International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

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# The Log-Gamma-Pareto Distribution

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**Abstract**

We study the log-gamma-Pareto distribution which includes as special cases two models such as gamma-Pareto and Pareto distributions. We demonstrate that its density function is an infinite linear combination of Pareto densities. Some mathematical properties of the new distribution are derived, such as moments, distribution of the order statistics, Shannon and Renyi entropies and maximum entropy characterization. We use maximum likelihood estimation to estimate model parameters and an application to a real data set illustrates its potentiality. We generate random numbers from the cdf of the distribution and obtain the mean, bias, mean square error, standard error, Kurtosis and Skewness for each parameter.

***Keywords:***Log-gamma-generated distributions; Pareto distribution; moments; order statistics; entropies.

## 1. Introduction

Statistical distributions are playing a very important role in the scientific researches, since recognizing the probability distribution of the sample study denotes the key world in many situations.

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Sometimes fitting a distribution to a given data set results in a poor fit. To deal with this problem, many statisticians attempt to generalize the distributions in order to produce a better fit for the data. The authors in [1] introduced two new families of distributions generated by log-gamma random variables. These two families of distributions has their cumulative distribution functions (cdfs) as

$$G(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_{0}^{F(x)}(-\log t)^{\alpha-1}t^{\beta-1}dt=\frac{\Gamma[-\log F(x),\alpha]}{\Gamma(\alpha)},\quad x\in R,\quad \alpha,\beta>0 \quad (1)$$

and

$$G(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\overline{F}(x)}(-\log t)^{\alpha-1}t^{\beta-1}dt,\quad x\in R,\quad \alpha,\beta>0,$$

respectively, where $\Gamma(.)$ is the complete gamma function, $\overline{F}(.)=1-F(.)$ is the survival function of $X$ and $\Gamma(x,\alpha)=\int_{x}^{\infty}x^{\alpha-1}e^{-x}dx$ is the upper incomplete gamma function. The cdf $F(x)$ is referred to as the parent distribution. The authors in [1] gave the corresponding two probability density functions (pdf) as

$$g(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}(-\log F(x))^{\alpha-1}(F(x))^{\beta-1}f(x),\quad x\in R,\quad \alpha,\beta>0 \quad (2)$$

and

$$g(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}(-\log \overline{F}(x))^{\alpha-1}(\overline{F}(x))^{\beta-1}f(x),\quad x\in R,\quad \alpha,\beta>0, \quad (3)$$

respectively, where $f(x)$ is the pdf of the parent distribution $F(x)$. The role of the parameters $\alpha$ and $\beta$ is to control tail weight and skewness of these distributions and give us the flexibility for modeling significantly skewed or tailed data. The authors in [1] then showed that the families (2) and (3) are generalized forms of distributions of $k$ record values and they are also obtained by applying the inverse probability integral transformation to the log-gamma distribution. In this note, we shall work only with the family of distributions defined by (2).

Consider the Pareto distribution with cdf (see the authors in [2])

$$F(x)=1-\left(\frac{\theta}{x}\right)^{k},\quad x>\theta,\quad \theta,k>0. \quad (4)$$

Inserting (4) in equation (1) yields the log-gamma-Pareto (LGP) distribution function as follows (see figure 1):

$$G(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{1-\left(\frac{\theta}{x}\right)^k} (-\log t)^{\alpha-1} t^{\beta-1} dt \tag{5}$$

Using the transformation $y = -\log t$, equation (5) reduces to

$$G(x) = \frac{1}{\Gamma(\alpha)} \int_{-\log\left[1-\left(\frac{\theta}{x}\right)^k\right]}^{\infty} (\beta y)^{\alpha-1} e^{-\beta y} d(\beta y) = \frac{\Gamma\left\{-\beta \log\left[1-\left(\frac{\theta}{x}\right)^k\right], \alpha\right\}}{\Gamma(\alpha)}, \quad x > \theta, \tag{6}$$

where $\alpha, \beta, k, \theta > 0$ and $\Gamma(.,.)$ is the upper incomplete gamma function defined above.

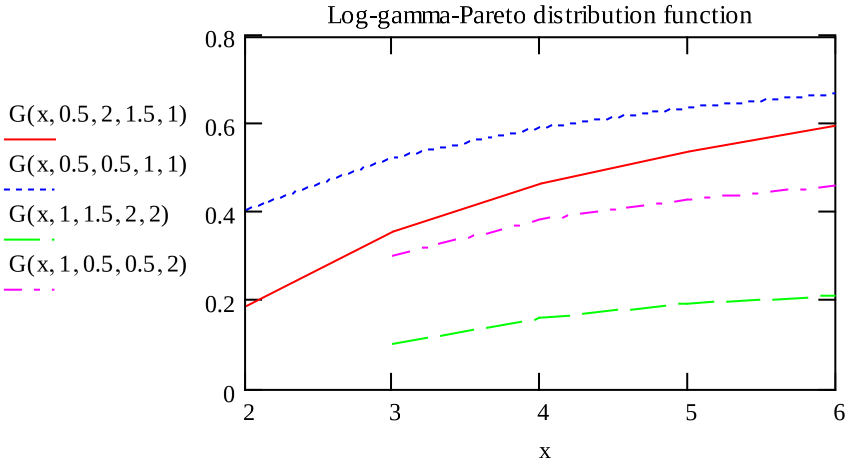


Figure 1: Possible shapes of LGP distribution function for different values of $\alpha, \beta, k$ and $\theta$.

