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Wijsman Rough Statistical Convergence on Triple Sequences

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Abstract

In this paper, using the concept of natural density, we introduce the notion of Wijsman rough statistical convergence of triple sequence. We define the set of Wijsman rough statistical limit points of a triple sequence spaces and obtain Wijsman statistical convergence criteria associated with this set. Later, we prove that this set is closed and convex and also examine the relations between the set of Wijsman rough statistical cluster points and the set of Wijsman rough statistical limit points of a triple sequences.

Keywords: Wijsman rough statistical convergence; natural density; triple sequence.

1. Introduction

The idea of statistical convergence was introduced by Steinhaus [17] and also independently by Fast [2] for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence. Let $\boldsymbol{K}$ be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) \in \boldsymbol{K}: \boldsymbol{m} \leq \boldsymbol{u}, \boldsymbol{n} \leq \boldsymbol{v}, \boldsymbol{k} \leq \boldsymbol{w}\}$ by $\boldsymbol{K}_{\boldsymbol{uvw}}$.

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Then the natural density of $\boldsymbol{K}$ is given by $\boldsymbol{\delta}(\boldsymbol{K}) = \boldsymbol{lim}_{uvw\to\infty} \frac{|K_{uvw}|}{uvw}$, where $|\boldsymbol{K}_{uvw}|$ denotes the number of elements in $\boldsymbol{K}_{uvw}$. Clearly, a finite subset has natural density zero, and we have $\boldsymbol{\delta}(\boldsymbol{K}^c) = \mathbf{1} - \boldsymbol{\delta}(\boldsymbol{K})$ where $\boldsymbol{K}^c = \mathbb{N}\backslash\boldsymbol{K}$ is the complement of $\boldsymbol{K}$. If $\boldsymbol{K}_1 \subseteq \boldsymbol{K}_2$, then $\boldsymbol{\delta}(\boldsymbol{K}_1) \leq \boldsymbol{\delta}(\boldsymbol{K}_2)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric $(\boldsymbol{X}, \boldsymbol{d})$. Consider a triple sequence $\boldsymbol{x} = (\boldsymbol{x}_{mnk})$ such that $\boldsymbol{x}_{mnk} \in \mathbb{R}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k} \in \mathbb{N}$.

A triple sequence $\boldsymbol{x} = (\boldsymbol{x}_{mnk})$ is said to be statistically convergent to $\mathbf{0} \in \mathbb{R}$, written as $\boldsymbol{st} - \boldsymbol{lim} \quad \boldsymbol{x} = \mathbf{0}$, provided that the set

$$\{(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) \in \mathbb{N}^3 : |\boldsymbol{x}_{mnk}, \mathbf{0}| \geq \boldsymbol{\varepsilon}\}$$

has natural density zero for any $\boldsymbol{\varepsilon} > \mathbf{0}$. In this case, $\mathbf{0}$ is called the statistical limit of the triple sequence $\boldsymbol{x}$.

If a triple sequence is statistically convergent, then for every $\boldsymbol{\varepsilon} > \mathbf{0}$, infinitely many terms of the sequence may remain outside the $\boldsymbol{\varepsilon} -$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $\boldsymbol{x} = (\boldsymbol{x}_{mnk})$ satisfies some property $\boldsymbol{P}$ for all $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}$ except a set of natural density zero, then we say that the triple sequence $\boldsymbol{x}$ satisfies $\boldsymbol{P}$ for almost all $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k})$ and we abbreviate this by a.a. $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k})$.

Let $\left(\boldsymbol{x}_{m_i n_j k_\ell}\right)$ be a sub sequence of $\boldsymbol{x} = (\boldsymbol{x}_{mnk})$. If the natural density of the set $\boldsymbol{K} = \left\{(\boldsymbol{m}_i, \boldsymbol{n}_j, \boldsymbol{k}_\ell) \in \mathbb{N}^3 : (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{\ell}) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(\boldsymbol{x}_{m_i n_j k_\ell}\right)$ is called a non thin subsequence of a triple sequence $\boldsymbol{x}$.

$\boldsymbol{c} \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $\boldsymbol{x} = (\boldsymbol{x}_{mnk})$ provided that the natural density of the set

$$\{(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) \in \mathbb{N}^3 : |\boldsymbol{x}_{mnk} - \boldsymbol{c}| < \boldsymbol{\varepsilon}\}$$

is different from zero for every $\boldsymbol{\varepsilon} > \mathbf{0}$. We denote the set of all statistical cluster points of the sequence $\boldsymbol{x}$ by $\boldsymbol{\Gamma}_x$.

