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Evaluating Value-at-Risk in BIST Using Copula Approach

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Abstract

Modelling dependence structure between variables is commonly investigated in literature. A large variety of methods have been improved in recent times. One of the methods is the copula which models accurately dependency regardless of marginal distributions. In this paper Value-at-Risk (VaR) is computed using the copulas. It is assumed that the dependency does not vary through time since small time interval is used. The study composes of two steps. In the first step the best fitted copula is determined by ML (Maximum likelihood). In the second step equal-weighted portfolio analysis is performed by joint distribution function obtained from the copula and maximum possible losses of the portfolio are evaluated. The main challenge in portfolio analysis is that joint distribution function for stocks cannot be correctly constructed by considering dependence structure among them. We obtain joint distribution function for stocks using the copula approach that has been commonly used in recent times. After the best fitted copula is determined using the criterions such as AIC (Akaike information criterion) and SBC (Schwarz's Bayesian Criterion), next-day maximum possible losses for the portfolio are evaluated by means of equal weighted portfolio technique.

Keywords: Copula; Dependence Structure; Stock Exchange; Value-at-Risk.

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1. Introduction

Copula, which means linking or relating, was first revealed statistically and mathematically by Sklar [1]. Copulas construct a link between multivariate distribution function and its marginals. Using the copulas provides a significant flexibly when dependence is considered. Multivariate distribution function composes of both marginals and dependence structure referring to random variables.

In modelling multivariate distributions, the copula approach provides a method to decompose dependence structure from marginal distributions. In general sense, copula is a function that redefine joint distribution function $I^2 \rightarrow I$ by means of marginal distribution functions when random variables are dependent. Copulas have played crucial role in statistics field including probability and Markov process, nonparametric distributions and multivariate distribution theory. Besides, copulas have theoretically a crucial position in statistics since it is an important tool in constructing bivariate or multivariate distribution families when marginal distributions are given. The authors in [2] suggest an estimator for distribution parameter based on Kendall distribution in Archimedean copulas. Eventually they compare their results with the authors in [3] estimation method on simulated data. The authors in [4] show that random variable vector having an Archimedean copula can be written as a product of simplex vector and radial variable. Taylor [5] compiles axioms that require to proving multivariate concordance measures. The authors in [3] suggest new estimators based on order statistics for multivariate Archimedean copulas. Berger [6] characterizes the effect of the basis copula on the risk analysis with special attention to Value-at-Risk (VaR) estimates. The authors in [7] suggest an Empirical Mode Decomposition Copula for evaluating the portfolio risk and forecasting Value-at-Risk (VaR). The authors in [8] utilize a serial dependence structure of financial assets to forecast flexibly risk values by means of pair-copula-construction (PCC). The copula approach constructs accurately a new joint distribution by considering dependence structure between variables. In this paper Value-at-Risk (VaR) is evaluated using the copula. This overcomes to difficulty of constructing joint distribution function for the portfolio. Five stocks in Istanbul Stock Exchange are taken to evaluate the Value-at-Risk. Uniform inputs are needed since the copula is defined in interval [0, 1]. For this purpose marginals are transformed into uniform by means of ECDF (Empirical Cumulative Distribution Function). Next the best fitted copula is determined by information criterion such as AIC and SBC. Finally, next-day maximum possible losses for the portfolio are evaluated using the joint distribution function constructed by means of the copula.

2. Theory of Copulas

2.1. Definition of Copula

Let $u = (u_1, u_2, \dots, u_n)$ with $u_i \in [0,1]$ be a vector of n-variables. A function $C(u)$: $[0,1]^n \rightarrow [0,1]$ is called as an *n*-dimensional copula if and only if it holds conditions as follows:

- $C(u) = u_k$ if all coordinates of u are 1, except u_k ;
- $C(u) = 0$ if at least one of the coordinates of u is zero;
- C is increasing in each coordinate of u.