The density function corresponding to (6) is given by (see figure 2)

$$g(x) = \frac{k\beta^{\alpha}}{\Gamma(\alpha)} \frac{\theta^k}{x^{k+1}} \left(-\log\left[1-\left(\frac{\theta}{x}\right)^k\right]\right)^{\alpha-1} \left[1-\left(\frac{\theta}{x}\right)^k\right]^{\beta-1}, \quad x > \theta, \quad \alpha, \beta, k, \theta > 0. \tag{7}$$

where $\Gamma(.)$ is the complete gamma function.

We motivate the use of this distribution in two ways. First, it extends the Pareto distribution and the gamma-Pareto distribution introduced by the authors in [3]. Second, for $\beta = 1, \alpha \in n$, we obtain the distribution of the lower record value from a sequence of independent and identically distributed random variables from a population with the Pareto distribution. If a random variable $Y$ follows the log-gamma distribution with parameters $\alpha$ and $\beta$, then the random variable $X = \theta/(1-Y)^{1/k}$ follows the LGP distribution with parameters

$\alpha, \beta, k$ and $\theta$. If a random variable $Y$ follows the gamma distribution with parameters $\alpha$ and $\beta$, then the random variable $X = \theta/(1-e^{-y})^{1/k}$ follows the LGP distribution with parameters $\alpha, \beta, k$ and $\theta$. A random variable $X$ having density (7) is denoted by $X \sim LGP(\alpha, \beta, k, \theta)$.

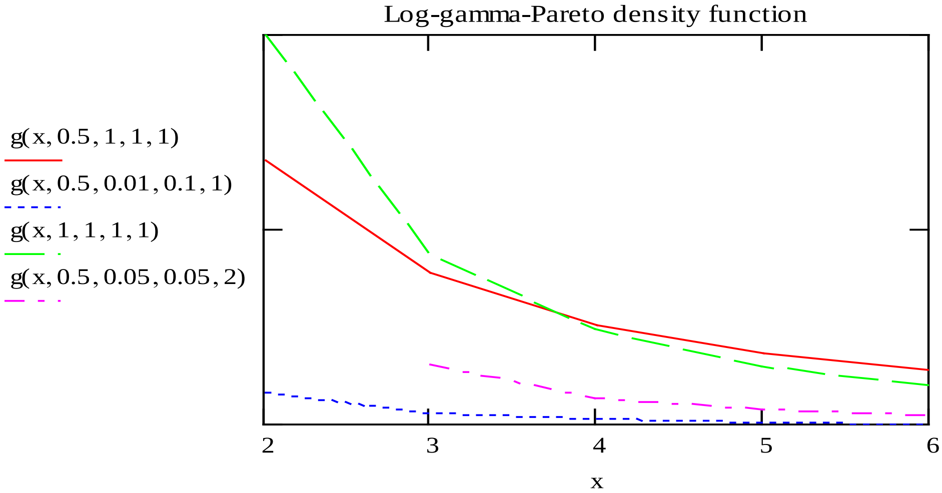


Figure 2: Possible shapes of LGP density function for different values of $\alpha, \beta, k$ and $\theta$.

The hrf of the LGP distribution is given by (see figure 3)

$$h(x) = \frac{k\beta^{\alpha}\theta^{k}\left[1-\left(\frac{\theta}{x}\right)^{k}\right]^{\beta-1}\left\{-\log\left[1-\left(\frac{\theta}{x}\right)^{k}\right]\right\}^{\alpha-1}}{x^{k+1}\Gamma\left\{-\log\left[1-\left(\frac{\theta}{x}\right)^{k}\right], \alpha\right\}}, \quad x > \theta, \quad \alpha, \beta, k, \theta > 0.$$

The LGP quantile function can be obtained by inverting (6) as

$$Q(\lambda) = \theta(1-e^{-z})^{\frac{-1}{k}},$$

where $0 < \lambda < 1$ and $z = \Gamma^{-1}[\lambda\Gamma(\alpha), \alpha]$. The inverse incomplete gamma function $\Gamma^{-1}(.,.)$ is already implemented in most used mathematical softwares (see the authors in [4]).

Using equation (7), two special cases can be obtained as follows:

1) For $\beta = 1$, the gamma-Pareto (GP) distribution is obtained

$$g(x) = \frac{k}{\Gamma(\alpha)} \frac{\theta^k}{x^{k+1}} \left( -\log\left[ 1 - \left( \frac{\theta}{x} \right)^k \right] \right)^{\alpha-1}, \quad x > \theta, \quad \alpha, k, \theta > 0$$

which introduced by the authors in [3].

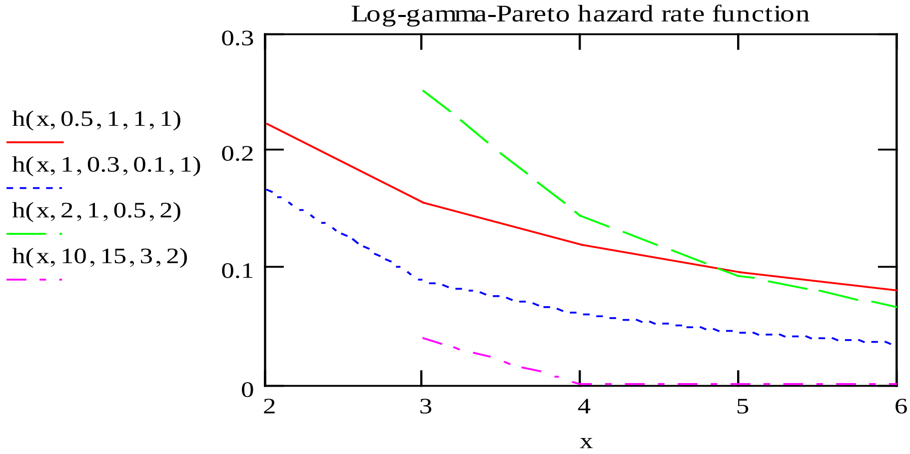


Figure 3: Possible shapes of LGP hazard rate function for different values of $\alpha, \beta, k$ and $\theta$.

