Minimum Convex and Differentiable Cost Flow Problem with Time Windows

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Abstract

This paper presents a new version of the Minimum Cost Flow Problem (MCFP). This version is the Minimum Convex and Differentiable Cost Flow Problem with Time Windows (MCDCFPTW). Given a directed graph $G = (V, A)$, where V is a set of vertices, A is a set of arcs. Each vertex $i \in V$ has a time-window $[a_i, b_i]$ within which the vertex may be visited with a non-negative service time $t_i \in T$ where, $a_i \leq t_i \leq b_i$. Each arc $(i, j) \in A$ is associated with three non-negative parameters: a positive capacity u_{ij}, an arbitrary transit cost c_{ij} and a transit time $t_{ij} \in T, i \neq j$. In this new version, we derive the optimality conditions for minimizing convex and differentiable cost functions which satisfy a condition of the time windows, and devise an algorithm based on the primal-dual algorithm commonly used in linear programming. The proposed algorithm minimizes the total convex and differentiable cost of flow by incrementing the network flow along augmenting paths of minimum cost from the source vertex s to the destination vertex d.

Keywords: Minimum Cost Flow Problem; Combinatorial Optimization; Network Optimization; Time windows

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1. Introduction

The Minimum Cost Flow Problem (MCFP) is a basic problem in network flow theory. The network flows are of fundamental importance in computer science, communication networks, industrial engineering, operation research, and many other areas. The standard formulation of the MCFP assumes that input data are known precisely. The Ahuja et al. in [1] textbook is an exhaustive reference on the subject. Like shortest path problem and maximum flow problem [9, 10, 11], the MCFP is a central problem in network flows. In [2] the MCFP with stochastic arc costs is studied and solution methods are developed based on two optimality concepts: cycle marginal cost, and network equilibrium. In [15] the MCFP with interval arc costs is considered and two solution methods are introduced based on extensions of some efficient combinatorial algorithms for the MCFP. In [13] the MCFP is established for fuzzy arc costs and, just as for the problem with interval arc costs, the proposed solution modifies the negative-cycle-canceling algorithm in order to allow the use of fuzzy numbers. Consequently, the MCFP has been studied extensively in the literature; [3, 18, 21].

Given $G = (V, A)$ be a directed flow network, where V is a set of vertices and $|V| = n$, A is a set of arcs, the two distinguished vertices, a source vertex $s \in V$ with time window $[a_s, b_s]$ and a departure time t_0, a destination vertex $d \in V$ with time window $[a_d, b_d]$. Each vertex $i \in V$ has a time-window $[a_i, b_i]$ within which the vertex may be visited and a non-negative service time $t_i \in T$ where, $a_i \leq t_i \leq b_i$. Each arc $(i, j) \in A$ is associated with three non-negative parameters: a positive capacity u_{ij}, an arbitrary transit cost c_{ij} and a transit time $t_{ij} \in T, i \neq j, i, j = 1, 2, ..., n,$ see [5, 6, 7, 8, 24]. The Minimum Convex and Differentiable Cost Flow Problem with Time Window (MCDCFPTW) asks to find an $s, d -$ path that leaves a source vertex s with time window $[a_s, b_s]$ at time t_0 and minimizes the total arrival time, a convex and differentiable cost flow at a destination vertex d with time window $[a_d, b_d]$, which satisfy the set of all constraints.

The maximum flow problem tries to find out a maximum allowed flow from a source vertex s to a destination vertex d in which each arc $(i, j) \in A$ has a maximum allowed flow of u_{ij}. Now we wish to associate a further parameter c_{ij} with each arc, where c_{ij} is the cost of sending a unit flow along (i, j). This has an obvious interpretation in any real network as well as any real application in practice where the unit cost may vary from arc to arc depending upon the nature of applications. The cost can be thought with a wide range of meaning. In a real network, it can a real cost to send a unit of data from one point to another point. It can also be other network parameters such as bandwidth, delay, probability. This extra consideration of costs poses a new problem. This is the problem of how to transport the units of flow across a network such that a minimum cost is incurred. The MCFP arises naturally in many contexts, including virtual circuit routing in communication networks, layout, scheduling, transportation, and hence has been studied extensively [16, 18, 20].

