COMMODITY PRICE MODELING THAT MATCHES CURRENT OBSERVABLES: A NEW APPROACH

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Abstract. We develop a stochastic model of the spot commodity price and the spot convenience yield such that the model matches the current term structure of forward and futures prices, the current term structure of forward and futures volatilities, and the inter-temporal pattern of the volatility of the forward and futures prices. We let the underlying commodity price be a geometric Brownian motion and we let the spot convenience yield have a mean-reverting structure. The flexibility of the model, which makes it possible to simultaneously obtain all these goals, comes from allowing the volatility of the spot commodity price, the speed of mean-reversion parameter, the mean-reversion parameter, and the diffusion parameter of the spot convenience yield all to be time-varying deterministic functions.

1. Introduction

An important aspect of building a stochastic model of the behavior of commodity prices is to make sure that the model matches the current observed term structure of forward and futures prices and their volatility structure. In addition, most commodity prices reflect mean reversion and seasonal behavior. It is an empirically stylized fact that most commodity price processes are mean reverting (Bessembinder, Coughenour, Seguin, and Smoller 1995) and show seasonal patterns. Standard no-arbitrage arguments completely determine the drift of the price processes under risk-neutral probabilities leaving no room for explicit modeling of mean reversion or seasonal effects via the drift of the spot commodity price. However, the spot convenience yield process enters the drift of the spot commodity price under risk-neutral probabilities. Hence, mean reversion and seasonal behavior is possible by manipulating volatilities and the correlation structure of the joint process of spot commodity prices and convenience yields. For example, a positive correlation between the spot commodity price and the spot convenience yield will have a mean reversion effect on the spot commodity price even under risk-neutral probabilities.

The main idea of the paper is to determine a stochastic model for the spot commodity price and the spot convenience yield such that the model matches the current term structure of forward and futures prices, the current term structure of forward and futures volatilities and the inter-temporal pattern of the volatility of the forward and futures prices. This idea is implemented more or less in the same way as in the Hull-White model (Hull and White 1993). We let the underlying commodity price be a geometric Brownian motion and we let the spot convenience yield have a Vasiček-alike structure, i.e. a generalized Ornstein-Uhlenbeck process. The flexibility of the model, which makes it possible to simultaneously match the current term structure of forward and futures prices, the current term structure of forward and futures volatilities, and the inter-temporal pattern of the volatility of the forward and futures prices, comes from allowing the volatility of the spot commodity price, the speed of mean-reversion parameter, the mean-reversion parameter, and the diffusion parameter of the spot convenience yield all to be explicit (and therefore deterministic) functions of time.

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The model presented in this paper is a no-arbitrage model in which the stochastic behavior of prices and convenience yields is exogenously given. The value of any contingent claim on the commodity can then be derived as a function of these primitives, imposing the condition that no arbitrage profits exist in perfect markets. A more complete equilibrium description of spot commodity prices and convenience yields can tie these variables to the aggregate inventory of the commodity. In this framework, the process for spot prices and convenience yields would be endogenous, rather than exogenously assumed. Brennan (1991) finds the empirical relationship between inventories of the commodity, spot prices, and convenience yields. When inventories are low, spot prices are relatively high, and convenience yields are also relatively high, since futures prices will not increase as much as the spot price, and vice versa, when inventories are high. Hence, there is empirical evidence of a consistent positive correlation between commodity prices and convenience yields for some commodities.

In section 2 we give an introduction to convenience yields and the modeling of convenience yields. In section 3, we develop the stochastic model for the spot commodity price and the spot convenience yield such that the model matches current observables. We illustrate our model with a simple example in section 4. Finally, section 5 concludes.

2. Convenience Yields

Before we start developing our model, we will quickly summarize some facts about convenience yields. The (net) convenience yield of a given commodity is defined as the (net) flow of services (per unit time and per monetary unit of the commodity) that accrues to a holder of the physical commodity, but not to a holder of a contract for future delivery of the same commodity (Brennan 1991).

Net convenience yields on a commodity can be thought of in the same way as dividend yields on a common stock. However, the dividends that the owner of a commodity receives are, in general, not just monetary units but more like *value* in a much broader or more abstract sense, e.g., the value of having a painting on the wall in the sense of the added value (utility) of having the option to enjoy looking at it. For most exchange traded commodities, as e.g. copper and oil, the convenience yields arise from the value of the flexibility of being able to use the physical commodity in a production process with short notice.

Net convenience can be separated into gross convenience and cost of carry. Gross convenience is the value of all the advantages of possessing the commodity, whereas the cost of carry is the cost of the disadvantages. The net convenience is the result of subtracting the cost of carry from the gross convenience and it can in many cases be negative.

Normally the convenience yield is quoted as a continuously compounded yield. That is, in a purely deterministic setting the following relation holds:

$$S(t) = e^{-r(T-t)} e^{\delta(T-t)} F(T) = e^{-r(T-t)} e^{c^+(T-t)} e^{-c^-(T-t)} F(T),$$

where

$S(t)$	is the price of the commodity at date t,
$F(T)$	is the (forward or future) price of the commodity at date T (which in this deterministic setting is predictable),
r	is the continuously compounded interest rate (yield) over the period from date t to date T,
δ	is the net convenience yield over the period from date t to date T,
c^+	is the gross convenience yield over the period from date t to date T, and
c^-	is the cost-of-carry yield over the period from date t to date T.

