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New Sequence Spaces with Respect to a Sequence of Modulus Functions

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Abstract

In this paper, we introduce the notions of A^{I} -invariant convergence, A^{I^*}-invariant convergence with respect to a sequence of modulus functions and establish some basic theorems. Furthermore, we give some properties of A^{I_σ} -Cauchy sequence and $A^{I_\sigma^*}$ -Cauchy sequence. We basically study some connections between A^{I}-invariant statistical convergence and A^{I} -invariant lacunary statistical convergence with respect to a sequence of modulus functions and between strongly A^{I} -invariant convergence and A^{I} -invariant lacunary statistical convergence with respect to a sequence of modulus functions. Also, we establish some inclusion relations between new concepts of $I_\sigma - \lambda$ statistically convergence and A^{I} –invariant statistically convergence with respect to a sequence of modulus functions.

Keywords: Lacunary invariant statistical convergence; Invariant statistical convergence; modulus function.

1. Introduction

The notion of statistical convergence of sequences of numbers was introduced by Fast [12]. Later on, statistical convergence turned out to be one of the most active areas of research in summability theory after the works of [15,29].

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The concept of lacunary statistical convergence was defined by [16]. Also, they gave the relationships between the lacunary statistical convergence and the Cesàro summability. Freedman and his colleagues established the connection between the strongly Cesàro summable sequences space σ_1 and the strongly lacunary summable sequences space N^{θ} in their work [1] published in 1978. The idea of λ-statistical convergence was introduced and studied by [20] as an extension of the $[V,\lambda]$ summability of Leindler [18]. The concept of *I*-convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal *I* of subsets of the set of natural numbers. P. Kostyrko and his colleagues [26] introduced the concept of *I*-convergence of sequences in a metric space and studied some properties of this convergence. Several authors including [24,25,22,5], and some authors have studied invariant convergent sequences. Nuray and his colleagues [10], defined the concepts of σ-uniform density of subsets *A* of the set $\mathbb{N}$, I_σ-convergence and investigated relationships between I_σ-convergence and invariant convergence also I_σ-convergence and $[V_\sigma]_p$ -convergence. The concept of strongly σ-convergence was defined by [21]. Reference [7] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Recently, the concept of strongly σ-convergence was generalized by [5]. Reference [30] investigated lacunary *I*-invariant convergence and lacunary *I*-invariant Cauchy sequence of real numbers. The notion of a modulus function was introduced by Nakano [11]. We recall that a modulus *f* is a function from $[0,\infty)$ to $[0,\infty)$ such that (i) $f(x)=0$ if and only if $x=0$, (ii) $f(x+y)=f(x)+f(y)$ for $x,y\geq 0$, (iii) f is increasing and (iv) f is continuous from the right at 0. It follows that *f* must be continuous on $[0,\infty)$. Connor [17,28,14,3,27,31] used a modulus function to construct sequence spaces. Now let $\mathcal{S}$ be the space of sequence of modulus function $F=(f_k)$ such that $\lim_{x\to 0^+} sup_k f_k(x)=0$. Throughout the paper we take $A=(a_{ki})$ as an infinite matrix of complex numbers and the set of all modulus functions determined by *F* and it will be denoted by $F=(f_k)\in\mathcal{S}$ for every $k\in\mathbb{N}$. First we recall some of the basic concepts which we will be used in this paper. A number sequence $x=(x_k)$ is said to be statistically convergent to the number *L* if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n: |x_k-L|\geq\varepsilon\}|=0.$$

In this case we write $st-limx_k=L$. By a lacunary sequence we mean an increasing integer sequence $\theta=\{k_r\}$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$.

A sequence $x=(x_k)$ is said to be lacunary statistically convergent to the number *L* if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r: |x_k-L|\geq\varepsilon\}|=0.$$

In this case we write $S_\theta-limx_k=L$ or $x_k\to L(S_\theta)$. The strongly lacunary summable sequences space N^{θ}, which is defined by

$$N_\theta=\left\{(x_k):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|x_k-L|=0\right\}.$$

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.

A sequence $x = (x_k)$ is said to be λ-statistically convergent or S_λ-convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n=1,2,\ldots$.

The generalized de la Valee-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k\in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $\lim_{n\to\infty} t_n(x) = L$. If $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$-summability.