In the conventional of the MCFP, the cost c_{ij} associated to each arc is normally a linear function of the flow carried by the arc. A large number of methods have been proposed for solving this MCFP. Among the most popular algorithms are the primal simplex method, primal-dual method, and the out-of-kilter method [1, 12, 15, 19]. However, in practice, it is possible that the cost incurred in each arc is nonlinear function of its flow.

A number of existing solutions can be applied to this convex cost flow problem. One possible way is to reduce the problem to a typical linear cost flow problem using piecewise linearization of the arc cost functions [17, 22]. This approach assumes that each of the convex functions is linear between successive integers, and then introduces a separate arc for each linear segment. In this way, the convex cost flow problem is transformed into the conventional MCFP, and solved by the existing MCFP algorithms. The convex cost flow problem has also been recently addressed by [11] with two approaches:

- the minimum cycle cancelling method,
- the cancel and tighten method for the MCFP based on [14] that proceeds by sending flows along negative cost cycles.

In this paper, we present a new version algorithm based on primal-dual algorithm used in linear programming to address this convex cost flow problem. We modify the optimality condition in primal-dual algorithm so that it can be applied to convex and differentiable cost functions with time windows. In particular, we show that using the new optimality condition, we can minimize the total cost of flow by incrementing the network flow along the augmenting paths of minimum cost.

In this paper, we give a new algorithm of a new version of the MCFP is MCDCFPTW and organized as follows. In Sections 2, we present the basic concepts and formulate the problem of a minimum cost flow with time windows in which the cost functions are strictly convex and differentiable. In Section 3, we give the optimality condition of the MCDCFPTW. A new algorithm of the MCDCFPTW is presented in Section 4. Finally, the conclusion is given in Section 5.

2. Basic concepts and formulation problem

Let $G = (V, A)$ be a directed graph, where V is a set of vertices, A is a set of arcs. The vertex set V consists of n vertices denoted by $1,2,...,n$, while the arc set A consists of all arcs $(i, j) \in A$. Each arc (i, j) has a cost c_{ij} that denotes the unit shipping cost along the arc (i, j). Each arc (i, j) is associated with an amount f_{ij} of flow on the arc and u_{ij} are the capacity also associated with arc (i, j). We associate a number b_i with each vertex $i \in V$, which indicates its available amount of supply or demand. Vertex i will be called a source, destination or transshipment vertex, depending on whether $b_i \succ 0, b_i \prec 0,$ or $b_i = 0,$ respectively. This way, a plethora of real-world applications (in logistics) requiring the flow of various products from warehouses (supply vertex) to markets (demand vertex) through a number of transfer points (transshipment vertex) can be efficiently

modeled. If $\sum_{i \in V} b_i = 0,$ then the network G will be a balanced network. Thus, the Minimum Convex and

Differentiable Cost Flow Problem with Time Window (MCDCFPTW) can be stated formally as follows:

$$z(f) = \sum_{(i,j) \in A} c_{ij}(f_{ij}) \qquad (1)$$

subject to: $\quad 0 \le f_{ij} \le u_{ij}, \quad \forall (i, j) \in A \qquad (2)$

$$\sum_{j:(i,j) \in A} f_{ij} - \sum_{j:(j,i) \in A} f_{ji} = b_i, \quad \forall i \in V \qquad (3)$$

$$t_{ij} + t_i \le t_j, \quad \forall i, j \in V \qquad (4)$$

$$a_i \le t_i \le b_i, \quad \forall i \in v \qquad (5)$$

where, $c_{ij}(f_{ij})$ are the convex and differentiable cost functions of flow f_{ij}. The constraints (2), are capacity

constraints of each arc $(i, j) \in A,$ constraints (3), are the flow conservation equations of each vertex $i \in V$