The present value of a contract for future delivery of the commodity at date T is $e^{-r(T-t)}F(T)$. Hence, the net convenience of holding the physical commodity from date t to date T as opposed to holding a contract for future delivery at date T (measured in date t present value terms) is

$$S(t) - e^{-r(T-t)}F(T) = e^{-r(T-t)}F(T)(e^{\delta(T-t)} - 1)$$

or $e^{\delta(T-t)} - 1$ per monetary unit of the commodity (in date t present value terms) or

$$\lim_{T\to t} \frac{1}{T-t}(e^{\delta(T-t)} - 1) = \delta$$

per unit time and per monetary unit of the commodity. This quoting of convenience *yields* parallels interest rate *yields* and dividend *yields*. In the same way

$$e^{-r(T-t)}F(T)(1 - e^{-c^-(T-t)})$$

represents the cost of carry and

$$S(t) - e^{-r(T-t)}e^{-c^-(T-t)}F(T)$$

represents the gross convenience.

In a non-deterministic setting the resemblance with interest rates and dividends is preserved. Stochastic models of commodity price behavior typically include both a stochastic process for the price of the commodity and a separate stochastic process for the convenience yield (Gibson and Schwartz 1990). Often the stochastic process for the convenience yield will be modeled in the same way as a stochastic interest rate process, e.g. a Vasiček or a Cox-Ingersoll-Ross model (Vasiček 1977, Cox, Ingersoll, and Ross 1985). For historical reasons the term *yield* in "convenience yield" is preserved in the transition to stochastic models where, strictly speaking, it is the instantaneous continuously compounded convenience *rate* that is modeled. Since the net convenience yield can typically be both positive and negative, a Vasiček type description is usually preferred. This is, e.g., the case in the Gibson-Schwartz model, where the stochastic behavior of the commodity price and its corresponding convenience yield under risk-neutral probabilities is described as

$$\begin{aligned} S_t &= S_0 + \int_0^t (r_u - \delta_u)S_u du + \int_0^t \sigma S_u dB_u, \\ \delta_t &= \delta_0 + \int_0^t \kappa(\theta - \delta_u)du + \int_0^t \gamma dZ_u, \end{aligned}$$

where

S_t	is the (stochastic) price of the commodity at date t,
r_t	is the (possibly stochastic) instantaneous continuously compounded interest rate at date t,
δ_t	is the (stochastic) instantaneous continuously compounded net convenience yield at date t,
σ	is the diffusion parameter (volatility) of the commodity price process,
κ	is the speed of mean-reversion parameter of the convenience yield process,
θ	is the mean-reversion level of the convenience yield process,
γ	is the diffusion parameter of the convenience yield process, and
B and Z	are (possibly correlated) Brownian motions.

Under risk-neutral probabilities the commodity price process must have the drift $(r_t - \delta_t)S_t$ in order to obey the no-arbitrage drift restriction. If the Brownian motions B and Z are positively correlated, an implicit mean-reversion effect on the commodity price process is introduced by the following line of reasoning: an increase in S from a positive dB typically resolves into a positive dZ which increases δ and thereby decreases the drift of S. It is a stylized fact that most commodity prices behave in a mean reverting way (Bessembinder, Coughenour, Seguin, and Smoller 1995). In the Gibson-Schwartz model the interest rate is assumed deterministic. An extension to stochastic interest rates is straightforward, e.g., by a Vasiček or Cox-Ingersoll-Ross description of the short term interest rate. In these types of models commodity forward and futures prices (Cox, Ingersoll, and Ross 1981) can be derived as

$$F(t,T) = \frac{E_t^*\left[e^{-\int_t^T r_s ds} S_T\right]}{B(t,T)} = E_t^T[S_T]$$

and

$$G(t,T) = E_t^*\left[S_T\right],$$

where

$F(t,T)$	is the date t forward price of the commodity for delivery at date T,
$B(t,T)$	is the date t price of a zero-coupon bond maturing at date T,
$G(t,T)$	is the date t futures price of the commodity for delivery at date T,
E_t^T	denotes the conditional expectation under T-forward probabilities given the information at date t, and
E_t^*	denotes the conditional expectation under risk-neutral probabilities given the information at date t.

With a Vasiček type short term interest rate description Schwartz (1997) derives closed form solutions for forward and futures prices.

The close resemblance between convenience yields and interest rates is emphasized when the term structure dimension is introduced in commodity pricing (Miltersen and Schwartz 1998). In a term structure of interest rate model (Heath, Jarrow, and Morton 1992) the instantaneous continuously compounded forward rate is defined, based on date t observations of zero-coupon bonds as a function on their maturity date T, by the following equation:

$$B(t,T) = e^{-\int_t^T f(t,s)ds}, \tag{1}$$

where

$f(t,s)$	is the instantaneous continuously compounded forward rate at date s as seen from date t.

This definition of the stochastic forward rate process extends the stochastic short term interest rate process in the following natural way:

$$f(t,t) = r_t. \tag{2}$$

By observing prices of zero-coupon bonds with different maturity dates the forward rates can be backed out, cf. equation (1), and from these observations the continuously compounded short term interest rate can then be derived as a limit, cf. equation (2).

In Heath, Jarrow, and Morton (1992) the instantaneous continuously compounded forward rate is modeled as the following stochastic differential equation (SDE):

$$f(t,s) = f(0,s) + \int_0^t \mu_f(u,s)ds + \int_0^t \sigma_f(u,s)\cdot dW_u, \tag{3}$$

where[1]

[1] "$\cdot$" denotes the standard Euclidean inner product of $\mathbb{R}^d$, and the corresponding norm is defined as $\|x\|^2 = x \cdot x$ for any $x \in \mathbb{R}^d$.