A triple sequence $\boldsymbol{x} = (\boldsymbol{x}_{mnk})$ is said to be statistically analytic if there exists a positive number $\boldsymbol{M}$ such that

$$\boldsymbol{\delta}\left(\{(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) \in \mathbb{N}^3 : |\boldsymbol{x}_{mnk}|^{1/m+n+k} \geq \boldsymbol{M}\}\right) = \mathbf{0}$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [8], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal and his colleagues [7] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence. Let $(\boldsymbol{X}, \boldsymbol{\rho})$ be a metric space. For any non empty closed subsets $\boldsymbol{A}, \boldsymbol{A_{mnk}} \subset \boldsymbol{X}(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k} \in \mathbb{N}^{\boldsymbol{3}})$, we say that the triple sequence $(\boldsymbol{A_{mnk}})$ is wijsman statistical convergent to $\boldsymbol{A}$ is the triple sequence $\left(\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{A_{mnk}})\right)$ is statistically convergent to $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{A})$, i.e., for $\boldsymbol{\varepsilon} > \boldsymbol{0}$ and for each $\boldsymbol{x} \in \boldsymbol{X}$

$$\boldsymbol{lim_{rst}} \frac{\boldsymbol{1}}{\boldsymbol{rst}} |\{\boldsymbol{m} \leq \boldsymbol{r}, \boldsymbol{n} \leq \boldsymbol{s}, \boldsymbol{k} \leq \boldsymbol{t} : |\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{A_{mnk}}) - \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{A})| \geq \boldsymbol{\varepsilon}\}| = \boldsymbol{0}.$$

In this case, we write $\boldsymbol{St} - \boldsymbol{lim_{mnk}} \boldsymbol{A_{mnk}} = \boldsymbol{A}$ or $\boldsymbol{A_{mnk}} \to \boldsymbol{A}(\boldsymbol{WS})$. The triple sequence $(\boldsymbol{A_{mnk}})$ is bounded if $\boldsymbol{sup_{mnk}} \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{A_{mnk}}) < \infty$ for each $\boldsymbol{x} \in \boldsymbol{X}$.

In this paper, we introduce the notion of Wijsman rough statistical convergence of triple sequences. Defining the set of Wijsman rough statistical limit points of a triple sequence, we obtain to Wijsman statistical convergence criteria associated with this set. Later, we prove that this set of Wijsman statistical cluster points and the set of Wijsman rough statistical limit points of a triple sequence.

A triple sequence (real or complex) can be defined as a function $\boldsymbol{x}: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner and his colleagues [9,10]*, *Esi and his colleagues [2-4]*, *Datta and his colleagues [5]*,*Subramanian and his colleagues [11]*, *Debnath and his colleagues [6]* and many others.

Throughout the paper let $\boldsymbol{r}$ be a nonnegative real number.

2. Definitions and Preliminaries

2.1 Definition

A triple sequence $x = (x_{mnk})$ is said to be Wijsman $r-$ convergent to A denoted by $A_{mnk} \to^r A$, provided that

$$\forall \varepsilon > 0 \quad \exists (m_\varepsilon, n_\varepsilon, k_\varepsilon) \in \mathbb{N}^3 : m \geq m_\varepsilon, n \geq n_\varepsilon, k \geq k_\varepsilon \Rightarrow lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |d(x, A_{mnk}) - d(x, A)| < r + \varepsilon\}| = 0$$

The set

$$LIM^r A = \{L \in \mathbb{R}^3 : A_{mnk} \to^r A\}$$

is called the Wijsman $r-$ limit set of the triple sequences.

2.2 Definition

A triple sequence $x = (x_{mnk})$ is said to be Wijsman $r-$ convergent if $LIM^r A \neq \phi$. In this case, r is called the Wijsman convergence degree of the triple sequence $x = (x_{mnk})$. For $r = 0$, we get the ordinary convergence.

2.3 Definition

A triple sequence (x_{mnk}) is said to be Wijsman $r-$ statistically convergent to A, denoted by $A_{mnk} \rightarrow^{rst} A$, provided that the set

$$lim_{rst} \frac{1}{rst} |\{(m,n,k) \in \mathbb{N}^3 : |d(x, A_{mnk}) - d(x, A)| \geq r + \varepsilon\}| = 0$$

has natural density zero for every $\varepsilon > 0$, or equivalently, if the condition

$$st - lim \quad sup \quad |d(x, A_{mnk}) - d(x, A)| \leq r$$

is satisfied.

In addition, we can write $A_{mnk} \rightarrow^{rst} A$ if and only if the inequality

$$lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |d(x, A_{mnk}) - d(x, A)| < r + \varepsilon\}| = 0$$

holds for every $\varepsilon > 0$ and almost all (m,n,k). Here r is called the wijsman roughness of degree. If we take $r = 0$, then we obtain the ordinary Wijsman statistical convergence of triple sequence.

In a similar fashion to the idea of classic Wijsman rough convergence, the idea of Wijsman rough statistical convergence of a triple sequence spaces can be interpreted as follows:

Assume that a triple sequence $y = (y_{mnk})$ is Wijsman statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or Wijsman statistically approximated) triple sequence $x = (x_{mnk})$ satisfying $|d(x-y, A_{mnk}) - d(x-y, A)| \leq r$ for all m,n,k (or for almost all (m,n,k), i.e.,

$$\delta\left(lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |d(x-y, A_{mnk}) - d(x-y, A)| > r\}|\right) = 0.$$