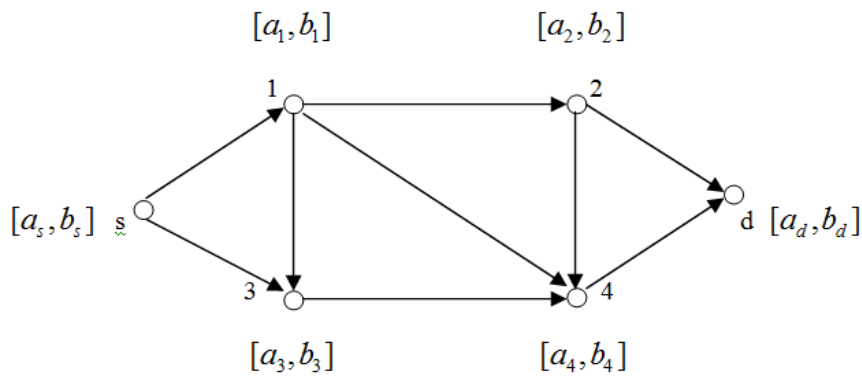and constraints (4), (5) are the time windows of each $i, j \in V.$

Fig. 1: Example of a directed network

For simplicity and without lost of generality, we assume in what follows that there is at most one arc associated

with each ordered pair of vertices $(i, j),$ and that all arc costs are non-negative. In addition, we consider only

the case where the cost functions $c_{ij}(f_{ij})$ are convex and differentiable by,

$$c_{ij}(\lambda x + (1-\lambda)y) \le \lambda c_{ij}(x) + (1-\lambda)c_{ij}(y), \quad \forall \lambda \in [0,1] \qquad (6)$$

We denoted the right-hand and the left-hand derivatives of $c_{ij}(x)$ by $c_{ij}^{+}(x)$ and $c_{ij}^{-}(x)$, respectively. The two conditions of the convex cost functions are given by the following:

- If the cost function $c_{ij}(x)$ is differentiable at every point in which it is defined, then

$$c_{ij}^{+}(x) = c_{ij}^{-}(x) \tag{7}$$

- The condition $0 \le f_{ij} \le u_{ij}, \ \forall (i, j) \in A$ needs to be satisfied, the cost function $c_{ij}(x)$ is defined only with positive x. The point x_0 minimizes $c_{ij}(x)$ if and only if $c_{ij}^{+}(x_0) \ge 0$ and $c_{ij}^{-}(x_0) \le 0$ for $x_0 \succ 0$, and $c_{ij}^{+}(x_0) \ge 0$ for $x_0 = 0$. $\tag{8}$

3. Optimality conditions of a convex and differentiable cost flow problem with time windows

The MCFP is a classical non-linear programming. We extend and modify the conditions of primal-dual method in linear programming [1, 23] to apply for the case of convex and differentiable cost functions with time windows. In our approach, we assume a potential p_i to each vertex $i \in V$. We shall show that an optimum optimized solution for the convex and differentiable cost flow problem with time windows can be found based on the optimality conditions as shown in the following theorem.

Theorem 1 A flow $\{f_{ij}\}$ satisfying the constraints (3), minimizes the total cost (1) with convex and differentiable cost functions with time windows $c_{ij}(f_{ij}) \succ 0$ if and only if there exist for each vertex $i \in V$ a potential p_i for which the following conditions are satisfied:

$$p_j - p_i \le c_{ij}^{+}(f_{ij}) \ \text{if} \ f_{ij} = 0 \tag{9}$$

$$c_{ij}^{-}(f_{ij}) \le p_j - p_i \le c_{ij}^{+}(f_{ij}) \ \text{if} \ f_{ij} \succ 0 \tag{10}$$

$$t_{ij} + t_i \le t_j, \ \forall i, j \in V \tag{11}$$

$$a_i \le t_i \le b_i, \ \forall i \in V \tag{12}$$

Proof: (i) Suppose that the conditions (9), (10), (11) and (12) are satisfied for a flow $\{f_{ij}\}$. We will show that $\{f_{ij}\}$ is optimum; that is, it minimizes the total cost of flow. A flow is not optimum if there exists an arc $(i, j) \in A$ such that $f_{ij} \succ 0$ and $f_{ji} \succ 0$; that is because we can reduce the cost by setting $f_{ij} = f_{ij} - f_{ji}$

and $f_{ji} = 0,$ an adjustment that still preserves the conservation of flow requirement at the vertex. We also impose $f_{ij} = 0,$ if arc (i, j) does not lie in G at all. Hence, the constraints (3) can be rewritten as follows

$$\sum_{j=1}^{n} f_{ij} - \sum_{j=1}^{n} f_{ji} = b_i, \ i = 1,2,...,n \tag{13}$$