$\mu_f(t,s)$ is the (possibly stochastic) drift of the forward rate process,
$\sigma_f(t,s)$ is the (possibly stochastic) (d-dimensional) diffusion of the forward rate process, and
W is a (d-dimensional) Brownian motion.

$f(0,s)$ is the initial term structure of interest rates observable at date zero. Under risk-neutral probabilities the drift, μ_f, of the forward rate process is determined by the so-called Heath-Jarrow-Morton drift restriction,

$$\mu_f(t,s) = \sigma_f(t,s) \cdot \int_t^s \sigma_f(t,v)dv.$$

Clearly, the SDE (3) induces a stochastic model of all bond prices as well as the short term interest rate via equations (1) and (2).

Like the short term interest rate, (spot) convenience yields cannot be directly observed from prices. Instead a term structure of forward (futures) prices can be observed at a given date t. That is, at a given date t, forward (futures) prices can be observed as a function of their maturity date T. Inspired by the usual definitions of instantaneous continuously compounded forward rates from interest rate models, Miltersen and Schwartz (1998) define the forward (future) convenience yields, $\delta(t,s)$ ($\epsilon(t,s)$), by the following equations:

$$F(t,T) = \frac{S_t}{B(t,T)} e^{-\int_t^T \delta(t,s)\,ds} = S_t e^{\int_t^T (f(t,s)-\delta(t,s))\,ds}, \tag{4}$$

$$G(t,T) = \frac{S_t}{B(t,T)} e^{-\int_t^T \epsilon(t,s)\,ds} = S_t e^{\int_t^T (f(t,s)-\epsilon(t,s))\,ds}, \tag{5}$$

where

$\delta(t,s)$ is the instantaneous continuously compounded forward convenience yield at date s as seen from date t, and
$\epsilon(t,s)$ is the instantaneous continuously compounded future convenience yield at date s as seen from date t.

These definitions of forward and future convenience yields extend the stochastic (spot) convenience yield in the following way:

$$\delta(t,t) = \epsilon(t,t) = \delta_t. \tag{6}$$

This is a natural extension of the spot convenience yield in the same way as the forward rate is a natural extension of the short term interest rate in equation (2). Miltersen and Schwartz (1998) model the future convenience yield and the spot commodity price as two governing stochastic processes to describe the behavior of the future term structure of forward and futures prices using the same basic ideas as (and as an extension to) the Heath-Jarrow-Morton model (Heath, Jarrow, and Morton 1992). The purpose of this model is to price derivatives based on (commodity) futures and forwards that are consistent with the observed term structure of forward and futures prices. The future convenience yield process is modeled as

$$\epsilon(t,s) = \epsilon(0,s) + \int_0^t \mu_\epsilon(u,s)du + \int_0^t \sigma_\epsilon(u,s) \cdot dW_u, \tag{7}$$

where

$\mu_\epsilon(t,s)$ is the (possibly stochastic) drift of the future convenience yield process,
$\sigma_\epsilon(t,s)$ is the (possibly stochastic) (d-dimensional) diffusion of the future convenience yield process, and
W is a (d-dimensional) Brownian motion.

The spot price of the underlying commodity is modeled as

$$S_t = S_0 + \int_0^t S_u \mu_S(u) du + \int_0^t S_u \sigma_S(u) \cdot dW_u, \tag{8}$$

where

$\mu_S(t)$ is the (possibly stochastic) drift of the commodity spot price process and
$\sigma_S(t)$ is the (possibly stochastic) (d-dimensional) diffusion of the commodity spot price process.

Possible correlation among the two processes comes via the specification of the diffusion terms (the σs), since it is the same vector Brownian motion, W, that is used in both SDEs. So far, the diffusion terms (the σs) are not specified further, however, they must fulfill certain regularity conditions, such that strong solutions of the stated SDEs exist. For example, they can be bounded predictable stochastic processes. Hence, state dependent correlation between the processes is certainly possible. Moreover, we must require that the initially observed future convenience yield, $\epsilon(0, s)$, is differentiable in the second time parameter, s.

Standard no-arbitrage restrictions imply that the drift of the spot commodity price process is determined as

$$\mu_S(t) = r_t - \epsilon(t, t)$$

under risk-neutral probabilities.

In Miltersen and Schwartz (1998) we derive that the drift of the future convenience yield process is given by

$$\begin{aligned} \mu_\epsilon(t, T) = {} & \sigma_f(t, T) \cdot \Big(\int_t^T \sigma_f(t, s) ds \Big) \\ & + \big(\sigma_f(t, T) - \sigma_\epsilon(t, T)\big) \cdot \Big(\sigma_S(t) + \int_t^T \big(\sigma_f(t, s) - \sigma_\epsilon(t, s)\big) ds \Big) \end{aligned}$$

under risk-neutral probabilities. In the special case where the term structure of interest rates is non-stochastic the drift restriction is reduced to

$$\mu_\epsilon(t, T) = \sigma_\epsilon(t, T) \cdot \Big(\int_t^T \sigma_\epsilon(t, s) ds - \sigma_S(t) \Big) \tag{9}$$

under risk-neutral probabilities.

Clearly, the SDEs (3), (7), and (8) induce stochastic models of all forward and futures prices as well as the spot convenience yield via equations (4), (5), and (6).

