Efficiently Realizing Interval Sequences*

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Abstract

We consider the problem of realizable interval-sequences. An interval sequence comprises of n integer intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq n-1$, and is said to be *graphic/realizable* if there exists a graph with degree sequence, say, $D = (d_1, \dots, d_n)$ satisfying the condition $a_i \leq d_i \leq b_i$, for each $i \in [1, n]$. There is a characterisation (also implying an $O(n)$ verifying algorithm) known for realizability of interval-sequences, which is a generalization of the Erdös-Gallai characterisation for graphic sequences. However, given any realizable interval-sequence, there is no known algorithm for computing a corresponding graphic certificate in $o(n^2)$ time.

In this paper, we provide an $O(n \log n)$ time algorithm for computing a graphic sequence for any realizable interval sequence. In addition, when the interval sequence is non-realizable, we show how to find a graphic sequence having minimum deviation with respect to the given interval sequence, in the same time. Finally, we consider variants of the problem such as computing the most regular graphic sequence, and computing a minimum extension of a length p non-graphic sequence to a graphic one.

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1 Introduction

The *Graph Realization problem* for a property P deals with the following existential question: Does there exist a graph that satisfies the property P? Its fundamental importance is apparent, ranging from better theoretical understanding, to network design questions (such as constructing networks with certain desirable connectivity properties). Some very basic, yet challenging, properties that have been considered in past are degree sequences [7, 16, 18], eccentricites [4, 22], connectivity and flow [14, 10, 8, 9].

One of the earliest classical problems studied in this domain is that of *graphic sequences*. A sequence of n positive integers, $D = (d_1, \dots, d_n)$, is said to be *graphic* if there exists an n vertex graph G such that D is identical to the sequence of vertex degrees of G. The problem

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of realizing graphic sequences and counting the number of non-isomorphic realizations of a given graphic-sequence, is particularly of interest due to many practical applications, see [25] and reference therein. In 1960, Erdös and Gallai [7] gave a characterization (also implying an $O(n)$ verifying algorithm) for graphic sequences. Havel and Hakimi [16, 18] gave a recursive algorithm that given a sequence D of integers computes a realizing graph, or proves that the sequence is non-graphic, in optimal time $O(\sum_i d_i)$. Recently, Tripathi et al. [26] provided a constructive proof of Erdös and Gallai's [7] characterization.

We consider a generalization of the graphic sequence problem where instead of specifying precise degrees, we are given a *range* (or interval) of possible degree values for each vertex. Formally, an *interval-sequence* is a sequence of n intervals $\mathcal{S} = ([a_1, b_1], \ldots, [a_n, b_n])$, also represented as $\mathcal{S} = (A, B)$, where $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and $0 \leq a_i \leq b_i \leq n-1$ for every i. It is said to be *realizable* if there exists a sequence $D = (d_1, \ldots, d_n)$ that is graphic and satisfies the condition $a_i \leq d_i \leq b_i$, for $1 \leq i \leq n$. Two questions that are natural to ask here are:

▶ **Question 1** (Verification). *Find an efficient algorithm for verifying the realizability of any given interval-sequence $\mathcal{S}$?*

▶ **Question 2** (Graphic Certificate). *Given a realizable interval-sequence $\mathcal{S}$, compute a certificate (that is, a graphic sequence D) realizing it.*

Cai et al. [5] extended Erdös and Gallai's work by providing an easy to verify characterization for realizable interval-sequences, thereby resolving Question 1. Their result crucially uses the (g, f)-Factor Theorem of Lovász [23]. Garg et al. [13] provided a constructive proof of the characterisation of Cai et al. [5] for realizable interval sequences. In [20], Hell and Kirkpatrick provided an algorithm based on Havel and Hakimi's work for computing a graph that realizes an interval sequence (if exists). For non-realizable interval sequences $\mathcal{S}$, their algorithm computes a graph whose deviation $\delta(D, \mathcal{S})$ (see Section 2 for definition) with respect to L1-norm is minimum. The time complexity of their algorithm is $O(\sum_{i=1}^{n} b_i)$ (which can be as high as $\Theta(n^2)$).

Our Contributions. In this paper we introduce a new approach for representing and analyzing the interval sequence realization problem. Our algorithms are based on a novel divide and conquer methodology, wherein we show that partitioning a realizable interval sequence along any levelled sequence (a new class of sequences introduced herein) guarantees that at least one of the new child interval sequences is also realizable. This enables us to present an $O(n \log n)$ time algorithm for computing a graphic certificate (if exists) for a given interval sequence. While the problem was well studied, to the best of our knowledge there was no known $o(n^2)$ time algorithm for computing graphic certificate. Also, there was no sub-quadratic time algorithm known for computing even the deviation $\delta(D, \mathcal{S})$. Specifically, we obtain the following result.

▶ **Theorem 1.** *There exists an algorithm that for any integer $n \geq 1$ and any length n interval sequence $\mathcal{S}$, computes a graphic sequence D realizing $\mathcal{S}$, if exists, in $O(n \log n)$ time.*

Moreover, when $\mathcal{S}$ is non-realizable, the algorithm outputs in same time a graphic sequence D minimizing the deviation $\delta(D, \mathcal{S})$.

Our new approach enables us to tackle also an optimization version of the problem in which it is required to compute the "most regular" sequence realizing the given interval sequence $\mathcal{S}$, using the natural measure of the minimum sum of pairwise degree differences, $\sum_{i,j} |d_i - d_j|$, as our regularity measure. To the best of our knowledge, this problem was not studied before and is not dealt with directly by the existing approaches to the interval sequence problem. Specifically, we obtain the following.

▶ **Theorem 2.** *There exists an algorithm that for any integer $n \geq 1$ and any length n realizable interval sequence $\mathcal{S}$, computes the most regular graphic sequence realizing interval sequence $\mathcal{S}$ (i.e., the one minimizing the sum of pairwise degree difference), in time $O(n^2)$.*

The tools developed in this paper allows us to study other interesting applications, such as computing a minimum extension of non-graphic sequences to graphic ones (see Section 6).

Related work. Kleitman and Wang [21], and Fulkerson-Chen-Anstee [2, 6, 12] solved the problem of degree realization for directed graphs, wherein, for each vertex both the in-degree and out-degree is specified. In [17], Nichterlein and Hartung proved the NP-completeness of the problem when the additional constraint of acyclicity is imposed. Over the years, various extensions of the degree realization problems were studied as well, cf. [1, 28]. The *Subgraph Realization problem* considers the restriction that the realizing graph must be a subgraph (*factor*) of some fixed input graph. For an interesting line of work on graph factors, refer to [27, 3, 19, 15]. The subgraph realization problems are generally harder. For instance, it is very easy to compute an n-vertex connected graph whose degree sequence consists of all values 2, however, the same problem for subgraph-realization is NP-hard (since it reduces to Hamiltonian-cycle problem).