As well known in linear programming [4], $\{f_{ij}\}$ and p_i are feasible solutions of primal and dual problems respectively. There is a relation between them

$$\sum_{i=1}^{n} p_i \sum_{1}^{n} f_{ji} = \sum_{i=1}^{n} p_j \sum_{1}^{n} f_{ij} \tag{14}$$

Substitute (13) into the total cost function (1) and use (14), we get

$$\sum_{i,j=1}^{n} c_{ij}(f_{ij}) = \sum_{i,j=1}^{n} c_{ij}(f_{ij}) + \sum_{i=1}^{n} p_i (\sum_{j=1}^{n} f_{ij} - \sum_{j=1}^{n} f_{ji} - b_i) = \sum_{i,j=1}^{n} (c_{ij}(f_{ij}) + (p_i - p_j)f_{ij}) - \sum_{i=1}^{n} p_i b_i$$

(15)

Since indexes i and j have the same role in the summation $\sum_{i=1}^{n} p_i \sum_{i=1}^{n} f_{ji}$. Therefore

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} f_{ji} = \sum_{j=1}^{n} p_j \sum_{i=1}^{n} f_{ij} = \sum_{i,j=1}^{n} p_j f_{ij} \tag{16}$$

Substitute (16) into (15), we have

$$\sum_{i,j=1}^{n} c_{ij}(f_{ij}) = \sum_{i,j=1}^{n} (c_{ij}(f_{ij}) + (p_i - p_j)f_{ij}) - \sum_{i=1}^{n} p_i b_i \tag{17}$$

Since the flow $\bar{f}_{ij}$ satisfies (9), (10), (11) and (12), we have

$$c_{ij}^-(\bar{f}_{ij}) \leq p_j - p_i \leq c_{ij}^+(\bar{f}_{ij}) \ \text{if} \ \bar{f}_{ij} \succ 0 \tag{18}$$

$$p_j - p_i \leq c_{ij}^+(\bar{f}_{ij}) \ \text{if} \ \bar{f}_{ij} = 0 \tag{19}$$

Moreover, let $k_{ij}(f_{ij}) = c_{ij}(f_{ij}) + (p_i - p_j)f_{ij},$ is a convex, we have

$$k_{ij}^{+}(f_{ij}) = c_{ij}^{+}(f_{ij}) + (p_i - p_j) \tag{20}$$

$$k_{ij}^{-}(f_{ij}) = c_{ij}^{-}(f_{ij}) + (p_i - p_j) \tag{21}$$

At the optimum, we can see that $k_{ij}^{+}(f_{ij}) \succ 0$ and $k_{ij}^{-}(f_{ij}) \prec 0$ for $\bar{f}_{ij} \succ 0$, and $k_{ij}^{+}(\bar{f}_{ij}) \geq 0$ for $\bar{f}_{ij} = 0$. Obviously, because cost function $c_{ij}(f_{ij})$ is convex, $k_{ij}(f_{ij}) = c_{ij}(f_{ij}) + (p_i - p_j)f_{ij}$ is also convex. We can apply condition (8) and conclude that the flow $\bar{f}_{ij}$ makes the cost function $k_{ij}(f_{ij})$ minimum for every arc $(i, j) \in A$. From (17), since $\sum_{i=1}^{n} p_i b_i$ is constant, this indicates that the total convex and differentiable cost with time windows is minimized. Hence, $\bar{f}_{ij}$, the flow which satisfies conditions (9), (10)), (11) and (12) is optimum.

