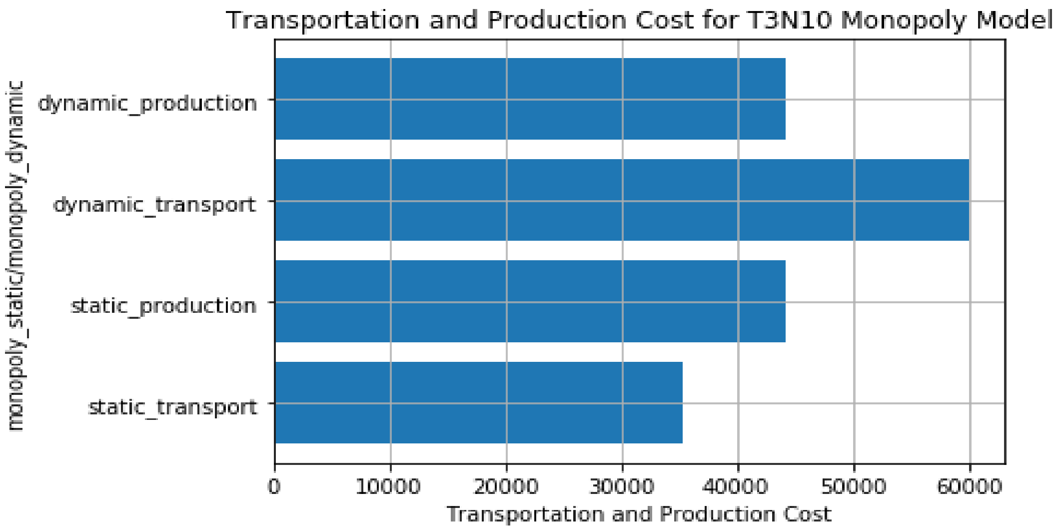

Figure 5.11: Transportation and production costs for monopoly model compared between static and dynamic models

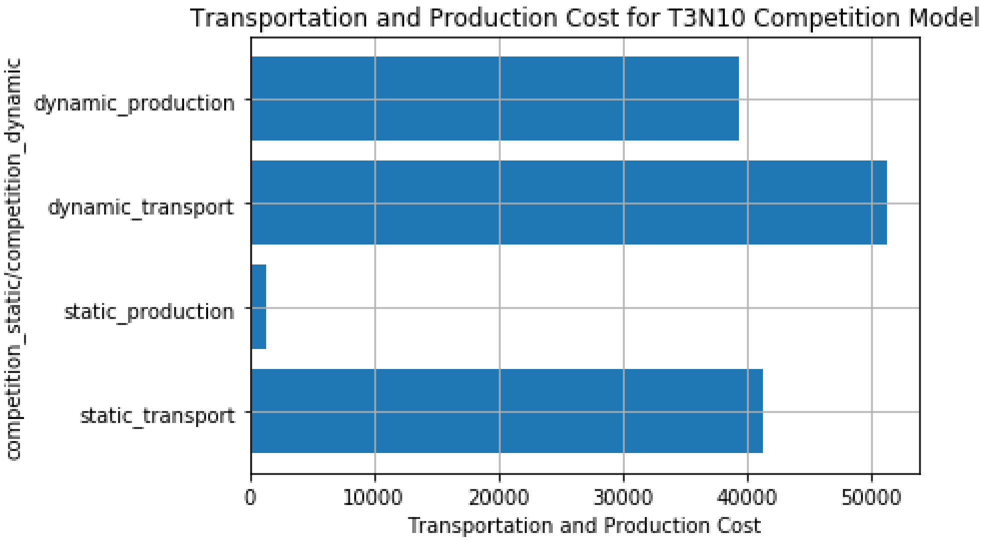

Figure 5.12: Transportation and production cost for Perfect Competitive market type compared between static and dynamic models

In Figure 5.11 and 5.12, Transportation cost for the dynamic model is higher for all the market types than it is for the static model. The production cost is fairly even across all models except for the competition of the static model. The extremely low value of production cost for the static competition model can be attributed the intention of the model to minimize production for the given planning horizon without consideration of the impact that decision would have on the rest of the scenario length.