Lesniak [22] provided a characterization for the sequence of eccentricities of an n-vertex graph. Behzad et al. [4] studied the problem of characterizing the set comprising of vertex-eccentricity values of general graphs (the sequence problem remains open). Fujishige et al. [11] considered the problem of realizing graphs and hypergraphs with given cut specifications.

Organization of the Paper. In Section 2, we present the notation and definitions. In Section 3, we discuss the main ideas and tools that help us to construct graph certificates for interval sequence problem. Section 4 presents our $O(n \log n)$ time algorithm for computing graphic certificate with minimum deviation. Section 5 provides a quadratic-time algorithm for computing the most regular certificate. We discuss the applications in Section 6.

2 Preliminaries

A *sequence* is defined to be an n-element vector whose entries are non-negative integers. For any sequence $D = (d_1, \ldots, d_n)$, define $\min(D) = \min_{i=1}^{n}\{d_i\}$, $\max(D) = \max_{i=1}^{n}\{d_i\}$, $\mathrm{sum}(D) = \sum_{i=1}^{n} d_i$, and $\mathrm{parity}(D) = \mathrm{sum}(D) \mod 2$. Given any two sequences $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, we say that $X \leq Y$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Any two sequences X and Y are said to be *similar* if they are identical up to permutation of the elements (i.e., their sorted versions are identical). A sequence D is said to *lie* in an interval-sequence (A, B), denoted by $D \in (A, B)$, if $A \leq D \leq B$. We define $\min(X, Y) = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$, and $\max(X, Y) = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$. The $\mathbf{L_1}$-distance of the pair (X, Y) is defined as $\mathbf{L_1}(X, Y) = \sum_{i=1}^{n} |y_i - x_i|$.

Denote by $\top$ and $\bot$ the n-length sequences all whose entries are respectively $n-1$ and 0. Given a sequence $D = (d_1, \ldots, d_n)$ and an integer $k \in [1, n]$, define the vectors $X(D)$ and $Y(D)$ by setting for $1 \leq k \leq n$:

$$X_k(D) \triangleq \sum_{i=1}^{k} d_i, \quad \text{and} \quad Y_k(D) \triangleq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k) \ .$$

For any sequence $D = (d_1, \ldots, d_n)$, the *spread* of D is defined as $\phi(D) = \sum_{1 \leq r < s \leq n} |d_r - d_s|$, and it always lies in the range $[0, n^3]$. A sequence D is said to be more *regular* than another sequence D' if $\phi(D) < \phi(D')$. For any two integers $x \leq y$, $[x, y] = \{x, x+1, \ldots, y\}$. For any

$I \subseteq [1, n]$, define $D[I]$ to be the subsequence of D consisting of elements d_i, for $i \in I$; and define E_I to be the *characteristic vector* of I, namely, the sequence $(e_1, e_2, \ldots, e_n)$ such that $e_i = 1$ if $i \in I$, and $e_i = 0$ otherwise. For any sequence $D = (d_1, \ldots, d_n)$ and an interval-sequence $\mathcal{S} = ([a_1, b_1], \ldots, [a_n, b_n])$, the upper and lower deviation of D, is respectively defined as

$$\delta_U(D, \mathcal{S}) = \sum_{i=1}^{n} \max\{0, (d_i - b_i)\}, \quad \text{and} \quad \delta_L(D, \mathcal{S}) = \sum_{i=1}^{n} \max\{0, (a_i - d_i)\} \ .$$

The *deviation* of D is defined as $\delta(D, \mathcal{S}) = \delta_U(D, \mathcal{S}) + \delta_L(D, \mathcal{S})$. For any vertex x in an undirected simple graph H, define $\deg_H(x)$ to be the degree of x in H, and define $N_H(x) = \{y \mid (x, y) \in E(H)\}$ to be the neighbourhood of x in H.

We next state the Erdös and Gallai [7] characterisation for realizable(graphic) sequences, and Cai et al. [5] characterisation for realizable interval sequences. An $O(n)$-time implementation of the both theorems is provided in the full version.

▶ **Theorem 3** (Erdös and Gallai [7]). *A non-increasing sequence $D = (d_1, \ldots, d_n)$ is graphic if and only if*

(i) *$X_n(D)$ is even, and*

(ii) *$X(D) \leq Y(D)$.*

▶ **Theorem 4** (Cai et al. [5]). *Let $\mathcal{S} = ([a_1, b_1], \ldots, [a_n, b_n]) = (A, B)$ be an interval-sequence such that A is non-increasing and for any index $1 \leq i < n$, $b_{i+1} \leq b_i$ whenever $a_i = a_{i+1}$. For each $k \in [1, n]$, define $W_k(\mathcal{S}) = \{i \in [k+1, n] \mid b_i \geq k+1\}$. Then $\mathcal{S}$ is realizable if and only if $X(A) \leq Y(B) - \varepsilon(\mathcal{S})$, where, $\varepsilon(\mathcal{S})$ is defined by setting*

$$\varepsilon_k(\mathcal{S}) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for } i \in W_k(\mathcal{S}) \text{ and } \sum_{i \in W_k(\mathcal{S})} (b_i + k|W_k(\mathcal{S})|) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

3 Main Tools

In this section, we develop some crucial tools that help us in efficient computation of certificate for a realizable interval-sequence. These tools will help us to search a graphic sequence in $O(n \log n)$ time using a clever divide and conquor methodology. Also they aid in searching for the maximally regular sequence in just quadratic time.

Levelling operation. Given a sequence $D = (d_1, \ldots, d_n)$ and a pair of indices $\alpha \neq \beta$ satisfying $d_\alpha > d_\beta$, we define $\pi(D, \alpha, \beta) = D^* = (d_1^*, \ldots, d_n^*)$ to be a sequence obtained from D by decrementing d_α by 1 and incrementing d_β by 1 (i.e., $d_\alpha^* = d_\alpha - 1$, $d_\beta^* = d_\beta + 1$, and $d_k^* = d_k$ for $k \neq \alpha, \beta$). This operation is called the *levelling operation* on D for the indices α and β. The operation essentially "levels" (or "flattens") the sequence D, making it more uniform.

We now discuss some properties of levelling operations.