2) For $\alpha = \beta = 1$, the Pareto distribution is obtained

$$g(x) = \frac{k\theta^k}{x^{k+1}}, \quad x > \theta, \quad k, \theta > 0.$$

which introduced by the authors in [2].

This article is organized as follows. In section 2, we express $g(x)$ in (7) as an infinite linear combination of Pareto density functions. In section 3, we obtain the moments and moment generating function. In section 4, we provide explicit expressions for the Shannon and Renyi entropies. Also, we propose suitable constraints for maximum entropy characterization of the log-gamma generated families in equation (2). In section 5, we discuss the distribution of the $i^{th}$ order statistic and its $r^{th}$ moment. We estimate the model parameters by maximum likelihood method in section 6. In section 7, we provide an application to a real dataset and generate a random number from the LGP distribution to illustrate the usefulness of the new model.

**2. Expansion for the LGP Density Function**

Consider the density function (7) and let $y = (\theta / x)^k$. Then equation (7) can be written as:

$$g(x) = \frac{k\beta^{\alpha}}{\Gamma(\alpha)} \frac{\theta^k}{x^{k+1}} (1-y)^{\beta-1} [-\log(1-y)]^{\alpha-1}.$$

Using the power series $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1} = \sum_{s=0}^{\infty} \frac{y^s}{s+2}$, where $i-1 = s$, we obtain

$$g(x) = \frac{k\beta^{\alpha}}{\Gamma(\alpha)} \frac{\theta^k}{x^{k+1}} y^{\alpha-1} (1-y)^{\beta-1} \left[ \sum_{m=0}^{\infty} \binom{\alpha-1}{m} y^m \left( \sum_{s=0}^{\infty} \frac{y^s}{s+2} \right)^m \right].$$

Next, using the binomial theorem $(1-y)^{\beta-1} = \sum_{r=0}^{\infty} (-1)^r \binom{\beta-1}{r} y^r$ and let $a_s = (s+2)^{-1}$ and consider the result on a power series raised to a positive integer (see the authors in [5]), we obtain

$$\left( \sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s ,$$

where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^{s} [m(l+1) - s] a_l b_{s-l,m}$ and $b_{0,m} = a_0^m$. We can write

$$g(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{r,m,s=0}^{\infty} (-1)^r \binom{\beta-1}{r} \binom{\alpha-1}{m} \frac{b_{s,m}}{(\alpha+r+m+s)} k(\alpha+m+s+r) \frac{\theta^{k(\alpha+m+s+r)}}{x^{k(\alpha+m+s+r)+1}}$$

and then

$$g(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{r,m,s=0}^{\infty} (-1)^r \binom{\beta-1}{r} \binom{\alpha-1}{m} \frac{b_{s,m}}{(\alpha+r+m+s)} f_*(x)$$

where $f_*(x) = f_*(x; k_*, \theta)$ denotes the density function of the Pareto distribution with shape parameter $k_* = k(\alpha+m+s+r)$ and scale parameter $\theta$. Using the same methodology of the authors in [4], define $V = \{(m,s,r) \in Z_+^3\}$ as an index set and the weights

$$w_{\upsilon} = \frac{(-1)^r \beta^{\alpha}}{\Gamma(\alpha)} \binom{\beta-1}{r} \binom{\alpha-1}{m} \frac{b_{s,m}}{m+r+s+\alpha} \quad for\ \upsilon \in V.$$

Hence, we can write

$$g(x)=\sum_{\upsilon\in V} w_\upsilon f_*(x) \tag{8}$$

Equation (8) reveals that the LGP density is a linear combination of Pareto densities. So, several of its mathematical properties can be immediately obtained from those of the Pareto distribution.

## 3. Moments and Moment Generating Function

Let $X$ be a random variable distributed according to equation (7) and $Z$ be a random variable with Pareto density function $f_*(x)$, i.e. $Z \sim Pareto(k_*,\theta)$.

### 3.1. Moments

It is known that, the r$^{th}$ moment about zero of Pareto distribution is $E(Z^r)=k_*\theta^r/(k_*-r)$. Then, by using equation (8), the r$^{th}$ moment about zero of the LGP distribution is given by

$$E(X^r)=\sum_{\upsilon\in V} w_\upsilon E(Z^r)=\sum_{\upsilon\in V} w_\upsilon \frac{k_*\theta^r}{k_*-r}.$$

As an alternative method to calculate ordinary moments of the LGP distribution without using equation (8), by using equation (7), we obtain

$$E(X^r)=\frac{k\beta^\alpha}{\Gamma(\alpha)}\int_\theta^\infty x^r \frac{\theta^k}{x^{k+1}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\beta-1}\left[-\log\left(1-\left(\frac{\theta}{x}\right)^k\right)\right]^{\alpha-1}dx$$

Making the substitution $u=-\log\left(1-\left(\frac{\theta}{x}\right)^k\right)$, we obtain the r$^{th}$ moment about zero as follows:

$$E(X^r)=\frac{\theta^r\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty (1-e^{-u})^{-r/k}u^{\alpha-1}e^{-\beta u}du. \tag{9}$$

