International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Numerical Solution for Solving Nonlinear Fuzzy Fractional Integral Equation by Using Approximate Method

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Abstract

In this paper we discus fractional order for fuzzy non-linear integral equation. The fractional integral is consider in the sense Riemann-liouville and establish the exists solution of nonlinear fuzzy fractional integral equation. Finally, Numerical examples are given to illustrate the results.

Keyword: fuzzy integral equation; fuzzy fractional integral equation; Riemann-liouville; successive approximate method.

1. Introduction

Theory of integral equations is an important part of pure and applied mathematics which plays a prominent role in many disciplines including engineering, physics, economics and biology. One of the significant branches of theory of integral equations is fractional integral equations that in recent years, has received considerable attention not only in mathematical research but also in other applied sciences. In fact, Fractional integral equations are the development of integral equations to real order. These type of integral equations can be applied to many real-world field such as polymer physics, viscoelastic materials, viscous damping and seismic analysis see [2,3,4].

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On the other hand, when one intends to analyze a real world phenomenon, it is also necessary to deal with uncertain factors. In this situation, the theory of fuzzy sets may be one of the best non-statistical or non-probabilistic approach, which leads us to investigate theory fuzzy fractional integral equations.

Recently, the topic of existence and uniqueness for the solutions to linear and nonlinear fuzzy fractional differential equations has been further investigated and discussed by many researchers in various aspects. For example, in [5] the existence and uniqueness of solutions of Riemann- Liouville fuzzy fractional differential equations has been proved by Arshad and in [9] the existence and uniqueness of solutions as well the approximate solutions to fuzzy fractional differential equations under the Liouville-Caputo H-differentiability has been studied by Salahshour and and his colleagues Further- more, the existence and uniqueness of solutions for fuzzy fractional differential equations under the Liouville-Caputo generalized Hukuhara differentiability has been investigated by Allahviranloo and his colleagues in [2].

In this paper, we intend to propose a new metric on the space of fuzzy continuous functions and we study the solutions to nonlinear fuzzy fractional integral equations under fractional generalized integration in the sense of the Liouville-Riemann fractional integral .

This paper organized as follows: basic concepts as well as fractional calculus are given in Section 2. Definition Fuzzy fractional 3, a new metric for the space of fuzzy continuous functions is introduce. 4. Fuzzy fractional nonlinear integral equation,5. numerical examples are given to illustrate the application of the results. Finally. Conclusion .

2. Preliminaries and Basic Definitions

introduces the concepts of fuzzy sets. The concept of fuzzy sets is a generalization of the crisp sets. Convex set, $\alpha -$ cut operation, cardinality of fuzzy number are also introduced.

Definition(2.1),[1]:-

A fuzzy number is a fuzzy set which is a map $\tilde{u}\colon \mathrm{R} \to [\mathrm{a},\mathrm{b}]$, that satisfies

(1) u is upper semi-continuous function,

(2) u(x) = 0 outside some interval $[\mathrm{a},\mathrm{d}]$

(3) There are a real numbers b,c such $\mathrm{a} \leq \mathrm{b} \leq \mathrm{c} \leq \mathrm{d}$ that is

(i) u(x)is a monotonic increasing function on $[\mathrm{a},\mathrm{b}]$.

(ii) u(x) is a monotonic decreasing function on $[\mathrm{c},\mathrm{d}]$.

(iii) u(x) = 1 for all $\mathrm{x} \in [\mathrm{b},\mathrm{c}]$.

The set of all fuzzy numbers is denoted by E^1 and is a convex cone.

Definition(2.2),[1]:-

A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u},\overline{u})$of function $\underline{u}(\alpha),\overline{u}(\alpha),\ 0 \le \alpha \le 1$, which satisfies the following conditions:

(i) . $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over [0, 1].

(ii). $\overline{u}(\alpha)$ is a bounded left continuous non- increasing function over [0, 1].

(ii) . $\underline{u}(\alpha) \le \overline{u}(\alpha)$,$0 \le \alpha \le 1$.