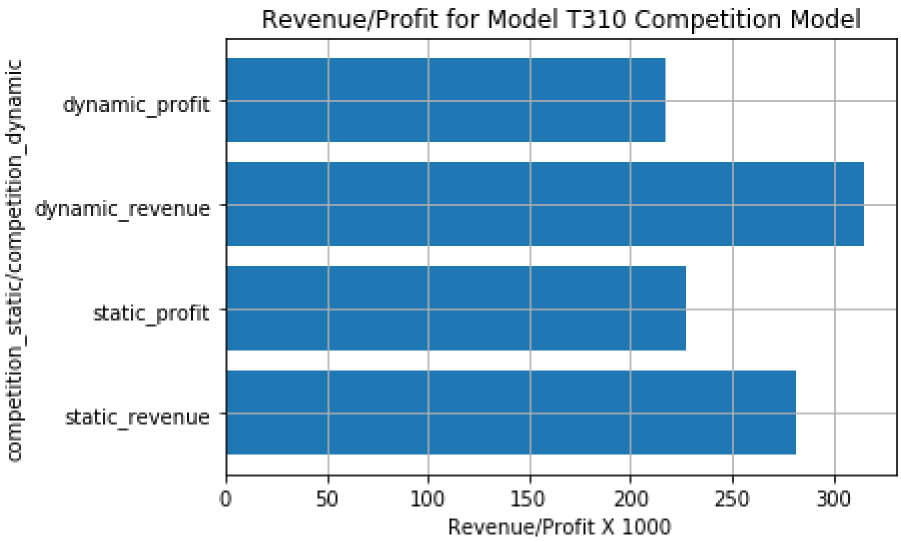

Figure 5.13: Revenue/Profit for Perfect Competitive market type compared between static and dynamic models

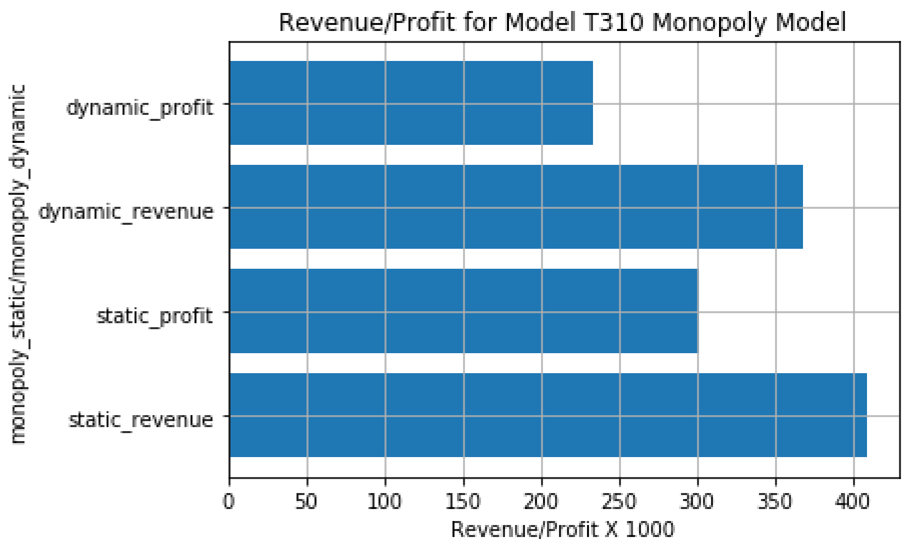

Figure 5.14: Revenue/Profit for Monopoly market type compared between static and dynamic models

In Figure 5.13 and 5.14, for the mincost model, the revenue and profit for the dynamic model are higher than the corresponding values for the static models. However, the profit max and monopoly models show higher profit for the static models than for the dynamic. This can be explained as being due to over-estimation of these values. The static model optimizes for the given planning horizon without consideration of the impact of such decisions on the next planning horizon and the rest of the scenario length. Such decisions may lead to minimization of inventory at the supplier and increased deliveries to the customers. In the next planning horizon, an increase in production would have been required resulting in increase in cost that is not being considered by the static model. In the perfect competition model we observe that the revenue is higher with the dynamic model than the static, while profit for the static model is higher.