Then the triple sequence x is not statistically convergent any more, but as the incluson

$$lim_{rst} \frac{1}{rst} \{|d(y, A_{mnk}) - d(y, A)| \geq \varepsilon\} \supseteq lim_{rst} \frac{1}{rst} \{|d(x, A_{mnk}) - d(x, A)| \geq r + \varepsilon\} \qquad (2.1)$$

holds and we have

$$\delta\left(lim_{rst}\frac{1}{rst}|\{(m,n,k)\in\mathbb{N}^3: |y_{mnk}-l|\geq\varepsilon\}|\right)=0,$$

i.e., we get

$$\delta\left(lim_{rst}\frac{1}{rst}|\{m\leq r, n\leq s, k\leq t: |d(x,A_{mnk})-d(x,A)|\geq r+\varepsilon\}|\right)=0,$$

i.e., the triple sequence spaces x is Wijsman $r-$ statistically convergent in the sense of definition (2.3)

In general, the Wijsman rough statistical limit of a triple sequence may not unique for the Wijsman roughness degree $r>0$. So we have to consider the so called Wijsman $r-$ statistical limit set of a triple sequence $x=(x_{mnk})$, which is defined by

$$st-LIM^r A_{mnk}=\{L\in\mathbb{R}: A_{mnk}\rightarrow^{rst} A\}.$$

The triple sequence x is said to be Wijsman $r-$ statistically convergent provided that $st-LIM^r A_{mnk}\neq\phi$. It is clear that if $st-LIM^r A_{mnk}\neq\phi$ for a triple sequence $x=(x_{mnk})$ of real numbers, then we have

$$st-LIM^r A_{mnk}=[st-lim \quad sup \quad A_{mnk}-r, st-lim \quad inf \quad A_{mnk}+r] \tag{2.2}$$

We know that $LIM^r=\phi$ for an unbounded triple sequence $x=(x_{mnk})$. But such a triple sequence might be Wijsman rough statistically convergent. For instance, define

$$d(x,A_{mnk})=\begin{Bmatrix}(-1)^{mnk}, & if\ (m,n,k)\neq(i,j,\ell)^2(i,j,\ell\in\mathbb{N}),\\ (mnk), & otherwise\end{Bmatrix}.$$

in $\mathbb{R}$. Because the set $\{1,64,739,\cdots\}$ has natural density zero, we have

$$st-LIM^r A_{mnk}=\begin{Bmatrix}\phi, & if \quad r<1,\\ [1-r,r-1], & otherwise\end{Bmatrix}$$

and $LIM^r A_{mnk}=\phi$ for all $r\geq 0$.

As can be seen by the example above, the fact that $st-LIM^r A_{mnk}\neq\phi$ does not imply $LIM^r A_{mnk}\neq\phi$. Because a finite set of natural numbers has natural density zero, $LIM^r A_{mnk}\neq\phi$ implies $st-LIM^r A_{mnk}\neq\phi$. Therefore, we get $LIM^r A_{mnk}\subseteq st-LIM^r A_{mnk}$. This obvious fact means $\{r\geq 0: LIM^r A_{mnk}\neq\phi\}\subseteq\{r\geq 0: st-LIM^r A_{mnk}\neq\phi\}$ in this language of sets and yields immediately

$$inf\{r\geq 0: LIM^r A_{mnk}\neq\phi\}\geq inf\{r\geq 0: st-LIM^r A_{mnk}\neq\phi\}.$$

Moreover, it also yields directly $diam(LIM^r A_{mnk})\leq diam(st-LIM^r A_{mnk})$.

3. Main Results

3.1 Theorem

For a Wijsman statistically triple sequence spaces $x = (x_{mnk})$, we have $diam(st - LIM^r A_{mnk}) \leq 2r$. In general $diam(st - LIM^r A_{mnk})$ has an upper bound.

Proof: Assume that $diam(st - LIM^r A_{mnk}) > 2r$. Then there exist $w, y \in st - LIM^r A_{mnk}$ such that $|w - y| > 2r$. Take $\varepsilon \in \left(0, \frac{|w-y|}{2} - r\right)$. Because $w, y \in st - LIM^r A_{mnk}$, we have $\delta(K_1) = 0$ and $\delta(K_2) = 0$ for every $\varepsilon > 0$ where

$$K_1 = lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(w, A)| \geq r + \varepsilon\}| = 0 \text{ and}$$

$$K_2 = lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(y, A)| \geq r + \varepsilon\}| = 0.$$

Using the properties of natural density, we get $\delta(K_1^c \cap \quad K_2^c) = 1$. Thus we can write

$$|w - y| \leq |d(x, A_{mnk}) - d(w, A)| + |d(x, A_{mnk}) - d(y, A)|$$

$$< 2(r + \varepsilon) = 2\left(\frac{|w - y|}{2}\right) = |w - y|$$

for all $(m, n, k) \in K_1^c \cap \quad K_2^c$, which is a contradiction.

Now let us prove the second part of the theorem. Consider a Wijsman triple sequence $x = (x_{mnk})$ such that $st - lim \quad A_{mnk} = A$. Let $\varepsilon > 0$. Then we can write

$$\delta\left(lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(x, A)| \geq \varepsilon\}|\right) = 0. \text{ We have}$$

$$|d(x, A_{mnk}) - d(y, A)| \leq |d(x, A_{mnk}) - d(x, A)| + |d(y, A) - d(y, A_{mnk})| \leq |d(x, A_{mnk}) - d(x, A)| + r$$

for each $y \in \bar{B}_r(A) = lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t: |d(y, A_{mnk}) - d(x, A)| \leq r\}|$.