2.2. Sklar's Theorem

According to Sklar's theorem, a joint cumulative distribution function H of random variables $\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_\mathrm{n}$, with continuous marginal distributions $F_1, F_2, \dots, F_n$ respectively can be qualified by a single *n*-dimensional dependency function or copula *C*, such that for all vectors $x \epsilon \overline{R^n}$:

$$H(x_1, x_2, \dots, x_n) = C\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big)$$

$$= P\{X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n\} \qquad (1)$$

Copula function C is uniquely defined if $F_1, F_2, \dots, F_n$ are continuous. In this case univariate marginals can be separated from the multivariate dependence structure represented by a copula. However it is not possible to assume that copula of Eq(1) is unique if $F_1, F_2, \dots, F_n$ are discrete. As an alternative representation, Eq(1) can be written in the inverted form. For any vector $u \ \epsilon \ [0,1]^n$:

$$C(u_1, u_2, \dots, u_n) = H\big(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_n^{-1}(u_n)\big) \qquad (2)$$

Where C denotes the copula associated with H and $H^{-1}(u) = \inf\{x\epsilon\mathbb{R} | F_i(x_i) \geq u\}$, $i = 1, \dots, n$, constitutes generalized inverse function of F. Then $x_1 = F_1^{-1}(u_1) \sim F_1, \dots, x_n = F_n^{-1}(u_n) \sim F_n$. I follows that an *n*-copula is a multivariate distribution with all *n* univariate margins distributed as uniform [9].

2.3. Definition

Under Eq.(1), h joint probability density function can be described as *n*-th order derivative of a copula:

$$h(x_1, x_2, \dots, x_n) = \frac{\partial^n C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))}{\partial F_1(x_1) \dots \partial F_n(x_n)} \prod_{i=1}^{n} f_i(x_i)$$

$$= c\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big) \prod_{i=1}^{n} f_i(x_i) \qquad (3)$$

Where F_i, f_i are marginal distribution and marginal density function, respectively. There are a large variety of copulas in literature. In this section crucial properties of elliptical copulas and Archimedean copula families are introduced.

2.4. Elliptical Copulas

In this subsection two elliptical copulas are outlined with important properties.

2.4.1. Normal Copula

Let $u(u_1, u_2, \dots, u_k) \sim \mathrm{U}(0,1)$ where $\mathrm{U}(0,1)$ is known as uniform distribution. Let Σ be correlation matrix with k(k-1)/2 parameters holding the positive semi-definiteness restriction. The normal copula is written as follows:

$$C_{\Sigma}(u_1, u_2, \dots, u_k) = \varphi_{\Sigma}\left(\varphi^{-1}(u_1), \dots, \varphi^{-1}(u_k)\right) \tag{4}$$

where φ is called as distribution function of a standard normal random variable and φ_{Σ} is k-variate standard normal distribution such that its mean vector is 0 and its covariance matrix is Σ. In other words, the distribution φ_{Σ} is $N_k(0, \Sigma)$.

2.4.2. Student's T Copula

Let $\omega = \{(v, \Sigma) \colon v \in (1, \infty),\ \Sigma \in \mathrm{R}^{kxk}\}$ and let t_v be a univariate t distribution. Where v represents the degress of freedom that is parameter of t distribution. The Student's t copula can be expressed as follows:

$$C_{\omega}(u_1, u_2, \dots, u_k) = t_{v,\Sigma}\left(t_v^{-1}(u_1), t_v^{-1}(u_2), \dots, t_v^{-1}(u_k)\right) \tag{5}$$

Where $t_{v,\Sigma}$ is the multivariate Student's t distribution. Σ that is correlation matrix and v being degrees of freedom are the parameters of the distributions.

2.5. Archimedean Copulas

Archimedean copulas are the most important class in copulas. "Archimedean" term was first used in 1965. Most of copulas are Archimedean and Archimedean copulas have a large variety and different dependence structure. In contrast to most of copula functions, these copulas are not constructed by using Sklar's Theorem. Archimedean copulas have a great numbers of application fields due to easily constructing, existing many copula families and having a broad range of features pertaining to the class. For detailed information on Archimedean copulas, Joe [10] and Nelsen [11] can be viewed. Archimedean copulas that are private class of copula family represent a variety of family. Some of these are shown below.