By using the series expansion $(1-e^{-u})^{-r/k}=\sum_{j=0}^\infty \binom{(r/k)+j-1}{j}e^{-ju}$ (see the authors in [6]), equation (9) reduces to

$$E(X^r)=\frac{\theta^r\beta^\alpha}{\Gamma(\alpha)}\sum_{j=0}^\infty \binom{(r/k)+j-1}{j}\int_0^\infty u^{\alpha-1}e^{-(\beta+j)u}du$$

$$=\frac{\theta^r\beta^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\binom{(r/k)+j-1}{j}\frac{1}{(\beta+j)^\alpha}\int_0^\infty((\beta+j)u)^{\alpha-1}e^{-(\beta+j)u}d((\beta+j)u)$$

$$=\frac{\theta^r\beta^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\binom{(r/k)+j-1}{j}\frac{\Gamma(\alpha)}{(\beta+j)^\alpha}. \tag{10}$$

In particular, the first two moments about zero can be derived by taking $r=1$ and $r=2$ in equation (10) as

$$E(X)=\theta\beta^\alpha\sum_{j=0}^{\infty}\binom{(1/k)+j-1}{j}(\beta+j)^{-\alpha},$$

and

$$E(X^2)=\theta^2\beta^\alpha\sum_{j=0}^{\infty}\binom{(2/k)+j-1}{j}(\beta+j)^{-\alpha}.$$

Hence, the variance of LGP distribution can be easily obtained.

### *3.2. Moment Generating Function*

It is known that the moment generating function (mgf) of the Pareto distribution is given by $M_Z(t)=E(e^{tZ})=k_*(-\theta t)^{k_*}\Gamma(-k_*,-\theta t)$. Then, mgf of the LGP distribution can be given by

$$M_X(t)=E(e^{tX})=\sum_{\upsilon\in V}^{\infty}w_\upsilon E(e^{tz})=\sum_{\upsilon\in V}w_\upsilon k_*(-\theta t)^{k_*}\Gamma(-k_*,-\theta t).$$

As an alternative method to calculate the mgf without using equation (8), we consider

$$M(t)=E(e^{tX})=\int_\theta^\infty e^{tx}g(x)dx=\sum_{i=0}^{\infty}\frac{t^i}{i!}\int_\theta^\infty x^i g(x)dx=\sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i) \tag{11}$$

By using equation (10) in equation (11), we obtain the moment generating function as

$$M_X(t)=\sum_{i=0}^{\infty}\left\{\frac{\theta^i t^i\beta^\alpha}{i!}\sum_{j=0}^{\infty}\binom{(i/k)+j-1}{j}(\beta+j)^{-\alpha}\right\}.$$

## 4. Some Entropies

The entropy represents a measure of uncertainty of a random variable. It is an important concept in many fields of science such as theory of communication, physics, engineering and economics (see the authors in [7]). We provide expressions for the Shannon entropy and Renyi entropy of the LGP distribution and present suitable constraints for the maximum entropy characterization of the log-gamma-generated family (2).

#### *4.1. Shannon Entropy*

The author in [7] introduced the most popular measure of entropy which is called Shannon entropy. For a continuous distribution $G(x)$ with density $g(x)$, the Shannon entropy is defined as

$$H_{Sh}[g(x)] = E\{-\log[g(X)]\} = -\int_{-\infty}^{+\infty}\{\log[g(x)]\}g(x)dx.$$

For the LGP distribution, the Shannon entropy is given by

$$H_{Sh}[g(x)] = \log\Gamma(\alpha) + (1-\alpha)\psi(\alpha) - \log\beta + (\beta-1)(\alpha/\beta) - \log(k/\theta) \\ + \frac{k+1}{k}\beta^{\alpha}\sum_{i=0}^{\infty}\frac{(\beta+i+1)^{-\alpha}}{(i+1)}.$$

where $\psi(\alpha) = \dfrac{d}{d\alpha}\log\Gamma(\alpha)$ is the digamma function.

#### *4.2. Renyi Entropy*

Another popular measure of entropy is the Renyi entropy (see the author in [8]) defined by

$$I_R(v) = (1-v)^{-1}\log\left[\int_{-\infty}^{+\infty}[g(x)]^v dx\right],$$

where $v > 0$ and $v \neq 1$. Raising equation (7) to the power $v$ and using similar expansions to those in section 2, we obtain

$$[g(x)]^v = \frac{\theta k^v \beta^{v\alpha}}{(\Gamma(\alpha))^v}\sum_{r,m,s=0}^{\infty}\binom{v(\beta-1)}{r}\binom{v(\alpha-1)}{m}\frac{(-1)^r b_{s,m}}{[kv\alpha+kr+ks+km-1]}x^{-v}k_{**}\frac{\theta^{k_{**}}}{x^{k_{**}}}$$

where $k_{**} = kv\alpha + ks + km + kr - 1$ and $b_{s,m}$ is defined in section 2. Integrating the above expression from $\theta$ to $\infty$ leads to a linear combination of Pareto moments of orders $(-v)$, the Renyi entropy of $X$ which follows the LGP distribution reduces to

$$I_R(v) = (1-v)^{-1}\log\left\{\frac{k^v\beta^{v\alpha}}{(\Gamma(\alpha))^v}\sum_{r,m,s=0}^{\infty}\binom{v(\alpha-1)}{m}\binom{v(\beta-1)}{r}\frac{(-1)^r b_{s,m}\theta^{1-v}}{(kv\alpha+kr+km+ks+v-1)}\right\}.$$