▶ **Lemma 5.** *Any levelling operation on a sequence D that results in a non-similar sequence, reduces its spread $\phi(D)$ by a value at least two.*

Proof. Let $D = (d_1, d_2, \ldots, d_n)$ and $Z = (z_1, \ldots, z_n) = \pi(D, \alpha, \beta)$, be a sequence obtained from D by performing a levelling operation on a pair of indices α, β such that $d_\alpha > d_\beta$. If $d_\alpha = d_\beta + 1$, then it is easy to verify that D and Z are similar. If $d_\alpha \geq d_\beta + 2$, then

$$\begin{aligned}
\phi(Z) &= |z_\alpha - z_\beta| + \sum_{s \neq \alpha,\beta} (|z_\alpha - z_s| + |z_\beta - z_s|) + \sum_{\substack{1 \leq r < s \leq n, \\ r,s \notin \{\alpha,\beta\}}} |z_r - z_s| \\
&= |d_\alpha - d_\beta| - 2 + \sum_{\substack{s \neq \alpha,\beta \text{ s.t.} \\ d_s \notin (d_\beta, d_\alpha)}} (|d_\alpha - d_s| + |d_\beta - d_s|) \\
&\quad + \sum_{\substack{s \neq \alpha, \\ \beta \text{ s.t.} d_s \in (d_\beta, d_\alpha)}} (|d_\alpha - d_s| + |d_\beta - d_s| - 2) + \sum_{\substack{1 \leq r < s \leq n, \\ r,s \notin \{\alpha,\beta\}}} |d_r - d_s| \\
&\leq \Big(\sum_{1 \leq r < s \leq n} |d_r - d_s| \Big) - 2 \;=\; \phi(D) - 2 \; .
\end{aligned}$$

Thus, the claim follows. ◀

▶ **Lemma 6** (Corollary 3.1.4, [24]). *The levelling operations preserves graphicity, that is, if we perform a levelling operation on a graphic sequence, then the resulting sequence is also graphic.*

Proof. Let $D = (d_1, \ldots, d_n)$ be a graphic sequence, and $\pi(D, \alpha, \beta) = D^* = (d_1^*, \ldots, d_n^*)$ for some indices α, β satisfying $d_\alpha > d_\beta$. If $d_\alpha = 1 + d_\beta$, then D^* is similar to D, and thus also graphic. So for the rest the proof let us focus on the case $d_\alpha \geq 2 + d_\beta$. Let $G = (V, E)$ be a graph realising the sequence D, and let x_α and x_β be two vertices in G having degrees respectively d_α and d_β. Since $|N_G(x_\alpha)| \geq 2 + |N_G(x_\beta)|$, there must exists at least one neighbour, say w, of vertex x_α that does not lie in set $\{x_\beta\} \cup N_G(x_\beta)$. Let $G^* = (V, E^*)$ be a graph obtained from G by deleting the edge (w, x_α), and adding a new edge (w, x_β). Observe that the degree of all vertices other than x_α and x_β are identical in graphs G and G^*, also $\deg_{G^*}(x_\alpha) = \deg_G(x_\alpha) - 1$, and $\deg_{G^*}(x_\beta) = \deg_G(x_\beta) + 1$. Therefore G^* is a graph realising the profile D^*, and thus the claim follows. ◀

Levelled sequences. A sequence D is said to be *levelled with respect to the integer-sequence* $\mathcal{S} = (A, B)$ if

(i) $A \leq D \leq B$, and

(ii) the spread of D cannot be decreased by a levelling operation, i.e., for any two indices $\alpha \neq \beta$ satisfying $d_\alpha > d_\beta$ and $A \leq \pi(D, \alpha, \beta) \leq B$, we have $\phi(\pi(D, \alpha, \beta)) = \phi(D)$.

See Figure 1.

The *volume* of a sequence D lying between A and B with respect to $\mathcal{S} = (A, B)$ is defined as

$$\textsc{vol}(D, \mathcal{S}) \triangleq \mathbf{L_1}(D, A) \; ,$$

and is invariant of levelling operations applied to D. In other words, applying a levelling operation to a sequence D may reduce its spread but preserves its volume. Note that the volume lies in the range $[0, \; \mathbf{L_1}(A, B)]$.

▶ **Lemma 7.** *For any $\mathcal{S} = (A, B)$, a sequence D satisfying $A \leq D \leq B$ can be transformed into a levelled sequence D^* having the same volume $\textsc{vol}(D, \mathcal{S})$ by a repeated application of (at most $O(n^3)$) levelling operations.*[1]

Proof. By Lemma 5, every levelling operation that results in a new (non-similar) sequence decreases the spread by at least two. Since the spread of any sequence D is always non-negative and finite (specifically, $O(n^3)$), it is possible to perform ($O(n^3)$) levelling operations on D so that the resultant sequence D^* is levelled. Since the levelling operation preserves the volume, $\textsc{vol}(D^*, \mathcal{S})$ must be same as $\textsc{vol}(D, \mathcal{S})$. ◀

Any graphic sequence D realizing the interval sequence $\mathcal{S} = (A, B)$ by Lemma 7 can be altered by $O(n^3)$ levelling operations to obtain a levelled sequence lying between A and B. The resultant sequence by Lemma 6 remains graphic, thus the following theorem is immediate.

▶ **Theorem 8.** *For any realizable interval sequence $\mathcal{S} = (A, B)$ there exists a graphic sequence realizing $\mathcal{S}$ which is a levelled sequence.*

Characterizing and Computing Levelled sequences. Given any interval sequence $\mathcal{S} = (A, B)$ and a real number $\ell \in [\min(A), \max(B)]$, let[2]

$$F(\ell, \mathcal{S}) \triangleq \sum_{i \in [1,n]} (\min\{\ell, b_i\} - \min\{\ell, a_i\}) \ .$$

Observe that $F(\cdot, \mathcal{S})$ is a non-decreasing function in the range $(\min(A), \max(B))$. Hence we may define the corresponding inverse function as $F^{-1}(L, \mathcal{S}) = \min\{\ell \mid F(\ell, \mathcal{S}) = L\}$.

Given any interval sequence $\mathcal{S} = (A, B)$, we define $I(\ell, \mathcal{S}) \triangleq \{i \in [1, n] \mid a_i < \ell < b_i\}$.

We conclude this section by providing the following theorems for characterising and computing levelled sequences (proofs are deferred to the full version).

▶ **Theorem 9.** *Consider an interval sequence $\mathcal{S} = (A, B)$. Let L be an integer in $[0, \mathbf{L_1}(A, B)]$ and $\ell \geq 0$ be such that $\ell = F^{-1}(L, \mathcal{S})$. Then the collection of levelled sequences that have volume L with respect to $\mathcal{S}$ is equal to the collection of sequences $D = (d_1, \ldots, d_n)$ satisfying the following three conditions:*

(a) *$d_i = b_i$ for any i satisfying $b_i \leq \ell$;*

(b) *$d_i = a_i$ for any i satisfying $a_i \geq \ell$; and*

(c) *Among all indices lying in set $I(\ell, \mathcal{S})$, exactly $F(\ell, \mathcal{S}) - F(\lfloor \ell \rfloor, \mathcal{S})$ indices i satisfy $d_i = \lceil \ell \rceil$, and the remaining indices i satisfy $d_i = \lfloor \ell \rfloor$.*

▶ **Theorem 10.** *Given an interval sequence $\mathcal{S} = (A, B)$ consisting of n-pairs, and an integer $L \in [0, \mathbf{L_1}(A, B)]$, a levelled sequence D having volume L with respect to $\mathcal{S}$ can be computed in $O(n)$ time.*

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[1] We remark that the algorithms presented later on generate a desired levelled sequence using more efficient methods than the one implicit in the proof, and are therefore faster.

