On Terminal Coalgebras Derived from Initial Algebras

Jiří Adámek
Department of Mathematics, Faculty of Electrical Engineering,
Czech Technical University in Prague, Czech Republic
j.adamek@tu-bs.de

Abstract

A number of important set functors have countable initial algebras, but terminal coalgebras are uncountable or even non-existent. We prove that the countable cardinality is an anomaly: every set functor with an initial algebra of a finite or uncountable regular cardinality has a terminal coalgebra of the same cardinality.

We also present a number of categories that are algebraically complete and cocomplete, i.e., every endofunctor has an initial algebra and a terminal coalgebra.

Finally, for finitary set functors we prove that the initial algebra μF and terminal coalgebra νF carry a canonical ultrametric with the joint Cauchy completion. And the algebra structure of μF determines, by extending its inverse continuously, the coalgebra structure of νF.

2012 ACM Subject Classification Theory of computation → Categorical semantics

Keywords and phrases terminal coalgebras, initial algebras, algebraically complete category, finitary functor, fixed points of functors

Digital Object Identifier 10.4230/LIPIcs.CALCO.2019.12

Funding *Jiří Adámek*: Supported by the Grant Agency of the Czech Republic under the grant 19-00902S.

Acknowledgements The referees helped improving the presentation of this paper by numerous valuable suggestions.

1 Introduction

Initial algebras for endofunctors are important in formal semantics and the theory of recursive domain equations. Further, for state based systems represented as coalgebras, Rutten [11] demonstrated that the terminal coalgebra formalizes behavior of states. If we work in the category of *cpo*'s as our base category, and if the given endofunctor is locally continuous, then Smyth and Plotkin proved in [12] that the initial algebra coincides with the terminal coalgebra. That is, the underlying objects are equal, and the structure maps are inverse to each other.

Is there a connection between initial algebras μF and terminal coalgebras νF for set functors F, too? In the case where F preserves limits of ω^{op}-chains, νF carries a canonical structure of a metric space and, whenever $F\emptyset \neq \emptyset$, this is the Cauchy completion of μF as its subspace, as proved by Barr [8]. But what can we say about general set functors? There are cases where μF is countable and νF is uncountable (e.g. $FX = A \times X + 1$, with $\mu F = A^*$ and $\nu F = A^\infty$) or νF does not exist:

▶ **Example 1** (see [4])**.** The following set functor F has a countable initial algebra but no terminal coalgebra:

$$FX = \{M \subseteq X \,;\, \mathrm{card}\ M \neq \aleph_0\}.$$

8th Conference on Algebra and Coalgebra in Computer Science (CALCO 2019).
Editors: Markus Roggenbach and Ana Sokolova; Article No. 12; pp. 12:1–12:21
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

To a function $f\colon X \to Y$ it assigns the function Ff turning $M \subseteq X$ to $f[M]$ if f restricted to M is monic or M is finite, else to $\emptyset$. Its initial algebra is that of the finite power-set functor (consisting of all hereditarily finite sets).

We are going to prove that the cardinal $\aleph_0$ is the only exception: whenever a set functor has a nonempty initial algebra of a finite or uncountable regular cardinality, then it has a terminal coalgebra of the same cardinality. See the Terminal-Coalgebra Theorem in Section 4. We also prove that the existence of a fixed point $FX \simeq X$ of an uncountable regular cardinality implies that the set functor F has a terminal coalgebra. This corresponds well with the result of [15] that every set functor with a fixed point has an initial algebra.

On the way to proving these results we present a number of categories that are algebraically complete and cocomplete. The concept of algebraic completeness, due to Freyd [9], means that every endofunctor has an initial algebra. But Freyd did not present any examples. It may seem that there are no "natural" examples since, as proved in [4], an algebraically complete category cannot be complete, unless it is equivalent to a preordered class. However, we prove that for every uncountable, regular cardinal λ the category $\mathbf{Set}_{\leq\lambda}$ of sets of cardinality at most λ is algebraically complete and cocomplete. That is, every endofunctor F has both μF and νF. For $\lambda > \aleph_1$ (the first uncountable cardinal) the category $\mathbf{Nom}_{\leq\lambda}$ of nominal sets of cardinality at most λ is also algebraically complete and cocomplete. Analogously, the category K-$\mathbf{Vec}_{\leq\lambda}$ of vector spaces of dimension at most λ, for any field K with $|K| < \lambda$, is algebraically complete and cocomplete. Finally, if G is a group, consider the category G-$\mathbf{Set}$ of sets with an action of G. For every group with $2^{|G|} < \lambda$ the category G-$\mathbf{Set}_{\leq\lambda}$ of G-sets of cardinality at most λ is algebraically complete and cocomplete. These results require assuming the Generalized Continuum Hypothesis.

Returning to metric structures on terminal coalgebras, we prove that for finitary set functors F with $F\emptyset \neq \emptyset$ the initial algebra and terminal coalgebra carry a canonical ultrametric such that the Cauchy completions of μF and νF coincide. And the coalgebra structure of νF is determined by the algebra structure ι of μF: it is the unique continuous extension of ι^{-1} to νF. This complements the above result of Barr [8].

2 Algebraically Cocomplete Categories

For a number of categories $\mathcal{K}$ we prove that the full subcategory $\mathcal{K}_{\leq\lambda}$ on objects of power at most λ is algebraically cocomplete. Power is a cardinal we introduce as follows:

▶ **Definition 2.** An object is called *connected* if it is non-initial and is not a coproduct of two non-initial objects. An object is said to have *power* λ if it is a coproduct of λ connected objects, but not of less than λ ones.

▶ **Example 3.** In $\mathbf{Set}$, connected objects are the singleton sets, and power of a set X is its cardinality $|X|$. In the category K-$\mathbf{Vec}$ of vector spaces over a field K the connected spaces are those of dimension one, and power means dimension. In the category $\mathbf{Set}^S$ of many-sorted sets the connected objects are those with precisely one element (in all sorts together), and the power of $X = (X_s)_{s\in S}$ is simply $|\coprod_{s\in S} X_s|$. A nominal set is connected in the category $\mathbf{Nom}$ of nominal sets and equivariant maps iff it consists of a single orbit.

▶ **Definition 4.** A category $\mathcal{K}$ is said to have *width* $w(\mathcal{K})$ if it has coproducts, every object is a coproduct of connected objects, and $w(\mathcal{K})$ is the smallest cardinal such that

(a) $\mathcal{K}$ has at most $w(\mathcal{K})$ connected objects up to isomorphism, and

(b) given an object K of power $\alpha \geq w(\mathcal{K})$, all quotients of K have power at most α, and there exist at most α morphisms from a connected object to K.

▶ **Example 5.**

(1) **Set** has width 1. More generally, $\mathbf{Set}^S$ has width $|S|$. Indeed, in Example 3 we have seen that the number of connected objects up to isomorphism is $|S|$, and (b) clearly holds.

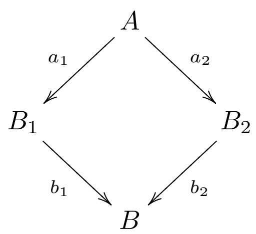
(2) K-**Vec** has width $|K| + \aleph_0$. Indeed, the only connected object, up to isomorphism, is K. For a space X of dimension α the number of morphisms from K to X is $|X|$. If K is infinite, then $\alpha \geq |K|$ implies $|X| = \alpha$ (and $|K| = |K| + \aleph_0$). For K finite, the least cardinal λ such that $|X| \leq \alpha$ holds for every α-dimensional spaces X with $\alpha \geq \lambda$ is $\aleph_0$ $(= |K| + \aleph_0)$.

(3) The category **Nom** of nominal sets has width $\aleph_0$.

(4) For every nontrivial finite group G the category G-**Set** of sets with an action of the group has width $\aleph_0$. For infinite groups the width of G-**Set** is at most λ if $2^{|G|} \leq \lambda$. For the proof of (3) and (4) see the Appendix.

We now present some technical results serving for the proof of Theorem 13 below. The following lemma is based on ideas of Trnková [14].

▶ **Lemma 6.** *Let a commutative square*



be given in a category $\mathcal{A}$. This is an absolute pullback, i.e., a pullback preserved by all functors with domain $\mathcal{A}$, provided that (1) b_1 and b_2 are split monomorphisms, and (2) there exist morphisms $\bar{b}_1 : B \to B_1$ and $\bar{a}_2 : B_2 \to A$ satisfying

$$\bar{b}_1 b_1 = \mathrm{id}\,, \quad \bar{a}_2 a_2 = \mathrm{id}, \quad \text{and} \quad a_1\bar{a}_2 = \bar{b}_1 b_2. \tag{2.1}$$

Proof. The given square is a pullback since given a commutative square

$$b_1 c_1 = b_2 c_2 \quad \text{for} \quad c_i \colon C \to B_i$$

there exists a unique c with $c_i = a_i \cdot c$ $(i = 1, 2)$. Uniqueness is clear since a_2 is split monic. Put $c = \bar{a}_2 \cdot c_2$. Then $c_1 = a_1 c$ follows from b_1 being monic:

$$\begin{aligned} b_1 c_1 &= b_1 \bar{b}_1 b_1 c_1 & \bar{b}_1 b_1 &= \mathrm{id} \\ &= b_1 \bar{b}_1 b_2 c_2 & b_1 c_1 &= b_2 c_2 \\ &= b_1 a_1 \bar{a}_2 c_2 & \bar{b}_1 b_2 &= a_1 \bar{a}_2 \\ &= b_1 a_1 c & c &= \bar{a}_2 c_2 \end{aligned}$$

And $c_2 = a_2 c$ follows from b_2 being monic:

$$\begin{aligned} b_2 c_2 &= b_1 c_1 & & \\ &= b_1 a_1 c & c_1 &= a_1 c \\ &= b_2 a_2 c & b_1 a_1 &= b_2 a_2 \end{aligned}$$

For every functor F with domain $\mathcal{A}$ the image of the given square satisfies the analogous conditions: Fb_1 and Fb_2 are split monomorphisms and $F\bar{b}_1$, $F\bar{a}_2$ verify (2.1). Thus, the image is an (absolute) pullback, too. ◀

▶ **Corollary 7** (See [14]). *Every set functor preserves nonempty finite intersections.*

Indeed if A in Lemma 6 is nonempty, choose an element $t \in A$ and define $\bar{b}_1$ and $\bar{a}_2$ by

$$\bar{b}_1(x) = \begin{cases} y & \text{if } b_1(y) = x \\ a_1(t) & \text{if } x \notin b_1[B_1] \end{cases}$$

and

$$\bar{a}_2(x) = \begin{cases} y & \text{if } a_2(y) = x \\ t & \text{if } x \notin a_2[A] \end{cases}$$

It is easy to see that (2.1) holds.