Definition(2.3),[10]:-

The arbitrary fuzzy number $\tilde{u} = (\underline{u}(\alpha),\overline{u}(\alpha))$,$\tilde{v} = (\underline{v}(\alpha),\overline{v}(\alpha))$, $0 \le \alpha \le 1$ and scalar k, the operations defended as follows:

(i) . $\underline{(u+v)}(\alpha)= (\underline{u}(\alpha)+\underline{v}(\alpha)),\overline{(u+v)}(\alpha) = (\overline{u}(\alpha)+\overline{v}(\alpha))$,

(ii). $\underline{(u-v)}(\alpha)= (\underline{u}(\alpha)-\underline{v}(\alpha)),\overline{(u-v)}(\alpha) = (\overline{u}(\alpha)-\overline{v}(\alpha))$,

(iii) . $k\check{u}=\begin{cases} Min\left(k\underline{u}(\alpha),k\overline{u}(\alpha)\right), & k \ge 0 \\ Max\left(k\underline{u}(\alpha),k\overline{u}(\alpha)\right), & k < 0 \end{cases}$

Definition (2.4),[10]:-

The family of all fuzzy number E^n, denotes all nonempty compact convex fuzzy subset of R^n. Let $I = [a,b]$ be a compact interval :

$$E^n = \{p: R^n \to I\}$$

such that p satisfies the following

i) p is normal
ii) p is fuzzy convex,
iii) p is upper semi continuous, i.e, the $\alpha-$level sets $[p]_\alpha$are closed for each $\alpha\in[0,1]$,
iv) $[p]^0=cl\{x\in R^n|p(x)>0\}$ is compact.

where the $\alpha-$level sets $[p]^\alpha$ is defined by $[p]^\alpha=\{x\in R^n|p(x)\ge\alpha\}$ for $0<\alpha\le1$

and $[p]^0$ for $\alpha=0$. Then from (i)-(iv), it follows that $[p]^\alpha\in E^n$for all

$0\le\alpha\le1$.

Definition (2.5),[1]:-

Let R be the set of real number and $\widetilde{P}(R)$ all fuzzy subsets defined on R , defined the fuzzy number $\tilde{a} \in E^1$ as follows :

1- $\tilde{a}$ is normal , that is there exists $x \in R$ such that $\mu_{\tilde{a}}(x) = 1$

2- For every $\alpha \in (0,1]$, $a_\alpha = \{x : \mu_{\tilde{a}}(x) \geq \alpha\}$ is closed interval , denoted by

$$[a^-{}_\alpha, a^+{}_\alpha]$$

Using Zaheh's notation $\tilde{a} \in F(R)$ is the fuzzy set on R defined by

$\tilde{a} = \cup_{\alpha\in[0,1]} a_\alpha = \cup_{\alpha\in[0,1]} \alpha[a^-{}_\alpha, a^+{}_\alpha]$

Definition(2.6),[1,10]:-

For any arbitrary Fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$

$D(\tilde{u}, \tilde{v}) = \max\{\sup_{\alpha\in[0,1]} \left|\underline{u}(\alpha) - \underline{v(\alpha)}\right|, \sup_{\alpha\in[0,1]} |\overline{u}(\alpha) - \overline{v}(\alpha)| \}$

Denoted the distance between $\tilde{u}$ and $\tilde{v}$, also (E^1, D) is a complete metric space .

Theorem(2.1),[8, 10]:-

(E^1, D) is a metric space

Theorem(2.2),[1]:-

If $\tilde{a}, \tilde{b}, \tilde{c} \in E^1$ then $D(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{c}) = D(\tilde{a}, \tilde{b})$

Proposition (2.1),[4]:-

For any $p, q, r, s \in E^n$ and $\varphi \in R$, then

(i) (E^n, D) is a complete metric space.

(ii) $D(\varphi p, \varphi q) = |\varphi| D(p, q)$.

(iii) $D(p + r, q + s) = D(p, q)$.

(iv) $D(p + q, r + s) \leq D(p, r) + D(q, s)$.