Table 1: Clayton copula family and features

Copula	$\boldsymbol{C_{\theta}(u,v) = max([u^{-\theta} + v^{-\theta} - 1]^{\frac{-1}{\theta}}, 0)}$	$\boldsymbol{\theta \in [-1, \infty) - \{0\}}$
Generator Function	$\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$	
Feature	It is an asymmetric copula family. Lower tail exhibits larger dependence than upper tail.	

Table 2: Frank copula family and features

Copula	$\boldsymbol{C_{\theta}(u,v) = -\frac{1}{\theta} ln\left[1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{(e^{-\theta} - 1)}\right]}$	$\boldsymbol{\theta \in (-\infty, \infty) - \{0\}}$
Generator Function	$\varphi(t) = -\ln\frac{\mathrm{e}^{-\theta t} - 1}{\mathrm{e}^{-\theta} - 1}$	
Feature	It is a symmetric Archimedean copula. It doesn't exhibit tail dependence.	

Table 3: Gumbel copula family and features

Copula	$\boldsymbol{C_\theta(u,v) = exp\left\{-[(-lnu)^\theta + (-lnv)^\theta]^{\frac{1}{\theta}}\right\}}$	$\boldsymbol{\theta \in [1,\infty)}$
Generator Function	$\varphi(t) = (-lnt)^\theta$	
Feature	It is an asymmetric Archimedean copula. Upper tail exhibits larger dependence than lower tail.	

2.6. Estimation

Portfolio analysis based on the copula composes of two steps. In the first step to calculate maximum loses of portfolio, the marginals of each stock are determined. For this purpose ECDF (empirical cumulative distribution function) can be used. In the second step, joint distribution function can be estimated via the copula. Copulas are functions that are defined in interval$[\mathbf{0},\mathbf{1}]$. Before estimating the parameter of copulas, uniform inputs are needed to obtain. To this end, observations are transformed into uniform distribution by implementing ECDF (empirical cumulative distribution function). In this paper, copulas with one parameter are examined. Copulas used for this aim are elliptical copulas e.g. Gaussian, Student's t and Archimedean copulas e.g. Clayton, Frank, Gumbel. After uniform inputs are obtained, parameter estimation is implemented by ML (Maximum Likelihood). The marginal distributions of vector elements $\boldsymbol{u_i = (u_{i1}, u_{i2}, \ldots, u_{im})}$ $\boldsymbol{i = 1, \ldots, n}$ are uniform inputs. Then the parameter $\boldsymbol{\theta}$ is estimated by maximum likelihood:

$$\theta = arg\max_{\theta\epsilon\Theta}\sum_{i=1}^{n} log\, c(u_{i1}, u_{i2}, \ldots, u_{im}; \theta) \tag{6}$$

In the next step new samples for each daily returns are simulated by means of the fitted joint distribution. After these assets are unified with a portfolio and its daily returns are evaluated. In the final step quantiles of simulated portfolio returns are investigated.

3. The Data and Empirical Results

3.1. The Data Description

The data set is composed of five stocks that increased values in İstanbul stock exchange: Goodyear tire & rubber company (GOODY), Doğuş automotive service and trade joint stock company (DOAS), Pegasus airlines company (PGSUS), TAV airports holding (TAVHL), Turkish airlines corporation (THYAO). The sample begins in 06/05/2014 to 11/10/2016 giving us 611 observations. Asset portfolio is created from this five stock obtained from İstanbul stock exchange. In this paper, Value-at-risk (VaR) of the portfolio including five stocks: GOODY, DOAS, PGSUS, TAVHL and THYAO are examined via the copula approach. Value-at-Risk (VaR) is a computation used in financial risk evaluation. The goal of this computation is to present some numerical perception to the risk level of an asset portfolio. This computation gives the maximum potential losses at a certain confidence level. The target of this study is to estimate the one-day later maximum potential losses for the portfolio of stocks constructed with the copula approach. In this sense one of approaches used for the estimation is to assume that the joint distribution of asset returns does not vary during time. In this case results are so close to truth if and only if a small time interval is utilized. Stocks interested in this study have increased in recent times. So, the main result expected is how many maximum loses of investors can be when they buy these stocks for the portfolio.