▶ Remark 8.

(a) We recall that for an infinite cardinal λ the *cofinality* is the smallest cardinal μ such that λ is a join of a μ-chain of smaller cardinals. And λ is *regular* if it is equal to its cofinality. The first non-regular cardinal is $\aleph_\omega = \bigvee_{n<\omega} \aleph_n$.

(b) For a set X of infinite cardinality λ a collection of subsets of cardinality λ is called *almost disjoint* if the intersection of arbitrary two distinct members has cardinality smaller than λ.
Tarski [13] proved that for every set X of infinite regular cardinality λ there exists an almost disjoint collection $Y_i \subseteq X$ $(i \in I)$ with $|I| > \lambda$. The argument is quite simple. Using the Maximality Principle (also known as Zorn's Lemma), we see that a maximum almost disjoint collection exists on X. Assuming that it has at most λ members, we derive a contradiction. We can index that collection by ordinals $i < \lambda$. Given an almost disjoint collection $(Y_i)_{i<\lambda}$, the following sets $Z_i = Y_i - \bigcup_{j<i} Y_j$ for $i < \lambda$ are clearly pairwise disjoint and, since λ is reguar, they have cardinality λ. We can find a choice set $Z^* \subseteq X$. From $|Z^* \cap Z_i| = 1$ it follows that $|Z^* \cap Y_i| < \lambda$ for every $i < \lambda$, thus, we can add Z^* to the given collection. This contradicts the maximality.

(c) Given an element $t \in X$ there exists a maximum almost disjoint collection Y_i, $i \in I$, with $t \in Y_i$ for all $i \in I$. Indeed, take any maximum collection $(Y_i)_{i\in I}$ and use $Y_i \cup \{t\}$ instead of Y_i (for $i \in I$).

▶ **Notation 9.** Let $\mathcal{K}$ be a category of width $w(\mathcal{K})$. For every infinite cardinal $\lambda > w(\mathcal{K})$ we denote by $\mathcal{K}_{\leq\lambda}$ the full subcategory of $\mathcal{K}$ on objects of power at most λ.

Our main technical tool is the following

▶ **Proposition 10.** *Let F be an endofunctor of $\mathcal{K}_{\leq\lambda}$ and $X = \coprod_{i\in I} X_i$ an object of $\mathcal{K}$ with all X_i connected and $|I| = \lambda$. Every morphism $b\colon B \to FX$, with B of power less than λ, factorizes through Fc for a coproduct injection $c\colon C \to X$ where $C = \coprod_{j\in J} X_j$ and $|J| < \lambda$.*

The proof can be found in the Appendix.

▶ **Proposition 11.** *Let λ be an uncountable regular cardinal. Every coalgebra for an endofunctor of $\mathcal{K}_{\leq\lambda}$ is a colimit of a λ-filtered diagram of coalgebras on objects of powers smaller than λ.*

Proof. Let $\beta : B \to FB$ be a coalgebra. Express $B = \coprod_{i \in I} B_i$ where B_i are connected and $|I| = \lambda$ (the case $|I| < \lambda$ is trival). For every set $J \subseteq I$ with $|J| < \lambda$ we are going to prove that there exists a set $J \subseteq J' \subseteq I$ with $|J'| < \lambda$ such that the summand

$$u_{J'} \colon B_{J'} = \coprod_{i \in J'} B_i \to B$$

carries a subcoalgebra. That is, there exists $\beta_{J'} \colon B_{J'} \to FB_{J'}$ for which $u_{J'}$ is a homomorphism. This proves the proposition: the diagram of all subcoalgebras of (B, β) on summands of less than λ components is clearly λ-filtered. And its canonical colimit is (B, β). This follows from the fact that colimits of coalgebras are formed on the level of the underlying category.

For every set $J \subseteq I$ with $|J| < \lambda$ we are to present a set $J \subseteq J' \subseteq I$ with $|J'| < \lambda$ such that $\beta_{j'}$ exists. Put $J' = \bigcup_{n<\omega} J_n$ for the following ω-chain of sets $J_n \subseteq I$ with $|J_n| < \lambda$. First, $J_0 = J$. Given J_n, define J_{n+1} as follows. For every subset $L \subseteq J$ denote by $u_L \colon \coprod_{i \in L} B_i \to B$ the coproduct injection. Given $j \in J_n$, apply Proposition 10 to $X = B$ and

$$b = B_j \xrightarrow{u_{\{j\}}} B \xrightarrow{\beta} FB.$$

There exists a set $L(j) \subseteq J$ with $|L(j)| < \lambda$ such that the morphism $\beta \cdot u_{\{j\}}$ factorizes through $Fu_{L(j)}$. Consequently, for the set $L_n = \bigcup_{j \in J_n} L(j)$ we see that $\beta \cdot u_{J_n}$ factorizes through Fu_{L_n}. That is, there exists a morphism $\beta^n \colon \coprod_{i \in J_n} B_i \to \coprod_{i \in L_n} B_i$ with $Fu_{L_n} \cdot \beta^n = \beta \cdot u_{J_n}$. Define $J_{n+1} = J_n \cup L_n$, then $|J_{n+1}| < \lambda + \coprod_{j \in J} \lambda \leq \lambda + \lambda^2 = \lambda$.

Thus, for the union $J' = \bigcup J_n$ we get $|J'| < \lambda$ because λ is uncountable and regular, therefore $|\coprod_{n<\omega} J_n| < \lambda$. And $u_{J'}$ carries the following subcoalgebra $\beta_{J'} \colon \coprod_{j \in J'} B_j \to F(\coprod_{j \in J'} B_j)$: Given $j \in J'$ let n be the least number with $j \in J_n$. Denote by $w \colon B_j \to \coprod_{i \in J_n} B_i$ and $v \colon \coprod_{i \in L_n} B_i \to \coprod_{j \in J'} B_j$ the coproduct injections. Then the j-th component of β' is the following composite

$$B_j \xrightarrow{w} \coprod_{i \in J_n} B_i \xrightarrow{\beta^n} F(\coprod_{i \in L_n} B_i) \xrightarrow{Fv} F(\coprod_{i \in J'} B)$$

To prove that the following square

$$\begin{array}{ccc}
\coprod_{j \in J'} B_j & \xrightarrow{\beta_{J'}} & F(\coprod_{j \in J'} B_j) \\
\Big\downarrow{\scriptstyle u_{J'}} & & \Big\downarrow{\scriptstyle Fu_{J'}} \\
\coprod_{i \in I} B_i & \xrightarrow[\beta]{} & F(\coprod_{i \in I} B_i)
\end{array}$$

commutes, consider the components for $j \in J'$ separately. The upper passage yields, since $u_{J'} \cdot v = u_{L_n} \colon \coprod_{i \in L_n} B_i \to \coprod_{i \in I} B_i$, the result

$$Fu_{J'} \cdot (Fv \cdot \beta^n \cdot w) = Fu_{L_n} \cdot \beta^n \cdot w = \beta \cdot u_{J_n} \cdot w\,.$$

The lower passage yields the same result. ◀

▶ Remark 12.

(a) Every ordinal α is considered as the set of all smaller ordinals. In particular $\aleph_0$ is the set of all natural numbers, and $\aleph_1$ the set of all countable ordinals.

(b) For cardinals λ and μ the power λ^μ is cardinality of the set of all functions from μ to λ.

(c) If an infinite cardinal λ has cofinality μ, then $\lambda^\mu > \lambda$, see [10], Corollary 1.6.4.

(d) Recall the *General Continuum Hypothesis (GCH)* which states that for every infinite cardinal λ the successor cardinal is 2^λ.

Under GCH every infinite regular cardinal λ fulfils $\lambda^\mu = \lambda \quad$ for all cardinals $\quad 1 \le \mu < \lambda$. See Theorem 1.6.17 in [10].