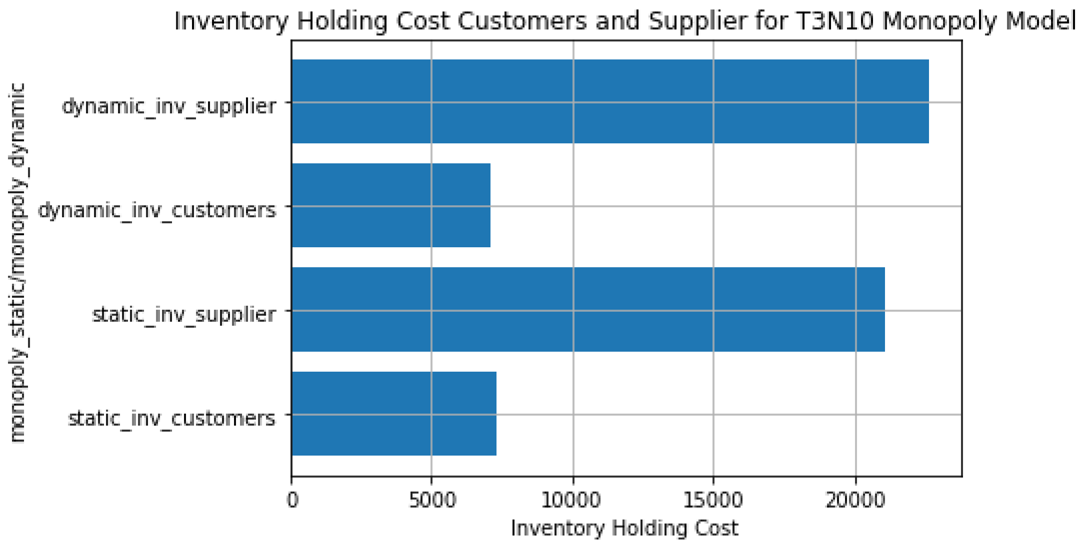

Figure 5.15: Supplier and customer's inventory holding costs for monopoly model compared between static and dynamic models

In Figure 5.15, and 5.16, the mincost model for the customer inventory holding cost is lower in the static model, which is understandable as the static model tries to minimize cost by reducing routing costs. This results in higher inventory remaining at the supplier location. The cumulative effect of this is higher supplier inventory cost for the static model than the dynamic model. In the case of the monopoly model there is less variation in the value of the inventory holding cost for both the static and the dynamic models. The reason may be that in both models the monopoly model requires the inventory holding cost to be high while it strives to maximize profit, and consequently, revenue (which is profit + cost).