Table 4: Descriptive statistics of the return series

	Goody	Doas	Pgsus	Tavhl	Thyao
Mean	0,000796	0,00031	-0,001446	-0,000333	-0,000376
SD	0,023914	0,02109	0,016776	0,016975	0,016826
Skewness	1,336	-1,0833	-0,3403	-0,6057	-0,7531
Kurtosis	16,1955	7,8681	6,5055	8,2171	10,4639
Min	-0,1204	-0,1124	-0,0939	-0,1186	-0,1237
5%	-0,032	-0,0316	-0,0284	-0,0269	-0,0274
25%	-0,0107	-0,009	-0,01	-0,0092	-0,0096
Median	0	0,0012854	-0,0009804	0	0,0008754
75%	0,009	0,0119	0,0085	0,0089	0,009
95%	0,0345	0,0302	0,0249	0,0259	0,0233
Observations	610	610	610	610	610

Table 4 gives descriptive statistics of the series. Non-normality of the data series is clear from the values for skewness and kurtosis.

In this case, the copula approach provides an efficient method to model the dependency among variables. Thus, joint distribution function obtained by means of copulas gives more robust results concerning the portfolio. Gaussian copula model is ignored due to non-normality of the data.

3.2. Empirical Results

In this section, only four copulas are investigated due to nonlinearity of the data. These copulas are Student's T from elliptical copula, Frank, Clayton and Gumbel from Archimedean copulas. Estimated results for these copulas are demonstrated in Table 5.

The choice of the best fitted copula model is based on Akaike information criterion (AIC) and Schwarz's Bayesian Criterion (SBC). The results obtained show that the best fitted copulas for the returns of five stocks is Student's T copula. AIC and SBC values of the copula are -976.4080 and -927.8780, respectively.

Table 5: The estimated results for copulas

Copulas	Parameter	SE	AIC	SBC
Student's T	10,5546	1,9961	-976,408	-927,878
Frank	2,5678	0,1191	-623,8091	-619,3973
Clayton	0,5933	0,0286	-723,7875	-719,3756
Gumbel	1,3402	0,0195	-585,004	-580,5922

The correlation matrix for the best fitted copula is given in Table 6. The results show that correlations between returns of the stocks are rather different. Under all samples there are positive correlations among returns. The strongest correlation is between Pgsus and Thyao. However the correlation between Goody and Doas is the lowest with respect to the best fitted copula. Plots of the returns and transform returns are given in Figure 1. While the returns that have the strongest dependence show denser diffusion along the diagonal, the returns of Doas and Goody that have the lowest dependence exhibit less denser diffusion along it. Dependence path of the best fitted copula are showed in Figure 2. This implies that data simulated from real data represents the same pattern with the returns. In this sense the estimations obtained from the best fitted copula give more robust results. Plots for other copulas are presented in annex (Figure A1. in Annex). After examined the correlation matrix, next step is to simulate data from the best fitted copula to construct the portfolio concerning five stocks. The number of the simulation can be taken the value of 12000.