3. Stochastic Modeling of Convenience Yields that Matches Current Observables

As emphasized in both the abstract and the introduction the main idea of the paper is to determine a stochastic model for the spot commodity price and the spot convenience yield such that the model matches the current term structure of forward and futures prices, the current term structure of forward and futures volatilities, and the inter-temporal pattern of the volatility of the forward and futures prices. This idea is implemented more or less in the same way as Hull and White (1993), who show how to make the Vasiček model match the initial term structure of interest rates by making the mean-reversion parameter under risk-neutral probabilities be a deterministic function of time. In our model we let the underlying commodity price be a geometric Brownian motion and we let the spot convenience yield have a Hull-White-alike structure but with all parameters time dependent, i.e. a generalized Ornstein-Uhlenbeck process. The flexibility of the model, which makes it possible to simultaneously match the current term structure of forward and futures prices, the current term structure of forward and futures volatilities, and the inter-temporal pattern of the volatility of the forward and futures prices, comes from allowing the volatility of the spot commodity price, the speed of mean-reversion parameter, the mean-reversion parameter, and the diffusion parameter of the spot convenience yield all to be explicit (and therefore deterministic) functions of time.

Empirically observed volatilities of the interest rates are orders of magnitude smaller than the empirically observed volatilities of the spot convenience yields and the spot commodity price (Schwartz 1997). This implies that the results obtained from a reduced model assuming a deterministic term structure of interest rates are very similar to the results obtained from the full three-factor stochastic model including a stochastic term structure of interest rates. That is, there is a very limited gain by introducing a stochastic term structure of interest rates which does not outweigh the costs in the sense of increased complexity of the model. Hence, for the rest of the paper we assume that the short term interest rate, r, is constant. This implies that

$$B(t,T) = e^{-r(T-t)}.$$

Since the term structure of interest rates is now non-stochastic, the forward convenience yield and the future convenience yield are the same, and also the forward and futures prices are the same.

As explained in the previous section, the stochastic model of future/forward price movements consists of two processes, the spot price of the underlying commodity and the term structure of future convenience yields. We are only concerned with the stochastic behavior of these two processes under risk-neutral probabilities. In this paper, we more explicitly model the volatility functions deterministically as

$$\sigma_S(t) = \sigma_S g_S(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{10}$$

and

$$\sigma_\epsilon(t,s) = \sigma_\epsilon g_\epsilon(t) e^{-\int_t^s \kappa(u)du} \begin{pmatrix} \rho(t) \\ \sqrt{1-\rho^2(t)} \end{pmatrix}. \tag{11}$$

That is, we are in the Gaussian case. The functions g_S, g_ϵ, ρ, and κ are deterministic functions of time. g_S and g_ϵ are used to explicitly model inter-temporal time variations such as seasonal patterns in the instantaneous volatility of the spot commodity price process and the future convenience yield process. The function ρ is used to explicitly model inter-temporal time variations in the instantaneous correlation between the spot commodity price process and the future convenience yield process. Finally, the function κ is used to explicitly model the term structure of convenience yield volatility as it will become clear by the end of this section, cf. equation (14).

In mathematical terms, we have the following structure of the diffusion terms of the model, here written up as quadratic variation terms,

$$\begin{aligned} d\langle S\rangle_t &= \sigma_S^2 g_S^2(t) S_t^2 dt, \\ d\langle \epsilon(\cdot,s)\rangle_t &= \sigma_\epsilon^2 g_\epsilon^2(t) e^{-2\int_t^s \kappa(u)du} dt, \end{aligned}$$

and

$$d\langle S, \epsilon(\cdot,s)\rangle_t = \sigma_S \sigma_\epsilon g_S(t) g_\epsilon(t) \rho(t) e^{-\int_t^s \kappa(u)du} S_t dt.$$

From the drift restriction, equation (9),

$$\begin{aligned}\mu_\epsilon(u,s) &= \sigma_\epsilon(u,s)\cdot\int_u^s \sigma_\epsilon(u,x)dx - \sigma_\epsilon(u,s)\cdot\sigma_S(u)\\ &= \sigma_\epsilon g_\epsilon(u)e^{-\int_u^s \kappa(v)dv}\int_u^s \sigma_\epsilon g_\epsilon(u)e^{-\int_u^x \kappa(v)dv}dx - \sigma_\epsilon g_\epsilon(u)e^{-\int_u^s \kappa(v)dv}\sigma_S g_S(u)\rho(u)\\ &= \sigma_\epsilon^2 g_\epsilon^2(u)e^{-\int_u^s \kappa(v)dv}\int_u^s e^{-\int_u^x \kappa(v)dv}dx - \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{-\int_u^s \kappa(v)dv}.\end{aligned}$$

That is, from equation (7), the future convenience yield process is modeled as

$$\begin{aligned}\epsilon(t,s) = \epsilon(0,s) &+ \int_0^t \sigma_\epsilon^2 g_\epsilon^2(u)e^{-\int_u^s \kappa(v)dv}\int_u^s e^{-\int_u^x \kappa(v)dv}dxdu\\ &- \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{-\int_u^s \kappa(v)dv}du + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{-\int_u^s \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u.\end{aligned}$$

From this description we would like to find the stochastic process for the spot convenience yield, $\delta_t = \epsilon(t,t)$. In order to do so, we have to be very careful, since now both time parameters change simultaneously. In order to cope with this problem we will apply a variation of the standard trick of how to show that an Ornstein-Uhlenbeck process is Gaussian. First, define a new stochastic process, Y, with a two-dimensional time-parameter set as

$$Y(t,s) = \epsilon(t,s)e^{\int_0^s \kappa(v)dv}.$$

Hence,

$$\begin{aligned}Y(t,s) = Y(0,s) &+ \int_0^t \sigma_\epsilon^2 g_\epsilon^2(u)e^{\int_0^u \kappa(v)dv}\int_u^s e^{-\int_u^x \kappa(v)dv}dxdu\\ &- \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{\int_0^u \kappa(v)dv}du + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u.\end{aligned}$$