By an ideal on a set X we mean a non-empty family of subsets of X closed under taking finite unions and subsets of its elements. In other words, a non-empty set $I \subset 2^{\mathbb{N}}$ is called an ideal on $\mathbb{N}$ if;

(i) For each $A, B \in I$ we have $A \cup B \in I$,

(ii) For each $A \in I$ and each $B \subseteq A$ we have $B \in I$.

If $\mathbb{N} \notin I$ then we say that this ideal is a proper ideal. Similarly an ideal is proper and also contains all finite subsets then we say that this ideal is admissible. Similarly, a non-empty set $\mathcal{F} \subset 2^{\mathbb{N}}$ is called a filter on $\mathbb{N}$ if;

(i) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,

(ii) (ii) For each $A \in \mathcal{F}$ and each $A \subseteq B$ we have $B \in \mathcal{F}$.

Proposition 1.1. If I is a non-trivial ideal in $\mathbb{N}$, then the family of sets

$$\mathcal{F}(I) = \{M \subset \mathbb{N} : (\exists A \in I), (M = X \setminus A)\}$$

is a filter in $\mathbb{N}$ and it is called the filter associated with the ideal. Filter is a dual notion of ideal and generally we will use ideals in our proofs but if the notion is more familiar for filters, we will use the notion of filter. Let $x = (x_k)$ be a real sequence. This sequence is said to be I-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set

$$A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

belongs to *I*. In this definition the number L is I-limit of the x. An admissible ideal $I \subset 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ..\}$ of mutually disjoint sets of *I*, there is sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1,2, ...)$ is finite and $\bigcup_{i=1}^{\infty} B_i \in I$. Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = (\sigma^{m-1}(n))$, $m = 1,2, ...$. A continuous linear functional on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if, (i) $\phi(x) \geq 0$, for all sequences $x = (x_n)$ with $x_n \geq 0$ for all *n*; (ii) $\phi(e) = 1$, where e=(1,1,1,…); (iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$. The mapping ϕ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers *n* and *m*, where $\sigma^m(n)$ denotes the *m*.th iterate of the mapping σ at *n*. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = limx$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences. It can be shown that

$$V_\sigma = \left\{x = (x_n) \in l_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(m)} = L\right\}, \text{uniformly in } m.$$

A bounded sequence $x = (x_k)$ is said to be strongly σ -convergent to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|x_{\sigma^k(m)} - L\right| = 0, \text{ uniformly in } m.$$

In this case we write $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly σ -convergent sequences.

A sequence $x = (x_k)$ is σ-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \left|k \leq m : \left|x_{\sigma^k(n)} - L\right| \geq \varepsilon\right|, \text{uniformly in } n.$$

In this case, we write $S_\sigma - limx = L$ or $x_k \to L(S_\sigma)$.

Nuray and his colleagues [10] introduced the concepts of σ -uniform density and I_σ -convergence.

Let $A \subset \mathbb{N}$ and

$$s_n = \min_m |A \cap \{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\}|$$

and

$$S_n = \max_m |A \cap \{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\}|.$$

If the following limits exists

$$\underline{V}(A) = \lim_{n\to\infty} \frac{s_n}{n}, \overline{V}(A) = \lim_{n\to\infty} \frac{S_n}{n}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then

$V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ -uniform density of A.

Denote by I_σ the class of all $A \subset \mathbb{N}$ with $V(A) = 0$.

A sequence $x = (x_k)$ is I_σ- convergent to the number L if for every $\varepsilon > 0$,

$$A_\varepsilon = \{k: |x_k - L| \geq \varepsilon\} \in I_\sigma,$$

that is $V(A_\varepsilon) = 0$. In this case, we write $I_\sigma - limx = L$.

Let $A = (a_{ki})$ be an infinite matrix of complex numbers. We write $Ax = \left(A_k(x)\right)$, if $A_k(x) = \sum_{i=1}^{\infty} a_{ki} x_k$ converges for each k.

In [19], the notion of $A^I - [V,\lambda]$ summability and $A^I - \lambda$ statistical convergence with respect to a sequence of modulus functions were introduced and some connections between $A^I - \lambda$ statistical convergence and A^I -statistically convergence were studied.

2. Main Results

In this section, we will give some new concepts, give the relationship between them and establish some basic theorems.