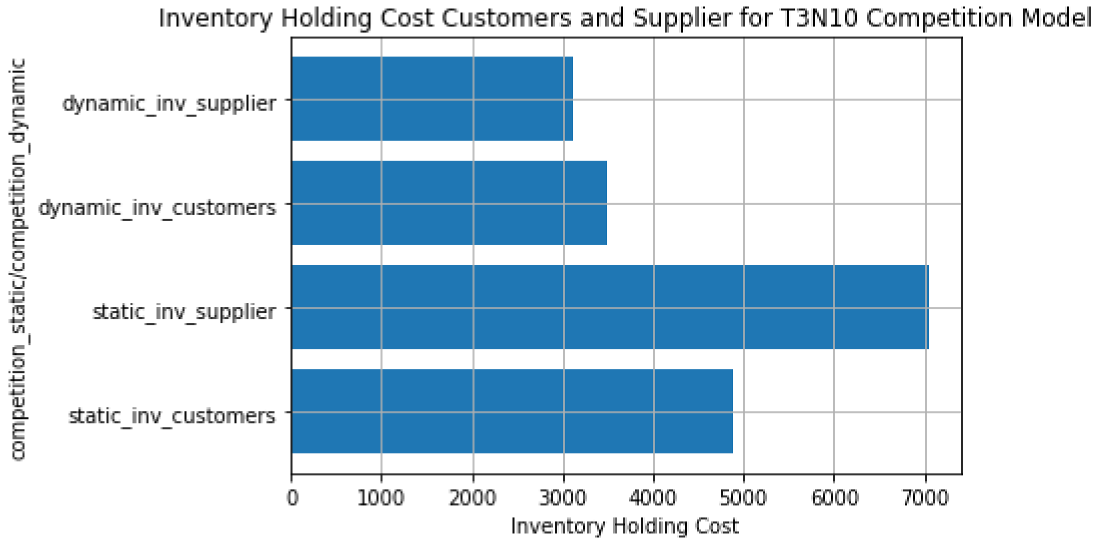

Figure 5.16: Inventory holding cost for consumer and supplier for Perfect Competitive market type compared between static and dynamic models

So in general, we can see that the static model, in its attempt to optimize the given objective over-estimates or under-estimates performance results based on the type of the market model. The static mincost model under-estimates the revenue, profit, transportation and inventory holding cost for the customers. The static profit_max model underestimates the transportation cost and inventory holding cost for the supplier, while the static monopoly model overestimates the revenue and profit and underestimates the transportation cost.

Finally, the static competition model underestimates the production cost, while over-estimating the supplier inventory holding cost. Table 5.11 below shows the impact on performance results when static models are used instead of dynamic models:

Model	Profit	Revenue	Total Cost	Transportation Cost	Inventory Holding Cost Customers	Inventory Holding Cost Supplier	Production Cost	Produced Amount	Shipped Amount	Consumed Amount	Time (seconds)
mincost_static	88950	190300	101340	34320	2880	19900	44230	19050	11160	19050	2,5
mincost_dynamic	187600	317467	129867	62145	6430	17054	44238	19050	18766	19050	209
profit_max_static	173710	282500	108780	41350	4880	8130	44230	19050	16410	19050	5,5
profit_max_dynamic	190424	315995	125571	59812	4574	18422	42763	18415	17939	18415	396
monopoly_static	301400	409450	108040	35420	7290	21080	44230	19050	11920	10380	110
monopoly_dynamic	234076	368238	134162	60173	7089	22662	44238	19050	18102	18289	26171
competition_static	227910	282500	54580	41350	4880	7050	1290	580	16410	19050	62,8
competition_dynamic	217804	315127	97323	51352	3498	3106	39367	17004	18387	18872	441

Table 5.11: static and dynamic model values for scenario T3N10 with 2 customers per route

Model	Revenue	Profit	Transportation Cost	Production Cost	Inventory Holding Cost Customers	Inventory Holding Cost Supplier
mincost	under	under	under	equal	under	equal
profit_max	equal	equal	under	equal	equal	under
monopoly	over	over	under	equal	equal	equal
competition	equal	equal	equal	under	equal	over

Table 5.12:Over and underestimation of performance results for the static model

5.4.1 Comparing the effect of number of break points

In this section, we compare the values of break points from our results, with those from the static model. Table 5.13 shows an overview of the total break point values, which are discussed below.

Revenue and Profit

As seen in Figure 5.17, revenue and profit in the dynamic model are higher for the corresponding values of the static model. Note that while the values of revenue and profit decrease with increase in the number of breakpoints for the static model, the opposite was observed for the dynamic model where both revenue and profit are observed to increase in values as the number of break points is increased from 5 to 15.