[2] One can think of $\mathcal{S}$ as representing a collection of n connected vessels, each in the shape of a unit column closed at both ends, then $F(\ell, \mathcal{S})$ is the amount of fluid that will fill this connected vessel system to level ℓ.

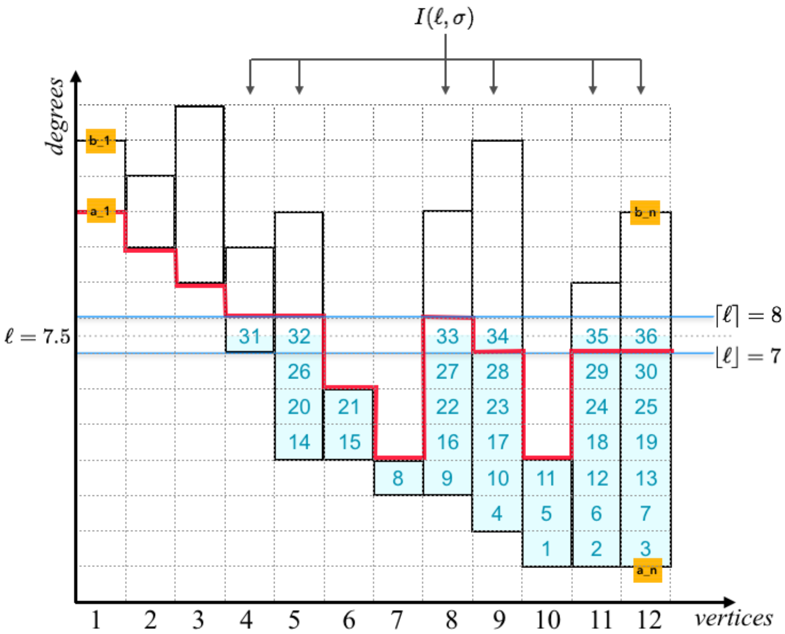

Figure 1 Illustration of a levelled sequence D (in red) satisfying $L = \textsc{vol}(D, \mathcal{S}) = 33$. For $\ell = 7.5$, $F(\ell = 7.5, \mathcal{S}) = 33$, $F(\lfloor \ell \rfloor = 7, \mathcal{S}) = 30$, and $F(\lceil \ell \rceil = 8, \mathcal{S}) = 36$. The segments contributing to $F(\ell = 7.5, \mathcal{S})$, i.e., the parts of the connected vessel system filled with fluid, are shown in blue. The values in D at all indices in set $I(\ell, \mathcal{S})$ differ by at most one as they lie in the set $\{\lfloor \ell \rfloor, \lceil \ell \rceil\}$.

4 An $O(n \log n)$ time algorithm for Graphic Certificate

In this section, we present an algorithm for computing a certificate for interval sequence that takes just $O(n \log n)$ time. If the input interval $\mathcal{S} = (A, B)$ is realizable, our algorithm computes a graphic sequence $D \in \mathcal{S}$, otherwise it computes a sequence minimizing the deviation value $\delta(D, \mathcal{S})$. We begin by considering the case where the sequence $\mathcal{S}$ is realizable (since it is simpler to understand given Theorems 9 and 10), and then we move to the case where $\mathcal{S}$ is non-realizable. Then characterization of [5] implies an $O(n)$ time verification algorithm for realizability of interval sequence. (For details refer to the full version).

4.1 Realizable Interval Sequences

First we show that any two levelled sequences after an appropriate reordering of their elements are coordinate-wise comparable.

▶ **Lemma 11.** *For any interval sequence $\mathcal{S} = (A, B)$, and any two levelled sequences $C, D \in \mathcal{S}$ satisfying $\textsc{vol}(D, \mathcal{S}) \le \textsc{vol}(C, \mathcal{S})$, the following holds.*

1. *$D' \le C$, for some sequence $D' \in \mathcal{S}$ similar to D.*
2. *$D \le C''$, for some sequence $C'' \in \mathcal{S}$ similar to C.*

Proof. We show how to transform $D = (d_1, \cdots, d_n)$ into sequence $D' = (d'_1, \cdots, d'_n) \in \mathcal{S}$ such that $D' \le C$. Let $\ell_D = F^{-1}(\textsc{vol}(D, \mathcal{S}), \mathcal{S})$ and $\ell_C = F^{-1}(\textsc{vol}(C, \mathcal{S}), \mathcal{S})$. Since $F(\cdot, \mathcal{S})$ is a non-decreasing function, we have that $\ell_D \le \ell_C$.

Let us first consider the case where ℓ_C and ℓ_D are both non-integral, and $\lfloor \ell_C \rfloor = \lfloor \ell_D \rfloor =$ (say ℓ_1) and $\lceil \ell_C \rceil = \lceil \ell_D \rceil =$ (say ℓ_2). By Theorem 9, for any index $i \in [1, n]$,

(i) $a_i \geq \ell_D$ (or $a_i \geq \ell_C$) implies $d_i = a_i = c_i$;
(ii) $b_i \leq \ell_D$ (or $b_i \leq \ell_C$) implies $d_i = b_i = c_i$.

Also, among indices in set $I_0 = I(\ell_D, \mathcal{S}) = I(\ell_C, \mathcal{S})$,

(i) exactly $L_D - F(\lfloor \ell_D \rfloor, \mathcal{S})$ indices i satisfy $d_i = \ell_2$ (let I_D denote the set of these indices) and the remaining indices i satisfy $d_i = \ell_1$;
(ii) exactly $L_C - F(\lfloor \ell_C \rfloor, \mathcal{S})$ indices i satisfy $c_i = \ell_2$ (let I_C denote the set of these indices) and the remaining indices i satisfy $c_i = \ell_1$.

Since $L_D \leq L_C$, it follows that $|I_D| \leq |I_C|$, however, observe that I_D need not be a subset of I_C. We set D' to be the sequence that satisfy the condition that

(i) $d'_i = d_i$, for each $i \notin I_0$, and
(ii) for indices in I_0, at any arbitrary $|I_D|$ indices lying in I_C, d'_i take the value ℓ_2, and at remaining $|I_0| - |I_D|$ indices d'_i take the value ℓ_1.