Definition 2.1 The sequence (x_k) is said to be A^I-invariant convergent to L with respect to a sequence of modulus functions if for every $\varepsilon > 0$ the set,

$$B(\varepsilon, x) = \{k: f_k(|A_k(x) - L|) \geq \varepsilon\}$$

belongs to I_σ. In this case, we write $x_k \to L\left({I_\sigma}^A, F\right)$.

Definition 2.2 The sequence (x_k) is said to be invariant convergent to L with respect to a sequence of modulus functions if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_k\left(A_k\left(x_{\sigma^k(m)}\right)\right) = L,$$

uniformly in m. In this case, we write $(x_k) \to L\left({V_\sigma}^A, F\right)$.

Theorem 2.1 Let (x_k) is bounded sequence. If (x_k) is A^I -invariant convergent to L with respect to a sequence of modulus functions, then (x_k) is invariant convergent to L with respect to a sequence of modulus functions.

Proof Let $m, n \in \mathbb{N}$ be arbitrary and every $\varepsilon > 0$. For each $x \in X$, we estimate

$$t(m,n,x) := \left| \frac{f_k\left(A_k\left(x_{\sigma(m)}\right)\right) + f_k\left(A_k\left(x_{\sigma^2(m)}\right)\right) + \cdots + f_k\left(A_k\left(x_{\sigma^n(m)}\right)\right)}{n} - L \right|.$$

Then, for each $x \in X$ we have $t(m,n,x) \leq t^1(m,n,x) + t^2(m,n,x)$, where

$$t^1(m,n,x) := \frac{1}{n} \sum_{\substack{k,j=1 \\ f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right)-L\right|\right) \geq \varepsilon}}^{n} f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right)$$

and

$$t^2(m,n,x) := \frac{1}{n} \sum_{\substack{k,j=1 \\ f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right)-L\right|\right) < \varepsilon}}^{n} f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right).$$

Therefore, we have $t^2(m,n,x) < \varepsilon$, for each $x \in X$ and for every m=1,2,… . The boundedness of (x_k) is implies that there exist $M>0$ such that for each $x \in X$,

$$f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right| \leq M, \qquad (j = 1,2,\ldots; m = 1,2,\ldots)\right),$$

for all $k \in \mathbb{N}$. This implies that

$$t^1(m,n,x) \leq \frac{M}{n}\left|\left\{1 < j < n: f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|$$

$$\leq M.\frac{max_m\left|\left\{1 < j < n: f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|}{n} = M\frac{S_n}{n}.$$

Hence, (x_k) is invariant convergent to L with respect to a sequence of modulus functions.

Definition 2.3 A sequence $x = (x_k)$ is said to be A^{I^*}-invariant convergent to $L \in X$ with respect to a sequence of modulus functions, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(I_\sigma)$ such that

$$\lim_{k\to\infty} f_k\left(A_k\left(x_{m_k}\right)\right) = L.$$

In this case, we write $x_k \to L\left(I_\sigma^{*A}, F\right)$.

Theorem 2.2 If a sequence $\mathrm{x} = (\mathrm{x_k})$ is A^{I^*}-invariant convergent to L, then this sequence is A^{I}-invariant convergent to L with respect to a sequence of modulus functions.

Proof. By assumption, there exists a set $H \in I_\sigma$ such that for $M = N \setminus H = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(I_\sigma)$ we have

$$\lim_{k\to\infty} f_k\left(A_k\left(x_{m_k}\right)\right) = L, \qquad (2.2.1)$$

Let $\varepsilon > 0$. By (2.2.1), there exists $k_0 \in \mathbb{N}$ such that

$$f_k\left(\left|A_k\left(x_{m_k}\right) - L\right|\right) < \varepsilon,$$

for each $k > k_0$. Then, obviously

$$\{k \in \mathbb{N}: f_k|A_k(x) - L| \geq \varepsilon\} \subset H \cup \left\{m_1 < m_2 < \cdots < m_{k_0}\right\}. \qquad (2.2.2)$$

Since I_σ is admissible, the set on the right-hand side of (2.2.2) belongs to I_σ. So $x = (x_k)$ is A^I-invariant convergent to L with respect to a sequence of modulus functions.

Theorem 2.3 Let I_σ be an admissible ideal with property (AP). If a sequence $\mathrm{x} = (\mathrm{x_k})$ is A^I-invariant convergent to L, then this sequence is A^{I^*}-invariant convergent to L with respect to a sequence of modulus functions.