▶ **Theorem 13.** *Assume GCH. If $\mathcal{K}$ is a cocomplete and cowellpowered category of width $w(\mathcal{K})$, then $\mathcal{K}_{\le\lambda}$ is algebraically cocomplete for all uncountable regular cardinals $\lambda > w(\mathcal{K})$.*

Proof. Let F be an endofunctor of $\mathcal{K}_{\le\lambda}$. Form a collection $a_i\colon A_i \to FA_i$ $(i \in I)$ representing all coalgebras of F on objects of power less than λ (up to isomorphism of coalgebras). We have $|I| \le \lambda$. Indeed, for every cardinal $n < \lambda$ let $I_n \subseteq I$ be the subset of all i with A_i having power n. Given $i \in I_n$, for every component $b\colon B \to A_i$ of A_i we know, since $\lambda > w(\mathcal{K})$, that there are at most λ morphisms from B to FA_i (recalling that FA_i has power at most λ), see (b) in Definition 4. Thus there are at most $n \cdot \lambda = \lambda$ morphisms from A_i to FA_i. And the number of objects A_i with n components is at most $w(\mathcal{K})^n < \lambda^n = \lambda$ (see Remark 12(d)). Thus, there are at most λ indexes in I_n. Since $I = \bigcup_{n<\lambda} I_n$, this proves $|I| \le \lambda^2 = \lambda$.

Consequently $A = \coprod_{i\in I} A_i$ is an object of $\mathcal{K}_{\le\lambda}$. We have the coalgebra structure $\alpha\colon A \to FA$ of a coproduct of (A_i, α_i) in $\mathbf{Coalg}\,F$. Let $e\colon A \to T$ be the wide pushout of all homomorphisms in $\mathbf{Coalg}\,F$ with domain (A, α) carried by epimorphisms of $\mathcal{K}$. Since $\mathcal{K}$ is cocomplete and cowellpowered, and since the forgetful functor from $\mathbf{Coalg}\,F$ to $\mathcal{K}$ creates colimits, this means that we form the corresponding pushout in $\mathcal{K}$ and get a unique coalgebra structure $\tau\colon T \to FT$ making e a homomorphism:

$$\begin{array}{ccc} \coprod A_i & \xrightarrow{\alpha} & F(\coprod A_i) \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle Fe} \\ T & \xrightarrow[\tau]{} & FT \end{array}$$

The power of T is at most λ since T is a quotient of A, see (b) in Definition 4. We are going to prove that (T, τ) is a terminal coalgebra.

For every coalgebra $\beta\colon B \to FB$ with B having power less than λ there exists a unique homomorphism into (T, τ). Indeed, the existence is clear: compose the isomorphism that exists from (B, β) to some (A_i, α_i) , the i-th coproduct injection of (A, α) and the above homomorphism e. To prove uniqueness, observe that by definition of (T, τ), this coalgebra has no nontrivial quotient: every homomorphism with domain (T, τ) whose underlying morphism is epic in $\mathcal{K}$ is invertible. Given homommorphisms u, $v\colon (B, \beta) \to (T, \tau)$

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle u}\downarrow\downarrow{\scriptstyle v} & & {\scriptstyle Fu}\downarrow\downarrow{\scriptstyle Fv} \\ T & \xrightarrow{\tau} & FT \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Fq} \\ Q & \dashrightarrow & FQ \end{array}$$

form their coequalizer $q\colon T \to Q$ in $\mathcal{K}$. Then Q carries the structure of a coalgebra making q a homomorphism. Thus, q is invertible, proving $u = v$.

From Proposition 11 we deduce that the same holds for *all* coalgebras, thus (T, τ) is terminal. ◀

3 Algebraically Complete Categories

All the concrete categories proved to be algebraically cocomplete above turn out to be algebraically complete, too. Moreover, General Continuum Hypothesis need not be assumed for this result.

▶ Remark 14. In this remark we assume that, for a given ordinal λ, all (co)limits mentioned below exist. We denote by 0 the intial object and by 1 the terminal one.

(a) Recall from [1] the *initial-algebra λ-chain* of an endofunctor F: its objects $F^i 0$ for all ordinals $i \le \lambda + 1$ and its connecting morphisms $w_{ij}\colon F^i 0 \to F^j 0$ for all $i \le j \le \lambda + 1$ are defined by transfinite recursion as follows: $F^0 0 = 0$, $F^{i+1}0 = F(F^i 0)$, and $F^j 0 = \underset{i<j}{colim} F^i 0$ for limit ordinals $j \le \lambda$. Analogously: $w_{01}\colon 0 \to F0$ is unique, $w_{i+1,j+1} = F w_{ij}$, and w_{ij} $(i < j)$ is a colimit cocone for every limit ordinal $j \le \lambda$.

(b) The initial-algebra chain *converges* at λ if the connecting map $w_{\lambda,\lambda+1}$ is invertible. In that case we get the initial-algebra

$$\mu F = F^\lambda 0$$

with the algebra structure $\iota = w^{-1}_{\lambda,\lambda+1}$

(c) In particular, if F preserves colimits of λ-chains for a limit ordinal λ, then $\mu F = F^\lambda 0$.

(d) Dually, the *terminal-coalgebra λ-chain* has objects $F^i 1$ (for $i \le \lambda + 1$) with $F^0 1 = 1$, $F^{i+1}1 = F(F^i 1)$ and $F^j 1 = \lim\limits_{i<j} F^i 0$ for limit ordinals $j \le \lambda$. Its connecting morphisms are denoted by v_{ij} $(i \ge j)$. If F preserves limits of λ^{op}-chains, then $\nu F = F^\lambda 1$. This was explicitly formulated by Barr [8].

(e) We say that a set functor F *preserves inclusion* if given a subset Y of X, then FY is a subset of FX, and for the inclusion map $i : Y \to X$ also Fi is the inclusion map . It follows that F preserves monomorphisms.

For every set functor F there exists a set functor G preserving inclusion and having the same initial-algebra chain as F for all infinite ordinals. Moreover, F and G coincide on all nonempty sets and functions and if $F\emptyset \neq \emptyset$, then $G\emptyset \neq \emptyset$. See [7, Theorem III.4.5] and[4, Remark 3]. We call G the *Trnková hull* of F.

▶ Remark 15. Let λ be an infinite regular cardinal. We recall from [6] that an object A of a category $\mathcal{K}$ is called *λ-presentable* if its hom-functor $\mathcal{K}(A, -)$ preserves λ-filtered colimits. This means that if a λ-filtered diagram D has a colimit cocone $b_i : B_i \to X (i \in I)$, then for every morphism $a : A \to X$ (i) a factorization through b_i exists for some $i \in I$ and (ii) given two factorizations $u, v : A \to B_i$ with $a = b_i \cdot u = b_i \cdot v$, some connecting morphism $d : B_i \to B_j$ of D fulfils $d \cdot u = d \cdot v$.

A category $\mathcal{K}$ is called *locally λ-presentable* if it is cocomplete and has a small full subcategory $\mathcal{D}$ consisting of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathcal{K}$. This implies that every object X is a canonical colimit of the diagram of all morphisms $a\colon A \to X$ with $A \in \mathcal{D}$. More precisely, of the λ-filtered diagram

$$D_X \colon \mathcal{D}/X \to \mathcal{D}\,, \quad D_X(A, a) = A\,.$$

In the case $\lambda = \aleph_0$ we speak about *locally finitely presentable* categories.

▶ **Definition 16** (See [5]). A *strictly locally λ-presentable* category is a locally λ-presentable category in which every morphism $b\colon B \to A$ with B λ-presentable has a factorization $b = b' \cdot f \cdot b$ for some morphisms $b'\colon B' \to A$ and $f\colon A \to B'$ with B' also λ-presentable.

▶ **Examples 17** (See [5]).
(a) The categories **Set**, K-**Vec** and G-**Set**, where G is a finite group, are strictly locally finitely presentable.
(b) **Nom** is strictly locally $\aleph_1$-presentable.
(c) $\mathbf{Set}^S$ is strictly locally λ-presentable for infinite $\lambda > |S|$.
(d) Given an infinite group G, the category G-**Set** is strictly locally λ-presentable if $\lambda > 2^{|G|}$.

▶ **Definition 18.** A category $\mathcal{K}$ has *strict width* $w(\mathcal{K})$ if it has width $w(\mathcal{K})$, coproduct injections are monic, and every connected object is λ-presentable for $\lambda = w(\mathcal{K}) + \aleph_0$.

▶ **Example 19.**
(1) The category $\mathbf{Set}^S$ has strict width $|S| + \aleph_0$, since connected objects (see Example 3) are finitely presentable.
(2) K-**Vec** has strict width $|K| + \aleph_0$: the only connected object K is finitely presentable.
(3) G-**Set** has strict width at most $2^{|G|} + \aleph_0$.
(4) **Nom** has strict width $\aleph_0$.

▶ **Lemma 20.** *If a category has strict width $w(\mathcal{K})$, then for every infinite regular cardinal $\lambda \geq w(\mathcal{K})$ its λ-presentable objects are precisely those of power less than λ.*

Proof. If X is λ-presentable and $X = \coprod_{i\in I} X_i$ with connected objects X_i, then in case card $I < \lambda$ we have nothing to prove. And if card $I \geq \lambda$, form the λ-filtered diagram of all coproducts $\coprod_{j\in J} X_j$ where J ranges over subsets of I with card $J < \lambda$. Since $\mathcal{K}(X,-)$ preserves this colimit, there exists a factorization of id_X through one of the colimit injections $v\colon \coprod_{j\in J} X_j \to \coprod_{i\in I} X_i$. Now v is monic (by the definition of strict width) and split epic, hence it is an isomorphism. Thus, $X \simeq \coprod_{j\in J} X_j$ has power at most card $J < \lambda$.