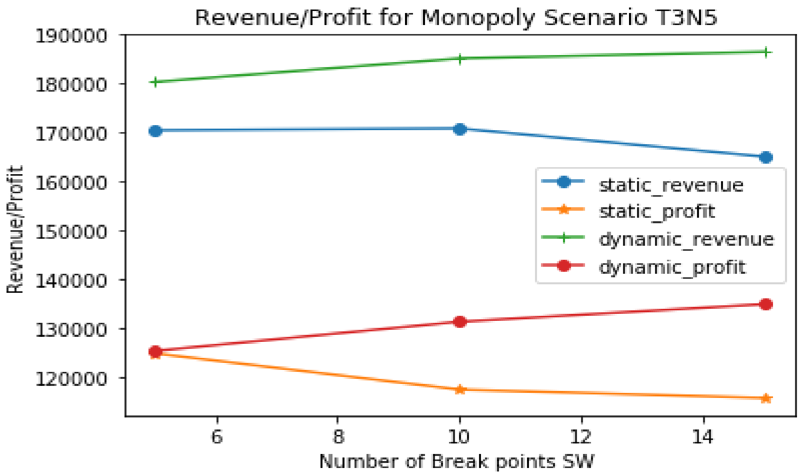

Figure 5.17: Break points of revenue and profit for Monopoly market type compared between static and dynamic models

Transportation and Production cost

Figure 5.18 shows that production cost is the same for both static and dynamic models while transportation cost reduces with increase in number of break points for dynamic model. Transportation cost does not show any distinct trend for the static model with increase in number of break points.

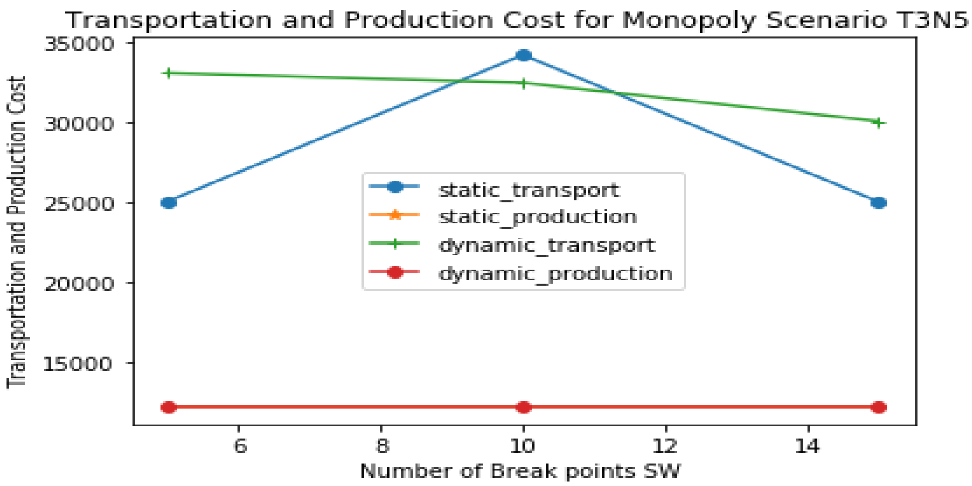

Figure 5.18: Break points of transportation and production costs for Monopoly market type compared between static and dynamic models

Inventory Holding Cost

As seen in Figure 5.19, the dynamic model shows a significantly higher inventory holding cost for the supplier than the static model. The values of the inventory holding cost for the supplier for both static and dynamic models are observed to reduce with increase in the number of break points. Meanwhile the inventory holding cost for the customers is slightly lower for the static model than the dynamic.

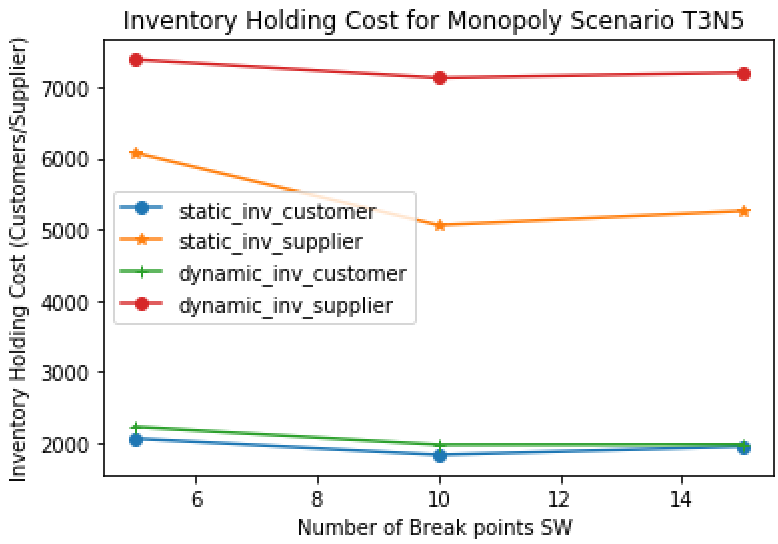

Figure 5.19: Break points of Inventory holding cost for consumer and supplier for Monopoly market type compared between static and dynamic models