Proof. Suppose that I_σ satisfies condition (AP). Let $\mathrm{x} = (\mathrm{x_k})$ is A^I -invariant convergent to L. Then

$$\{k \in \mathbb{N}: f_k(|A_k(x) - L|) \geq \varepsilon\} \in I_\sigma.$$

for each $\varepsilon > 0$. Put

$$E_1 = \{k \in \mathbb{N}: f_k(|A_k(x) - L|) \geq 1\}$$

and

$$E_n = \left\{k \in \mathbb{N}: \frac{1}{n} \leq f_k(|A_k(x) - L|) < \frac{1}{n-1}\right\}$$

for $n \geq 2$ and $n \in \mathbb{N}$. Obviously $E_i \cap E_j = \emptyset$ for $i \neq j$. By condition (AP) there exists a sequence of sets

$\{F_n\}_{n\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and

$$F = \bigcup_{j=1}^{\infty} F_j \in I_\sigma.$$

It is sufficient to prove that for $M = \mathbb{N} \setminus \mathcal{F}, M = \{m = (m_i): m_i < m_{i+1}, i \in \mathbb{N}\} \in \mathcal{F}(I_\sigma)$ we have

$$\lim_{k\to\infty} f_k\left(A_k(x_{m_k})\right) = L, k \in M. \qquad (2.3.1)$$

Let λ>0. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \lambda$. Then

$$\{n \in \mathbb{N}: f_k(|A_k(x) - L|) \geq \lambda\} \subset \bigcup_{j=1}^{k+1} E_j$$

Since $E_j \Delta F_j$, *j=1,2,...,n+1* are finite sets, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{k+1} F_j\right) \cap \{k \in \mathbb{N}: k > k_0\} = \left(\bigcup_{j=1}^{k+1} E_j\right) \cap \{k \in \mathbb{N}: k > k_0\} \qquad (2.3.2)$$

If $k > k_0$ and $k \notin F$, then $k \notin \bigcup_{j=1}^{n+1} F_j$ and by (2.3.2) $k \notin \bigcup_{j=1}^{n+1} E_j$.

But then $f_k(|A_k(x) - L|) < \frac{1}{n+1} < \lambda$; so (2.3.1) holds and we have $\lim_{k\to\infty} f_k\left(A_k(x_{m_k})\right) = L$.

Now, we define the concepts of *I*-invariant Cauchy sequence and I^*-invariant Cauchy sequence of real numbers with respect to a sequence of modulus functions.

Definition 2.4 Let I_σ be an admissible ideal in ℕ. A sequence $(\mathrm{x_k})$ is said to be I_σ-Cauchy sequence if for each $\varepsilon > 0$, there exists a number $\mathrm{N} = \mathrm{N}(\varepsilon)$ such that

$$A(x, \varepsilon) = \left\{k: \left|f_k\left(A_k(x_k)\right) - f_k\left(A_k(x_N)\right)\right| \geq \varepsilon\right\}$$

belongs to I_σ.

Definition 2.5 Let I_σ be an admissible ideal in ℕ. A sequence $(\mathrm{x_k})$ is said to be I_σ^*-Cauchy sequence if there exists a set $M = \{m = (m_i): m_i < m_{i+1}, i \in \mathbb{N}\} \in \mathcal{F}(I_\sigma)$, such that

$$\lim_{k,p\to\infty} \left|f_k\left(A_k(x_{m_k})\right) - f_k\left(A_k\left(x_{m_p}\right)\right)\right| = 0.$$

We give following theorems which show relationships between I_σ-convergence, I_σ-Cauchy sequence and I_σ^*-Cauchy sequence.

Theorem 2.4 If a sequence $(\mathrm{x_k})$ is I_σ-convergent, then $(\mathrm{x_k})$ is an I_σ-Cauchy sequence.

Theorem 2.5 If a sequence $(\mathrm{x_k})$ is I_σ^* -Cauchy sequence, then $(\mathrm{x_k})$ is I_σ-Cauchy sequence.

Theorem 2.6 Let I_σ has property (AP). Then the concepts I_σ^* -Cauchy sequence and I_σ -Cauchy sequence coincides.