### *4.3. Maximum Entropy Characterization*

The author in [9] showed that the Shannon entropy may also be used to identify a probabilistic model (see the author in [4]). Suppose a class of probability densities under a set of constraints

$$F = \{g(x) | E[T_i(X)] = \alpha_i, i = 1,2,\dots,m\},$$

for which all expectations are assumed to exist and be finite. One should choose a member from $F$ as the density function for a random variable $X$ if it maximizes the Shannon entropy. The chosen density is called the maximum entropy distribution. The authors in [3] provided suitable constraints for the gamma-generated distributions such that the maximum entropy distribution is unique. For the log-gamma-generated distributions defined in (2), we propose the following constraints:

$$E[\log(-\log F(x))] = \psi(\alpha) - \log\beta \text{ and } E[\log f(x)] = E_Y[\log f(F^{-1}(e^{-y}))],$$

where $Y \sim Gamma(\alpha,\beta)$ and $\psi(.)$ denotes the digamma function. It can be shown that under the above constraints, the maximum entropy distribution is unique. The proof of this statement is very similar to Zografos and Balakrishnan gamma-generated case, and thus it is omitted.

## 5. The Distribution of the i$^{th}$ Order Statistic

It is proved in the authors in [6] that the density function of the i$^{th}$ order statistic of the GEE distribution is an infinite weighted sum of GEE density functions and according to the authors in [4] the density function of the i$^{th}$ order statistic of the GEW distribution is an infinite weighted sum of Weibull densities. We shall prove that, for any log-gamma-generated distribution with density (2), the density of the i$^{th}$ order statistic in a random sample of size *n* can be expressed as an infinite weighted sum of log-gamma-generated densities defined in (2).

Consider $X_1,\dots,X_n$ i.i.d random variables distributed according to (2). The pdf of the i$^{th}$ order statistic, say $X_{i:n}$, is given by

$$g_{i:n}(x) = \frac{n!\,g(x)}{(n-i)!(i-1)!}[1-G(x)]^{n-i}(G(x)^{i-1}.$$

Using the binomial theorem and equation (1), we obtain

$$g_{i:n}(x) = \frac{n!\,g(x)}{(n-i)!(i-1)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left(\frac{\Gamma[\alpha,-\log F(x)]}{\Gamma(\alpha)}\right)^{i+j-1}. \qquad (12)$$

Since $\Gamma[\alpha,-\log F(x)] = \Gamma(\alpha) - \gamma(\alpha,-\log F(x))$ and using the power series from the authors in [10].

$$\gamma(x,\alpha) = \sum_{\tau=0}^{\infty}\frac{(-1)^\tau x^{\tau+\alpha}}{(\tau+\alpha)\tau!}.$$

we can express the pdf $g_{i:n}(x)$ as

$$g_{i:n}(x) = \frac{n!\,g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{l=0}^{i+j-1}\binom{n-i}{j}\binom{i+j-1}{l}\frac{(-1)^{j+l}(-\log F(x))^{\alpha l}}{(\Gamma(\alpha))^l}$$
$$\times\left\{\sum_{\tau=0}^{\infty}\frac{(-1)^\tau(-\log F(x))^\tau}{\tau!(\tau+\alpha)}\right\}^l \qquad (13)$$

Following the authors in [4], let $c_\tau = \dfrac{(-1)^\tau}{\tau!(\alpha+\tau)}$ and use the power series raised to a positive integer, as in section 2, to write

$$\left[\sum_{\tau=0}^{\infty}c_\tau[-\log F(x)]^\tau\right]^l = \sum_{\tau=0}^{\infty}d_{\tau,l}[-\log F(x)]^\tau$$

where $d_{0,l} = c_0^l$, $d_{\tau,l} = (\tau c_0)^{-1}\sum_{w=1}^{\tau}[lw-\tau+w]c_w d_{\tau-w,l}$ and Substituting from equation (2) in equation (13), we obtain

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{l=0}^{i+j-1}\sum_{\tau=0}^{\infty}\binom{n-i}{j}\binom{i+j-1}{l}\frac{(-1)^{j+l}d_{\tau,l}}{(\Gamma(\alpha))^{l+1}}\frac{\Gamma(\alpha_*)}{\beta^{\alpha l+\tau}}g_{\alpha_*}(x) \qquad (14)$$

where $\alpha_* = \alpha(l+1)+\tau$ , and $g_{\alpha_*}(x)$ denotes the log-gamma-generated family (2) with two parameters $\alpha_*$ and $\beta$ . Then, it is obtained from (14) that the pdf of the i$^{th}$ order statistic from a random sample of size *n* for any log-gamma-generated family with density (2) can be expressed as an infinite sum of log-gamma-generated densities (2).

Equations (8) and (14) can be used to write the pdf of the i$^{th}$ order statistic from a sample of size *n* of the log-gamma-Pareto distribution as a linear combination of Pareto densities as follows:

$$g_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!} \sum_{j=0}^{n-i} \sum_{l=0}^{i+j-1} \sum_{\tau=0}^{\infty} \sum_{\upsilon 1 \in V} \binom{n-i}{j} \binom{i+j-1}{l}$$
$$\times \frac{(-1)^{j+l} w_{\upsilon 1} d_{\tau} \Gamma(\alpha_*)}{\beta^{\alpha l+\tau} (\Gamma(\alpha))^{l+1}} f_{*1}(x)$$

where $f_{*1}(x)$ is Pareto density with scale parameter $\theta$ and shape parameter $k_{*1} = k(\alpha_* + r + m + s)$ and the weights

$$w_{\upsilon 1} = \frac{(-1)^r \beta^{\alpha_*}}{\Gamma(\alpha_*)} \binom{\beta-1}{r} \binom{\alpha_*-1}{m} \frac{b_{s,m}}{m+r+s+\alpha_*} \quad for\ \upsilon 1 \in V$$

