International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

A Generalization of Notion Group as Dynamical Groups

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Abstract

In this paper the concept of groups will be extended by a dynamical system to the dynamical groups and we will investigate some results about them. Also notions coset and qoutient dynamical group are introduced.

***Keywords*: ** Dynamical groups; group; qoutient dynamical group.

1. Introduction

Some generalizations of notion group are presented sofar. For example generalized group is introduced by Molaie [2] and is studied in [1-6]. We assume the reader is familiar with the definition of dynamical system [7]. In this paper we introduce a new generalization of group by notion dynamical system that we call dynamical group. The paper is organized as follows. In Section 2 the notion of dynamical groups is introduced, also we define dynamical subgroups and give some properties and examples about dynamical groups. In Section 3 we define and study notions coset and qoutient dynamical groups.

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2. Dynamical groups

Definition 2.1. Let G be a non-empty set and o be an associative operation on G. We say $(\mathrm{G}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{o})$ is a dynamical group if the functions $\mathrm{f}: \mathrm{G} \rightarrow \mathrm{G}$, $\mathrm{g}: \mathrm{G} \rightarrow \mathrm{G}$ and $\mathrm{h}: \mathrm{G} \rightarrow \mathrm{G}$ be injective and satisfies the following conditions:

i) There exists $\mathrm{e} \in \mathrm{G}$ such that for every $\mathrm{x} \in \mathrm{G}$,

$$\mathrm{f}(\mathrm{x}) \mathrm{o} \mathrm{g}(\mathrm{e}) = \mathrm{f}(\mathrm{e}) \mathrm{o} \mathrm{g}(\mathrm{x}) = \mathrm{h}(\mathrm{x}), \qquad (1)$$

Which e is called the identity element of $(\mathrm{G}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{o})$

ii) For every $\mathrm{x} \in \mathrm{G}$ there exists $\mathrm{x}' \in \mathrm{G}$ such that

$$\mathrm{f}(\mathrm{x}) \mathrm{o} \mathrm{g}(\mathrm{x}') = \mathrm{f}(\mathrm{x}') \mathrm{o} \mathrm{g}(\mathrm{x}) = \mathrm{h}(\mathrm{e}), \qquad (2)$$

Which x' is called the inversion element of x in $(\mathrm{G}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{o})$. Note that every function of the functions $\mathrm{f}, \mathrm{g}, \mathrm{h}$ is considered as a dynamical system.

Remark 2.2. The notion of dynamical groups is a generalization of groups. In fact if $\mathrm{f} = \mathrm{g} = \mathrm{h}$ and the functions are injective and surjective then (G, o) is a group because if $\mathrm{x} \in \mathrm{G}$ and $\bar{\mathrm{e}} = \mathrm{f}(\mathrm{e})$ then there exists $\bar{\mathrm{x}} \in \mathrm{G}$ such that $\mathrm{x} = \mathrm{f}(\bar{\mathrm{x}})$ and we can write

$$\begin{aligned} \bar{\mathrm{e}} \mathrm{o} \mathrm{x} &= \mathrm{f}(\mathrm{e}) \mathrm{o} \mathrm{f}(\bar{\mathrm{x}}) \\ &= \mathrm{f}(\mathrm{e}) \mathrm{o} \mathrm{g}(\bar{\mathrm{x}}) \\ &= \mathrm{h}(\bar{\mathrm{x}}) = \mathrm{f}(\bar{\mathrm{x}}) = \mathrm{x}, \quad (3) \end{aligned}$$

Also

$$\begin{aligned} \mathrm{x} \mathrm{o} \bar{\mathrm{e}} &= \mathrm{f}(\bar{\mathrm{x}}) \mathrm{o} \mathrm{f}(\mathrm{e}) \\ &= \mathrm{f}(\bar{\mathrm{x}}) \mathrm{o} \mathrm{g}(\mathrm{e}) \\ &= \mathrm{h}(\bar{\mathrm{x}}) = \mathrm{f}(\bar{\mathrm{x}}) = \mathrm{x}. \quad (4) \end{aligned}$$