Since $s \mapsto Y(0,s)$ is differentiable, we can write

$$Y(0,s) = Y(0,0) + \int_0^s Y_T(0,u)du, \tag{12}$$

where $Y_T(t,s)$ denotes $\frac{\partial}{\partial s}Y(t,s)$. Similarly, we will let $\epsilon_T(t,s)$ denote $\frac{\partial}{\partial s}\epsilon(t,s)$. By applying equation (12) we get

$$\begin{aligned}
Y(t,s) &= \epsilon(0,0) + \int_0^s \Big(\epsilon_T(0,u)e^{\int_0^u \kappa(v)dv} + \epsilon(0,u)\kappa(u)e^{\int_0^u \kappa(v)dv}\Big)du \\
&\quad + \int_0^t \sigma_\epsilon^2 g_\epsilon^2(u)e^{\int_0^u \kappa(v)dv}\int_u^s e^{-\int_u^x \kappa(v)dv}dxdu - \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{\int_0^u \kappa(v)dv}du \\
&\quad + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u \\
&= \epsilon(0,0) + \int_0^s e^{\int_0^u \kappa(v)dv}\Big(\epsilon_T(0,u) + \epsilon(0,u)\kappa(u)\Big)du + \int_0^s\int_0^x \sigma_\epsilon^2 g_\epsilon^2(u)e^{\int_0^x \kappa(v)dv}e^{-2\int_u^x \kappa(v)dv}dudx \\
&\quad - \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{\int_0^u \kappa(v)dv}du + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u \\
&= \epsilon(0,0) + \int_0^s e^{\int_0^u \kappa(v)dv}\Big(\epsilon_T(0,u) + \epsilon(0,u)\kappa(u)\Big)du + \int_0^s\int_0^u \sigma_\epsilon^2 g_\epsilon^2(x)e^{\int_0^u \kappa(v)dv}e^{-2\int_x^u \kappa(v)dv}dxdu \\
&\quad - \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{\int_0^u \kappa(v)dv}du + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u \\
&= \epsilon(0,0) + \int_0^s e^{\int_0^u \kappa(v)dv}\Big(\epsilon_T(0,u) + \epsilon(0,u)\kappa(u) + \int_0^u \sigma_\epsilon^2 g_\epsilon^2(x)e^{-2\int_x^u \kappa(v)dv}dx\Big)du \\
&\quad - \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{\int_0^u \kappa(v)dv}du + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{Y}_t \equiv Y(t,t) &= \epsilon(0,0) + \int_0^t e^{\int_0^u \kappa(v)dv}\Big(\epsilon_T(0,u) + \epsilon(0,u)\kappa(u) \\
&\quad + \int_0^u \sigma_\epsilon^2 g_\epsilon^2(x)e^{-2\int_x^u \kappa(v)dv}dx\Big)du - \int_0^t \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)e^{\int_0^u \kappa(v)dv}du \\
&\quad + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u \\
&= \epsilon(0,0) + \int_0^t e^{\int_0^u \kappa(v)dv}\Big(\epsilon_T(0,u) + \epsilon(0,u)\kappa(u) + \int_0^u \sigma_\epsilon^2 g_\epsilon^2(x)e^{-2\int_x^u \kappa(v)dv}dx - \sigma_S\sigma_\epsilon g_S(u)g_\epsilon(u)\rho(u)\Big)du \\
&\quad + \int_0^t \sigma_\epsilon g_\epsilon(u)e^{\int_0^u \kappa(v)dv}\begin{pmatrix}\rho(u)\\ \sqrt{1-\rho^2(u)}\end{pmatrix}\cdot dW_u.
\end{aligned}$$

That is, $\tilde{Y}$ is an Itô process since there is no t dependence neither in the drift term nor in the volatility term.

Recall that the spot convenience yield can be written as

$$\delta_t = \tilde{Y}_t e^{-\int_0^t \kappa(v)dv}.$$

Hence, by Itô's lemma

$$\begin{aligned}
\delta_t = \tilde{Y}_0 + \int_0^t \Big(& \epsilon_T(0,u) + \epsilon(0,u)\kappa(u) + \int_0^u \sigma_\epsilon^2 g_\epsilon^2(x) e^{-2\int_x^u \kappa(v)dv} dx \\
& - \sigma_S \sigma_\epsilon g_S(u) g_\epsilon(u) \rho(u) - \tilde{Y}_u \kappa(u) e^{-\int_0^u \kappa(v)dv} \Big) du \\
& + \int_0^t \sigma_\epsilon g_\epsilon(u) \begin{pmatrix} \rho(u) \\ \sqrt{1-\rho^2(u)} \end{pmatrix} \cdot dW_u \\
= \delta_0 + \int_0^t \Big(& \epsilon_T(0,u) + \epsilon(0,u)\kappa(u) + \int_0^u \sigma_\epsilon^2 g_\epsilon^2(x) e^{-2\int_x^u \kappa(v)dv} dx - \sigma_S \sigma_\epsilon g_S(u) g_\epsilon(u) \rho(u) - \delta_u \kappa(u) \Big) du \\
& + \int_0^t \sigma_\epsilon g_\epsilon(u) \begin{pmatrix} \rho(u) \\ \sqrt{1-\rho^2(u)} \end{pmatrix} \cdot dW_u \\
= \delta_0 + \int_0^t & \kappa(u)(\hat{\theta}_u - \delta_u) du + \int_0^t \sigma_\epsilon g_\epsilon(u) \begin{pmatrix} \rho(u) \\ \sqrt{1-\rho^2(u)} \end{pmatrix} \cdot dW_u .
\end{aligned}$$

Thus, the instantaneous spot convenience yield is mean reverting to the following time dependent variable

$$\hat{\theta}_t = \frac{1}{\kappa(t)} \epsilon_T(0,t) + \epsilon(0,t) + \frac{\sigma_\epsilon^2}{\kappa(t)} \int_0^t g_\epsilon^2(x) e^{-2\int_x^t \kappa(v)dv} dx - \frac{\sigma_S \sigma_\epsilon}{\kappa(t)} g_S(t) g_\epsilon(t) \rho(t). \tag{13}$$

Note that the mean-reversion parameter has two separate terms:

- a term determined by the initial future convenience yield, $\epsilon(0,\cdot)$, and
- a term determined by the variance-covariance structure of the joint process of the spot commodity price and the convenience yield.