Definition (2.7),[10]:-

A function $F: I \to E^n$ is called bounded if there exists a constant $M > 0$ such that $D(F(x), \tilde{0}) \leq M$ for all $x \in I$.

Definition (2.8),[10]:-

A function $F: I \rightarrow E^n$ is said to be continuous if for arbitrary fixed $x_0 \in I$ and

$\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ then $D(F(x), F(x_0)) < \varepsilon$

Definition (2.9),[9]:-

Let : $I \rightarrow E^n$, the integral of Fover I which is levelwise continuous is denoted by $\int_I F(x)dx$ or $\int_a^b F(x)dx$, also

$[\int_I F(x)dx]^\propto = \int_I F(x)_\alpha dx \quad \{\int_I f(x)dx \mid f: I \rightarrow R^n$ is measurable function for $F(x)_\alpha$ for all $0 \le \alpha \le 1\}$

Definition(2.10) :- Let $f: I \rightarrow R^n$, the integral of f over I, denote $\int_I f(t)dt$ is defined levelwise by

$[\int_I f(t)dt]^\alpha = \int_I f_\alpha(t)dt = \{\int_I f(t)dt \mid f: I \rightarrow E^n \text{ is a masurable selection for } f_\alpha\}$

For all $\alpha \in [0,1]$

A mapping $f: I \rightarrow E^n$ is strong measurable if, for all $\alpha \in (0,1]$, the set valued mapping $f_\alpha: I \rightarrow p(R^n)$ denoted by $f_\alpha = [f(t)]^\alpha$ if $f: I \rightarrow E^n$ is continuous, then it is integrable

Proposition (2.2),[6]:-

Let $F, G: I \rightarrow E^n$ be a integrable functions and $\varphi \in R$. Then :

(i) $\int(F + G) = \int F + \int G$.
(ii) $\int \varphi F = \varphi \int F$
(iii) $D(F, G)$ is integrable.
(iv) $D(\int F, \int G) \le \int D(F, G)$

Definition(2.11),[7] :- The Riemann-Liouville fractional integral of order α for a function f is denoted by

$I_{a+}^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds$, where a=0 , we can write $I^\alpha f(t)$

3. Fuzzy fractional nonlinear integral equation

The fuzzy fractional nonlinear integral equation with complexity fuzzy kernel functions which represented completely as follows :

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \tilde{I} \tilde{k}(x, \int_a^t \tilde{F}(s, u(s))ds) \qquad (3.1)$$

where $\lambda > 0$, $\tilde{f}(x)$ is a fuzzy function of x such that I= $a \le x \le b$, also $\tilde{k}(x, \int_a^t \tilde{F}(s, u(s))ds)$: $I\times I \times E^1 \to E^1$,

are analytic functions, $\tilde{F}(s, u(s))$: $I\times E^1 \to E^1$,

are fuzzy continuous functions. Consider the pairs $\tilde{f}(x) = (\underline{f}(x,\alpha), \bar{f}(x,\alpha))$ and

$\tilde{u}(t) = (\underline{u}(x,\alpha), \overline{u}(x,\alpha))$, $0 \le \alpha \le 1$ and t ,s∈ [a,b] then, the parametric form of Eq. (3.1)

as follow:

$$\left.\begin{aligned} \underline{u}(x,\alpha) &= \underline{f}(x,\alpha) + \lambda \underline{I^{\alpha_1}}\, \underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds, \int_a^t \overline{F}\left(s, \overline{u}(s,\alpha)\right) ds)) \\ \overline{u}(x,\alpha) &= \bar{f}(x,\alpha) + \lambda \overline{I^{\alpha_1}}\, \overline{k}(x, \int_a^t \overline{F}\left(s, \overline{u}(s,\alpha)\right) ds, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right)) \end{aligned}\right\}$$

Where

$$\underline{I^{\alpha_1}}\, \underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds) = 1/\lceil(\alpha_1) \int_a^t \frac{\underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt$$

$$\overline{I^{\alpha_1}}\, \overline{k}(x, \int_a^t \overline{F}\left(s, \overline{u}(s,\alpha)\right) ds \ ) = 1/\lceil(\alpha_1) \int_a^t \frac{\overline{k}(x, \int_a^t \overline{F}(s, \overline{u}(s,\alpha)) ds}{(t-s)^{1-\alpha_1}} dt$$