The r$^{th}$ moment about zero of the i$^{th}$ order statistic from the log-gamma-Pareto distribution is given by

$$E(X_{i:n}^r) = \frac{n!}{(n-i)!(i-1)!} \sum_{j=0}^{n-i} \sum_{l=0}^{i+j-1} \sum_{\tau=0}^{\infty} \sum_{\upsilon 1 \in V} \binom{n-i}{j} \binom{i+j-1}{l}$$
$$\times \frac{(-1)^{j+l} w_{\upsilon 1} d_{\tau,l} \Gamma(\alpha_*)}{\beta^{\alpha l+\tau} (\Gamma(\alpha))^{l+1}} \frac{k_{*1} \theta^r}{k_{*1} - r}.$$

**6. Maximum-likelihood Estimation**

In this section, we consider the estimation of the unknown parameters by method of maximum likelihood. For $x_1, x_2, ..., x_n$ a random sample of size *n* from the $LGP(\alpha, \beta, k, \theta)$ distribution, the log-likelihood function based on the given random sample is

$$\log L = n \log k + n\alpha \log \beta - n \log \Gamma(\alpha) + nk \log \theta - (k+1) \sum_{i=1}^{n} \log x_i$$
$$+ (\beta - 1) \sum_{i=1}^{n} \log \left( 1 - \left( \frac{\theta}{x_i} \right)^k \right) + (\alpha - 1) \sum_{i=1}^{n} \log \left[ -\log \left( 1 - \left( \frac{\theta}{x_i} \right)^k \right) \right].$$

The first partial derivatives of the log-likelihood function with respect to the parameters $\alpha, \beta, k$ and $\theta$ are, respectively,

$$\frac{\partial \log L}{\partial \alpha} = n \log \beta - n\psi(\alpha) + \sum_{i=1}^{n} \log \left[ -\log \left( 1 - \left( \frac{\theta}{x_i} \right)^k \right) \right],$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n\alpha}{\beta} + \sum_{i=1}^{n} \log\left(1 - \left(\frac{\theta}{x_i}\right)^k\right),$$

$$\frac{\partial \log L}{\partial k} = \frac{n}{k} + n\log\theta - \sum_{i=1}^{n}\log x_i - (\beta-1)\sum_{i=1}^{n}\frac{(\theta/x_i)^k \log(\theta/x_i)}{(1-(\theta/x_i)^k)}$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{(\theta/x_i)^k \log(\theta/x_i)}{(1-(\theta/x_i)^k)\log(1-(\theta/x_i)^k)},$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{nk}{\theta} - (\beta-1)\sum_{i=1}^{n}\frac{k(\theta/x_i)^k}{\theta(1-(\theta/x_i)^k)}$$
$$-(\alpha-1)\sum_{i=1}^{n}\frac{k(\theta/x_i)^k}{\theta(1-(\theta/x_i)^k)\log(1-(\theta/x_i)^k)}. \quad (15)$$

Equating the system of equations (15) with zero, we obtain

$$0 = n\log\hat{\beta} - n\psi(\hat{\alpha}) + \sum_{i=1}^{n}\log\left[-\log\left(1-\left(\frac{\hat{\theta}}{x_i}\right)^{\hat{k}}\right)\right],$$

$$0 = \frac{n\hat{\alpha}}{\hat{\beta}} + \sum_{i=1}^{n}\log\left(1-\left(\frac{\hat{\theta}}{x_i}\right)^{\hat{k}}\right),$$

$$0 = \frac{n}{\hat{k}} + n\log\hat{\theta} - \sum_{i=1}^{n}\log x_i - (\hat{\beta}-1)\sum_{i=1}^{n}\frac{(\hat{\theta}/x_i)^{\hat{k}}\log(\hat{\theta}/x_i)}{(1-(\hat{\theta}/x_i)^{\hat{k}})}$$
$$+(\hat{\alpha}-1)\sum_{i=1}^{n}\frac{(\hat{\theta}/x_i)^{\hat{k}}\log(\hat{\theta}/x_i)}{(1-(\hat{\theta}/x_i)^{\hat{k}})\log(1-(\hat{\theta}/x_i)^{\hat{k}})},$$

and

$$0 = \frac{n\hat{k}}{\hat{\theta}} - (\hat{\beta}-1)\sum_{i=1}^{n}\frac{\hat{k}(\hat{\theta}/x_i)^{\hat{k}}}{\hat{\theta}(1-(\hat{\theta}/x_i)^{\hat{k}})}$$
$$-(\hat{\alpha}-1)\sum_{i=1}^{n}\frac{\hat{k}(\hat{\theta}/x_i)^{\hat{k}}}{\hat{\theta}(1-(\hat{\theta}/x_i)^{\hat{k}})\log(1-(\hat{\theta}/x_i)^{\hat{k}})}. \quad (16)$$