Recall that this is the mean-reversion parameter under risk-neutral probabilities. Under the physical probabilities a third term coming from the market price of convenience yield risk is added.

From Miltersen and Schwartz (1998) we know that the instantaneous volatilities of the futures prices are

$$\begin{aligned}
\sigma_{G_T}(t) &= \sigma_S(t) - \int_t^T \sigma_\epsilon(t,s) ds \\
&= \begin{pmatrix} \sigma_S g_S(t) - \sigma_\epsilon g_\epsilon(t) \rho(t) \int_t^T e^{-\int_t^s \kappa(v)dv} ds \\ -\sigma_\epsilon g_\epsilon(t) \sqrt{1-\rho^2(t)} \int_t^T e^{-\int_t^s \kappa(v)dv} ds \end{pmatrix}.
\end{aligned}$$

Hence, the instantaneous variance of percentage changes of the futures prices can be derived as

$$\|\sigma_{G_T}(t)\|^2 = \sigma_S^2 g_S(t)^2 + \sigma_\epsilon^2 g_\epsilon(t)^2 \Big(\int_t^T e^{-\int_t^s \kappa(v)dv} ds \Big)^2 - 2\sigma_S \sigma_\epsilon g_S(t) g_\epsilon(t) \rho(t) \int_t^T e^{-\int_t^s \kappa(v)dv} ds. \tag{14}$$

The seasonality functions can be concretized to

$$g_S(t) = 1 + A_S \sin\big(2\pi(t + B_S)\big) \tag{15}$$

and

$$g_\epsilon(t) = 1 + A_\epsilon \sin\big(2\pi(t + B_\epsilon)\big). \tag{16}$$

The calibration exercise can now be split up into the following parts:[2]

(1) Calibration of $s \mapsto \epsilon(0,s)$. Parameterize the current term structure of forward or futures prices, $T \mapsto G(0,T)$. Derive the initial term structure of future convenience yields, $s \mapsto \epsilon(0,s)$, using equation (5) and its derivative $\epsilon_T(0,s)$.

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[2]Parts (3) and (4) must be done simultaneously.

The spot commodity price at date zero	S_0	145
The base volatility of the spot commodity price	σ_S	40%
The relative amplitude of the spot commodity price volatility	A_S	10%
The time adjustment of the spot commodity price volatility	B_S	–0.25
The convenience yield at date zero	δ_0	10%
The base volatility of the convenience yield	σ_ϵ	50%
The relative amplitude of the convenience yield volatility	A_ϵ	10%
The time adjustment of the convenience yield volatility	B_ϵ	–0.25
The riskless interest rate	r	5%
The speed of mean reversion parameter for the convenience yield	κ	1.8
The instantaneous correlation coefficient	ρ	0.75

Table 1. Parameter values for the fictitious commodity. These values are based on empirical estimates of crude oil prices (Schwartz 1997).

(2) Estimation of σ_S and $t \mapsto g_S(t)$. Estimate the inter-temporal seasonal pattern of the underlying spot commodity price process volatility, $\sigma_S g_S(t)$. Use the parameterization

$$\sigma_S + \sigma_S A_S \sin\big(2\pi(t + B_S)\big).$$

(3) Calibration of $t \mapsto \kappa(t)$. Parameterize the current term structure of future price volatilities, $T \mapsto \|\sigma_{G_T}(0)\|$. Derive the time dependent speed of mean-reversion function, κ, using equation (14).

(4) Estimation of σ_ϵ, $t \mapsto g_\epsilon(t)$, and $t \mapsto \rho(t)$. Estimate the inter-temporal seasonal pattern of the long term futures price process volatility, $t \mapsto \|\sigma_{G_{t+T}}(t)\|$, for a given time to maturity, $T > 0$. Use the parameterization

$$\sigma_S^2 g_S(t)^2 + \sigma_\epsilon^2 g_\epsilon(t)^2 \Big(\int_t^T e^{-\int_t^s \kappa(v)dv} ds\Big)^2 - 2\sigma_S \sigma_\epsilon g_S(t) g_\epsilon(t) \rho(t) \int_t^T e^{-\int_t^s \kappa(v)dv} ds,$$

where g_S and g_ϵ are given by equations (15) and (16). You will also need to find some kind of parameterization for ρ.

(5) Finding $\hat{\theta}_t$. The time-dependent mean-reversion level of the spot convenience yield, $\hat{\theta}_t$, is now given by equation (13).

