# Combined shape and topology optimization of 3D structures 

Christiansen, Asger Nyman; Bærentzen, Jakob Andreas; Nobel-Jørgensen, Morten; Aage, Niels; Sigmund, Ole<br>Published in:<br>Computers \& Graphics

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# Combined Shape and Topology Optimization of 3D Structures 

Asger N. Christiansen, J. Andreas Bærentzen, Morten Nobel-Jørgensen, Niels Aage, Ole Sigmund<br>Technical University of Denmark, Denmark


#### Abstract

We present a method for automatic generation of 3D models based on shape and topology optimization. The optimization procedure, or model generation process, is initialized by a set of boundary conditions, an objective function, constraints and an initial structure. Using this input, the method will automatically deform and change the topology of the initial structure such that the objective function is optimized subject to the specified constraints and boundary conditions. For example, this tool can be used to improve the stiffness of a structure before printing, reduce the amount of material needed to construct a bridge, or to design functional chairs, tables, etc. which at the same time are visually pleasing.

The structure is represented explicitly by a simplicial complex and deformed by moving surface vertices and relabeling tetrahedra. To ensure a well-formed tetrahedral mesh during these deformations, the Deformable Simplicial Complex method is used. The deformations are based on optimizing the objective, which in this paper will be maximizing stiffness. Furthermore, the optimization procedure will be subject to constraints such as a limit on the amount of material and the difference from the original shape.


Keywords: Topology optimization, shape optimization, Deformable Simplicial Complex method, structural design

## 1. Introduction

Topology optimization is the discipline of finding the ${ }_{3}$ optimal shape and topology of a structure [1][2]. It can be used to solve a wide variety of design problems arising when producing such diverse products as cars, ${ }_{6}$ houses, computer chips and antennas. The manufacturers are often concerned with finding the stiffest structure, the lightest structure which does not break, the structure with the highest cooling effect, or the structure with the best flow or highest efficiency.

With the advances in 3D printing technology, topol12 ogy optimization is not just of interest to manufacturers, but to anyone who has access to a 3D printer.
${ }_{4}$ Most consumers lack formal training in structural me${ }_{5}$ chanics, which can hinder the process with many iterations and costly failed attempts. Consumers can underengineer a design unsuitable for the intended load, or over-engineer a design that wastes expensive construction material. Topology optimization offers consumers a tool for designing shapes that meet their structural needs while using minimal construction resources.

In this paper, we present a fully automated design tool ${ }_{3}$ for designing structurally sound structures which can be manufactured, constructed or printed. The modeler only

5 has to specify boundary conditions, the optimization ${ }_{6}$ objective, constraints and an initial structure. In other ${ }_{27}$ words, the designer specifies a set of requirements (the ${ }_{8}$ functionality of the structure and not the structure itself) and the method automatically designs a structure which so fits those requirements. Note that this design process is significantly different from today where a designer man32 ually models a structure and requirements are taken into ${ }_{3}$ account during this design process.
${ }_{34}$ The proposed method for topology optimization is 5 based on the Deformable Simplicial Complex (DSC) method [3]. The DSC method represents a solid structure with a conforming tetrahedral mesh (a simplicial complex) whose tetrahedral elements either lie entirely inside or outside the structure. The interface between solid and void (the surface) is represented explicitly by the triangular faces shared by an interior and exterior tetrahedral element. Furthermore, the DSC method en${ }_{3}$ sures well-formed tetrahedral elements by constantly ${ }_{44}$ performing mesh improvement routines while the sur45 face is being deformed. Finally, it provides adaptive res6 olution, allowing fine details where and when needed.
The method uses two optimization strategies:

## ${ }_{8}$ Discrete optimization



Figure 1: Given a few input parameters, the proposed method automatically optimizes the shape and topology of a 3D structure. Here is an example of optimizing a bridge. The initial structure is seen to the upper left along with supports (green) and loads (red). This structure is optimized such that stiffness is maximized and the amount of material is minimized. A few iterations of the method are depicted along with the result.
${ }_{65}$ These optimization strategies are iterated until changes ${ }_{66}$ are small. An example is seen in Figure 1.
${ }_{67}$ We will show that this tool is of interest to both engi${ }_{68}$ neers and designers. For example, we show that it can 9 be used to improve stiffness and balance of a 3D model, ${ }_{70}$ to save material and to generate functional as well as, in our opinion, visually pleasing designs.

### 1.1. Related work

Recent trends in the computer graphics society are to add mechanical properties to 3D models. Prévost et al. have been concerned with the balance of printed models [15], Skouras et al. about printing deformable characters using a stiff and soft material [16] and several research 78 teams have focused on self-supporting masonry structures [17][18][19].