It is easy to verify that D and D' are similar, and $D' \leq C$.

The remaining case is when $\lceil \ell_D \rceil \leq \lfloor \ell_C \rfloor$. For any index $i \in [1, n]$, $d_i \leq \lceil \ell_D \rceil$ and $c_i \geq \lfloor \ell_C \rfloor$, implies $d_i \leq c_i$. Observe that by Theorem 9,

(i) for an index i, $d_i > \lceil \ell_D \rceil$ implies $d_i = a_i (\leq c_i)$; and
(ii) for an index i, $c_i < \lfloor \ell_C \rfloor$ implies $c_i = b_i (\geq d_i)$.

Therefore, for each index i, $d_i \leq c_i$. So in this case, we set D' to be D. The construction of sequence C'' follows similarly. ◀

Next lemma shows significance of partitioning an interval-sequence using a levelled sequence.

▶ **Lemma 12.** *Let C and D be any two levelled sequences lying in an interval sequence $\mathcal{S} = (A, B)$, and having volume L_C and L_D, respectively. Also assume D is a graphic sequence. Then,*

(a) *$L_D \leq L_C$ implies (A, C) is a realizable interval sequence.*
(b) *$L_D \geq L_C$ implies (C, B) is a realizable interval sequence.*

Proof. We provide proof of the case $L_D \leq L_C$ (the proof of part (b) will follow in a similar fashion). By Lemma 11, we can transform $D = (d_1, \cdots, d_n)$ into another levelled sequence $D' = (d'_1, \cdots, d'_n) \in \mathcal{S}$ such that D' is similar to D and $D' \leq C$. Since $D' \leq C$, and D' is a graphic sequence, it follows that (A, C) is realizable interval sequence. ◀

From Lemma 12, and the fact that each realizable interval-sequence contains a levelled graphic sequence (see Theorem 8), we obtain following.

▶ **Theorem 13.** *For any realizable interval sequence $\mathcal{S} = (A, B)$, and any levelled sequence $C \in \mathcal{S}$, at least one of the interval-sequences (A, C) and (C, B) is realizable.*

The above theorem provides a divide-and-conquer strategy to search for a levelled graphic sequence for realizable interval-sequences as shown in Algorithm 1. Let (A_0, B_0) be initialized to (A, B). We compute a levelled sequence C_0 having volume $\lfloor \mathbf{L_1}(A_0, B_0)/2 \rfloor$ using Theorem 9. It follows from Theorem 13, either (A_0, C_0) or (C_0, B_0) must be a realizable interval-sequence. If (A_0, C_0) is realizable then we replace B_0 by C_0; otherwise (C_0, B_0) must be realizable, so we replace A_0 by C_0. We continue this process (of replacements) until $\mathbf{L_1}(A_0, B_0)$ decreases to a value smaller than 2. In the end, the interval sequence (A_0, B_0) contains at most two sequences, namely A_0 and B_0. If A_0 is graphic then we return A_0, otherwise we return B_0. The correctness of the algorithm is immediate from the description.

Algorithm 1 CERTIFICATE-REALIZABLE(A, B).

Initialize interval sequence (A_0, B_0) to (A, B);
while $\mathbf{L_1}(A_0, B_0) \geq 2$ **do**
  $C_0 \leftarrow$ a levelled sequence of volume $\lfloor \mathbf{L_1}(A_0, B_0)/2 \rfloor$;
  if (Interval-sequence (A_0, C_0) is realizable) **then** $B_0 \leftarrow C_0$;
  else $A_0 \leftarrow C_0$;
if A_0 *is graphic* **then** Return A_0;
else Return B_0;

To analyze the running time, observe that the $\mathbf{L_1}$-distance between A_0 and B_0 decreases by (roughly) a factor of 2 in each call of the while loop, so it follows that number of iterations is $O(\log n)$. Verifying if an interval sequence is realizable, or a sequence D is graphic can be performed in $O(n)$ time. Also in $O(n)$ time we can generate a levelled sequence of any given volume L by Theorem 10. Thus, the total time complexity of the algorithm is $O(n \log n)$.

We obtain the following result:

▶ **Theorem 14.** *There exists an algorithm that for any integer $n \geq 1$ and any n-length interval sequence $\mathcal{S} = (A, B)$, computes a graphic sequence $D \in (A, B)$, if it exists, in $O(n \log n)$ time.*

4.2 Non-Realizable Sequences

In this subsection we consider the scenario where $\mathcal{S}$ is non-realizable, our goal is to compute a graphic sequence D minimizing the deviation $\delta(D, \mathcal{S})$ with respect to the given interval sequence $\mathcal{S}$.

As a first step, we show that in order to search a sequence D minimizing $\delta(D, \mathcal{S})$, it suffices to search a sequence $D \geq A$ that minimizes the value $\delta_U(D, \mathcal{S})$.

▶ **Lemma 15.** $\min\{\delta(D, \mathcal{S}) \mid D$ *is graphic* $\} = \min\{\delta_U(D, \mathcal{S}) \mid D$ *is graphic,* $D \geq A\}$*, for any interval sequence* $\mathcal{S} = (A, B)$.

Proof. Let $D = (d_1, \ldots, d_n)$ be a graphic sequence minimizing the value $\delta(D, \mathcal{S})$, and in case of ties take that D for which $\delta_L(D, \mathcal{S})$ is the lowest. Let us suppose there exists an index $i \in [1, n]$ such that $d_i < a_i$. Consider the graph G realizing the sequence D, and let v_i denote the ith vertex of G, so that, $\deg(v_i) = d_i$. Observe that $|N_G(v_i)| \neq n - 1$, since $d_i < a_i \leq n - 1$. For any vertex $v_j \notin N_G(v_i)$, $d_j = \deg(v_j)$ must be at least b_j, because otherwise adding (v_i, v_j) to G reduces $\delta(D, \mathcal{S})$. Thus for any vertex $v_j \notin N_G(v_i)$, adding (v_i, v_j) to G, decreases $\delta_L(D, \mathcal{S})$ and increases $\delta_U(D, \mathcal{S})$ by a value exactly 1. However, by our choice D was a sequence minimizing $\delta_L(D, \mathcal{S})$, thus $\delta_L(D, \mathcal{S})$ must be zero. The claim follows from the fact that $D \geq A$ and $\delta(D, \mathcal{S}) = \delta_U(D, \mathcal{S})$. ◀

By the previous lemma, our goal is to find a graphic sequence D in the interval sequence $(A, \top)$ minimizing $\delta(D, \mathcal{S})$. Notice that if D is graphic, then the interval sequence (A, R), where $R = \max(D, B)$, is realizable. Also, $\delta(D, S) = \mathrm{sum}(R - B)$. Hence, in order to compute a graphic sequence with minimum deviation, we define $\mathcal{R}$ to be the set of all sequence $R \in [B, \top]$ such that

(i) the interval sequence (A, R) is realizable, and
(ii) $\mathrm{sum}(R - B)$ is minimized.