Now let $\mathrm{y} \in \mathrm{G}$. Since f is surjective there exists $\mathrm{x} \in \mathrm{G}$ such that $\mathrm{y} = \mathrm{f}(\mathrm{x})$. By the property (2), there exists $\mathrm{x}' \in \mathrm{G}$ such that: (let $\mathrm{y}' = \mathrm{f}(\mathrm{x}')$)

$$\mathrm{y} \mathrm{o} \mathrm{y}' = \mathrm{f}(\mathrm{x}) \mathrm{o} \mathrm{f}(\mathrm{x}') = \mathrm{f}(\mathrm{x}) \mathrm{o} \mathrm{g}(\mathrm{x}')$$

$$= h(e) = f(e) = \bar{e}, \qquad (5)$$
$$y'oy = f(x')of(x) = f(x')og(x)$$
$$= h(e) = f(e) = \bar{e}. \qquad (6)$$

So (G, o) is a group.

Example 2.3. Let G be an abelian group and $a, b \in G$. If $f(x) = ax$, $g(x) = b.x$ and $h(x) = a.b.x$. Then we have

$$f(x).g(e) = a.x.(b.e)$$
$$= a.x.b = a.b.x$$
$$= h(x), \qquad (7)$$
$$f(e).g(x) = ae.(b.x)$$

$$= a.b.x = h(x). \quad (8)$$

where e is the identity element of G.

On the other hand let $x \in G$ and x' be the Inversion element of x in G, then we can write

$$f(x).g(x') = a.x.b.x'$$
$$= a.b.x.x'$$
$$= a.b.e = h(e), \qquad (9)$$
$$f(x').g(x) = a.x'.b.x$$
$$= a.b.x'.x$$

$$= a.b.e = h(e). \quad (10)$$

Hence (G, f, g, h) is a dynamical group.

Example 2.4. Let $G = \{a, b, c\}$. We define an operation on G by

$$aa = a, \; ab = b, \; ac = a,$$

$$ba = a, \; bb = c, \; bc = b,$$

$$ca = c, \; cb = a, \; cc = c. \quad (11)$$

Also let the functions f, g, h are defined by

$$f(a) = b, \quad f(b) = a, \; f(c) = c,$$

$$g(a) = b, \; g(b) = c, \; g(c) = a,$$

$$h(a) = b, \; h(b) = a, \; h(c) = c. \quad (12)$$

Then (G, f, g, h) is a dynamical group because

i) b is the identity elelment because

$$f(a)g(b) = bc = ab = f(b)g(a) = b = h(a),$$

$$f(b)g(b) = ac = f(b)g(b) = a = h(b),$$

$$f(c)g(b) = cc = ca = f(b)g(c) = c = h(c). \qquad (13)$$

ii) Inverse element: we can write

$$a' = c, \qquad b' = b, \qquad c' = a$$

because

$$f(a)g(a') = ba = cb = f(a')g(a) = a = h(b),$$

$$f(b)g(b') = ac = f(b')g(b) = a = h(b),$$

$$f(c)g(c') = cb = ba = f(c')g(c) = a = h(b). \quad (14)$$

Proposition 2.5. Let (G, f, g, h) be a dynamical group, then $(x')' = x$.

Proof. It can be deduced from property 2.2.

Definition 2.6. Let (G, f, g, h) be a dynamical group. A non-empty subset H of G is called a dynamical subgroup of G if $(H, f|_H, g|_H, h|_H, o)$be a dynamical group.

Since f, g, h are injective, $f|_H, g|_H, h|_H$ are injective.

Definition 2.7. We say two dynamical groups (G, f, g, h, o), $(\overline{G}, \overline{f}, \overline{g}, \overline{h}, \overline{o})$ are isomorphic if there exists a bijective function $\varphi: G \to \overline{G}$ such that

i) $\varphi(xoy) = \varphi(x)\overline{o}\varphi(y)$,

ii) $\varphi of = \overline{f} o\varphi$, $\varphi og = \overline{g} o\varphi$, $\varphi oh = \overline{h} o\varphi$.

Then φ is called an isomorphism.

Proposition 2.8. Let (G, f, g, h, o), $(\overline{G}, \overline{f}, \overline{g}, \overline{h}, \overline{o})$ be two dynamical groups and $\varphi: G \to \overline{G}$ be an isomorphism. Then

i) If e is the identity element of (G, f, g, h, o), then $\varphi(e)$ is the identity element of $(\bar{G}, \bar{f}, \bar{g}, \bar{h}, \bar{o})$,

ii) If $x'$is the inversion element of x in (G, f, g, h, o), then $\varphi(x')$ is the inversion element of $\varphi(x)$ in $(\bar{G}, \bar{f}, \bar{g}, \bar{h}, \bar{o})$.