A major concern has been to improve the stiffness of 3D models. Umetani et al. perform a cross-sectional 2 structural analysis and visualize the result [20]. A user ${ }_{3}$ can then manually edit the model to improve the stiffness while getting almost instant feedback. The instant feedback is only possible because the analysis is limited to cross-sections. Stava et al. presents a more automated method for improving stiffness [21]. They perform a complete worst-case structural analysis on a tetrahedral mesh to determine the structurally weak regions. Based on this analysis, it is decided whether to improve the model by thickening, hollowing or adding a strut. Finally, Zhou et al. [22] also perform a worst-case structural analysis with more precise determination of the 4 worst-case loads than in [21]. Furthermore, they conclude that solving a shape optimization problem to minimize stress is impractical due to the non-linearity and
non-convexity of the problem. Therefore, they make do ${ }_{98}$ with visualizing the structurally weak regions.

Topology optimization problems are indeed non100 convex. However, the topology optimization commu-

## 186 2. Method

The proposed method uses a simplicial complex to 188 represent the shape of a structure. A simplicial com${ }_{189}$ plex discretizes a domain into tetrahedral elements. In 3D it consists of the simplices; nodes (points), edges (line pieces), faces (triangles) and tetrahedra (triangular pyramids). Furthermore, the tetrahedra do not overlap and any point in the discretized domain is either inside


Figure 2: Rotation of a cube using the Deformable Simplicial Complex method. The interface between solid and void (the surface of the cube) is depicted in turquoise. Furthermore, all edges of the simplicial complex are drawn in black.
a tetrahedron or on the boundary between tetrahedra. In addition, all tetrahedra are labeled as being either void (no material) or solid (filled with material). Therefore, the interface between solid and void (the surface) is represented by the faces that are sandwiched between a tetrahedron labeled void and a tetrahedron labeled solid. Figure 2 depicts a cube represented by a simplicial complex. The tetrahedral mesh generator TetGen [34] is used to generate the initial mesh.

Apart from the shape representation, the tetrahedral elements of the simplicial complex can be used for physical computations using the finite element method. Since the finite element analysis will produce large errors if used with nearly degenerate tetrahedra, it is important to sustain a high quality mesh.

### 2.1. Deformable Simplicial Complex method

To ensure a high quality mesh, we use the Deformable Simplicial Complex (DSC) method [3] ${ }^{1}$. The DSC method ensures high quality tetrahedral elements during deformation of a model embedded in a simplicial complex as illustrated in Figure 2. Low quality tetrahedra (slivers, wedges, caps and needles) are removed by continuously performing a set of mesh operations while the surface is being deformed. The tetrahedron quality measure is $\frac{6 \sqrt{2} V}{\left(\frac{1}{6} \sum_{i} l_{i}^{2}\right)^{3 / 2}}$ [35] where $V$ is the volume of the tetrahedron and $l_{i}$ is the length of edge $i$. Note that the DSC method only improves the mesh quality where necessary (often near the surface). Furthermore, the DSC method also handles topology changes by removing low quality tetrahedra which are sandwiched between two surfaces. This is illustrated by two objects colliding in Figure 3.

[^0]

Figure 3: Illustration of topology changes using the Deformable Simplicial Complex method. Here, only edges having both end nodes on the surface are drawn. As the objects approach each other the tetrahedra between the objects get squeezed. When a tetrahedron between the two surfaces is squeezed too much, this tetrahedron will be collapsed. Consequently, the only thing separating the two objects is a face. However, this face has tetrahedra which are labeled solid on both sides and it is therefore no longer part of the surface. Consequently, the two objects are now merged into one.

In addition to ensuring high quality tetrahedral elements, the DSC method also controls the level of detail of both the surface and the tetrahedral mesh. In practice, the DSC method attempts to collapse too small simplices and split too large simplices. Consequently, we always attain a mesh of the desired complexity, described by the discretization parameter $\delta$ (corresponding to the average edge length). More importantly, the detail control allows for mesh adaptivity. This means that smooth regions on the surface are represented by a more coarse discretization than regions with small features.
The mesh operations used are smoothing [36] (not performed on surface nodes), edge split [37], edge collapse [37], edge removal [38] and multi-face removal [38]. The latter two use the flips illustrated in Figure 4. Consequently, these two mesh operations do not change the position of any nodes, only the connectivity. The quality of the mesh is improved by all five operations, whereas the detail level of the mesh is controlled through the operations edge split and edge collapse. Note that changes have been made compared to [3]. The multi-face retriangulation, optimization-based smoothing, null-space smoothing and tetrahedron relabeling operations have not been necessary for this application. Removing these operations has resulted in a significant speed-up. Also, the edge removal operation on the surface and boundary is an addition since [3].

The strategy for moving the surface nodes is to first compute a destination $p_{n}^{*}$ for each surface node $n$ currently at position $p_{n}$. The destination $p_{n}^{*}$ is computed using a user-defined velocity function which, for the case of topology optimization, will be described later. Afterwards, all surface nodes are moved from $p_{n}$ to $p_{n}^{*}$ using the strategy illustrated in Figure 5.