Conversely, if X has power less than λ, then it is λ-presentable because every coproduct of less than λ objects which are λ-presentable is λ-presentable. ◀

▶ Remark 21. In every locally λ-presentable category $\mathcal{K}$ all hom-functors of λ-presentable objects collectively reflect λ-filtered colimits. That is, given a λ-filtered diagram D with objects D_i ($i \in I$), then a cocone $c_i\colon D_i \to C$ of D is a colimit iff for every λ-presentable object Y the following holds: (i) every morphism $f\colon Y \to C$ factorizes through some c_i and (ii) given two such factorizations $u, v\colon Y \to C$, $c_i \cdot u = c_i \cdot v$, there exists a connecting morphism $d\colon D_i \to D_j$ of D with $d \cdot u = d \cdot v$. This is proved for $\lambda = \aleph_0$ in [5, Lemma 2.7], the general case is completely analogous.

▶ **Theorem 22.** *Let $\mathcal{K}$ be a strictly locally α-presentable category with a strict width. Then $\mathcal{K}_{\leq\lambda}$ is algebraically complete for every cardinal $\lambda \geq \max(\alpha, w(\mathcal{K}))$.*

Proof. Following Remark 14, it is sufficient to prove that $\mathcal{K}_{\leq\lambda}$ has colimits of i-chains for all limit ordinals $i \leq \lambda$, and every endofunctor of $\mathcal{K}_{\leq\lambda}$ preserves colimits of λ-chains.

(1) $\mathcal{K}_{\leq\lambda}$ has for every limit ordinal $i \leq \lambda$ colimits of i-chains $(B_j)_{j<i}$. In fact, let X be the colimit of that chain in $\mathcal{K}$, then we verify that X has power at most λ. Indeed, each B_j is a coproduct of at most λ connected objects, thus, $\coprod_{j<i} B_j$ is a coproduct of at most $i \cdot \lambda = \lambda$ connected objects. The same holds for X, since it is a quotient of $\coprod_{j<i} B_j$.

(2) For every endofunctor F of $\mathcal{K}_{\leq\lambda}$ and every λ-chain B_i $(i < \lambda)$ in $\mathcal{K}_{\leq}$ we prove that F preserves the colimit

$$X = \mathop{colim}_{i \in I} B_i \quad \text{(with cocone } b_i \colon B_i \to X,\ i < \lambda).$$

Let us choose a small subcategory $\mathcal{D}$ of $\mathcal{K}$ as in Remark 15. We verify that the functor $B \colon \lambda \to \mathcal{D}/X$ given by $i \mapsto (B_i, b_i)$ is cofinal, i.e., for every object (A, a) of $\mathcal{D}/X$ (a) there exists a morphism of $\mathcal{D}/X$ into some (B_i, b_i) and (b) given a pair of morphisms u, $v \colon (A, a) \to (B_i, b_i)$, there exists $j \geq i$ with u and v merged by the connecting morphism $b_{ij} \colon B_i \to B_j$ of our chain. Indeed, since A is λ-presentable, the morphism $a \colon A \to \mathop{colim}_{i<\lambda} B_i$ factorizes through b_i for some $i < \lambda$. And since u, v above fulfil $b_i \cdot u = b_i \cdot v$ $(= a)$, some connecting morphism b_{ij} also merges u and v, see Remark 15.

Consequently, in order to prove that F preserves the colimit $X = \mathop{colim} B_i$, it is sufficient to verify that it preserves the colimit of the codomain restriction $D'_X \colon \mathcal{D}/X \to \mathcal{K}_{\leq}$ of D_X (see Remark 15). Indeed, since $B \colon \lambda \to \mathcal{D}/X$ is cofinal, the colimits of the diagrams $F \cdot D'_X$ and $(FB_i)_{i<\lambda}$ coincide. We apply Remark 21 and verify the conditions (i) and (ii) for the cocone $Fa \colon FA \to FX$ of $F \cdot D_X$ (in $\mathcal{K}$). Thus $FX = \mathop{colim} F \cdot D_X$ in $\mathcal{K}$ which implies $FX = \mathop{colim} FD'_X$ in $\mathcal{K}_{\leq\lambda}$.

Ad (i) Given a morphism $f \colon Y \to FX$ with Y λ-presentable, then Y has power less than λ, thus, by Proposition 10 there exists a coproduct injection $c \colon C \to X$ with C λ-presentable such that f factorizes through Fc (which is a member of our cocone).

Ad (ii) Let u, $v \colon Y \to FA$, with A λ-presentable, fulfil $Fa \cdot u = Fa \cdot v$. We are to find a connecting morphism

$$h \colon (A, a) \to (B, b) \quad \text{in } \mathcal{D}/X \qquad \textit{with} \quad Fh \cdot u = Fh \cdot v.$$

By the strictness of $\mathcal{K}$, since A is λ-presentable, for $a \colon A \to X$ there exist morphisms $b \colon B \to X$ and $f \colon X \to B$ with B λ-presentable and $a = b \cdot f \cdot a$. It is sufficient to put $h = f \cdot a \colon A \to B$. Then h is a morphism of $\mathcal{D}/X$ since $b \cdot h = a$, and $Fa \cdot u = Fb \cdot v$ implies $Fh \cdot u = Fh \cdot v$, as desired. ◀

▶ **Example 23.**

(1) For every uncountable regular cardinal λ the category $\mathbf{Set}_{\leq\lambda}$ is algebraically complete (by Theorem 22) and, assuming GCH, algebraically cocomplete (by Theorem 13). The former was already proved in [3], Example 14, using an entirely different method.

(2) The category $\mathbf{Set}_{\leq\aleph_0}$ of countable sets is algebraically complete, but not algebraically cocomplete. Indeed, the restriction $\mathcal{P}_f$ of the finite power-set functor to it does not have a terminal coalgebra. Assuming that a (countable) terminal coalgebra T is given, we find a contradiction as follows. For every subset A of $\mathbb{N}$ denote by C_A the tree with root r_A obtained from an infinite path by adding, for every number $n \in A$, a leaf of height $n + 1$. These trees are, as coalgebras for $\mathcal{P}_f$, clearly pairwise non-bisimilar. Consequently, the unique homomorphisms $h_A : C_A \to T$ have the property that the elements $h_A(r_A)$ are pairwise distinct. This is the desired contradiction: T is countable, but the number of all A's is uncountable.

▶ **Example 24.** Let λ be an uncountable regular cardinal. The following categories are algebraically complete and, assuming GCH, algebraically cocomplete:

(a) $\mathbf{Set}^S_{\leq\lambda}$ whenever $\lambda > |S|$,

(b) K-$\mathbf{Vec}_{\leq\lambda}$ whenever $\lambda > |K|$,

(c) $\mathbf{Nom}_{\leq\lambda}$ whenever $\lambda > \aleph_1$, and
(d) G-$\mathbf{Set}_{\leq\lambda}$ for groups G with $\lambda > 2^{|G|}$.
This follows from Theorems 13 and 22.

4 Terminal Coalgebras Derived from Initial Algebras

In this section we prove that a set functor F with a non-empty initial algebra of regular cardinality λ (see Remark 12) has a terminal coalgebra of the same cardinality λ – with one exception: $\lambda = \aleph_0$. We first formulate a fixed-point theorem.

A *fixed point* of an endofunctor F is an object X isomorphic to FX.

▶ **Theorem 25** (The Fixed-Point Theorem)**.** *Assume GCH. A set functor with a nonempty fixed point of a finite or regular uncountable cardinality λ has a terminal coalgebra of cardinality at most λ.*

Proof.

(1) Without loss of generality we can assume $F\emptyset = \emptyset$. Indeed, otherwise we prove the theorem for the Trnková hull G, see Remark 14. The terminal coalgebras for F and G are the same.
F restricts to an endofunctor F_0 of $\mathbf{Set}_{\leq\lambda}$. Indeed, if A is a fixed point with $|A| = \lambda$, then every object $Y \neq \emptyset$ of $\mathbf{Set}_{\leq\lambda}$ is a split subobject of A, hence, FY is a split subobject of FA, proving that $|FY| \leq |FA| = \lambda$. We know from Theorem 13 that F_0 has a terminal coalgebra. We prove that this is also terminal for F. For that, it is sufficient to prove every coalgebra for F is a colimit of coalgebras for F_0 in $\mathbf{Coalg}\, F$.

(2) Suppose that λ is finite. Then we verify that the terminal coalgebra is obtained as the limit of the following ω^{op}-chain

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} F^2 1 \xleftarrow{\;F^2 1\;} \cdots$$

Indeed, since by (1) we have $|F^n 1| \leq \lambda$ for all n, there exists $k \leq \lambda$ such that some infinite set $A \subseteq \mathbb{N}$ fulfils $|F^n 1| = k \quad \text{for every} \quad n \in A$. Observe that the connecting maps of our chain are all epic. Hence, given $n \geq m$ in A, the connecting map from $F^n 1$ to $F^m 1$ is invertible: it is monic due to $|F^n 1| = |F^m 1|$. Thus, the limit of the cofinal subchain $F^n 1$ $(n \in A)$ is absolute, since this subchain consists of isomorphisms. Hence, the original chain also has an absolute limit. This implies by Remark 14(d), that $\lim_{n<\omega} F^n 1$ is a terminal coalgebra of F. It has $k \leq \lambda$ elements.