$where\ \ \alpha_1\ is$ function defined by integral and $0 \le \alpha_1 < 1$

Now we will have the system of fuzzy nonlinear fractional intergral equation

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + 1/\lceil(\alpha_1) \int_a^t \frac{\underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt + 1/\lceil(\alpha_1) \int_a^t \frac{\overline{k}(x, \int_a^t \overline{F}\left(s, \overline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt$$

$$\overline{u}(x,\alpha) = \bar{f}(x,\alpha) + 1/\lceil(\alpha_1) \int_a^t \frac{\overline{k}(x, \int_a^t \overline{F}\left(s, \overline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt + 1/\lceil(\alpha_1) \int_a^t \frac{\underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt$$

Where

$\alpha \in [0,1]$ and $\alpha_1 \in [0,1)$

4. Fuzzy successive approximation method

Consider the following fractional fuzzy nonlinear integral equation

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + 1/\lceil(\alpha_1) \int_a^t \frac{\underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt + 1/\lceil(\alpha_1) \int_a^t \frac{\overline{k}(x, \int_a^t \overline{F}\left(s, \overline{u}(s,\alpha)\right) ds}{(t-s)^{1-\alpha_1}} dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + 1/\Gamma(\alpha_1)\int_a^t \frac{\overline{k}(x,\int_a^t \overline{F}(s,\overline{u}(s,\alpha))ds}{(t-s)^{1-\alpha_1}}dt + 1/\Gamma(\alpha_1)\int_a^t \frac{\underline{k}(x,\int_a^t \underline{F}\left(s,\underline{u}(s,\alpha)\right)ds}{(t-s)^{1-\alpha_1}}dt$$

Where

$\alpha \in [0,1]$ *and* $\alpha_1 \in [0,1)$, $a \le x, s \le b$.

The successive approximation method introduces the recurrence relation:

$$\underline{u_{n+1}}(x,\alpha) = \underline{f}(x,\alpha) + 1/\Gamma(\alpha_1)\int_a^t \frac{\underline{k}(x,\int_a^t \underline{F}\left(s,\underline{u_n}(s,\alpha)\right)ds}{(t-s)^{1-\alpha_1}}dt + 1/\Gamma(\alpha_1)\int_a^t \frac{\overline{k}(x,\int_a^t \overline{F}(s,\overline{u_n}(s,\alpha))ds}{(t-s)^{1-\alpha_1}}dt$$

$$\overline{u_{n+1}}(x,\alpha) = \overline{f}(x,\alpha) + 1/\Gamma(\alpha_1)\int_a^t \frac{\overline{k}(x,\int_a^t \overline{F}(s,\overline{u_n}(s,\alpha))ds}{(t-s)^{1-\alpha_1}}dt + 1/\Gamma(\alpha_1)\int_a^t \frac{\underline{k}(x,\int_a^t \underline{F}\left(s,\underline{u_n}(s,\alpha)\right)ds}{(t-s)^{1-\alpha_1}}dt$$

We can choose the $zeroth - component$ either [$\underline{0}$,$\overline{\overline{0}}$], [$f(x,\alpha)$, $\overline{f}(x,\alpha)$]

The exact solution $\tilde{u}(x,\alpha)$=[$\underline{u}(x,\alpha)$, $\overline{u}(x,\alpha)$].

where

$\lim_{n\to\infty} \underline{u}_{n+1} = \underline{u}(x,\alpha)$ *and* $\lim_{n\to\infty} \overline{u}_{n+1} = \overline{u}(x,\alpha)$