The following lemma shows the significance of the set $\mathcal{R}$ in computing a certificate with minimum deviation.

▶ **Lemma 16.** *For any $R \in \mathcal{R}$, and any graphic sequence D_0 lying in the interval sequence (A, R), we have $\delta(D_0, \mathcal{S}) = \min\{\delta(D, \mathcal{S}) \mid D$ is graphic $\} = \mathrm{sum}(R - B)$.*

Proof. Let D^* be a graphic sequence minimizing the value $\delta(D, \mathcal{S})$. By Lemma 15, we may assume that D^* belongs to $(A, \top)$. Observe that $\delta(D^*, \mathcal{S}) = \mathrm{sum}(R^* - B)$, where $R^* = \max\{B, D^*\}$. By the choice of D^* we have that $\mathrm{sum}(R^* - B) = \delta(D^*, \mathcal{S}) \leq \delta(D_0, \mathcal{S}) = \delta_U(D_0, \mathcal{S}) \leq \mathrm{sum}(R - B)$, where the last inequality follows from the fact that $D_0 \in (A, R)$. By definition of $\mathcal{R}$, we have that $\mathrm{sum}(R^* - B) \geq \mathrm{sum}(R - B)$, and therefore $\delta(D^*, \mathcal{S}) = \delta(D_0, \mathcal{S}) = \mathrm{sum}(R - B) = \mathrm{sum}(R^* - B)$. Thus R^* also lies in the set $\mathcal{R}$. The lemma follows from the fact that $\delta(D^*, \mathcal{S}) = \min\{\delta(D, \mathcal{S}) \mid D$ is graphic $\}$. ◀

Next, let $\mathcal{R}_L$ be the set of all levelled sequences in $\mathcal{R}$ with respect to interval sequence $(B, \top)$.

▶ **Lemma 17.** $\mathcal{R}_L \neq \emptyset$.

Proof. Clearly, $\mathcal{R} \neq \emptyset$. Consider any sequence $R = (r_1, \ldots, r_n) \in \mathcal{R}$. Suppose there exists $\alpha, \beta \in [1, n]$ such that $r_\alpha - r_\beta \geq 1$ and $R' = \pi(R, \alpha, \beta) \in [B, \top]$. Observe that $\mathrm{sum}(R' - B) = \mathrm{sum}(R - B)$. It remains to show that (A, R') is realizable. Indeed, if $D \in (A, R)$ is a graphic sequence, then either

(i) $D = (d_1, \ldots, d_n)$ lies in (A, R'), or

(ii) $d_\alpha - d_\beta \geq 1$ and $D' = \pi(D, \alpha, \beta)$ lies in (A, R').

Since levelling operation preserves graphicity, D' is graphic. Thus $R' \in \mathcal{R}$, which shows that $\mathcal{R}$ is closed under the levelling operation, and hence $\mathcal{R}_L$ is non-empty. ◀

Algorithm 2 CERTIFICATE-NON-REALIZABLE(A, B).

$(M_1, M_2) \leftarrow (B, \top)$;
while $\mathbf{L_1}(M_1, M_2) \geq 2$ **do**
  $M_0 \leftarrow$ a levelled sequence of volume $\lfloor \mathbf{L_1}(M_1, M_2)/2 \rfloor$;
  if (Interval-sequence (A, M_0) is realizable) **then** $M_2 \leftarrow M_0$;
  else $M_1 \leftarrow M_0$;
if (A, M_1) *is realizable* **then** $R \leftarrow M_1$;
else $R \leftarrow M_2$;
Return CERTIFICATE-REALIZABLE(A, R)

We now describe the algorithm for computing a graphic sequence with minimum deviation (refer to Algorithm 2 for a pseudocode). Recall that we assume that (A, B) is a non-realizable interval sequence. The first step is to compute a levelled sequence $R \in \mathcal{R}_L$, and the second is to use Algorithm 1 to find a graphic sequence in (A, R).

We initialize two sequences M_1 and M_2, resp., to B and $\top$, and these sequences serve as lower and upper boundaries for sequence R. The pair (M_1, M_2) is updated as long as $\mathrm{sum}(M_2 - M_1) \geq 2$ as follows. We compute a levelled sequence M_0 having volume $\lfloor \mathbf{L_1}(M_1, M_2)/2 \rfloor$ with respect to the interval sequence (M_1, M_2) using Theorem 10. There are two cases:

Case 1. (A, M_0) is realizable.

Consider any sequence $R \in (M_1, M_2)$ that lies in $\mathcal{R}_L$. Since (A, M_0) is realizable, from the definition of R it follows that $\text{sum}(R-B) \leq \text{sum}(M_0 - B)$. As R and M_0 both belong to (M_1, M_2), by Lemma 11, there exists a sequence R_0 similar to R lying in interval $(M_1, M_2) \subseteq (B, \top)$ such that $R_0 \leq M_0$. It is easy to check that $R_0 \in \mathcal{R}_L$, thus the search range of R which was (M_1, M_2) can be narrowed down to (M_1, M_0), so we reset M_2 to M_0.

Case 2. (A, M_0) is not realizable.

Consider any $R \in \mathcal{R}_L$, we first show that $\text{sum}(R - B) > \text{sum}(M_0 - B)$. Let us assume on the contrary, $\text{sum}(R-B) \leq \text{sum}(M_0 - B)$. In such a case, by Lemma 11, there exists a sequence R' similar to R lying in interval $(M_1, M_2) \subseteq (B, \top)$ such that $R' \leq M_0$. Also $R' \in \mathcal{R}_L$. Since, by definition of $\mathcal{R}_L$, (A, R') is realizable, it violates the fact that (A, M_0) is not realizable. Now as R, M_0 both belong to (M_1, M_2), by Lemma 11, there exists a sequence R_0 similar to R lying in interval $(M_1, M_2) \subseteq (B, \top)$ such that $R_0 \geq M_0$. Also $R_0 \in \mathcal{R}_L$, thus the search range of R can be narrowed down to (M_0, M_2), so we reset M_1 to M_0.

We continue the process of shrinking the range (M_1, M_2) until $\mathbf{L_1}(M_1, M_2)$ decreases to a value smaller than 2. Finally there exists in range (M_1, M_2) at most two sequences, namely M_1 and M_2. If (A, M_1) is graphic then we set R to M_1, otherwise we set R to M_2.