Definition 2.6 The sequence $(\mathrm{x_k})$ is said to be *p*-strongly invariant convergent to L with respect to a sequence of modulus functions, if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|^p\right) = 0,$$

uniformly in *m*, where 0<*p*<∞. In this case, we write $(x_k) \to L[V_\sigma^A, F]_p$.

Theorem 2.7 Let I_σ be an admissible ideal and 0<*p*<∞.

i. If $(x_k) \to L[V_\sigma^A, F]_p$, then $(x_k) \to L(I_\sigma^A, F)$.

ii. If $x \in m(X)$, the space of all bounded sequences of X and $(x_k) \to L(I_\sigma^A, F)$, *then* $(x_k) \to L[V_\sigma^A, F]_p$.

iii. If $x \in m(X)$, *then* (x_k) *is* I_σ^A *-convergent if and only if* $(x_k) \to L[V_\sigma^A, F]_p$.

Proof. (i) Let $\varepsilon > 0$ and $(x_k) \to L[V_\sigma^A, F]_p$. Then we can write

$$\sum_{j=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|^p\right) \geq \sum_{\substack{j=1 \\ f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right)-L\right|\right)\geq\varepsilon}}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|^p\right)$$

$$\geq \varepsilon^p.\left|\left\{j \leq n: f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \varepsilon^p.max_m\left|\left\{j \leq n: f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|,$$

and

$$\sum_{j=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|^p\right) \geq \varepsilon^p.\frac{max_m\left|\left\{1 < j < n: f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|}{n} = \varepsilon^p.\frac{S_n}{n}$$

for every *m=1,2,…*. This implies $\lim_{n\to\infty} \frac{S_n}{n} = 0$ and so $(x_k) \to L(I_\sigma^A, F)$.

(ii) Suppose that $x \in m(X)$ and $(x_k) \to L(I_\sigma^A, F)$. Let $\varepsilon > 0$. Since (x_k) is bounded, (x_k) implies that there exist M>0 such that for each $x \in X$,

$$f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \leq M,$$

for all *j* and *m*. Then, we have

$$\frac{1}{n}\sum_{j=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|^p\right)$$

$$= \frac{1}{n}\left(\sum_{\substack{j=1 \\ f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right)-L\right|\right)\geq\varepsilon}}^{n} f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|^p\right) + \sum_{\substack{j=1 \\ f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right)-L\right|\right)<\varepsilon}}^{n} f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|^p\right)\right)$$

$$\leq M.\frac{max_m\left|\left\{1 < j < n : f_k\left(\left|A_k\left(x_{\sigma^j(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|}{n} + \varepsilon^p < M.\frac{S_n}{n} + \varepsilon^p,$$

for each $x \in X.$

Hence, for each $x \in X$ we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|^p\right) = 0,$$

uniformly in m.

(iii) This is immediate consequence of (i) and (ii).

Definition 2.7 A sequence $x = (x_k)$ is said to be A^I -invariant lacunary statistically convergent to $L \in X$ with respect to a sequence of modulus functions, for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I_\sigma, \text{uniformly in } m.$$

Definition 2.8. A sequence $x = (x_k)$ is said to be strongly A^I -invariant lacunary convergent to $L \in X$ with respect to a sequence of modulus functions, if, for each $\varepsilon > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\} \in I_\sigma, \text{uniformly in } m.$$

We shall denote by $S_{\sigma\theta}^{A}(I,F)$, $N_{\sigma\theta}^{A}(I,F)$ the collections of all A^I -invariant lacunary statistically convergent and strongly A^I –invariant lacunary convergent sequences, respectively.

Theorem 2.8 Let $A = (a_{ki})$ be an infinite matrix of complex numbers, $\theta = \{k_r\}$ be a lacunary sequence and $F = (f_k)$ be a sequence of modulus function in S.

i. If $x_k \to L(N_{\sigma\theta}^{A}(I,F))$ then $x_k \to L\left(S_{\sigma\theta}^{A}(I,F)\right)$.

ii. If $x \in m(X)$, the space of all bounded sequences of X and $x_k \to L\left(S_{\sigma\theta}^{A}(I,F)\right)$ *then* $x_k \to L(N_{\sigma\theta}^{A}(I,F))$.

iii. $S_{\sigma\theta}^{A}(I,F) \cap m(X) = N_{\sigma\theta}^{A}(I,F) \cap m(X)$.