Proof.

i) By definition 2.1 it is sufficient to show that

$$\bar{f}(x)\bar{g}\big(\varphi(e)\big) = \bar{f}\big(\varphi(e)\big)\bar{g}(x) = \bar{h}(x). \qquad (15)$$

From definition 2.7 we have

$$\begin{aligned}\bar{f}(x)\bar{g}\big(\varphi(e)\big) &= \bar{f}(x)\varphi\big(g(e)\big)\\ &= \varphi\big(f(\varphi^{-1}x)\big)\varphi\big(g(e)\big)\\ &= \varphi\Big(f(\varphi^{-1}x)\big(g(e)\big)\Big)\\ &= \varphi\big(h(\varphi^{-1}x)\big)\\ &= (\varphi o h o \varphi^{-1})(x) = \bar{h}(x). \qquad (16)\end{aligned}$$

Also

$$\begin{aligned}\bar{f}(\varphi(e))\bar{g}(x) &= \varphi\big(f(e)\big)(\varphi o g o \varphi^{-1})(x)\\ &= \varphi\big(f(e)\big)\varphi(g(\varphi^{-1}x))\\ &= \varphi\Big(\big(f(e)\big)(g(\varphi^{-1}x))\Big)\\ &= \varphi\big(h(\varphi^{-1}x)\big)\\ &= (\varphi o h o \varphi^{-1})(x) = \bar{h}(x). \qquad (17)\end{aligned}$$

ii) We show that

$$\bar{f}(\varphi(x))\bar{g}(\varphi(x')) = \bar{f}(\varphi(x'))\bar{g}(\varphi(x)) = \bar{h}(\varphi(e)). \qquad (18)$$

We can write

$$\begin{aligned}\bar{f}(\varphi(x))\bar{g}(\varphi(x')) &= \varphi\big(f(x)\big)\varphi\big(g(x')\big) = \varphi\big(f(x)g(x')\big)\\ &= \varphi\big(h(e)\big) = (\varphi o h o \varphi^{-1})(\varphi(e))\end{aligned}$$

$$= \bar{h}(\varphi(e)). \qquad (19)$$

Also

$$\bar{f}(\varphi(x'))\bar{g}(\varphi(x)) = \varphi\big(f(x')\big)\varphi\big(g(x)\big)$$

$$= \varphi\big(f(x')g(x)\big) = \varphi\big(h(e)\big)$$

$$= (\varphi o h o \varphi^{-1})(\varphi(e)) = \bar{h}(\varphi(e)). \qquad (20)$$

3. Coset and qoutient dynamical group

Definition 3.1. Let (G, f, g, h, o) be a dynamical group. Also let H be a dynamical subgroup of G and $c \in G$. We define left and right cosets of H in G as

$$cH = \{f(c)g(h) | h \in H\}, \quad (21)$$

$$Hc = \{f(h)g(c) | h \in H\}. \qquad (22)$$

Remark 3.2. Let H be a dynamical subgroup of dynamical group (G, f, g, h, o) and e be the identity element of G. Then we have

$$eH = He = h(H). \qquad (23)$$

Because

$$eH = \{f(e)g(x) | x \in H\} = \{f(x)g(e) | x \in H\}$$

$$= He = \{h(x) | x \in H\} = h(H). \; (24)$$

Lemma 3.3. Let H be a dynamical subgroup of dynamical group (G, f, g, h). Also let for each $x, y \in G$

$$f(x)g(y) = g(y)f(x),$$

$$f(xy) = f(x)f(y),$$

$$g(xy) = g(x)g(y). \quad (25)$$

Then for every $c, b \in G$

$$(cH)(bH) = (cb)H. \qquad (26)$$

Proof.