The running time analysis is similar to the one for Algorithm 1. Since the $\mathbf{L_1}$-distance between M_1 and M_2 decreases by a factor of 2 in each successive call of the while loop of the algorithm, it follows that number of times the while loops run is $O(\log n)$. Verifying if an interval sequence is realizable, or a sequence D is graphic can be performed in $O(n)$ time. Also it takes $O(n)$ time to generate a levelled sequence of any given volume L by Theorem 10. Finally, the running time of Algorithm 1 is $O(n \log n)$. Thus, the total time complexity of algorithm is $O(n \log n)$.

This completes the proof of Theorem 1.

5 Most Regular Certificate in $O(n^2)$ time

In this section, we present an $O(n^2)$-time algorithm for computing a most-regular certificate with respect to a given interval sequence $\mathcal{S} = (A, B)$. We assume that $\mathcal{S}$ is realizable. Our algorithm involves a subroutine that given an integer $z \in [\min(A), \max(B) - 1]$, computes a most-regular graphic-sequence, say D, satisfying the condition $z \leq \ell = F^{-1}(\text{VOL}(D, \mathcal{S}), \mathcal{S}) \leq z + 1$. The following lemma is immediate from Theorem 9.

▶ **Lemma 18.** *Any levelled sequence $\bar{D} = (\bar{d}_1, \ldots, \bar{d}_n)$ of volume L with respect to interval sequence $\mathcal{S} = (A, B)$, satisfies $z \leq \ell = F^{-1}(L, \mathcal{S}) \leq z+1$ if and only if $\bar{d}_i = a_i$ for $a_i \geq z+1$, $\bar{d}_i = b_i$ for $b_i \leq z$, and $\bar{d}_i \in \{z, z+1\}$ for remaining indices i.*

We partition the set $[1, n]$ into three sets I_1, I_2, I_3 such that $I_1 = \{i \in [1, n] \mid a_i \geq z+1\}$, $I_2 = \{i \in [1, n] \mid a_i \leq z \text{ and } z + 1 \leq b_i\}$, and $I_3 = \{i \in [1, n] \mid b_i \leq z\}$. Also, using integer sort in linear time, we rearrange the pairs in (A, B) along with the corresponding sets I_1, I_2, I_3 so that

(i) for any $i \in I_1$, $j \in I_2$, $k \in I_3$, we have $i < j < k$, and

(ii) the sub-sequences $A[I_1]$ and $B[I_3]$ are sorted in the non-increasing order.

We initialize $D_z = (d_{z,1}, d_{z,2}, \ldots, d_{z,n})$ by setting $d_{z,i}$ to : a_i if $i \in I_1$, z if $i \in I_2$, and b_i if $i \in I_3$. The sequence D_z is sorted in non-increasing order, since the sub-sequences $A[I_1]$ and $B[I_3]$ are sorted in non-increasing order. Let $\alpha = |I_1|$ and $\beta = |I_1| + |I_2|$, so that $I_2 = [\alpha+1, \alpha+2, \ldots, \beta]$. We would search all those indices $i \in [\alpha, \beta]$ such that on incrementing $d_{\alpha+1}, \ldots, d_i$ to value $z+1$, the resulting sequence is graphic; or equivalently, the sequence $D_z + E_{[\alpha+1,i]}$ is graphic. Note that for any index $i \in [\alpha, \beta]$,

(i) the sequence $D_z + E_{[\alpha+1,i]}$ is non-increasing, and
(ii) $A \leq D_z + E_{[\alpha+1,i]} \leq B$.

The next lemma, which follows from the definition of ϕ, will be used to compute $\phi(D_z + E_{[\alpha+1,i]})$ from $\phi(D_z)$. (The proof is deferred to the full version).

▶ **Lemma 19.** *For any index $i \in [\alpha+1, \beta]$, $\phi(D_z + E_{[\alpha+1,i]}) = \phi(D_z) + (i-\alpha)(n-i-\alpha)$.*

For each z we compute the vectors $X(D_z)$ and $Y(D_z)$. For each integer $k \in [1, n]$, let

$$\textsc{Avoid}(k) = \left\{ i \in [\alpha, \beta] \;\middle|\; X_k\big(D_z + E_{[\alpha+1,i]}\big) > Y_k\big(D_z + E_{[\alpha+1,i]}\big) \right\}, \text{ and}$$

$$\textsc{Avoid} = \textstyle\bigcup_{k=1}^{n} \textsc{Avoid}(k) \ .$$

By Theorem 3, for any $i \in [\alpha, \beta]$, the sequence $D_z + E_{[\alpha+1,i]}$ is graphic if and only if i does not lie in the set $\textsc{Avoid}$, and $\text{parity}(D_z + E_{[\alpha+1,i]}) = 0$. The following lemma (whose proof is deferred to the full version) shows that the set $\textsc{Avoid}(k)$, for any index k, is computable in $O(1)$ time.

▶ **Lemma 20.** *For each $k \in [1, n]$, $\textsc{Avoid}(k)$ is a contiguous sub-interval of $[1, n]$, and is computable in $O(1)$ time.*

Algorithm 3 presents the procedure for computing the most-regular certificate. For each $k \in [1, n]$, $\textsc{Avoid}(k)$ is a contiguous sub-interval of $[1, n]$, therefore, the union $\textsc{Avoid} = \bigcup_{k=1}^{n} \textsc{Avoid}(k)$ can be computed in linear time using simple stack based data-structure, once the intervals are sorted in order of their endpoints[3] using integer sort. Let $\mathcal{I}_z$ denote the set obtained by removing from $[\alpha, \beta] \setminus \textsc{Avoid}$ each index i for which $\text{parity}(D_z + E_{[\alpha+1,i]}) = \text{parity}(\text{sum}(D_z) + (i-\alpha))$ is non-zero. Since $\text{sum}(D_z)$ (or $\text{parity}(D_z)$) is computable in $O(n)$, the set $\mathcal{I}_z$ can be computed in $O(n)$ time as well. Note that $D_z + E_{[\alpha+1,i]}$ is graphic if and only if $i \in \mathcal{I}_z$. By Lemma 19, for any index $i \in \mathcal{I}_z$, the value $\phi(D_z + E_{[\alpha+1,i]})$ is computable in $O(1)$ time, once we know $\phi(D_z)$. This shows that in just $O(n)$ time, we can compute the spread of all the levelled sequences D satisfying $z \leq F^{-1}(\textsc{vol}(D), \mathcal{S}) < z+1$, and also find a sequence having the minimum spread. All that remains is to efficiently computing $\phi(D_z)$ for each $z \in [\min(A), \max(B)]$. Observe that $D_{\min(A)} = A$, and so $\phi(D_{\min(A)}) = \sum_{1 \leq r < s \leq n} |a_r - a_s|$ is computable in $O(n^2)$ time. Next by Lemma 19, for any $z \in [\min(A), \max(B) - 1]$, $\phi(D_{z+1}) = \phi(D_z) + (\beta - \alpha)(n - \beta - \alpha)$ is computable in $O(1)$ time. Since z can take $\max(B) - \min(A) - 1$ values, our algorithm in total takes $O(n^2 + n(\max(B) - \min(A) - 1)) = O(n^2)$ time.