Proof. (i) Let $\varepsilon > 0$ and $x_k \to L(N_{\sigma\theta}^{A}(I,F))$. Then we can write

$$\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \sum_{\substack{k\in I_r \\ f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right)$$
$$\geq \varepsilon.\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|.$$

So for given $\delta > 0$,

$$\frac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta \mapsto \frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\,.\,\delta,$$

i.e.

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon.\delta\right\}.$$

Since $x_k \to L(N_{\sigma\theta}^{A}(I,F))$, the set on the right-hand side belongs to I_σ and so it follows that $x_k \to L\left(S_{\sigma\theta}^{A}(I,F)\right)$.

(ii) Suppose that $x \in m(X)$ and $x_k \to L\left(S_{\sigma\theta}^{A}(I,F)\right)$.

Then we can assume that

$$f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \leq M$$

for each $x \in X$ and all *k*. Given $\varepsilon > 0$, we get

$$\frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)$$

$$=\frac{1}{h_r}\left(\sum_{\substack{k\in I_r\\ f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\right.$$

$$\left.+\sum_{\substack{k\in I_r\\ f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)<\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\right)\leq\frac{M}{h_r}\left|\left\{k\in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\right\}\right|+\varepsilon.$$

Note that

$$A(\varepsilon)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\right\}\right|\geq\frac{\varepsilon}{M}\right\}$$

belongs to I_σ. If $r\in\left(A(\varepsilon)\right)^c$ then

$$\frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)<2\varepsilon.$$

Hence

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq 2\varepsilon\right\}\subset A(\varepsilon)$$

and so belongs to I_σ. This shows that $\mathrm{x_k}\rightarrow \mathrm{L}\left(N_{\sigma\theta}^{A}(I,F)\right)$. This completes the proof. (iii) This is an immediate consequence of (i) and (ii).

Definition 2.9 The sequence (x_k) is A^I –invariant statistically convergent to L if for each $\varepsilon>0$, for each $x\in X$ and $\delta>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\right\}\right|\geq\delta\right\}$$

belongs to I_σ. $\left(\text{denoted by } x_k\rightarrow L\left(S(I_\sigma^A,F)\right)\right)$.

Theorem 2.9 If $\theta=\{k_r\}$ be a lacunary sequence with $\lim inf_r\, q_r>1$, then

$$x_k\rightarrow L\left(S(I_\sigma^A,F)\right)\Rightarrow\ x_k\rightarrow L\left(S_{\sigma\theta}^{A}(I,F)\right).$$

Proof. Suppose first that $\lim inf_r\, q_r>1$, then there exists a $\alpha>0$ such that $q_r\geq 1+\alpha$ for sufficiently large r, which implies that $\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}$.

If $x_k \to L(S(I_\sigma^A, F))$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k \leq k_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \frac{1}{k_r}\left|\left\{k \in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{\alpha}{1+\alpha} \cdot \frac{1}{h_r}\left|\left\{k \in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|;$$

Then for any $\delta > 0$, we get

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\left|\left\{k \in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{r \in \mathbb{N}: \frac{1}{k_r}\left|\left\{k \leq k_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \frac{\delta\alpha}{1+\alpha}\right\}$$

belongs to I_σ. This completes the proof.

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in \mathcal{F}(I_\sigma)$,

$$\bigcup\{n: k_{r-1} < n \leq k_r, r \in C\} \in \mathcal{F}(I_\sigma).$$

Theorem 2.10 *If* $\theta = \{k_r\}$ *be a lacunary sequence with* $\lim sup_r q_r < \infty$, *then*

$$x_k \to L\left(S_{\sigma\theta}^A(I, F)\right) \Rightarrow x_k \to L(S(I_\sigma^A, F)).$$

Proof. If $\lim sup_r q_r < \infty$ then without any loss of generality we can assume that there exists a $0 < M < \infty$ such that $q_r < M$ for all $r \geq 1$.

Suppose that $x_k \to L\left(S_{\sigma\theta}^A(I, F)\right)$ and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{r \in \mathbb{N}: \frac{1}{h_r}\left|\left\{k \in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k < n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in \mathcal{F}(I_\sigma)$, the filter associated with the ideal I_σ. Further observe that

$$K_j = \frac{1}{h_j}\left|\left\{k \in I_j: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$.