The system of equations (16) can be solved numerically using statistical packages and the MLEs $\hat{\alpha}, \hat{\beta}, \hat{k}$ and $\hat{\theta}$ can be obtained. Under suitable regularity conditions, the distribution of the maximum-likelihood estimator $\hat{\phi} = (\hat{\alpha}, \hat{\beta}, \hat{k}, \hat{\theta})$ of the vector parameter $\phi = (\alpha, \beta, k, \theta)$ is given by

$$\sqrt{n}(\hat{\phi} - \phi) \sim N_4(0, I^{-1}(\phi)),$$

where $I(\phi)$ is the Fisher information matrix given by

$$I(\phi) = -E\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha k} & I_{\alpha\theta} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta k} & I_{\beta\theta} \\ I_{k\alpha} & I_{k\beta} & I_{kk} & I_{k\theta} \\ I_{\theta\alpha} & I_{\theta\beta} & I_{\theta k} & I_{\theta\theta} \end{bmatrix}.$$

where

$$I_{\alpha\alpha} = \frac{\partial^2 \log L}{\partial\alpha\partial\alpha} = -n\psi'(\alpha), I_{\alpha\beta} = I_{\beta\alpha} = \frac{\partial^2 \log L}{\partial\alpha\partial\beta} = \frac{n}{\beta}, I_{\beta\beta} = \frac{\partial^2 \log L}{\partial\beta\partial\beta} = -\frac{n\alpha}{\beta^2},$$

$$I_{\alpha k} = I_{k\alpha} = \frac{\partial^2 \log L}{\partial\alpha\partial k} = -\sum_{i=1}^{n} \frac{(\theta / x_i)^k \log(\theta / x_i)}{[1-(\theta / x_i)^k]\{\log[1-(\theta / x_i)^k]\}},$$

$$I_{\alpha\theta} = I_{\theta\alpha} = \frac{\partial^2 \log L}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n} \frac{(k/\theta)(\theta / x_i)^k}{[1-(\theta / x_i)^k]\{\log[1-(\theta / x_i)^k]\}},$$

$$I_{\beta k} = I_{k\beta} = \frac{\partial^2 \log L}{\partial\beta\partial k} = -\sum_{i=1}^{n} \frac{(\theta / x_i)^k \log(\theta / x_i)}{[1-(\theta / x_i)^k]},$$

$$I_{\beta\theta} = I_{\theta\beta} = \frac{\partial^2 \log L}{\partial\beta\partial\theta} = -\sum_{i=1}^{n} \frac{(k/\theta)(\theta / x_i)^k}{[1-(\theta / x_i)^k]},$$

$$I_{kk} = \frac{\partial^2 \log L}{\partial k\partial k} = -\frac{n}{k^2} - (\beta-1)\sum_{i=1}^{n} \frac{(\theta / x_i)^k (\log(\theta / x_i))^2}{[1-(\theta / x_i)^k]^2}$$
$$-(\alpha-1)\sum_{i=1}^{n} \frac{(\theta / x_i)^k (\log(\theta / x_i))^2 \{(\theta / x_i)^k + \log[1-(\theta / x_i)^k]\}}{[1-(\theta / x_i)^k]^2 \{\log[1-(\theta / x_i)^k]\}^2},$$

$$I_{k\theta}=I_{\theta k}=\frac{\partial^2\log L}{\partial k\partial\theta}=\frac{n}{\theta}-(\beta-1)\sum_{i=1}^{n}\frac{(k/\theta)(\theta/x_i)^k\log(\theta/x_i)+(1/\theta)(\theta/x_i)^k[1-(\theta/x_i)^k]}{[1-(\theta/x_i)^k]^2}-(\alpha-1)\times$$
$$\sum_{i=1}^{n}\frac{(\theta/x_i)^k(k/\theta)\log(\theta/x_i)\{(\theta/x_i)^k+\log[1-(\theta/x_i)^k]\}+(1/\theta)(\theta/x_i)^k\log[1-(\theta/x_i)^k][1-(\theta/x_i)^k]}{[1-(\theta/x_i)^k]^2\{\log[1-(\theta/x_i)^k]\}^2},$$

and

$$I_{\theta\theta}=\frac{\partial^2\log L}{\partial\theta\partial\theta}=\frac{-nk}{\theta^2}-(\beta-1)\sum_{i=1}^{n}\frac{(k/\theta)^2(\theta/x_i)^k-(k/\theta^2)(\theta/x_i)^k[1-(\theta/x_i)^k]}{[1-(\theta/x_i)^k]^2}-(\alpha-1)\times$$
$$\sum_{i=1}^{n}\frac{(k/\theta)^2(\theta/x_i)^k\{(\theta/x_i)^k+\log[1-(\theta/x_i)^k]\}-(k/\theta^2)(\theta/x_i)^k\log[1-(\theta/x_i)^k][1-(\theta/x_i)^k]}{[1-(\theta/x_i)^k]^2\{\log[1-(\theta/x_i)^k]\}^2} \quad (17)$$

**7. Application**

The LGP distribution is compared to the gamma distribution, the Pareto distribution, the gamma-Pareto (GP) distribution introduced by the authors in [3]. Each distribution was fitted to the dataset using computer facilities (Mathcad 2001). We calculated the MLEs for each model parameters and their standard errors, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the logarithm of the maximized likelihood for each fit. We calculated the Kolomogrov-Smirnov test statistic and its *p*-value to decide whether the LGP distribution or its special cases provided a superior fit.

The data used in this application is from Colorado Climate Center, Colorado State University given inthe authors in [6]. The results of these data are presented in table 1. From these results, we can observe that the LGP distribution provides a very good fit to these data followed by the gamma, GP and Pareto distributions.