5. Numerical Example

Consider the fuzzy fractional nonlinear integral equation

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + 1/\Gamma(\alpha_1)\int_a^t \frac{\underline{k}(x,\int_a^t \underline{F}\left(s,\underline{u}(s,\alpha)\right)ds}{(t-s)^{1-\alpha_1}}dt + 1/\Gamma(\alpha_1)\int_a^t \frac{\overline{k}(x,\int_a^t \overline{F}(s,\overline{u}(s,\alpha))ds}{(t-s)^{1-\alpha_1}}dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + 1/\Gamma(\alpha_1)\int_a^t \frac{\overline{k}(x,\int_a^t \overline{F}(s,\overline{u}(s,\alpha))ds}{(t-s)^{1-\alpha_1}}dt + 1/\Gamma(\alpha_1)\int_a^t \frac{\underline{k}(x,\int_a^t \underline{F}\left(s,\underline{u}(s,\alpha)\right)ds}{(t-s)^{1-\alpha_1}}dt$$

Where

$\underline{f}(x,\alpha) = x(\alpha+1) - .4550376649x^{\frac{21}{10}}\alpha - .4550376649x^{\frac{21}{10}} - 1.051137006x^{\frac{11}{10}} - 4.095338984x^{\frac{31}{10}}\alpha^2 +$
$2.730225989x^{\frac{31}{10}}\alpha - .4550376649x^{\frac{31}{10}} + 4.095338984(x-1.)^{\frac{1}{10}}x^3\alpha^2 + .4095338984(x-1.)^{\frac{1}{10}}x^2\alpha^2 -$
$2.7302259(x-1)^{\frac{1}{10}}x^3\alpha + 0.2252436442(x-1.)^{\frac{1}{10}}x\alpha^2 - .2730225989(x-1.)^{\frac{1}{10}}x^2\alpha + .4550376649(x -$

$1.)^{\frac{1}{10}}x^3 - .1501624295(x-1.)^{\frac{1}{10}}x\alpha + 0.04550376649(x-1.)^{\frac{1}{10}}x^2 + 0.02502707157(x-1.)^{\frac{1}{10}}x$

$$\overline{f}(x,\alpha) = x^2(3\alpha-1) - 4.095338984x^{\frac{31}{10}}\alpha^2 + 2.730225989x^{\frac{31}{10}}\alpha - .4550376649x^{\frac{31}{10}} + .4550376649x^{\frac{21}{10}}\alpha - 0.45503769x^{\frac{21}{10}} - 1.051137006x^{\frac{11}{10}} - 0.02502707(x-1.)^{\frac{1}{10}}\alpha - 0.045503766(x-1.)^{\frac{1}{10}}\alpha x - 0.4550376649(x-1.)^{\frac{1}{10}}x^2\alpha + 0.02502707157(x-1.)^{\frac{1}{10}} + 1.096640772(x-1.)^{\frac{1}{10}}x + 0.4550376649(x-1.)^{\frac{1}{10}}x^2$$

$\underline{k}(x, \int_a^t \underline{F}\left(s, \underline{u}(s,\alpha)\right) ds = x - \int_0^t \underline{u}\,(s,\alpha)^2 ds$

$x * \int_0^t \overline{u}\,(s,\alpha)^2 ds\ \overline{k}(x, \int_a^t \overline{F}(s, \overline{u}(s,\alpha)) ds =$

The exact solution are

$\underline{u}(x,\alpha) = x(\alpha + 1)$

$\overline{u}(x,\alpha) = x^2(3\alpha - 1)$

$Where\ \alpha \in [0,1]\ and\ 0 \le \alpha_1 < 1$

By using successive method for solving our problem formulation , we get

$\underline{u}_0(x,\alpha) = \underline{f}(x,\alpha)$

$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha)$

$\vdots$

$\underline{u}_n(x,\alpha) = \underline{f}(x,\alpha)$

$\overline{u}_n(x,\alpha) = \overline{f}(x,\alpha)$

Where n=3

$\lim_{n\to\infty} \underline{u}_3(x,\alpha) = \underline{u}(x,\alpha)$

$\lim_{n\to\infty} \overline{u}_3(x,\alpha) = \overline{u}(x,\alpha)$

Table 1: calculate the results between the exact solution and approximate solution and also calculate the absolute error with differente value of α and different value of fraction order of α_1. For lower and upper side