Table 6: Correlations between returns for the whole sample

	Goody	Doas	Pgsus	Tavhl	Thyao
Goody	1	0,29963	0,43151	0,3385	0,44601
Doas	0,29963	1	0,44102	0,31492	0,50803
Pgsus	0,43151	0,44102	1	0,43052	0,74103
Tavhl	0,3385	0,31492	0,43052	1	0,46039
Thyao	0,44601	0,50803	0,74103	0,46039	1

Table 7: Descriptive statistics of the created portfolio

Statistics	Value
Mean	-0,00032
SD	0,01625
Variance	0,00026
Median	0,00016
Range	0,30149
Skewness	-1,85993
Kurtosis	21,0503
MSE	0,00014

Descriptive statistics of the portfolio created is presented in Table 7. The value of MSE is quite small. This implies that the model selected is appropriate for the portfolio that will be created. The inference from this model is rather robust for calculating the risk regarding the portfolio. In this step of the paper the equally weighted next day portfolio return is evaluated. Firstly each of the returns is transformed into nominal scale. After all returns are given equal weights, the result is transformed into a net return by subtracting one. Eventually empirical quantiles of simulated daily portfolio return are evaluated.

Table 8: The return quantiles of the created portfolio

Quantile	Estimate
100 % Max	0,102695
99%	0,040545
95%	0,02153
90%	0,015967
75 % Q3	0,000785

Table 8 shows that the with 99 % maximum potential losses based on an equally weighted portfolio for the next day does not surpass 4 %. In other words, with % 99 maximum potential losses for the portfolio constructed based on the copula approach are 4 %. Maximum potential losses of the portfolio with 95 % and 90 % are 2.1 and 1.5, respectively. For Frank, Clayton and Gumbel copula, the return quantiles of the created portfolio are given in annex (Tables A1-A3).

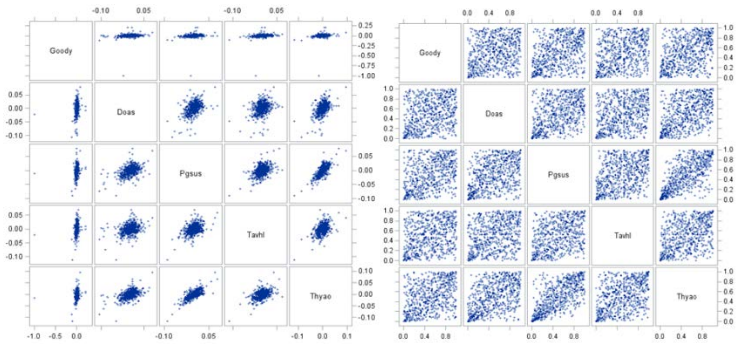

Figure 1: Dependence path of the returns and the transformed returns, respectively.

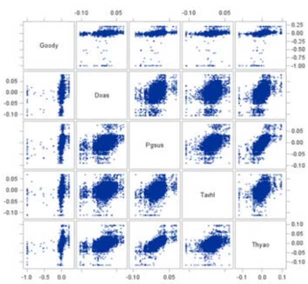

Figure 2: Dependence path of the Student's T copula with empirical margins.

4. Conclusion

This study investigated the modelling of dependency. It is quite difficult to determine dependence structure when relationship is nonlinear and distribution of variables is not appropriate for normal distribution. In this case copulas are commonly used and play crucial role in modelling the dependence structure between variables regardless of marginals. In this paper portfolio analysis was performed by considering dependence structure between variables. Five stocks that have increased in value recently were investigated and the portfolio concerning these stocks was constructed. Marginals of each stock were transformed into joint distribution function by taking into consideration dependence structure between variables. The portfolio of these stocks was constructed by means of new joint distribution function obtained. In this analysis each stock was given an equal weight to create the portfolio. Next-day maximum possible losses of the portfolio were computed using the best fitted copula. In consequence of the study it was seen that the copula approach provides crucial flexibilities to construct the joint distribution function. In general the values of stocks are distributed as heavy-tailed. In this case a tool that can be model the best for the stocks is needed. İrrespective of marginals, copulas constructs the joint distribution to evaluate Value-at-Risk. Besides, the copula does not require too much assumption compared to other methods modelling the dependence structure. Since the dependency was modelled correctly, the results obtained using the copulas are rather robust and reliable. On the whole, buying the portfolio of stocks considered will be benefit for investors since the portfolio indicates low risk. In this sense this paper leads the investor to evaluate risk level for the portfolio. Moreover the study guides to investigations in economics field such as banking, insurance in evaluating risk and determining investment tools.