Then we get $|d(y, A) - d(y, A_{mnk})| < r + \varepsilon$ for each

$(m, n, k) \in lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(x, A)| < \varepsilon\}| = 0$. Because the triple sequence spaces A_{mnk} is Wijsman statistically convergent to A, we have

$$\delta\left(lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(x, A)| < \varepsilon\}|\right) = 1.$$

Therefore we get $y \in st - LIM^r A_{mnk}$. Hence, we can write

$$st - LIM^r A_{mnk} = \bar{B}_r(A).$$

Because $diam(\bar{B}_r(A)) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $st - LIM^r A_{mnk}$ is not an lower bound.

3.2 Theorem

Let $r > 0$. Then a triple sequence $x = (x_{mnk})$ is Wijsman $r-$ statistically convergent to A if and only if there exists a triple sequence $y = (y_{mnk})$ such that $st - lim \quad A_{mnk} = A$ and $|d(x - y, A_{mnk}) - d(x - y, A)| \leq r$ for each $(m, n, k) \in \mathbb{N}^3$.

Proof: Necessity: Assume that $A_{mnk} \rightarrow^{rst} A$. Then we have

$$st - lim \quad sup \quad |d(x, A_{mnk}) - d(x, A)| \leq r. \qquad (3.1)$$

Now, define

$$d(y, A_{mnk}) = \begin{cases} A, \quad if \quad |d(x, A_{mnk}) - d(x, A)| \leq r, \\ d(x, A_{mnk}) + r\left(\dfrac{d(x, A) - d(x, A_{mnk})}{|d(x, A_{mnk}) - d(x, A)|}\right), \quad otherwise, \end{cases}$$

Then, we we write

$$|d(y, A_{mnk}) - d(y, A)| = \begin{cases} |A - A|, \quad if \quad |d(x, A_{mnk}) - d(x, A)| \leq r, \\ |d(x, A_{mnk}) - d(x, A)| + r\left(\dfrac{|A - A| - |d(x, A_{mnk}) - d(x, A)|}{|d(x, A_{mnk}) - d(x, A)|}\right), \quad otherwise, \end{cases}$$

$$\text{(i.e)} \ |d(y, A_{mnk}) - d(y, A)| = \begin{cases} 0, \quad if \quad |d(x, A_{mnk}) - d(x, A)| \leq r, \\ |d(x, A_{mnk}) - d(x, A)| - r\left(\frac{|d(x, A_{mnk}) - d(x, A)|}{|d(x, A_{mnk}) - d(x, A)|}\right), \quad otherwise, \end{cases}$$

$$\text{(i.e)} \ |d(y, A_{mnk}) - d(y, A)| = \begin{cases} 0, \quad if \quad |d(x, A_{mnk}) - d(x, A)| \leq r, \\ |d(x, A_{mnk}) - d(x, A)| - r, \quad otherwise. \end{cases}$$

We have $|d(y, A_{mnk}) - d(y, A)| \geq |d(x, A_{mnk}) - d(x, A)| - r \Rightarrow |d(x, A_{mnk}) - d(x, A) - d(y, A_{mnk}) + d(y, A)| \leq r$

$$|d(x - y, A_{mnk})| \leq r \qquad (3.2)$$

for all $m, n, k \in \mathbb{N}^3$. By equation (3.1) and by definition of y_{mnk}, we get $st - limsup|d(y, A_{mnk}) - d(y, A)| =$

0.

$\Rightarrow st - lim \quad A_{mnk} \to^r A.$

Sufficiency: Because $st - lim \quad A_{mnk} = A$, we have

$$\delta\left(lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(y, A_{mnk}) - d(y, A)| \ge \varepsilon\}|\right) = 0$$

for each $\varepsilon > 0$. It is easy to see that the inclusion

$$\{m \le r, n \le s, k \le t: |d(y, A_{mnk}) - d(y, A)| \ge \varepsilon\} \supseteq \{m \le r, n \le s, k \le t: |d(x, A_{mnk}) - d(x, A)| \ge r + \varepsilon\}$$

holds. Because $\delta\left(lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(y, A_{mnk}) - d(y, A)| \ge \varepsilon\}|\right) = 0$, we get

$$\delta\left(lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(x, A_{mnk}) - d(x, A)| \ge r + \varepsilon\}|\right) = 0.$$

3.3 Remark

If we replace the condition $|d(x - y, A_{mnk})| \le r$ for all $m, n, k \in \mathbb{N}^3$ in the hypothesis of the Theorem (3.2) with the condition

$$\delta\left(lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(x - y, A_{mnk})| > r\}|\right) = 0$$

is valid.

3.4 Theorem

For an arbitrary $c \in \Gamma_x$ of Wijsman statistically triple sequence $x = (x_{mnk})$ we have $|A - c| \le r$ for all $A \in st - LIM^r A_{mnk}$.