(ii) Let $\bar{f}_{ij}$ be a minimum flow. We will show that there exist a potential set p_i that satisfies conditions (9), (10), (11) and (12). First, we assign an appropriate potential set as follows:

Consider the sub-graph $\overline{G}(V, \overline{A})$ of $G(V, A)$ which has the same set of vertices as G and the arc set

$$\overline{A} = \{(i, j) : c_{ij}^{+}(\bar{f}_{ij}) = c_{ij}^{-}(\bar{f}_{ij}) = c_{ij}^{/}(\bar{f}_{ij})\} \tag{22}$$

Obviously, on every arc (i, j) of $\overline{G}$ the flow $f_{ij} \succ 0$, since $f_{ij} = 0$ the left derivative $c_{ij}^{-}(f_{ij})$ is not defined. Assume the graph $\overline{G}$ consists of a number of connected sub-graph. We choose an arbitrary vertex from each connected sub-graph, and assign the potential 0. We then assign the potentials to all other vertices in the following manner:

Let p_i be the potential already assigned to vertex i. We assign

$$p_j = p_i + c_{ij}^{/}(\bar{f}_{ij}) \text{ if } \bar{f}_{ij} \succ 0, \tag{23}$$

$$p_j = p_i - c_{ij}^{/}(\bar{f}_{ij}) \text{ if } \bar{f}_{ji} \succ 0. \tag{24}$$

The potential p_j is said to be coordinate to j with time window $[a_j, b_j]$ along the arc $(i, j) \in A$. Now, each vertex has been provided with a potential. We now show that in this way, the potential set assigned will satisfy conditions (9), (10), (11) and (12).

Assume that in $\overline{G}$ there exist an arc (u,v) with $\bar{f}_{uv} \succ 0$ and $p_v - p_u \neq c'_{uv}(\bar{f}_{uv})$; this violates condition (10). According to the way we assign the potentials, the potential p_u has been coordinated to u with time window $[a_u, b_u]$ along a chain $(i_0, i_1, ..., i_m = u)$ and the potential p_v to v with time window $[a_v, b_v]$ along a chain $(i_0 = i_p, i_{p+1}, ..., i_{m+1} = v)$. Together with the arc (u,v) we obtain a coordinated cycle:

$$\alpha = (i_0, i_1, ..., i_m = u, i_{m+1} = v, ..., i_p = i_0) \tag{25}$$

The flow going on each arc of α is always positive. Now, we modify the flows on the arcs of α slightly in the following way:

On the arcs of α^+ (arcs on which the flow goes in the same direction as the orientation of α), we increase the flows by a value h; on the arcs of α^-, we decrease the flows by the value h. The flows on all the other arcs remain unchanged. Obviously, the vertex constraint (3) remains unaffected.

Consider the cost difference resulting between the old and the new flows

$$\mu = \sum_{(i,j)\in\alpha^+}(c_{ij}(\bar{f}_{ij}) - c_{ij}(\bar{f}_{ij} + h)) + \sum_{(i,j)\in\alpha^-}(c_{ij}(\bar{f}_{ij}) - c_{ij}(\bar{f}_{ij} - h)) \tag{26}$$

By the Taylor series, we get

$$\mu = h(-\sum_{(i,j)\in\alpha^+}(c'_{ij}(\bar{f}_{ij}) + \sum_{(i,j)\in\alpha^-}(c'_{ij}(\bar{f}_{ij})) + o(h) \tag{27}$$

According to the formulation rules (23), (24) for p_i, we can say that if there is a flow from i to j, then $c'_{ij}(f_{ij}) = p_j - p_i$. Hence, equation (27) assumes the following form

$$\mu = h(p_{i0} - p_{i1} + p_{i1} - ... + p_{m-1} - p_u - c'_{uv}(\bar{f}_{ij}) + p_v - p_{m+2} + ... - p_{i0}) + o(h)$$
$$= h(p_v - p_u - c'_{uv}(\bar{f}_{uv})) + o(h) \tag{28}$$

For sufficiently small $h \succ 0$, however, this expression is certainly greater than zero if $p_v - p_u \succ c'_{uv}(\bar{f}_{uv})$. But if $p_v - p_u \prec c'_{uv}(\bar{f}_{uv})$, we modify the flow in the opposite way with the above; that is, on the arcs α^+, we decrease the flow by h; on that of α^-, we increase it by h. The resulting difference can be obtained in an analogous way. It is easy to see that by modifying the flow in this way, the total cost of flow can be lower.