This completes the proof of Theorem 2.

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[3] We say $[r, s] \leq [r', s']$ if either
(i) $r < r'$, or
(ii) $r = r'$ and $s \leq s'$.

Algorithm 3 MOST-REGULAR-CERTIFICATE(A, B).

OPT $\leftarrow \infty$;
foreach $z \in [\min(A), \max(B) - 1]$ **do**
  $I_1 \leftarrow \{i \in [1, n] \mid a_i \geq z + 1\}$;
  $I_2 \leftarrow \{i \in [1, n] \mid a_i \leq z, z + 1 \leq b_i\}$;
  $I_3 \leftarrow \{i \in [1, n] \mid b_i \leq z\}$;
  Rearrange the pairs in (A, B) along with the corresponding sets I_1, I_2, I_3 so that (i) for any triplet (i, j, k) satisfying $i \in I_1$, $j \in I_2$, $k \in I_3$, we have $i < j < k$, and (ii) the sub-sequences $A[I_1]$ and $B[I_3]$ are sorted in the non-increasing order;
  Initialize $D_z = (d_1, d_2, \ldots, d_n)$, where for $i \in I_1$, $d_i = a_i$; for $i \in I_2$, $d_i = z$; and for $i \in I_3$, $d_i = b_i$;
  if $z = \min(A)$ **then** set $\phi(D_z) = \sum_{1 \leq r < s \leq n} |a_r - a_s|$;
  Compute $X(D_z), Y(D_z)$;
  Let $\alpha = |I_1|$ and $\beta = |I_1| + |I_2|$;
  for $k = 1$ *to* n **do**
    AVOID$_1(k) = [\alpha, \beta] \cap [\alpha + 1 + Y_k(D_z) - X_k(D_z), k]$;
    if $\max\{k - \alpha, 0\} + X_k(D_z) > Y_k(D_z)$ *and* $k \leq z$ **then** AVOID$_2(k) = [\alpha, \beta] \cap [k, n]$;
    else if $\max\{k - \alpha, 0\} + X_k(D_z) \leq Y_k(D_z)$ **then** AVOID$_2(k) = \emptyset$;
    else AVOID$_2(k) = [\alpha, \beta] \cap [k, \max\{k, \alpha\} + X_k(D_z) - Y_k(D_z) - 1]$;
    AVOID$(k) =$ AVOID$_1(k) \cup$ AVOID$_2(k)$;
  Compute AVOID $= \bigcup_{k=1}^{n}$ AVOID(k);
  foreach $i \in [\alpha, \beta] \setminus$ AVOID **do**
    if $(\text{parity}(D_z) = (i - \alpha) \mod 2)$ **then**
      Compute $\phi(D_z + E_{[\alpha+1, i]}) = \phi(D_z) + (i - \alpha)(n - i - \alpha)$;
      OPT $= \min\{$OPT$, \phi(D_z + E_{[\alpha+1, i]})\}$;
  Set $\phi(D_{z+1}) = \phi(D_z) + (\beta - \alpha)(n - \beta - \alpha)$;
Return OPT and the corresponding graphic sequence;

6 Applications and Extensions

In this section, we discuss some related problems whose solutions follow as immediate application of our interval sequence work.

▶ **Problem 1** (Minimum Graphic extensions). *Given a sequence $A = (a_1, \ldots, a_p)$ find the minimum integer $n (\geq p)$ such that a super sequence $D = (a_1, \ldots, a_p, d_{p+1}, d_{p+2}, \ldots, d_n)$ of sequence A is realizable.*

Solution: Let M denote the value $\max(A) = \max_{i \in [1,p]} a_i$. For any $n \geq p$, let $\mathcal{S}_n = ([a_1, a_1], \ldots, [a_p, a_p], [1, n], \ldots, [1, n])$ denote the sequence obtained by appending $n - p$ copies of interval $[1, n]$ to interval sequence (A, A). Let n_0 the denote the length of a minimum graphic extension of A. Observe that $n_0 \in [\max\{p, M\}, p + M]$. The lower limit is due to the fact that the length of minimum graphic extension of A must be at least $\max\{p, M\}$; the upper limit holds since one can have a bipartite graph with partitions $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, Y_M\}$ of length p and M, and for $i \in [1, p]$, connect the vertex x_i to vertices $y_1, \ldots, y_{a_i}$. It turns out that we need to find the smallest integer $n \in [\max\{p, M\}, p + M]$

such that $\mathcal{S}_n$ is graphic. The minimum n can be obtained by a binary search over the range $[\max\{p, M\}, p + d]$ and using Theorem 4; this takes $O(\max\{p, M\} \log \max\{p, M\})$ time. Once n_0 is known, the optimal graphic extension can be computed using Theorem 14 for searching graphic certificate in $O(\max\{p, M\} \log \max\{p, M\})$ time.

▶ **Problem 2.** *Given $A = (a_1, \ldots, a_n)$, find a graphic sequence $D = (d_1, \ldots, d_n)$ whose chebyshev distance (L_∞ distance) from A is minimum.*

Solution: The above problem can be reduced to interval sequence problem, as we need to find smallest non-negative integer $c \in [1, n]$ such that $\mathcal{S}_c = ([a_1 - c, a_1 + c], \ldots, [a_n - c, a_n + c])$ is realizable. To find the minimum c, we do a binary search with help of Theorem 4 for verification; this takes $O(n \log n)$ time. Once optimal c is known, the sequence D can be computed using Theorem 14 to search graphic certificate in $\mathcal{S}_c$, thus the time complexity for computing sequence D is $O(n \log n)$.

▶ **Problem 3.** *Given $A = (a_1, \ldots, a_n)$, find minimum fraction ϵ and a graphic sequence $D = (d_1, \ldots, d_n)$ satisfying $a_i(1 - \epsilon) \leq d_i \leq a_i(1 + \epsilon)$.*

Solution: Again we need to find smallest non-negative fraction ϵ such that the interval sequence $\mathcal{S}_\epsilon = ([a_1(1 - \epsilon), a_1(1 + \epsilon)], \ldots, [a_n(1 - \epsilon), a_n(1 + \epsilon)])$ is realizable. To find the minimum ϵ, we do a binary search with help of Theorem 4; this takes $O(n \log n)$ time. Once ϵ is known, using Theorem 14, sequence D can be computed in $O(n \log n)$ time.

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