(3) From now on we assume that λ is uncountable. For every coalgebra $\alpha\colon A \to FA$ and every subset $b\colon B \hookrightarrow A$ with $|B| < \lambda$ a subset $b'\colon B' \hookrightarrow A$ exists which contains b, fulfils $|B'| < \lambda$, and carries the structure $\beta'\colon B' \to FB'$ of a subcoalgebra (i.e., $b'\colon (B', \beta') \to (A, \alpha)$ is a coalgebra homomorphism). This is proved precisely as Proposition 11. It then follows that the diagram of all subcoalgebras of (A, α) on less then λ elements (and all coalgebra homomorphisms carried by inclusion maps) has the canonical λ-filtered colimit (A, α) in $\mathbf{Coalg}\, F$. Indeed, the forgetful functor U from $\mathbf{Coalg}\, F$ to $\mathbf{Set}$ creates colimits, and A is (in $\mathbf{Set}$) a canonical λ-filtered colimit of all subsets of less than λ elements. The subdiagram of all subalgebras of less than λ elements is cofinal in the above diagram, hence, it also has the canonical colimit A. And U creates that colimit. ◀

▶ **Example 26.** None of the assumptions of the Fixed-Point Theorem can be left out, as we now demonstrate.

(1) Assuming the negation of the Continuum Hypothesis, i.e. $\aleph_1 < 2^{\aleph_0}$, we present a set functor F with the fixed point $\aleph_1$ (the set of all countable ordinals) which has no terminal coalgebra. Define F on objects by

$$FX = X \times \aleph_1 + \{Y \subseteq X; |Y| > \aleph_1 \text{ or } Y = \emptyset\}\,.$$

For every morphism $f\colon X \to X'$ the left-hand summand of Ff is $f \times \mathrm{id}_{\aleph_1}$, and the right-hand one is given by $Ff(\emptyset) = \emptyset$ and$Ff(Y) = f[Y]$ if f restricted to Y is monic, else $\emptyset$.
Then $\aleph_1$ is a fixed point of F: $F\aleph_1 = \aleph_1 \times \aleph_1 + \{\emptyset\} \cong \aleph_1$. But assuming that a terminal coalgebra $\tau\colon \nu F \to F(\nu F)$ exists, we derive a contradiction. It is clear that every fixed point of F has power $\aleph_1$, thus $|\nu F| = \aleph_1$.
For every function $\varphi\colon \mathbb{N} \to \aleph_1$ define a coalgebra $A_\varphi = (\mathbb{N}, \alpha_\varphi)$ for F as follows: α_φ maps n to the element $(n+1, \varphi(n))$ of the left-hand summand of FX. We have a unique coalgebra homomorphism $h_\varphi\colon A_\varphi \to \nu F$. Since $\tau \cdot h_\varphi = Fh_\varphi \cdot \alpha_\varphi$, for every $n \in \mathbb{N}$ we get

$$\tau\big(h_\varphi(n)\big) = Fh_\varphi\big(n+1, \varphi(n)\big) = \big(h_\varphi(n+1), \varphi(n)\big)\,. \tag{4.1}$$

We conclude that the elements $h_\varphi(0) \in \nu F$ for all $\varphi\colon \mathbb{N} \to \aleph_1$ are pairwise distinct: assuming $h_\varphi(0) = h_{\varphi'}(0)$, we prove $\varphi = \varphi'$. Indeed, it is sufficient to verify that $h_\varphi(n) = h_{\varphi'}(n)$ by induction on n. This is trivial, the induction hypothesis yields, due to (4.1), $\big(h_\varphi(n+1), \varphi(n)\big) = \big(h_{\varphi'}(n+1), \varphi'(n)\big)$, and the left-hand components prove $h_\varphi(n+1) = h_{\varphi'}(n+1)$.
This is a desired contradiction: all elements $h_\varphi(0) \in T$ form a set of power at most $|T| = \aleph_1$, but all $\varphi\colon \mathbb{N} \to \aleph_1$ form a set of power $|\aleph_1^{\mathbb{N}}| \geq 2^{\aleph_0} > \aleph_1$.

(2) For the non-regular uncountable cardinal $\aleph_\omega = \bigvee_{n<\omega} \aleph_n$ we present a set functor F with the fixed point $\aleph_\omega$ not having a terminal coalgebra. Since by Remark 12(b) we have $|\aleph_\omega^{\mathbb{N}}| > \aleph_\omega$, this is completely analogous to the preceding example: put $FX = X \times \aleph_\omega + \{Y \subseteq X; |Y| > \aleph_\omega \text{ or } Y = \emptyset\}$.

(3) The Fixed-Point Theorem does not hold for $\aleph_0$, see Example 1.

▶ **Theorem 27** (The Terminal-Coalgebra Therorem). *Assume GCH. If a set functor F has a nonempty initial algebra of a finite or regular uncountable cardinality, then it also has a terminal coalgebra of the same cardinality. Shortly:*

$$\mu F \simeq \nu F\,.$$

Proof. In [7], Theorem 3.10, it is proved that the existence of μF implies that the initial-algebra chain $F^i 0$ ($i \in \mathrm{Ord}$), see Remark 14, converges, thus $\mu F = F^\rho 0$ for some ordinal ρ. Without loss of generality we assume $\rho \geq \omega$. Put $\lambda = |\mu F|$.

By the Fixed-Point Theorem we have a terminal coalgebra $\tau\colon T \to FT$ with $|T| \leq \lambda$. The algebra (T, τ^{-1}) yields, as proved in [1], a unique cocone $\alpha_i\colon F^i 0 \to T$ of the initial-algebra chain satisfying $\alpha_{i+1} = \tau^{-1} \cdot F\alpha_i$ for every ordinal i. To prove $|T| \geq \lambda$, we verify that α_i is monic for all ordinals $i \geq \omega$. Thus $|T| \geq |F^i 0|$ for all i, proving $|T| \geq |F^\rho 0| = \lambda$.

Let F preserve monomorphisms. Then all α_i ($i \in \mathrm{Ord}$) are monic. This is easily seen by transfinite induction, since $\alpha_0\colon \emptyset \to A$ is monic, and $\alpha_{i+1} = \tau^{-1} \cdot F\alpha_i$ is monic whenever α_i is.

For a functor F not preserving monomorphisms we have $F\emptyset \neq \emptyset$, since all nonempty monomorphisms split. We apply the result of Remark 14(e) that the Trnková hull is a monic-preserving set functor G which coincides with F on all nonempty sets (and functions) and whose initial-algebra chain is, from the ordinal ω onwards, the same as that for F. Thus, $\mu G \simeq \mu F$. We also have $\nu G \simeq \nu F$ since, due to $F\emptyset \neq \emptyset \neq G\emptyset$, F and G have the same coalgebras. ◀

▶ **Example 28.**

(a) For set functors with countable initial algebras nothing can be deduced about the terminal coalgebras. As we have seen in Example 1, νF need not exist. And for every infinite cardinal λ there exists a set functor with a terminal coalgebra such that $|\mu F| = \aleph_0$ and $|\nu F| > \lambda$. Define F as the following subfunctor of the functor of Example 1:

$$FX = \mathcal{P}_f X + \{M \subseteq X; \aleph_0 < |M| \leq \lambda\}.$$

This functor has a terminal coalgebra because it preserves colimits of λ^+-chains (see Remark 14). And νF is uncountable. Indeed, P_f has an uncountable terminal coalgebra: the argument is as in Example 23(2). Since $\mathcal{P}_f$ is a subfunctor of F, the terminal-coalgebra chain of P_f is also a subfunctor of the terminal-coalgebra chain of F, from which we conlcude $|\nu P_f| \leq |\nu F|$.
Furthermore, $\mu F \cong \mu \mathcal{P}_f$ is countable. And for every uncountable fixed point $X \simeq FX$ we clearly have $X \simeq \{M \subseteq X;\ |M| = \lambda\}$, therefore $|X| = |X|^\lambda$ from which it follows, by Remark 12, that $|X| > \lambda$. Hence, $|\nu F| > \lambda$.

(b) For many-sorted sets the Terminal-Coalgebra Theorem does not hold. Indeed, given any cardinal λ there is an endofunctor F of $\mathbf{Set} \times \mathbf{Set}$ that fulfils: μF exists and has λ elements, but νF does not exist. Put

$$F(X, Y) = \begin{cases} (\emptyset, \lambda) & \text{if } X = \emptyset, \\ (X, \lambda + \mathcal{P}Y) & \text{else.} \end{cases}$$

Given a morphism $(f, g)\colon (X, Y) \to (X', Y')$ with X nonempty, put $F(f, g) = (f, id + Pg)$. It is easy to see that the initial algebra of F is $(\emptyset, \lambda)$. If F would have a terminal coalgebra $\nu F = (A, B)$, then $A = \emptyset$ (since otherwise (A, B) is not a fixed point of F). But for any coalgebra $\alpha\colon (X, Y) \to (X, \lambda + \mathcal{P}Y)$, with $X \neq \emptyset$, no morphism into $(\emptyset, B)$ exists, a contradiction.

(c) Moreover, for every pair $\lambda_\mu \leq \lambda_\nu$ of infinite cardinals there exists an endofunctor F of $\mathbf{Set} \times \mathbf{Set}$ with μF of λ_μ elements and νF of λ_ν elements. On objects put $F(X, Y) = (\emptyset, \lambda_\mu)$ if $X = \emptyset$, else $(1, \lambda_\mu)$ To every morphism F assigns the (obvious) inclusion map.

Both the initial-algebra chain and the terminal-coalgebra chain converge in one step and yield $\mu F = (\emptyset, \lambda_\mu)$ and $\nu F = (1, \lambda_\nu)$.