Now we have

$$
\begin{aligned}
\frac{1}{n}\left|\left\{k \leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| &\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\left|\left\{k \in I_1: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| + \frac{1}{k_{r-1}}\left|\left\{k \in I_2: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \\
&+ \cdots + \frac{1}{k_{r-1}}\left|\left\{k \in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \\
&= \frac{k_1}{k_{r-1}}.\frac{1}{h_1}\left|\left\{k \in I_1: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \\
&+ \frac{k_2 - k_1}{k_{r-1}}.\frac{1}{h_2}\left|\left\{k \in I_2: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| + \cdots \\
&+ \frac{k_r - k_{r-1}}{k_{r-1}}.\frac{1}{h_r}\left|\left\{k \in I_r: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \\
&= \frac{k_1}{k_{r-1}}.K_1 + \frac{k_2 - k_1}{k_{r-1}}.K_2 + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}.K_r \leq \left\{sup_{j \in C} K_j\right\}.\frac{k_r}{k_{r-1}} < M\delta.
\end{aligned}
$$

Choosing $\delta_1 = \frac{\delta}{M}$ and in view of the fact that $\bigcup\{n: k_{r-1} < n \leq k_r, r \in C\} \subset T$ where $C \in \mathcal{F}(I_\sigma)$.

It follows from our assumption on θ that the set T also belongs to $\mathcal{F}(I_\sigma)$ and this completes the proof of the theorem. Combining Theorem 2.9 and Theorem 2.10 we have,

Theorem 2.11 If $\theta = \{k_r\}$ be a lacunary sequence with $1 < liminf_r q_r < limsup_r q_r < \infty$, then

$$x_k \to L\left(S_{\sigma\theta}^A(I, F)\right) \Leftrightarrow x_k \to L\left(S(I_\sigma^A, F)\right).$$

Proof. This is an immediate consequence of Theorem 2.9 and Theorem 2.10.

Definition 2.10 The sequence $x = (x_k)$ is said to be strongly Cesàro I_σ -summable to L with respect to a sequence of modulus functions, if for each $\varepsilon > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}$$

belongs to I_σ. (denoted by $(x_k) \to L[C_1^A(I_\sigma, F)]$).

Definition 2.11 The sequence $x = (x_k)$ is said to be strongly λ_I -invariant convergent to L with respect to a sequence of modulus functions, if for each $\varepsilon > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}$$

belongs to I_σ, *where* $I_n = [n - \lambda_n + 1, n]$. (denoted by $(x_k) \to L\left(V_\lambda^A(I_\sigma, F)\right)$.

Theorem 2.12 If $(\mathrm{x_k}) \to \mathrm{L}\left(\mathrm{V}_\lambda^{\mathrm{A}}(I_\sigma, F)\right)$ is then $(x_k) \to L[C_1^A(I_\sigma, F)]$).

Proof Assume that $(\mathrm{x_k}) \to \mathrm{L}\left(\mathrm{V}_\lambda^{\mathrm{A}}(I_\sigma, F)\right)$ and $\varepsilon > 0$. Then,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) &= \frac{1}{n}\sum_{k=1}^{n-\lambda_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) + \frac{1}{n}\sum_{k \in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \\ &\leq \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) + \frac{1}{\lambda_n}\sum_{k \in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \\ &\leq \frac{2}{\lambda_n}\sum_{k \in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right)\end{aligned}$$

and so,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k \in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \frac{\varepsilon}{2}\right\} \in I_\sigma.$$

Hence $(x_k) \to L[C_1^A(I_\sigma, F)]$).

Definition 2.12 The sequence $\mathrm{x} = (\mathrm{x_k})$ is said to be $I_\sigma - \lambda$ statistically convergent to L with respect to a sequence of modulus functions, if for each $\varepsilon > 0$, for each $\delta > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\left|k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right| \geq \delta\right\}$$

belongs to I_σ. (denoted by $(x_k) \to L\left(S_\lambda^A(I_\sigma, F)\right)$.

Theorem 2.13 Let $\lambda = (\lambda_n)$ and I_σ is an admissible ideal in $\mathbb{N}$. If $(\mathrm{x_k}) \to \mathrm{L}\left(\mathrm{V}_\lambda^{\mathrm{A}}(\mathrm{I}_\sigma, \mathrm{F})\right)$, then $(x_k) \to L\left(S_\lambda^A(I_\sigma, F)\right)$.