Model	Approx. Profit	Approx. Revenue	Total Cost	Transportation Cost	Inventory Holding Cost Customers	Inventory Holding Cost Supplier	Penalty Cost	Production Cost	Produced Amount	Shipped Amount	Consumed Amount	Time (seconds)
monopoly_static	12476	17038	4561	2505	206	608	20	1222	579	473	421	0,39
monopoly_static	124760	170380	45610	25050	2060	6080	20	12220	5790	4730	4210	3,9
monopoly_dynamic	125268	180229	54961	33100	2225	7388	20	12228	5790	5473	5497	337
monopoly_static	11737	17073	5334	3423	183	506	0	1222	579	603	617	0,8
monopoly_static	117370	170730	53340	34230	1830	5060	0	12220	5790	6030	6170	8
monopoly_dynamic	131227	185039	53812	32482	1972	7130	0	12228	5790	5473	5522	547
monopoly_static	11567	16499	4536	2504	195	526	89	1222	579	574	576	1,58
monopoly_static	115670	164990	45360	25040	1950	5260	89	12220	5790	5740	5760	15,8
monopoly_dynamic	134792	186370	51578	30081	1975	7205	89	12228	5790	5473	5519	1332

Table 5.13: Break point comparison for static and dynamic model values for scenario T3N5 with 2 customers per route

6. CONCLUDING REMARKS AND FUTURE WORK

The previous work done on profit maximization of an inventory routing problem has been extended in this work by the following:

- Inclusion of dynamic aspects thereby changing the problem from a static one to a dynamic IRP.
- The use of a rolling horizon methodology which was necessary for addressing the problem of overestimation and underestimation of various results for static model such as under-estimation of revenue and profit for mincost model, over-estimation of these same values for monopoly and underestimation of production cost for competition and the study of the impact of variation of the length of the planning horizon on the profitability of an inventory routing problem.

Overall, an increase in profit is observed when a dynamic model is used in place of the static model. In addition, it shows that the increase in profit does not change significantly as shown in the static model due to reduction in variation when problems are solved using a rolling horizon approach.

Similar to the static model, an increase in profitability was observed as the length of planning horizon is increased for the four different market condition-based models, however, this increase was accompanied by a significant increase in computational time especially for the monopoly market situation of the dynamic model. The effect of the number of breakpoints for the 5-customer monopoly scenario was studied and the results showed an increase in revenue and profit with increase in the number of breakpoints for the dynamic model. This is the contrary to the result for the static model as demonstrated in (Zaitseva 2017)

Regarding future work it is important to address the main problem observed with the model, which was the increase in computational time associated with increase in number of customers and length of the planning horizon and infeasibilities that were observed while testing some of the scenarios. This is a common problem with the use of MIP solvers when solving combinatorial problems. As a result, a heuristic approach is recommended for solving such dynamic inventory routing problems.

Another area for future work would be the use of available optimization packages in combination with a programming language for the solving of non-linear problems. This is because of the cumbersome nature of the current method being used for solving nonlinear problems that involves separation of non-linear equation and linearization, which is an approximate method that is dependent on the number of break points which negatively impacts on the computation time as demonstrated in this work and in Zaitseva 2017.

For this work, we arbitrarily chose a linear demand curve to represent the market situation for the suppliers. In the future, it would be interesting to see how the profitability of the firm behaves with an iso-elastic demand curve representing the market situation.

Stochasticity in the demand can be introduced in the future to observe how the different models will perform when demand is uncertain and generated from forecasting.

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