5 Finitary Set Functors

In the preceding section we have established, for some set functors F, an isomorphism $\mu F \simeq \nu F$. But that concerned only the underlying sets! In the generality of that section, nothing can be derived about the relationship of the algebra structure $\iota\colon F(\mu F) \to \mu F$ and the coalgebra structure $\tau\colon \nu F \to F(\nu F)$. For finitary set functors F (i.e., those preserving filtered colimits) with $F\emptyset \neq \emptyset$ we can say more. Firstly, μF and νF exist, and μF (considered as a coalgebra via ι^{-1}) is a subcoalgebra of νF. Second, there is a canonical ultrametric on νF, such that for the metric subspace μF we prove that

(a) μF and νF share the same Cauchy completion,
and

(b) ι determines τ as the unique continuous extension of ι^{-1}.

This generalizes the result of Barr [8] that, in case F moreover preserves limits of ω^{op}-chains, νF is the Cauchy completion of μF.

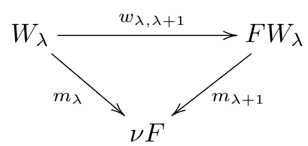
▶ **Proposition 29** (μF as a subcoalgebra of νF)**.** *If a set functor F has a terminal coalgebra, then it also has an initial algebra carried by a subset $\mu F \subseteq \nu F$ such that the inclusion map $m\colon \mu F \hookrightarrow \nu F$ is the unique coalgebra homomorphism, i.e., $\tau \cdot m = Fm \cdot \iota^{-1}$.*

Proof.

(1) Assume first that F preserves monomorphisms. There exists a unique cocone of the initial-algebra chain with codomain νF, $m_i\colon F^i 0 \to \nu F \quad (i \in \mathrm{Ord})$ determined by the condition below:

$$m_{i+1} \equiv F(F^i 0) \xrightarrow{Fm_i} F(\nu F) \xrightarrow{\tau^{-1}} \nu F \quad (i \in \mathrm{Ord})\,.$$

Easy transfinite induction verifies that m_i is monic for every i. Since νF has only a set of subobjects, there exists an ordinal λ such that all m_i with $i \geq \lambda$ represent the same subobject. Thus the commutative triangle below



implies that $w_{\lambda,\lambda+1}$ is invertible. Consequently, the following algebra

$$F(F^\lambda 0) \xrightarrow{w^{-1}_{\lambda,\lambda+1}} F^\lambda 0$$

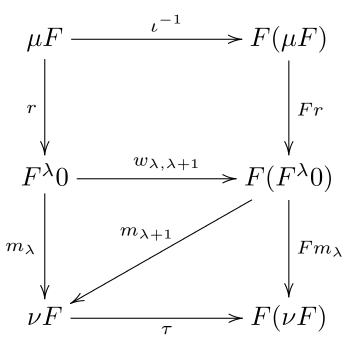
is initial, see Remark 14.
For the monomorphism $m_\lambda\colon F^\lambda 0 \to \nu F$ put

$$\mu F = m_\lambda[F^\lambda 0] \subseteq \nu F\,.$$

Choose an isomorphism $r\colon \mu F \to F^\lambda 0$ such that $m = m_\lambda \cdot r\colon \mu F \to \nu F$ is the inclusion map. Then there exists a unique algebra structure $\iota\colon F(\mu F) \to \mu F$ for which r is an isomorphism of algebras:

$$r\colon (\mu F, \iota) \xrightarrow{\sim} (F^\lambda 0, w^{-1}_{\lambda,\lambda+1})\,.$$

The following commutative diagram



proves that $m = m_\lambda \cdot r$ is the unique coalgebra homomorphism, as required.

(2) Let F not preserve monomorphisms. Therefore $F\emptyset \neq \emptyset$. Our proposition holds for the Trnková hull G of Remark 14(e). Since F and G agree an all nonempty sets and $F\emptyset \neq \emptyset \neq G\emptyset$, they have the same terminal coalgebras. Since the initial algebra of G is, as we have just seen, obtained via the initial-algebra chain, and F has from ω onwards the same initial-algebra chain, F and G have the same initial algebras. Thus, our proposition holds for F too. ◀

The fact that every set functor with a terminal coalgebra has an initial algebra was proved in [15]. Our proof above uses ideas of that paper.

Next we recall the behaviour of the terminal-coalgebra chain, see Remark 14, for finitary set functors:

▶ **Theorem 30** (Worrell [16]). *For every finitary set functor F the terminal-coalgebra chain converges at $\omega + \omega$: $\nu F = F^{\omega+\omega}1$. Moreover, every connecting morphism $v_{i,\omega}$ $(i \geq \omega)$ is monic.*

▶ Remark 31.

(a) Consequently, νF is a canonical subset of $F^\omega 1 = \lim_{n<\omega} F^n 1$. And this endows νF with a canonical ultrametric, as our next lemma explains. Recall that a metric d is called an *ultrametric* if for all elements x, y, z the triangle inequality can be strengthened to $d(x, z) \leq \max(d(x, y), d(y, z))$.

(b) For every set functor F there exists a unique morphism $\bar{u}\colon F^\omega 0 \to F^\omega 1$ with $\bar{u} \cdot w_{n,\omega} = v_{\omega,n} \cdot F^n 1$ (where $!\colon 0 \to 1$ is unique). See [2], Lemma 2.4.

(c) The homomorphism m of Proposition 29 fulfils $\bar{u} = v_{\omega+\omega,\omega} \cdot m$.
Indeed, since F is finitary, we have $\mu F = F^\omega 0$ and $\iota = w^{-1}_{\omega,\omega+1}$. Thus m being a coalgebra homomorphism states precisely that

$$v^{-1}_{\omega+\omega+1,\omega+1} \cdot m = Fm \cdot w_{\omega,\omega+1}$$

or, $m = v_{\omega+\omega+1,\omega+\omega} \cdot Fm \cdot w_{\omega,\omega+1}$. The squares defining $\bar{u}$ in Remark 31(b) thus commute when $\bar{u}$ is substituted by $v_{\omega+\omega,\omega} \cdot m$ $(= v_{\omega+\omega+1,\omega} \cdot Fm \cdot w_{\omega,\omega+1})$. That is, we claim that

$$v_{\omega,n}\big[v_{\omega+\omega+1,\omega} \cdot Fm \cdot w_{\omega,\omega+1}\big) \cdot w_{n,\omega} = F^n!$$

This is clear for $n = 0$. If this holds for n, i.e., if

$$v_{\omega+\omega+1,n} \cdot Fm \cdot w_{n,\omega+1} = F^n!\,,$$

then it also holds for $n + 1$: just apply F to that equation. Thus, $\bar{u} = v_{\omega+\omega} \cdot m$.

▶ **Lemma 32.** *Every limit L of an ω^{op}-chain in **Set** carries a complete ultrametric: assign to $t \neq s$ in L the distance 2^{-n} where n is the least natural number with $p_n(t) \neq p_n(s)$ for the limit projections p_n.*

Proof. Let $l_n\colon L \to A_n$ $(n \in \mathbb{N})$ be a limit cone. For the above function

$$d(x, y) = 2^{-n}$$

where $l_n(x) \neq l_n(y)$ and n is the least such number we see that d is symmetric. It satisfies the ultrametric inequality

$$d(x, z) \leq \max\big(d(x, y), d(y, z)\big) \quad \text{for all} \;\; x, y, z \in L\,.$$

This is obvious if the three elements are not pairwise distinct. If they are, the inequality follows from the fact that if l_n separates two elements, then so do all l_m with $m \geq n$.

It remains to prove that the space $F^\omega 1$ is complete. Given a Cauchy sequence $x_r \in L$ ($r \in \mathbb{N}$), for every $k \in \mathbb{N}$ there exists $r(k) \in \mathbb{N}$ with

$$d(x_{r(k)}, x_n) < 2^{-k} \quad \text{for every} \quad n \geq r(k)\,.$$

Choose $r(k)$'s to form an increasing sequence. Then $d(x_{r(k)}, x_{r(k+1)}) < 2^{-k}$, i.e., $l_k(x_{r(k)}) = l_k(x_{r(k+1)})$. Therefore, the elements $y_k = l_k(x_{r(k)})$ are compatible: we have $a_{k+1}(y_{k+1}) = y_k$ for all $k \in \mathbb{N}$. Consequently, there exists a unique $y \in L$ with $l_k(y) = y_k$ for all $k \in \mathbb{N}$. That is, $d(y, x_{r(k)}) < 2^{-k}$. Thus, y is the desired limit:

$$y = \lim_{k\to\infty} x_{r(k)} \quad \text{implies} \quad y = \lim_{n\to\infty} x_n\,. \qquad ◀$$

We conclude that for a finitary set functor both νF and μF carry a canonical ultrametric: νF as a subspace of $F^\omega 1$ via $v_{\omega+\omega,\omega}\colon \nu F \to F^\omega 1$, and μF as a subspace of νF via m. Or, equivalently, a subspace of $F^\omega 1$ via $\bar{u}$, see Remark 31. Given $t \neq s$ in νF we have $d(t,s) = 2^{-n}$ for the least $n \in \mathbb{N}$ with $v_{\omega+\omega,n}(t) \neq v_{\omega+\omega,n}(s)$.