4. Numerical Example

To illustrate the method consider the following example of a fictitious commodity. The parameters of the model can be found in table 1. These values are based on empirical estimates of crude oil prices (Schwartz 1997). Assume that the term structure of futures prices observed at date zero is as illustrated in figure 1. The implied term structure of future convenience yields, $T \mapsto \epsilon(0, T)$, at date zero can then be derived based on equation (5). This is illustrated as the red curve in figure 4. In addition we assume the instantaneous volatility of the commodity price and the convenience yield are given by the deterministic functions illustrated in figure 2. In figure 2 the red curve illustrates the volatility of the commodity price and the blue curve illustrates the volatility of the convenience yield. Seasonal patterns in the instantaneous volatility are quite common for a lot of reasons. For crude oil it is mostly driven by the demand cycle at the Northern hemisphere. The implied instantaneous term structure of futures price volatilities, $T \mapsto \|\sigma_{G_T}(0)\|$, at date zero can then be derived based on equation (14). This curve is illustrated in figure 3. Note that this curve does not have a seasonal pattern. This is because we

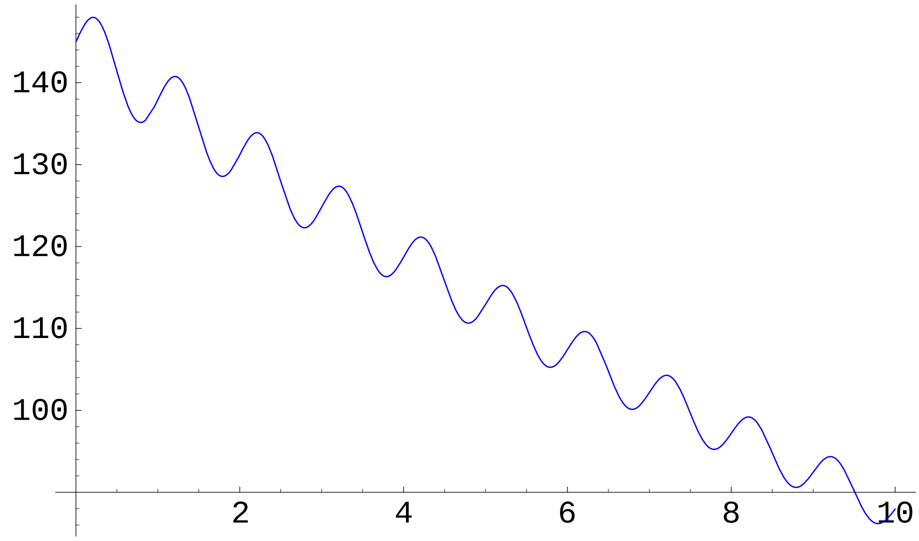

FIGURE 1. The term structure of futures prices, $T \mapsto G(0,T)$, for a fictitious commodity at date zero. The figure is based on the parameter values from table 1.

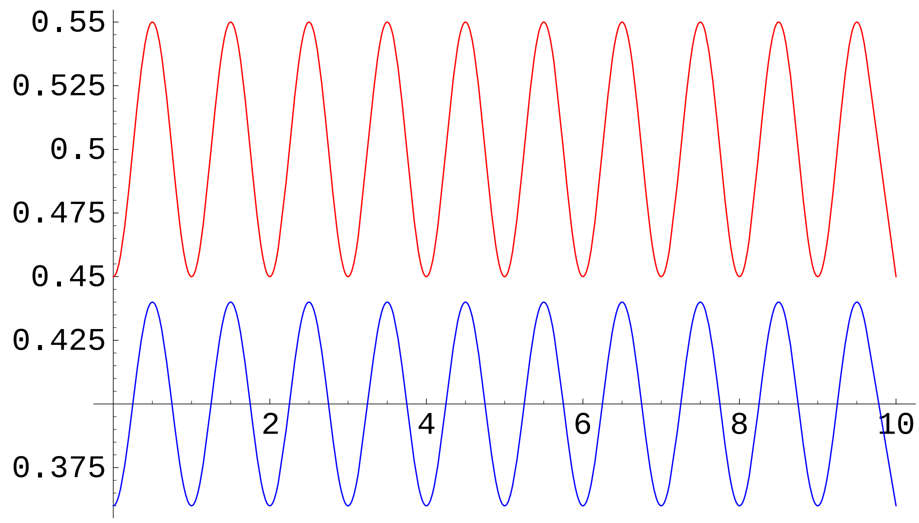

FIGURE 2. The instantaneous volatility of the commodity price (blue curve), $\sigma_S g_S(t)$, and the convenience yield (red curve), $\sigma_\epsilon g_\epsilon(t)$, as function of the calendar time, t. The figure is based on the parameter values from table 1. g_S and g_ϵ are given by equations (15) and (16).

have assumed that the speed of mean reversion function, κ, is just a constant. The seasonal pattern of the volatility of the spot commodity price and the spot convenience yield does not by itself lead to a seasonal pattern in the implied instantaneous term structure of futures price volatilities at date zero since the volatility of the spot commodity price and the spot convenience yield only play a role at the date when the contract is fixed, cf. equation (14). Finally, in this simple example we have assumed that the instantaneous correlation function between the spot commodity price and the convenience yield, ρ, is a constant. Based on all this the implied mean reversion level, $\hat{\theta}$, can be derived from equation (13). This is illustrated as the blue curve in figure 4. Note how the mean reversion level must have a very strong