Proof Assume that $(\mathrm{x_k}) \to \mathrm{L}\left(\mathrm{V}_\lambda^{\mathrm{A}}(I_\sigma, \mathrm{F})\right)$ *and* $\varepsilon > 0$. *Then,*

$$\sum_{k\in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \sum_{\substack{k\in I_n \\ f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)$$

$$\geq \varepsilon.\left|\left\{k\in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\right\}\right|$$

and so,

$$\frac{1}{\varepsilon.\lambda_n}\sum_{k\in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \frac{1}{\lambda_n}\left|\left\{k\in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\right\}\right|.$$

Then for any $\delta > 0$,

$$\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\left\{k\in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\subseteq\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon\delta\right\}.$$

Since right hand belongs to I_σ then left hand also belongs to I_σ and this completes the proof.

Theorem 2.14 Let $\lambda\in\Lambda$ and I_σ is an admissible ideal in $\mathbb{N}$. *If* (x_k) *is bounded and* $(x_k)\to L\left(S_\lambda^A(I_\sigma,F)\right)$ *then* $(x_k)\to L\left(V_\lambda^A(I_\sigma,F)\right)$.

Proof Let (x_k) is bounded sequence and $(x_k)\to L\left(S_\lambda^A(I_\sigma,F)\right)$. Then there is an M such that

$$f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\leq M,$$

for all k. *For each* $\varepsilon > 0$,

$$\frac{1}{\lambda_n}\sum_{k\in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)$$

$$=\frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)$$

$$+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)<\varepsilon}} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)$$

$$\leq M.\frac{1}{\lambda_n}\left|\left\{k\in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right)\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2}$$

Then,

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\} \in I_\sigma.$$

Therefore $(x_k) \to L\left(V_\lambda^A(I_\sigma, F)\right)$.

Theorem 2.15 If $liminf \frac{\lambda_n}{n} > 0$ then $(x_k) \to L\left(S^A(I_\sigma, F)\right)$ implies $(x_k) \to L\left(S_\lambda^A(I_\sigma, F)\right)$.

Proof Assume that $liminf \frac{\lambda_n}{n} > 0$ there exists a $\delta > 0$ such that $\frac{\lambda_n}{n} \geq \delta$ for sufficiently large *n*.

For given $\varepsilon > 0$ we have,

$$\frac{1}{n}\left\{k \leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\} \supseteq \frac{1}{n}\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}.$$

Therefore,

$$\frac{1}{n}\left|\left\{k \leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \frac{1}{n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{\lambda_n}{n} . \frac{1}{\lambda_n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta . \frac{1}{\lambda_n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right|$$

then for any η>0 we get

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \eta\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \eta\delta\right\}$$
$$\in I_\sigma$$

and this completes the proof.

Theorem 2.16 If $\lambda = (\lambda_n) \in \Delta$ be such that $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$, then $S_\lambda^A(I_\sigma, F) \subset S^A(I_\sigma, F)$.

Proof Let $\delta > 0$ be given. Since $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$, we can choose $M \in \mathbb{N}$ such that $\left|\frac{\lambda_n}{n} - 1\right| < \frac{\delta}{2}$, for all $n \geq m$. Now observe that, for $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k \leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| \\
&\qquad = \frac{1}{n}\left|\left\{k \leq n-\lambda_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| + \frac{1}{n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| \\
&\qquad \leq \frac{n-\lambda_n}{n} + \frac{1}{n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| \\
&\qquad \leq 1-\left(1-\frac{\delta}{2}\right) + \frac{1}{n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| \\
&\qquad = \frac{\delta}{2} + \frac{1}{n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right|,
\end{aligned}$$

for all $n \geq m$. Hence

$$\begin{aligned}
&\left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \leq n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \\
&\qquad \subset \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\left|\left\{k \in I_n: f_k\left(\left|A_k\left(x_{\sigma^k(m)}\right)-L\right|\right) \geq \varepsilon\right\}\right| \geq \frac{\delta}{2}\right\} \cup \{1,2,\ldots,m\}.
\end{aligned}$$

If (x_k) is $I_\sigma - \lambda$ statistically convergent to L, then the set on the right hand side belongs to I_σ and so the set on the left hand side also belongs to I_σ. This shows that (x_k) is I_σ -statistically convergent to L.

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