▶ **Notation 33.** Given a finitary set functor F with $F\emptyset \neq \emptyset$, choose an element $p\colon 1 \to F\emptyset$. This defines the following morphisms for every $n \in \mathbb{N}$:

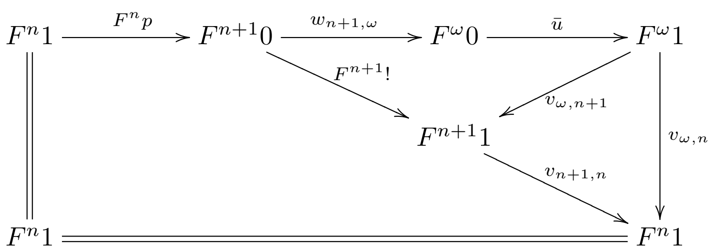
$$e_n = \bar{u} \cdot w_{n+1,\omega} \cdot F^n p : F^n 1 \to F^\omega 1.$$

We also put $r_n = e_n \cdot v_{\omega,n} : F^\omega 1 \to F^\omega 1$.

▶ **Observation 34.** Denote by! : $\emptyset \to 1$ the unique map. For every $n \in \mathbb{N}$ we have

(a) $v_{n,n+1} \cdot F^{n+1}! \cdot F^n p = id_{F^n 1}$.
This is obvious for $n = 0$. The induction step just applies F to the given square.

(b) $v_{\omega,n} \cdot e_n = \mathrm{id}_{F^n 1}$. Indeed, in the following diagram



the upper right-hand part commutes by the definition of $\bar{u}$, see Remark 31(b), the left-hand one does by (a), and the lower right-hand triangle is clear.

(c) $v_{\omega,n} \cdot r_n = v_{\omega,n}$. This follows from (b): precompose it with $v_{\omega,n}$.

▶ **Theorem 35.** *For a finitary set functor F with $F\emptyset \neq \emptyset$ the Cauchy completions of the ultrametric spaces μF and νF coincide. And the algebra structure ι determines the coalgebra structure τ as the unique continuous extension of ι^{-1}.*

Proof.

(1) Assume first that F preserves inclusion, see Remark 14(e).

(a) We prove that the subset $\bar{u} = v_{\omega+\omega,\omega} \cdot m\colon \mu F \to F^\omega 1$ of Remark 31(b) is dense in $F^\omega 1$, thus, the complete space $F^\omega 1$ is a Cauchy completion of both $\bar{u}[\mu F]$ and $v_{\omega+\omega,\omega}[\nu F]$.

For every $x \in F^\omega 1$ the sequence $r_n(x)$ lies in the image of $e_n \cdot v_{\omega,n}$ which, in view of the definition of e_n, is a subset of the image of $\bar{u}$. And we have $x = \lim_{n\to\infty} r_n(x)$ because Observation 34 (c) yields $v_{\omega,n}(x) = v_{\omega,n}(r_n(x))$. Thus $d\big(x, r_n(x)\big) < 2^{-n}$ for all $n \in \mathbb{N}$.

(b) We have ultrametric subspaces μF and νF of $F^\omega 1$, hence, the bijections

$$F(\mu F) \xrightarrow{\iota} \mu F \quad \text{and} \quad \nu F \xrightarrow{\tau} F(\nu F)$$

make also $F(\mu F)$ and $F(\nu F)$ ultrametric spaces. The continuous map ι^{-1} has at most one continuous extension to νF, since μF is dense in νF (even in $F^\omega 1$). And τ is such an extension: it is not only continuous, it is an isometry. And it extends ι^{-1} by Proposition 29: choose an inclusion map m with $\tau \cdot m = Fm \cdot \iota^{-1}$. Since Fm is an inclusion map, τ is an extension of ι^{-1}.

(2) Once we have established (a) and (b) for inclusion-preserving finitary functors, it holds for all finitary functors F. Indeed, use the Trnková hull G that agrees with F on all nonempty set and functions with $G\emptyset \neq \emptyset$ provided that $F\emptyset \neq \emptyset$, see Remark 14(e). Consequently, the coalgebras for F and G coincide. And the initial-algebra chains coincide for infinite ordinals, in particular $F^\omega 0 = G^\omega 0$, that is, F and G have the same initial algebra. ◀

▶ **Example 36.**

(1) For the set functor $FX = X \times \Sigma + 1$ (of dynamic systems with inputs from Σ and deadlock states) the terminal coalgebra is obtained in ω steps, since F preserves limits of ω^{op}-sequences. It can be described as the coalgebra $\nu F = \Sigma^\infty$ of all finite and infinite words over Σ. The distance of distinct words u and v is 2^{-n} for the largest n such that u and v have the same prefix of length n.
The initial algebra Σ^* is dense in Σ^∞: every infinite word is the limit of the sequence of its finite prefixes. The algebra structure $\iota : \Sigma^* \times \Sigma + 1 \to \Sigma^*$ is given by concatenation on the left-hand summand, and the empty word on the right-hand one. Its inverse has a unique continuous extension to Σ^∞ assigning to every nonempty word u the pair $(head(u), tail(u))$. This is indeed the coalgebra structure of νF.

(2) For the finite power-set functor $\mathcal{P}_f$ the initial algebra can be described as $\mu\mathcal{P}_f =$ all finite extensional trees (where trees are considered up to isomorphism), see [16]. Recall that a tree is called *extensional* if for every node x the maximum subtrees of x are pairwise non-isomorphic. And it is called *strongly extensional* if it has no nontrivial tree bisimulation; for finite trees these two concepts are equivalent. Worrell proved in [16] that the terminal coalgebra $\nu\mathcal{P}_f$ consists of all finitely branching strongly extensional trees, whereas $\mathcal{P}_f^\omega$ consists of all strongly extensional trees. The metric on $\mathcal{P}_f^\omega 1$ assigns to trees $t \neq s$ the distance $d(t, s) = 2^{-n}$, where n is the least number with $\partial_n t \neq \partial_n s$. Here $\partial_n t$ is the extensional tree obtained from t by cutting it at level n and forming the extensional quotient of the resulting tree.

The algebraic structure $\iota\colon \mathcal{P}_f(\mu\mathcal{P}_f) \to \mathcal{P}_f$ assigns to a set $\{t_1, \dots, t_n\}$ of finite trees the tree-tupling (consisting of a new root and n maximum subtrees $t_1, \dots, t_n$). The coalgebraic structure $\tau\colon \nu\mathcal{P}_f \to \mathcal{P}_f(\nu\mathcal{P}_f)$ assigns to a tree $t \in \nu\mathcal{P}_f$ the finite set of its maximum subtrees. This is indeed a continuous extension of ι^{-1}.

6 Conclusions and Open problems

Whereas a set functor is known to have an initial algebra iff it has a fixed point, for terminal coalgebras fixed points are not sufficient in general. However, we have proved that a non-empty fixed point of a finite or regular cardinality λ implies that a terminal coalgebra exists and has at most λ elements – with a single exception, $\lambda = \aleph_0$. From this fixed-point result we have derived that every set functor F with a nonempty initial algebra μF whose cardinality is finite or regular uncountable has a terminal coalgebra $\nu F \cong \mu F$.

We have also presented a number of categories that are algebraically complete and cocomplete, i.e., every endofunctor has a terminal coalgebra and an initial algebra. Examples include (for sufficiently large regular cardinals λ) the category $\mathbf{Set}_{\leq\lambda}$ of sets of power at most λ, $\mathbf{Nom}_{\leq\lambda}$ of nominal sets of power at most λ, K-$\mathbf{Vec}_{\leq\lambda}$ of vector spaces of dimension at most λ, and G-$\mathbf{Set}_{\leq\lambda}$ of G-sets (where G is a group) of power at most λ.

All these results assumed the General Continuum Hypothesis. It is an open question what could be proved without this assumption. Another question is whether the above relationship $\nu F \cong \mu F$ can, under suitable side conditions, be proved for more general base categories than **Set**.

For finitary set functors F with $F\emptyset \neq \emptyset$ we have presented a sharper result: both μF and νF carry a canonical ultrametric and these two spaces have the same Cauchy completion. Moreover, by inverting the algebra structure of μF we obtain the coalgebra structure of νF as the unique continuous extension.

References

1. J. Adámek. Free algebras and automata realizations in the language of categories. *Comment. Math. Univ. Carolinae*, 15:589–602, 1974.
2. J. Adámek. Final coalgebras are ideal completions of initial algebras. *J. Logic Comput.*, 12:217–242, 2002.
3. J. Adámek and V. Koubek. Least fixed point of a functor. *J. Comput. System Sciences*, 19:163–168, 1979.
4. J. Adámek and V. Koubek. On the greatest fixed point of a set functor. *Theoret. Comput. Sci.*, 150:57–75, 1995.
5. J. Adámek, S. Milius, L. Sousa, and T. Wissmann. On finitary functors and finitely presentable algebras. *CoRR*, 2019. arXiv:1902.05788.
6. J. Adámek and J. Rosický. *Locally presentable and accessible categoreis.* Cambridge University Press, 1994.
7. J. Adámek and V. Trnková. *Automata and Algebras in Categories.* Kluwer Acad. Publ., London, 1990.
8. M. Barr. Terminal coalgebras in well-founded set theory. *Theoret. Comput. Sci.*, 114:299–315, 1993.
9. P. Freyd. Algebraically complete categories. *Lecture Notes in Math.*, 1488:95–104, 1970.
10. T. Jech. *Set theory.* Academic Press, 1978.
11. J. Rutten. Universal coalgebra. *Theoret. Comput. Sci.*, 249:3–80, 2000.
12. M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM J. Comput.*, 11:761–788, 1982.
13. A. Tarski. Sur la de composition des ensembles en sous-ensembles prèsque disjoint. *Fund. Math.*, 14:205–2015, 1929.
14. V. Trnková. On descriptive classification of set-functors I., II. *Comment. Math. Univ. Carolinae*, 12:143–175 and 345–357, 1971.
15. V. Trnková, J. Adámek, V. Koubek, and J. Reiterman. Free algebras, input processes and free monads. *Comment. Math. Univ. Carolinae*, 15:589–602, 1974.
16. J. Worrell. On the final sequence of a finitary set functor. *Theoret. Comput. Sci.*, 338:184–199, 2005.