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Appendix

Table A1: The return quantiles of the created portfolio with respect to Frank copula

Quantile	Estimate
100 % Max	0,079326
99%	0,032772
95%	0,020922
90%	0,016128
75 % Q3	0,008527

Table A2: The return quantiles of the created portfolio with respect to Clayton copula

Quantile	Estimate
100 % Max	0,069381
99%	0,029145
95%	0,018158
90%	0,014387
75 % Q3	0,008213

Table A3: The return quantiles of the created portfolio with respect to Gumbel copula

Quantile	Estimate
100 % Max	0,111311
99%	0,041415
95%	0,022286
90%	0,015478
75 % Q3	0,006445

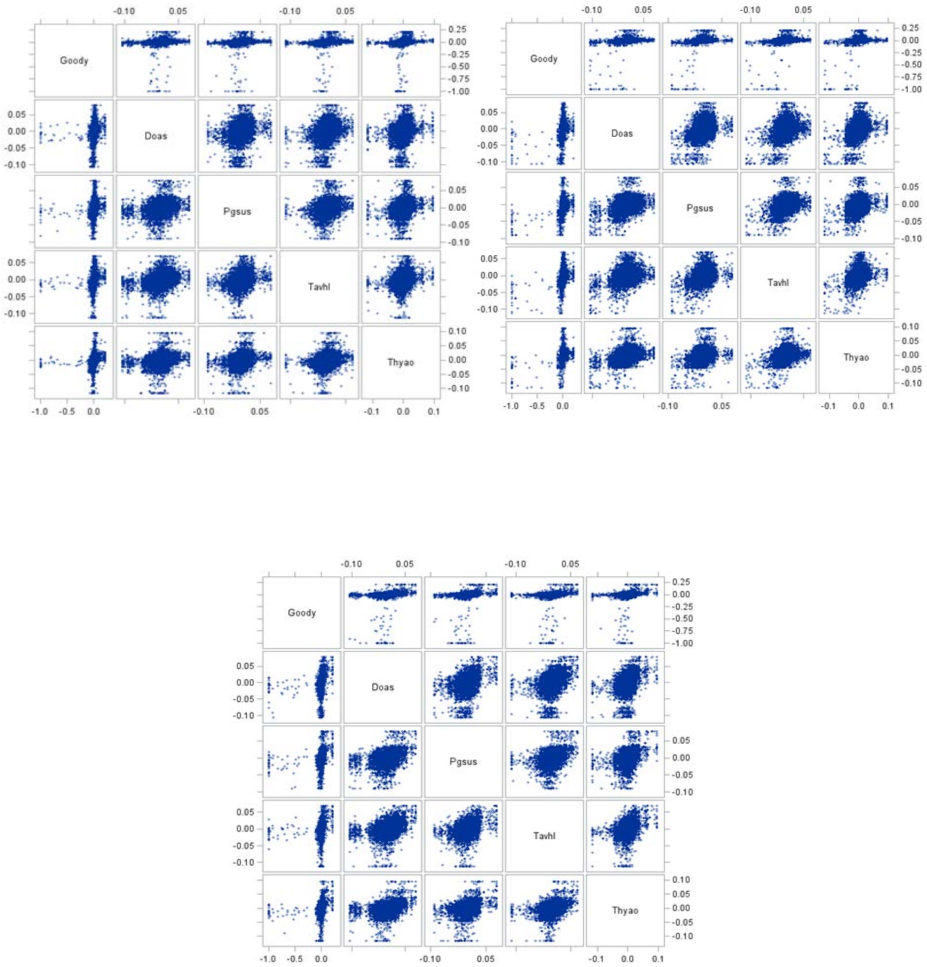

Figure A1: Dependence path of Frank, Clayton, Gumbel copula with empirical margins, respectively.