However, this contradicts with the fact that $\bar{f}_{ij}$ is an optimum flow. Hence, we can conclude our assumption that there exists an arc (u,v) with $\bar{f}_{uv} \succ 0$ and $p_v - p_u \neq c'_{uv}$ is not true. In other words, the potential set we defined satisfies conditions (9), (10), (11) and (12).

Remark: The capacity u_{ij} of arc (i, j) into account, since when $f_{ij} = 0$ or $f_{ij} = u_{ij}$, only either the right-hand or left-hand derivative of $c_{ij}(f_{ij})$ exists; hence the cost function $c_{ij}(f_{ij})$ is only differentiable for $0 \prec f_{ij} \prec u_{ij}$. From condition (7), we have $c^+_{ij}(f_{ij}) = c^-_{ij}(f_{ij})$ for $0 \prec f_{ij} \prec u_{ij}$.

4.   Algorithm of a minimum convex and differentiable cost flow problem with time windows

Theorem 2 Let the cost functions be convex and differentiable with time windows, then a flow f_{ij} is optimum if and only if there exists a potential set p_i such that

$$p_j - p_i \leq c^+_{ij}(f_{ij}) \text{ for } f_{ij} = 0 \tag{29}$$

$$p_j - p_i = c^+_{ij}(f_{ij}) \text{ for } 0 \prec f_{ij} \prec u_{ij} \tag{30}$$

$$p_j - p_i \geq c^-_{ij}(f_{ij}) \text{ for } f_{ij} = u_{ij} \tag{31}$$

$$t_{ij} + t_i \leq t_j, \ \forall i, j \in V \tag{32}$$

$$a_i \leq t_i \leq b_i, \ \forall i \in v \tag{33}$$

According to the optimal conditions obtained in the previous section, we can now present the proposed primal dual algorithm for solving the convex and differentiable cost flow problem with time windows given by:

Begin

Find a feasible flow;

While p_i of vertex i with time window $[a_i, b_i]$ not satisfying optimality conditions *Do*

Begin

Build $\overline{G}(V, \overline{A})$ where $\overline{A} = \{(i, j) : c^+_{ij}(f_{ij}) = c^-_{ij}(f_{ij}) = c'_{ij}(f_{ij})\}$;

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Select a vertex u with time window $[a_u, b_u]$ in every connected sub-graph in $\overline{G}$ and set $p_u = 0$;

For every vertex j with time window $[a_j, b_j]$ of arc (i, j) that p_i is assigned ***Do***

Begin

  If $f_{ij} \succ 0$ ***then*** $p_j = p_i + c'_{ij}(f_{ij})$;

  If $f_{ij} \succ 0$ ***then*** $p_j = p_i - c'_{ij}(f_{ij})$;

End

If there exist p_u and p_v not satisfying optimality conditions ***then***

Begin

  Find the coordinated cycle α that contains u and v;

  Modify the flow along α;

End

End

End

The present algorithm uses a labeling process, we assign the potential set according to criteria (23), (24); that is, the potential of vertex j with time windows $[a_j, b_j]$ is assigned based on the potential p_i of vertex i with time windows $[a_i, b_i]$ which has previously been assigned. The potential p_j is said to be coordinated with the vertex j along the arc (i, j). If there exist vertices u and v that p_u and p_v do not satisfy the optimality conditions, we will find the coordinated cycle $\alpha = (i_0, i_1, ..., i_m = u, i_{m+1} = v, ..., i_p = i_0)$ where p_{i_l} is coordinated to i_l along the arc (i_{l-1}, i_l). The flow of α is modified to lower the cost as follows. On the arcs where the flow goes in the same or opposite direction as the orientation of α, the flow is increased or reduced by a value h. The value h is fund to minimize the cost incurred in the cycle α.

5.   Conclusion

In this paper, we present a new algorithm of a new version of the MCFP. This algorithm is a modification and combination of the maximum flow algorithm by Ford & Fulkerson and primal-dual algorithm commonly used in linear programming. The proposed algorithm minimizes the total convex and differentiable cost of flow by incrementing the network flow along augmenting paths of minimum cost from the source vertex s to the destination vertex d satisfy the time window condition. The class of this problem has a crucial of many applications on a large and complicated in network optimization.

Acknowledgements

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