Proof: Assume on the contrary that there exist a point $c \in \Gamma_x$ and $A \in st - LIM^r A_{mnk}$ such that $|A - c| > r$. Define $\varepsilon := \frac{|A-c|-r}{3}$. Then

$$\{m \le r, n \le s, k \le t: |d(x, A) - c| < \varepsilon\} \subseteq \{m \le r, n \le s, k \le t: |d(x, A_{mnk}) - d(x, A)| \ge r + \varepsilon\}. \quad (3.3)$$

Since $c \in \Gamma_x$, we have

$$\delta\left(lim_{rst}\frac{1}{rst}\{m \le r, n \le s, k \le t: |d(x, A_{mnk}) - c| < \varepsilon\}\right) \ne 0.$$

Hence, by (3.3), we get

$\delta\left(lim_{rst}\frac{1}{rst}|\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(x, A)| \geq r + \varepsilon\}|\right) \neq 0,$

which contradicts the fact $A \in st - LIM^r A_{mnk}$.

3.5 Proposition

If a Wijsman statistically triple sequence $x = (x_{mnk})$ is analytic, then there exists a non-negative real number r such that $st - LIM^r A_{mnk} \neq \phi$.

Proof: If we take the Wijsman statistically triple sequence is to be Wijsman statistically analytic, then the of proposition holds. Thus we have the following theorem.

3.6 Theorem

A triple sequence $x = (x_{mnk})$ is Wijsman statistically analytic if and only if there exists a non-negative real number r such that $st - LIM^r A_{mnk} \neq \phi$.

Proof: Since the triple sequence x is Wijsman statistically analytic, there exists a positive real number M such that

$$\delta\left(lim_{rst}\frac{1}{rst}\left|\left\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk})|^{1/m+n+k} \geq M\right\}\right|\right) = 0.$$

Define

$$r' = sup\left\{|d(x, A_{mnk})|^{1/m+n+k}: (m, n, k) \in K^c\right\}, \text{ where}$$

$$K = lim_{rst}\frac{1}{rst}\left|\left\{m \leq r, n \leq s, k \leq t: |x_{mnk}|^{1/m+n+k} \geq M\right\}\right|.$$

Then the set $st - LIM^{r'} A_{mnk}$ contains the origin of $\mathbb{R}$. So we have $st - LIM^r A_{mnk} \neq \phi$.

If $st - LIM^r A_{mnk} \neq \phi$ for some $r \geq 0$, then there exists $d(x, A)$ such that $A \in st - LIM^r A_{mnk}$, i.e.,

$\delta\left(lim_{rst}\frac{1}{rst}\left|\left\{m \leq r, n \leq s, k \leq t: |d(x, A_{mnk}) - d(x, A)|^{1/m+n+k} \geq r + \varepsilon\right\}\right|\right) = 0.$

for each $\varepsilon > 0$. Then we say that almost all x_{mnk} are contained in some ball with any radius greater than r. So the triple sequence spaces x is Wijsman statistically analytic.

3.7 Remark

If $x' = \left(x_{m_i, n_j, k_\ell}\right)$ is a sub sequence of $x = (x_{mnk})$, then $LIM^r A_{mnk} \subseteq LIM^r A'_{mnk}$. But it is not valid for

Wijsman statistical convergence. For

Example: Define

$$d(x, A_{mnk}) = \begin{Bmatrix} (mnk), & if \quad (m,n,k) = (i,j,\ell)^2 (i,j,\ell \in \mathbb{N}), \\ 0, & otherwise \end{Bmatrix}$$

of real numbers. Then the triple sequence spaces $x' = (1,64,739,\cdots)$ is a sub sequence of x. We have $st - LIM^r A_{mnk} = [-r, r]$ abd $st - LIM^r A'_{mnk} = \phi$.

3.8 Theorem

Let $x' = \left(x_{m_i, n_j, k_\ell}\right)$ is a non thin sub sequence of Wijsman statistically triple sequence spaces $x = (x_{mnk})$, then $st - LIM^r A_{mnk} \subseteq st - LIM^r A'_{mnk}$.

Proof: Omitted.

3.9 Theorem

The Wijsman $r-$ statistical limit set of a triple sequence $x = (x_{mnk})$ is closed.

Proof: If $st - LIM^r A_{mnk} \neq \phi$, then it is true. Assume that $st - LIM^r A_{mnk} \neq \phi$, then we can choose a triple sequence spaces $(y_{mnk}) \subseteq st - LIM^r A_{mnk}$ such that $d(y, A_{mnk}) \to^r d(y, A)$ as $m, n, k \to \infty$. If we prove that $A \in st - LIM^r A_{mnk}$, then the proof will be complete.

Let $\varepsilon > 0$ be given. Because $d(y, A_{mnk}) \to^r d(y, A), \exists \left(m_{\frac{\varepsilon}{2}}, n_{\frac{\varepsilon}{2}}, k_{\frac{\varepsilon}{2}}\right) \in \mathbb{N}^3$ such that

$|d(y, A_{mnk}) - d(y, A)| < \frac{\varepsilon}{2}$ for all $m > m_{\frac{\varepsilon}{2}}, n > n_{\frac{\varepsilon}{2}}, k > k_{\frac{\varepsilon}{2}}$.