Table (1). Estimates and relative goodness-of-fit measures for the one rain gauge data.

| Distributions | Estimates | | | | LL | AIC | BIC | K-S | *p*-value |
|---|---|---|---|---|---|---|---|---|---|
| Gamma $(\alpha,\beta)$ | 5.28 | 0.03 | - | - | -568.97 | 1142 | 1147 | 0.062 | 0.925 |
| | (0.219) | (0.0013) | | | | | | | |
| Pareto $(k,\theta)$ | 0.999 | 58.56 | - | - | -607.17 | 1218 | 1224 | 0.29 | 0 |
| | (2.924) | (171.55) | | | | | | | |
| GP $(\alpha,k,\theta)$ | 0.347 | 3.14 | 59.93 | - | -598.16 | 1202 | 1210 | 0.241 | 0 |
| | (0.122) | (1.08) | (0.23) | | | | | | |
| LGP $(\alpha,\beta,k,\theta)$ | 17.94 | 50.52 | 0.459 | 10.79 | -565.11 | 1138 | 1149 | 0.041 | 0.996 |
| | (55.13) | (104.83) | (1.058) | (29.221) | | | | | |

where the values in the parentheses is the standard errors of the MLE's of the parameters**.**

The second partial derivatives (17) are then evaluated at the resulted estimates of the LGP distribution's parameters in order to be used in finding the following variance-covariance matrix for the parameters $\alpha, \beta, k$ and $\theta$:

$$I^{-1}(\phi) = \begin{bmatrix} 3039 & -1947 & -57.508 & 547.761 \\ -1947 & 10990 & 53.82 & -3057 \\ -57.508 & 53.82 & 1.119 & -15.014 \\ 547.761 & -3057 & -15.014 & 853.882 \end{bmatrix}$$

Also, we will generate 1000 samples of sizes 15, 20…, 30 from LGP distribution for different values of the parameters $\alpha, \beta, k$ and $\theta$ using proposed random number generator, and then the maximum likelihood estimates for each sample will be obtained along with the mean, biases, mean square error, standard error, skewness and kurtosis of those estimates for different sample sizes.

In table 2, we list the mean, biases, mean square error, standard error, skewness and kurtosis for the MLEs of the parameters $\alpha, \beta, k$ and $\theta$ for 1000 random samples of sizes 15, 20…, 30. The table shows that the mean square error and the bias for the parameters $\alpha, \beta, k$ and $\theta$ decrease as the sample sizes increase. We used a statistical package called Mathcad 2001.

Table (2).Means, Biases, Mean Square Errors, Standard Errors, Skewness and Kurtosis for the estimates of LGP distribution's parameters for different values of the parameters.

| $\alpha = 1.223$ $\beta = 0.512$ $k = 0.00821$ $\theta = 0.000215$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| *n* | | Mean | Bias | MSE | Standard error | Skewness | Kurtosis |
| 15 | $\alpha$ | 1.4889 | -0.2659 | 0.0884 | 0.0089 | -1.4995 | 1.9391 |
| | $\beta$ | 4.2216 | -3.7096 | 15.6062 | 0.0905 | -3.0194 | 9.3352 |
| | $k$ | 0.1102 | -0.102 | 0.0351 | 0.0105 | 3.1466 | 9.926 |
| | $\theta$ | 0.01102 | -0.0109 | 0.0009 | 0.0019 | 3.1617 | 9.9975 |
| 20 | $\alpha$ | 1.5835 | -0.3605 | 0.1307 | 0.0014 | -0.0943 | -1.7644 |
| | $\beta$ | 4.6253 | -4.1133 | 16.9222 | 0.0027 | 1.6026 | 1.0845 |
| | $k$ | 0.0549 | -0.0467 | 0.00022 | 0 | 1.7797 | 1.4123 |
| | $\theta$ | 0.0025 | -0.0022 | 0 | 0 | 1.6357 | 1.0726 |

| $\alpha=1.223$ $\beta=0.512$ $k=0.00821$ $\theta=0.000215$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | Mean | Bias | MSE | Standard error | Skewness | Kurtosis |
| 25 | $\alpha$ | 1.5201 | -0.2971 | 0.1214 | 0.0073 | -3.0583 | 9.5116 |
| | $\beta$ | 4.5303 | -4.0183 | 16.1966 | 0.009 | -3.1489 | 9.9384 |
| | $k$ | 0.0586 | -0.0504 | 0.0027 | 0.0005 | 3.1623 | 10 |
| | $\theta$ | 0.0024 | -0.0022 | 0 | 0 | -3.1353 | 9.8751 |
| 30 | $\alpha$ | 1.5191 | -0.2961 | 0.0995 | 0.0036 | -2.2602 | 5.5044 |
| | $\beta$ | 4.8427 | -4.3307 | 19.4123 | 0.027 | 3.1373 | 9.8895 |
| | $k$ | 0.0574 | -0.0492 | 0.0024 | 0.0002 | 1.6392 | 1.2098 |
| | $\theta$ | 0.0024 | -0.0021 | 0 | 0 | -3.1141 | 9.7624 |

## 8. Conclusion

We introduced and studied the log-gamma-Pareto (LGP) distribution and developed, from previous partial proofs, two general results on the log-gamma-generated family (2), the distribution of the $i^{th}$ order statistic and maximum entropy characterization. The general properties of the new distribution can be obtained by expressing its density function as a linear combination of Pareto density functions. It is very likely that the pdf of any member of the log-gamma-generated family of distributions (2) and (3) can be expressed as a linear combination of the parent pdf which will be generalized using these families.

The real dataset example suggests the LGP distribution as an improved alternative to the Pareto distribution as a distribution of incomes and in reliability.

**Acknowledgements.** We are grateful for the contribution, comments and suggestions made by the anonymous referees, which improved the presentation and content of this paper. We also thank the editor of the journal for his time and attention.

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