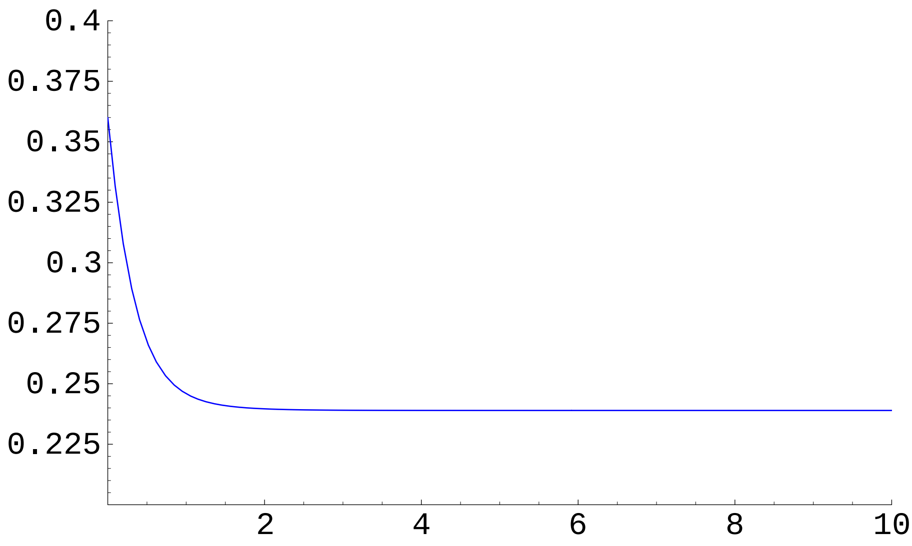

FIGURE 3. The implied instantaneous term structure of futures price volatilities, $T \mapsto \|\sigma_{G_T}(0)\|$, at date zero. The figure is based on the parameter values from table 1. $\|\sigma_{G_T}(0)\|$ is calculated based on equation (14).

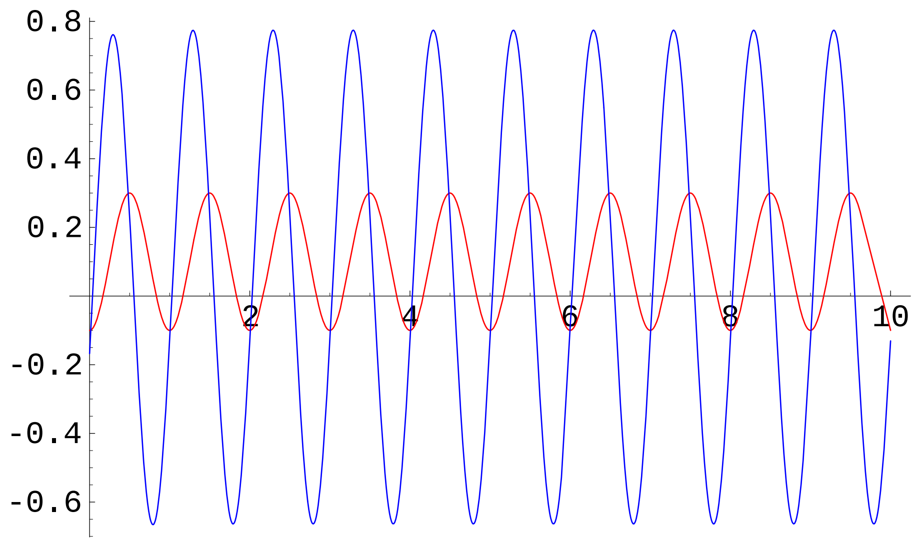

FIGURE 4. The implied term structure of future convenience yields (red curve), $T \mapsto \epsilon(0,T)$, at date zero and the implied mean reversion level for the spot convenience yield process (blue curve), $\hat{\theta}_t$, as function of the calendar time, t. The figure is based on the parameter values from table 1. $\epsilon(0,T)$ is calculated based on equation (5) and $\hat{\theta}_t$ is calculated based on equation (13).

seasonal pattern in order to 'drag' the stochastic spot convenience yield up and down so that the initially observed term structure of futures prices (cf. figure 1) (and implied future convenience yields (cf. the red curve in figure 4)) can be consistent with the stochastic behavior of the processes.

5. Conclusion

In this paper, we have shown how to build a stochastic model of commodity price behavior under risk-neutral probabilities that fulfills all no-arbitrage conditions and at the same time matches all observables. That is, the stochastic model will match the current term structure of forward and futures prices, the current term structure of forward and futures volatilities, and the inter-temporal pattern of the volatility of the forward and futures prices. Moreover, we show how to build a model that can cope with mean reversion and seasonal effects in the commodity price behavior.

Intuitively the spot convenience yield is dragged around by its deterministically determined time dependent mean-reversion parameter, $\hat{\theta}_t$, under risk-neutral probabilities in order to match the observed initial term structure of forward and futures prices, cf. equation (13). This is the same basic idea as in Hull and White (1993). One can argue that it is somewhat strange that the mean-reversion parameter of the spot convenience yield should have such a peculiar time-dependent behavior. However, it should be emphasized that this is the derived mean-reversion parameter *under risk-neutral probabilities*. The mean-reversion parameter under the physical probabilities has another term coming from the market price of convenience yield risk. Hence, it is possible to have, e.g., a constant mean-reversion parameter of the spot convenience yield process under the physical measure. In that case, it is simply the market price of risk that is time-varying. Economically it makes sense to think in this direction. That is, the current term structure of forward and futures prices, the current term structure of forward and futures volatilities, and the inter-temporal pattern of the volatility of the forward and futures prices simply tell us how the market price of risk, i.e. the investors' preferences, should be specified based on the observables.

Finally, the model developed in this paper can also be used to model the stochastic behavior of more abstract underlying *securities* such as weather, commodity quantities, etc. The main point is that as soon as the market has determined a term structure of forward and/or futures prices, it is no longer necessary for the underlying security to have any relation to a traded asset. However, for these *quanto* style products the future as well as the spot convenience yield is nothing but an artificial variable that tells us how to determine the drift under risk-neutral probabilities based on the observed term structure of forward and/or futures prices.

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