Now choose an $(m_0, n_0 k_0) \in \mathbb{N}^3$ such that $m_0 > m_{\frac{\varepsilon}{2}}, n_0 > n_{\frac{\varepsilon}{2}}, k_0 > k_{\frac{\varepsilon}{2}}$. Then we can write

$\left|d\left(y, A_{m_0 n_0 k_0}\right) - d(y, A)\right| < \frac{\varepsilon}{2}$.

On the other hand, because $(y_{mnk}) \subseteq st - LIM^r A_{mnk}$, we have $y_{m_0 n_0 k_0} \in st - LIM^r A_{mnk}$, namely,

$$\delta \left(lim_{rst} \frac{1}{rst} \left|\left\{m \leq r, n \leq s, k \leq t : \left|d(x, A_{mnk}) - d\left(y, A_{m_0 n_0 k_0}\right)\right| \geq r + \frac{\varepsilon}{2}\right\}\right|\right) = 0. \quad (3.4)$$

Now let us show that the inclusion

$$\{|d(x, A_{mnk}) - d(x, A)| < r + \varepsilon\} \supseteq \left\{\left|d(x, A_{mnk}) - d\left(y, A_{m_0 n_0 k_0}\right)\right| < r + \frac{\varepsilon}{2}\right\} \quad (3.5)$$

holds. Take $(i,j,\ell) \in \left\{m \le r, n \le s, k \le t: \left|d(x,A_{mnk}) - d\left(y, A_{m_0 n_0 k_0}\right)\right| < r + \frac{\varepsilon}{2}\right\}$. Then we have

$$\left|d(x,A_{mnk}) - d\left(y, A_{m_0 n_0 k_0}\right)\right| < r + \frac{\varepsilon}{2}$$

and hence

$$\left|d\left(x,A_{ij\ell}\right) - d(y,A)\right| \le \left|d\left(x,A_{ij\ell}\right) - d\left(y,A_{m_0 n_0 k_0}\right)\right| + \left|d\left(y,A_{m_0 n_0 k_0}\right) - d(y,A)\right| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < r + \varepsilon$$

i.e., $(i,j,\ell) \in \{m \le r, n \le s, k \le t:: |d(x,A_{mnk}) - d(x,A)| < r + \varepsilon\}$ which proves the equation (3.5). Hence the natural density of the set on the LHS of equation (3.5) is equal to 1. So we get $\delta\left(lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(x,A_{mnk}) - d(x,A)| \ge r + \varepsilon\}|\right) = 0$.

3.10 Theorem

The Wijsman $r-$ statistical limit set of a triple sequence is convex.

Proof: Let $A_1, A_2 \in st - LIM^r A_{mnk}$ for the triple sequence $x = (x_{mnk})$ and let $\varepsilon > 0$ be given. Define $K_1 = lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(x,A_{mnk}) - d(x,A_1)| \ge r + \varepsilon\}| = 0$ and

$K_2 = lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(x,A_{mnk}) - d(x,A_2)| \ge r + \varepsilon\}| = 0$. Because $A_1, A_2 \in st - LIM^r A_{mnk}$, we have $\delta(K_1) = \delta(K_2) = 0$. Thus we have

$|d(x,A_{mnk}) - [(1-\lambda)d(x,A_1) + \lambda d(x,A_2)]| = \left|(1-\lambda)\left(d(x,A_{mnk}) - d(x,A_1)\right) + \lambda\left(d(x,A_{mnk}) - d(x,A_2)\right)\right| < r + \varepsilon$, for each $(m,n,k) \in (K_1^c \cap K_2^c)$ and each $\lambda \in [0,1]$. Because $\delta(K_1^c \cap K_2^c) = 1$, we get

$$\delta\left(lim_{rst}\frac{1}{rst}|\{m \le r, n \le s, k \le t: |d(x,A_{mnk}) - [(1-\lambda)d(x,A_1) + \lambda d(x,A_2)]| \ge r + \varepsilon\}|\right) = 0,$$

i.e., $[(1-\lambda)d(x,A_1) + \lambda d(x,A_2)] \in st - LIM^r A_{mnk}$, which proves the convexity of the set $st - LIM^r A_{mnk}$.

3.11 Theorem

A triple sequence $x = (x_{mnk})$ of Wijsman statistically converges to A if and only if $st - LIM^r\left(d(x,A_{mnk})\right) = \bar{B}_r\left(d(x,A)\right)$.

Proof: We have proved the necessity part of this theorem in proof of the Theorem (3.1).