A Full proofs

PROOF OF EXAMPLES 5 (3) and (4).

(1) For 5(4) recall that objects of **G-Set** are pairs $(X, \cdot)$ where X is a set and $\cdot$ is a function from $G \times X$ to X such that

$$h(gx) = (hg)x \qquad \text{for} \quad h, g \in G \text{ and } x \in X\,,$$

and

$$ex = x \qquad \text{for} \quad x \in X \ (e \text{ neutral in } G)\,.$$

An important example is given by any equivalence relation $\sim$ on G which is *equivariant*, i.e., fulfils

$$g \sim g' \Rightarrow hg \sim hg' \quad \text{for all} \quad g, g', h \in G\,.$$

Then the quotient set $G/\sim$ is a G-set (of equivalence classes $[g]$) w.r.t. the action $g[h] = [gh]$. This G-set is clearly connected.

(2) Let $(X, \cdot)$ be a G-set. For every element $x \in X$ we obtain a subobject of $(X, \cdot)$ on the set

$$Gx = \{gx; g \in G\} \qquad (\text{the orbit of } \ x)\,.$$

The equivalence on G given by

$$g \sim g' \quad \text{iff} \quad gx = g'x$$

is equivariant, and the G-sets Gx and $G/\sim$ are isomorphic. Moreover, two orbits are disjoint or equal: given $gx = hy$, then $x = (g^{-1}h)y$, thus, $Gx = Gy$.

(3) Every object $(X, \cdot)$ is a coproduct of at most $|X|$ connected objects: if X_0 is a choice class of the equivalence $x \equiv y$ iff $Gx = Gy$, then

$$X = \coprod_{x \in X_0} Gx\,.$$

(4) The number of connected objects, up to isomorphism, is at most $2^{|G|} + \aleph_0$. Indeed, it follows from the above that the connected objects are represented by precisely all $G/\sim$ where $\sim$ is an equivariant equivalence relation. If $|G| = \beta$ then we have at most β^β equivalence relations. For β infinite, this is equal to $2^{|G|}$, for β finite, this is smaller than $2^{|G|} + \aleph_0$.

(5) The number of morphisms from $G/\sim$ to an object $(X, \cdot)$ is at most $|X| \leq \max(\alpha, 2^{|G|}+\aleph_0)$ where $\alpha = |X_0|$ in (3) above. Indeed, every morphism p is determined by the value $x_0 = p([e])$ since $p([g]) = p(g[e]) = g \cdot x_0$ holds for all $[g] \in G/\sim$.

(6) Finally, for 5(3)the proof is completely analogous: in (2) each orbit $S_f(\mathbb{A})/\!\sim\, \simeq S_f(\mathbb{A})x$ is a nominal set. And the number of all such orbits up to isomorphism is $\aleph_0$, see Lemma A1 in [5]. In (5) we have $|X| \leq \alpha \cdot \aleph_0 = \alpha$ for all $\alpha \geq \aleph_0$. ◀

PROOF OF PROPOSITION 10. It is sufficient to prove this in case X has power precisely λ (otherwise put $c = \mathrm{id}_X$). And we can assume that B is connected. In the general case we have $B = \coprod_{k \in K} B_k$ with $|K| < \lambda$, and find for each k a summand $c_k \colon C_k \to X$ corresponding to the k-th component of b. Then we let $c : C \to X$ be the least summand containing each c_k. (C has power less than λ since each C_k does and $|K| < \lambda$.)

Since $\lambda > w(K)$, in the coproduct of λ connected objects representing X at least one, say R, must appear λ times. Thus X has the form

$$X = \coprod_{\lambda} R + X_0$$

for objects R and X_0, with R connected. Let $\bar{X}_0$ be the coproduct of the same components as in X_0, but each taken precisely once. Thus

(a) $\bar{X}_0$ has power at most $w(\mathcal{K})$, and

(b) we have a coproduct injection

$$m \colon \bar{X}_0 \to X_0$$

which has an (obvious) splitting

$$\hat{m} \colon X_0 \to \bar{X}_0 \,, \quad \hat{m} \cdot m = \mathrm{id} \,.$$

Put

$$Y = \coprod_{\lambda} R + \bar{X}_0$$

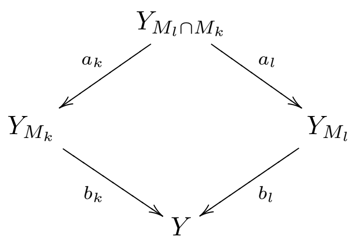
and for every set $M \subseteq \lambda$ put

$$Y_M = \coprod_{M} R + \bar{X}_0 \,.$$

(c) By Remark 8(b) we can choose $t \in \lambda$ and an almost disjoint collection of sets $M_k \subseteq \lambda$, $k \in K$, with

$$t \in M_k \,, \quad |M_k| = \lambda \quad \text{and} \quad |K| > \lambda \,.$$

Consider the following square of coproduct injections for any pair $k, l \in K$:



This is an absolute pullback. Indeed, it obviously commutes. And b_k and b_l are split monomorphisms: define

$$\bar{b}_k \colon Y \to Y_{M_k}$$

as identity on the summand $\bar{X}_0$, whereas the i-th copy of R is sent to copy i, if $i \in M_k$, and to copy t else. Then

$$\bar{b}_k b_k = \mathrm{id} \,.$$

Analogously for b_l. Next define

$$\bar{a}_l \colon Y_{M_l} \to Y_{M_k \cap M_l}$$

as identity on the summand $\bar{X}_0$, whereas the i-th copy of R is sent to copy i, if $i \in M_k$, and to copy t else. Then clearly

$$\bar{a}_k a_l = \mathrm{id} \quad \text{and} \quad a_k \bar{a}_l = \bar{b}_k b_l \,.$$

Thus, the above square is an absolute pullback by Lemma 6.

(d) We are ready to prove that for a connected object B every morphism

$$b\colon B \to FX$$

has the required factorization. For every $k \in K$ since $|M_k| = \lambda$ we have an isomorphism

$$y_k\colon Y = \coprod_{\lambda} R + \bar{X}_0 \longrightarrow \coprod_{M_k} R + \bar{X}_0 = Y_{M_k}$$

which composed with $b_k\colon Y_{M_k} \to Y_\lambda$ yields an endomorphism

$$z_k = b_k \cdot y_k\colon Y \to Y\,.$$

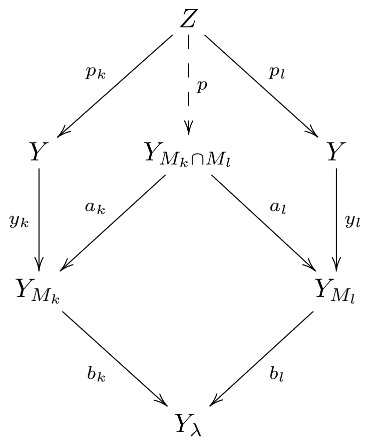
We use (b) above and precompose z_k with $\tilde{m} = \mathrm{id} + \hat{m}\colon X \to Y$ to get the following morphisms

$$B \xrightarrow{b} FX \xrightarrow{F\tilde{m}} FY \xrightarrow{Fz_k} FY \quad (k \in K)\,.$$

They are not pairwise distinct because $|K| > \lambda$, whereas FY has at most λ components (since F is an endofunctor of $\mathcal{K}_{\leq\lambda}$) so that (b) in Definition 4 implies that $\mathcal{K}(B, FY)$ has cardinality at most λ. Choose $k \neq l$ in K with

$$Fz_k \cdot F\tilde{m} \cdot b = Fz_l \cdot F\tilde{m} \cdot b\,. \tag{A.1}$$

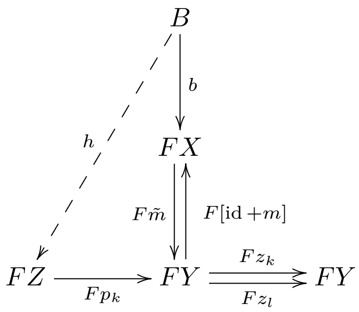
Compare the pullbacks Z of z_k and z_l and $Y_{M_k \cap M_l}$ of b_k and b_l:



Since y_k and y_l are isomorphisms, the connecting morphism p between the above pullbacks is an isomorphism, too. We know that $|M_k \cap M_l| < \lambda$ since M_k, M_l are of our almost disjoint family, thus the object

$$C = Y_{M_k \cap M_l} = \coprod_{M_k \cap M_l} R + \bar{X}_0$$

has less than λ summands, as required. And, due to (c), the pullback of z_k and z_l is absolute. The equality (A.1) thus implies that $F\tilde{m} \cdot b$ factorizes through Fp_k:



Consequently, from $\hat{m}m = \text{id}$ we obtain

$$b = F\tilde{m} \cdot Fp_k \cdot h = F\tilde{m} \cdot Fp_k \cdot Fp^{-1} \cdot Fp \cdot h\,.$$

Thus, for the coproduct injection

$$c \equiv \tilde{m} \cdot p_k \cdot p^{-1} \colon C \to X\,,$$

we get the desired factorization $b = Fc \cdot (Fp \cdot h)$. ◀