$$(cH)(bH) = \{f(c)g(h)|h \in H\}\{f(b)g(h')|h' \in H\}$$

$$= \{f(c)g(h)f(b)g(h')|h, h' \in H\}$$

$$= \{f(c)f(b)g(h)g(h')|h, h' \in H\}$$

$$= \{f(cb)g(hh')|h, h' \in H\}$$

$$= (cb)H. \qquad (27)$$

Proposition 3.4. Let H be a dynamical subgroup of dynamical group (G, f, g, h), such that Also let for each $c \in G$, $cH = Hc$. Also let for every $x, y \in G$,

$$f(x)g(y) = g(y)f(x)$$

$$f(xy) = f(x)f(y)$$

$$g(xy) = g(x)g(y). \qquad (28)$$

Then the family of left (right) cosets of H in (G, f, g, h)which is shown by $(\frac{G}{H}, f', g', h')$is a dynamical group called the quotient dynamical group of G by H such that f', g', h' are defined by

$$f'(cH) = f(c)H,$$

$$g'(cH) = g(c)H,$$

$$h'(cH) = h(c)H. \qquad (29)$$

Proof.

It is necessary to check the axioms of dynamical group on $\frac{G}{H}$.

f', g', h' are injective functions. If e is the identity element in (G, f, g, h), then eH is the identity element in $(\frac{G}{H}, f', g', h')$ because

$$f'(cH)g'(eH) = f(c)Hg(e)H$$

$$= f(c)g(e)H$$

$$= h(c)H = h'(cH), \qquad (30)$$

$$f'(eH)g'(cH) = f(e)Hg(c)H$$

$$= f(e)g(c)H$$

$$= h(c)H = h'(cH). \qquad (31)$$

Now Let c' be the inverse of c in (G, f, g, h). We show that $c'H$ is the inverse of cH in $\left(\frac{G}{H}, f', g', h'\right)$.

$$f'(cH)g'(c'H) = f(c)Hg(c')H$$

$$= f(c)g(c')H$$

$$= h(e)H = h'(eH), \qquad (32)$$

$$f'(c'H)g'(cH) = f(c')Hg(c)H$$

$$= f(c')g(c)H$$

$$= h(e)H = h'(eH). \qquad (33)$$

So the axioms of dynamical group are satisfied in $\left(\frac{G}{H}, f', g', h'\right)$.

Example 3.5. Let (G, f, g, h) be the dynamical group of example 2.3 with $aa = a$ and $bb = b$. Also let H be a dynamical subgroup of dynamical group G such that for each $c \in G$, $cH = Hc$. Then $(\frac{G}{H}, f', g', h')$ is a quotient dynamical group because by proposition 3.4 we can write

$$f(x)g(y) = (ax)(by)$$

$$= (ay)(bx) = g(y)f(x), \qquad (34)$$

$$f(xy) = axy = aaxy$$

$$= (ax)(ay) = f(x)f(y), \qquad (35)$$

$$g(xy) = bxy = bbxy$$

$$= (bx)(by) = g(x)g(y). \, (36)$$

4. Conclusion

We introduced the notion dynamical groups as a generalization of concept group. Also we defined notions coset and qoutient dynamical group and we presented some propositions and examples about them.

References

[1] J. O. Adeniran, J. T. Akinmoyewa, A. R. T. Solarin and T. G. Jaiyeola. "On Some Algebraic Properties of Generalized Groups." Acta Mathematica Academiae Paedagogicae Nyregyh aziensis, Vol. 27, pp. 23-30, 2011.

[2] M. R. Molaei. "Generalized actions." Proceedings of the First International Conference on Geometry, Integrability and Quantization. Coral Press Scientific Publishing Proceedings of the First International Conference on Geometry, pp. 175-180, 1999.

[3] M.R. Molaei. "Generalized groups." Bull. Inst. Polit. Di. Iase Fasc, 3(4), PP. 21-24, 1999.

[4] M.R. Molaei. "Topological generalized groups." Int. Jour. Appl. Math, 2(9), PP. 1055-1060, 2000.

[5] M.R. Molaei and A. Tahmoresi. "Connected topological generalized groups." Mathematics, 12(1), PP. 13-22, 2004.

[6] M. R. Mehrabi and A. Oloomi. "Generalized subgroups and homomorphisms." Arabs Jour. Math. Sc, Vol. 6, PP. 1-7, 2000.

[7] Ya. G. Sinai. "Dynamical Systems." Aarhus Univ, Vol. 1, 1970.