Sufficiency: Because $st - LIM^r\left(d(x,A_{mnk})\right) = \bar{B}_r\left(d(x,A)\right) \neq \phi$, then by Theorem (3.5) we can say that the triple sequence spaces $d(x,A_{mnk})$ is Wijsman statistically analytic. Assume on the contrary that the triple sequence spaces $\left(d(x,A)\right)$ has another Wijsman statistical cluster point $\left(d(x,A)\right)'$ different from $d(x,A)$. Then the point

$$\left(d(x,A)\right) = d(x,A) + \frac{r}{\left|d(x,A) - \left(d(x,A)\right)'\right|}\left(d(x,A) - \left(d(x,A)\right)'\right)$$

satisfies

$$\left(d(x,A)\right) - \left(d(x,A)\right)' = d(x,A) - \left(d(x,A)\right)' + \frac{r}{\left|d(x,A) - \left(d(x,A)\right)'\right|}\left(d(x,A) - \left(d(x,A)\right)'\right)$$

$$\left|\left(d(x,A)\right) - \left(d(x,A)\right)'\right| = \left|d(x,A) - \left(d(x,A)\right)'\right| + \frac{r}{\left|d(x,A) - \left(d(x,A)\right)'\right|}\left(d(x,A) - \left(d(x,A)\right)'\right)$$

$$\left|\bar{d}(x,A) - \left(d(x,A)\right)'\right| = \left|d(x,A) - \left(d(x,A)\right)'\right| + r > r.$$

Because $\left(d(x,A)\right)'$ is a Wijsman statistical cluster point of the triple sequence spaces $d(x,A_{mnk})$, by Theorem (2.4) this inequality implies that $\bar{d}(x,A) \notin st - LIM^r\left(d(x,A)\right)$. This contradicts the fact $\left|\bar{d}(x,A) - d(x,A)\right| = r$ and $st - LIM^r\left(d(x,A)\right) = \bar{B}_r\left(d(x,A)\right)$. Therefore, $d(x,A)$ is the unique Wijsman statistical cluster point of the triple sequence spaces $d(x,A_{mnk})$. Hence the Wijsman statistical cluster point of a Wijsman statistically analytic triple sequence spaces is unique, then the triple sequence spaces $d(x,A_{mnk})$ is wijsman statistically convergent to $d(x,A)$.

3.12 Theorem

Let $(\mathbb{R}^3, |.,.|)$ be a strictly convex space and $x = (x_{mnk})$ be a triple sequence spaces if there exist $y_1, y_2 \in st - LIM^r\left(d(x,A)\right)$ such that $|d(y_1,A) - d(y_2,A)| = 2r$ then this triple sequence is Wijsman statistically convergent to $\frac{1}{2}\left(d(y_1,A) + d(y_2,A)\right)$.

Proof: Assume that $z \in \Gamma_{(d(x,A_{mnk}))}$. Then $y_1, y_2 \in st - LIM^r\left(d(x,A_{mnk})\right)$ implies that

$$\left|\left(d(y_1,A)\right) - z\right| \leq r \quad and \quad \left|\left(d(y_2,A)\right) - z\right| \leq r, \qquad (3.6)$$

by Theorem 3.4. On the other hand, we have

$$2r = \left|\left(d(y_1,A)\right) - \left(d(y_2,A)\right)\right| \leq \left|\left(d(y_1,A)\right) - z\right| + \left|\left(d(y_2,A)\right) - z\right| \qquad (3.7)$$

combining the inqualities (3.6) and (3.7), we get $\left|\left(d(y_1,A)\right) - z\right| = \left|\left(d(y_2,A)\right) - z\right| = r$. Because

$$\frac{1}{2}\left(\left(d(y_2,A)\right) - \left(d(y_1,A)\right)\right) = \frac{1}{2}\left[\left(z - \left(d(y_1,A)\right)\right) + \left(-z + \left(d(y_2,A)\right)\right)\right] \qquad (3.8)$$

and $\left|\left(d(y_1,A)\right)-\left(d(y_2,A)\right)\right|=2r$, we get $\left|\frac{1}{2}\left(\left(d(y_2,A)\right)-\left(d(y_1,A)\right)\right)\right|=r$. By the strict convexity of the space and from the equality 3.8, we get

$\frac{1}{2}\left(\left(d(y_2,A)\right)-\left(d(y_1,A)\right)\right)=\left(z-\left(d(y_1,A)\right)\right)=\left(-z+\left(d(y_2,A)\right)\right)$ which implies that $z=\frac{1}{2}\left(\left(d(y_1,A)\right)+\left(d(y_2,A)\right)\right)$. Hence z is the unique Wijsman statistical cluster point of the triple sequence spaces $\left(d(x,A_{mnk})\right)$.

On the other hand, the assumption $y_1,y_2\in st-LIM^r\left(d(x,A_{mnk})\right)$ implies that $st-LIM^r x\neq 0$. By Theorem 3.6, the triple sequence $\left(d(x,A_{mnk})\right)$ is statistically analytic. Consequently, the statistical cluster point of a Wijsman statistically analytic triple sequence spaces is unique, then the triple sequence spaces is Wijsman statistically convergent, i.e., $st-lim\ \left(d(x,A_{mnk})\right)=\frac{1}{2}\left(\left(d(y_1,A)\right)+\left(d(y_2,A)\right)\right)$.