x	Exact solution $\underline{u}(x,\alpha)$	Approximate solution $\underline{u}(x,\alpha)$	Absolute error $\underline{u}(x,\alpha)$
$\boldsymbol{\alpha = 0.1}$ **and** $\boldsymbol{\alpha_1}$**=0.1**			
0	0.00	0.000	0.000
0.2	0.22	0.217	0.003
0.4	0.44	0.430	0.010
0.6	0.66	0.578	0.082
0.8	0.66	0.651	0.009
1.0	1.1	1.098	0.002
	α=0.6 and α_1=0.5		
0	0.00	0.00	0.00
0.2	0.32	0.301	0.019
0.4	0.64	0.556	0.084
0.6	0.69	0.674	0.016
0.8	1.28	1.159	0.121
1.1	1.6	1.585	0.015

x	Exact solution $\underline{u}(x,\alpha)$	Approximate solution $\underline{u}(x,\alpha)$	Absolute error $\underline{u}(x,\alpha)$
$\boldsymbol{\alpha = 0.1}$ **and** $\boldsymbol{\alpha_1}$**=0.1**			
0	0.00	0.000	0.000
0.2	-0.028	-0.019	0.009
0.4	-0.112	-0.107	0.005
0.6	-0.252	-0.162	0.09
0.8	-0.448	-0.449	0.001
1.0	-0.7	-0.652	0.048
	α=0.6 and α_1=0.5		
0	0.00	0.00	0.00
0.2	0.032	0.030	0.02
0.4	0.128	0.119	0.009
0.6	0.288	0.277	0.011
0.8	0.512	0.458	0.054
1.1	0.8	0.781	0.019

6. Conclusion

In this paper, incorporation between volterra- Fredholm fuzzy fractional nonlinear integral equation is so complicity equation we also solve our problem formula by using approximate method specially successive approximate method . The accuracy of proposed illustrative by calculate the proposed absolute error, is clearly when show the value for table (1)of the basic function increasing . the mention method is very useful and fast to get the converge result , and given accuracy can be obtained by chosen the value of α *and the fraction value* α_1 .

References

[1] A.M.A. El-Sayed, A.-G. Ibrahim, Set-valued integral equations of fractional-orders. Applied Mathematics and Computation 118 (2001), 113–121.

[2] T. Allahviranloo, A. Armand and Z. Gouyandeh, Fuzzy fractional differential equations under generalized fuzzy Caputo derivative. Journal of Intelligent & Fuzzy Systems. 26 (2014), 1481- 1490.

[3] B. Bede, Mathematics of fuzzy sets and fuzzy logic. Springer-Verlag Berlin Heidelberg (2013).

[4] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, Fractional Calculus Models and Numer- ical Methods, in: Series on Complexity, Nonlinearity and Chaos. World Scientific, Singapore, (2012).

[5] K. Deimling, Nonlinear Functional Analysis. Springer–Verlag, Berlin– Heidelberg, 1985.

[5] K. Diethelm, The Analysis of Fractional Differential Equations. Springer, 2004.

[6] M. Amawi and N. Qatanani, Numerical Methods for Solving Fuzzy Fredholm Integral Equation of the Second Kind, International Journal of Applied Mathematics, Vol.28, No.3, 2015, 177-195 .

[7] M. Dehghan, and B. Hashemi, Iterative Solution of Fuzzy Linear systems, Applied Mathematics and Computation, Vol. 175, 2006, 645-674

[8] P. Diamond, P. Kloeden, Metric Spaces of Fuzzy Sets. World Scientific, Singapore, 1994.

[9] S. Arshad, on existence and uniqueness of solution of fuzzy fractional differential equations. Iranian Journal of Fuzzy Systems. Vol. 10. No. 6 (2013), 137–151.

[10] W. Cong-xin, and M. Ming, Embedding problem of fuzzy number space: part 1, Fuzzy sets and systems, Vol. 44, Issue 1, 1991, 33-38.