3.13 Theorem

(a) If $c\in\Gamma_x$ then

$$st-LIM^r\left(d(x,A_{mnk})\right)\subseteq\bar{B}_r(c) \quad (3.9)$$

(b)

$$st-LIM^r\left(d(x,A_{mnk})\right)=\cap_{c\in\Gamma_x}\bar{B}_r(c)=\left\{\left(d(x,A)\right)\in\mathbb{R}^3:\Gamma_x\subseteq\bar{B}_r\left(\left(d(x,A)\right)\right)\right\} \quad (3.10)$$

Proof: (a) Assume that $\left(d(x,A)\right)\in st-LIM^r\left(d(x,A_{mnk})\right)$ and $c\in\Gamma_x$. Then by Theorem 3.4, we have

$\left|\left(d(x,A)\right)-c\right|\leq r;$

other wise we get

$\delta\left(lim_{rst}\frac{1}{rst}\left\{m\leq r,n\leq s,k\leq t:\left|\left(d(x,A_{mnk})\right)-\left(d(x,A)\right)\right|\geq r+\varepsilon\right\}\right)\neq 0$

for $\varepsilon=\frac{\left|\left(d(x,A)\right)-c\right|-r}{3}$. This contradicts the fact $\left(d(x,A)\right)\in st-LIM^r\left(d(x,A_{mnk})\right)$.

(b) By the equation (3.9), we can write

$$st-LIM^r\left(d(x,A_{mnk})\right)\subseteq\cap_{c\in\Gamma_x}\bar{B}_r(c). \quad (3.11)$$

Now assume that $A\in\cap_{c\in\Gamma_x}\bar{B}_r(c)$. Then we have

$\left|\left(d(y,A)\right)-c\right|\leq r$

for all $c \in \Gamma_x$, which is equivalent to $\Gamma_x \subseteq \bar{B}_r\left(\left(d(y,A)\right)\right)$, i.e.,

$$\cap_{c \in \Gamma_x} \bar{B}_r(c) \subseteq \left\{\left(d(y,A)\right) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_r\left(\left(d(y,A)\right)\right)\right\}. \quad (3.12)$$

Now let $\left(d(y,A)\right) \notin st - LIM^r\left(d(x,A_{mnk})\right)$. Then there exists an $\varepsilon > 0$ such that

$$\delta\left(lim_{rst} \frac{1}{rst}\left\{m \leq r, n \leq s, k \leq t : \left|\left(d(x,A)\right) - \left(d(y,A)\right)\right| \geq r + \varepsilon\right\}\right) \neq 0$$

the existence of a Wijsman statistical cluster point c of the triple sequence spaces $\left(d(x,A_{mnk})\right)$ with $\left|\left(d(y,A)\right) - c\right| \geq r + \varepsilon$, i.e., $\Gamma_x Ú \bar{B}_r\left(\left(d(y,A)\right)\right)$ and

$$y \notin \left\{l \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_r\left(\left(d(x,A)\right)\right)\right\}.$$

Hence $y \in st - LIM^r\left(d(x,A_{mnk})\right)$ follows from $y \in \left\{\left(d(x,A)\right) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_r\left(\left(d(x,A)\right)\right)\right\}$, i.e.,

$$\left\{\left(d(x,A)\right) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_r\left(\left(d(x,A)\right)\right)\right\} \subseteq st - LIM^r\left(d(x,A_{mnk})\right). \quad (3.13)$$

Therefore the inclusions (3.11)-(3.13) ensure that (3.10) holds.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

References

[1] S. Aytar 2008. Rough statistical Convergence, Numerical Functional Analysis Optimization, 29(3), 291-303.

[2] A. Esi , On some triple almost lacunary sequence spaces defined by Orlicz functions, Research and Reviews:Discrete Mathematical Structures, 1(2), (2014), 16-25.

[3] A. Esi and M. Necdet Catalbas,Almost convergence of triple sequences, Global Journal of Mathematical Analysis, 2(1), (2014), 6-10.

[4] A. Esi and E. Savas, On lacunary statistically convergent triple sequences in probabilistic normed space,Appl.Math.and Inf.Sci., 9 (5) , (2015), 2529-2534.

[5] A. J. Datta A. Esi and B.C. Tripathy,Statistically convergent triple sequence spaces defined by Orlicz function , Journal of Mathematical Analysis, 4(2), (2013), 16-22.

[6] S. Debnath, B. Sarma and B.C. Das ,Some generalized triple sequence spaces of real numbers , Journal

of nonlinear analysis and optimization, Vol. 6, No. 1 (2015), 71-79.

[7] S.K. Pal, D. Chandra and S. Dutta 2013. Rough ideal Convergence, Hacee. Jounral Mathematics and Statistics, 42(6), 633-640.

[8] H.X. Phu 2001. Rough convergence in normed linear spaces, Numerical Functional Analysis Optimization, 22, 201-224.

[9] A. Sahiner, M. Gurdal and F.K. Duden, Triple sequences and their statistical convergence, Selcuk J. Appl. Math. , 8 No. (2)(2007), 49-55.

[10] A. Sahiner, B.C. Tripathy , Some I related properties of triple sequences, Selcuk J. Appl. Math. , 9 No. (2)(2008), 9-18.

[11] N. Subramanian and A. Esi, The generalized tripled difference of χ^3 sequence spaces, Global Journal of Mathematical Analysis, 3 (2) (2015), 54-60.