Figure 4: Illustrations of 2-3, 3-2 and 4-4 flips inspired by the illustration in [38].


286 tetrahedra $\boldsymbol{m}$ and can be calculated by

$$
\begin{equation*}
\boldsymbol{K}_{t}(\boldsymbol{m}, \boldsymbol{p})=\int_{V_{t}} \boldsymbol{B}_{t}^{T}(\boldsymbol{p}) \boldsymbol{E}_{t}(\boldsymbol{m}) \boldsymbol{B}_{t}(\boldsymbol{p}) \partial(x, y, z) \tag{1}
\end{equation*}
$$

287 288 289 re

$$
\boldsymbol{E}=\frac{E}{(1+v)(1-2 v)}\left[\begin{array}{cccccc}
1-v & v & v & 0 & 0 & 0 \\
v & 1-v & v & 0 & 0 & 0 \\
v & v & 1-v & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2 v}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2 v}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2 v}{2}
\end{array}\right]
$$

Figure 5: Illustration of how the surface (red) is moved in 2D. The same principle applies to 3D. A filled arrow indicates the destination $\boldsymbol{p}_{n}^{*}$ of the surface node $n$. One of the nodes cannot move to its destination without creating low quality tetrahedra and it is therefore only moved as depicted by the unfilled arrow. The other two are moved to their destinations. Then, mesh operations are applied to improve the mesh quality and the node that did not reach its destination is moved again. This is repeated until all nodes have reached their destinations.

As described previously, a domain is discretized into high quality tetrahedral elements which are analyzed using the finite element method. Using quadratic basis functions solves a well-known issue with a jagged surface when using the analysis as a basis for nonparametric shape optimization [11][12]. Consequently, quadratic basis functions are chosen instead of linear to interpolate the tetrahedral elements. Therefore one control point $c$ is associated with each node and edge of a tetrahedron. Furthermore, the positions of all control points are assembled in a vector termed $\boldsymbol{p}=$
$\left[\ldots, \boldsymbol{p}_{c}^{T}, \ldots\right]^{T}$. In addition, each tetrahedron $t$ has an $\left[\ldots, \boldsymbol{p}_{c}^{T}, \ldots\right]^{T}$. In addition, each tetrahedron $t$ has an associated material $m_{t}$ with material parameters density $\rho_{t}$, Young's modulus $E_{t}$ and Poisson's ratio $v_{t}$. Finally, the materials of the tetrahedra are also assembled in a vector $\boldsymbol{m}=\left[\ldots, m_{t}, \ldots\right]^{T}$.

The local stiffness matrix $\boldsymbol{K}_{t}$ contains information on he stiffness of tetrahedron $t$. It depends on both the positions of the control points $\boldsymbol{p}$ and the materials of the
In this paper, we will optimize the topology of physically valid structures in static equilibrium. In order to achieve physical validity, structural analyses using the finite element method are performed. This implies considering the discretization, boundary conditions and equilibrium which are the topics of this section. 313
314 314 ${ }_{316}^{315}$ tid

Here, $\boldsymbol{g}=[0,-9.8,0]^{T} m / s^{2}$ is a vector of the gravitational acceleration and $a_{i}$ is a scale factor computed by a mass lumping scheme for each element $i$. Furthermore, $\rho_{i}$ is the density and $V_{i}(\boldsymbol{p})$ is the volume of tetrahedral element $i$ which is adjacent to control point $c$. Consequently, the global force vector is

$$
\begin{equation*}
\boldsymbol{f}(\boldsymbol{m}, \boldsymbol{p})=\left[\ldots, \boldsymbol{f}_{c}^{T}+\boldsymbol{w}_{c}^{T}(\boldsymbol{m}, \boldsymbol{p}), \ldots\right]^{T} \tag{3}
\end{equation*}
$$

Since we desire a structure in static equilibrium, the sum of the forces on all particles must be zero (Newton's first law). Consequently, we will utilize the equilibrium equations

$$
\begin{equation*}
K(m, p) u=f(m, p) \tag{4}
\end{equation*}
$$

These equations are used to calculate the global displacement vector $\boldsymbol{u}=\left[\ldots, \boldsymbol{u}_{c}, \ldots\right]$. At each control
point $c, \boldsymbol{u}_{c}$ represents the displacement caused by the forces $\boldsymbol{f}$ applied to the structure. Note that, since $\boldsymbol{K}$ and $\boldsymbol{f}$ are functions of $\boldsymbol{p}$ and $\boldsymbol{m}$, so is $\boldsymbol{u}$.
Solving the equilibrium equations is the most time consuming part of the optimization. Furthermore, the number of equations scales linearly with the number of degrees of freedom. Consequently, the sparse solver CHOLMOD [40], which is a part of the SuiteSparse library [41], is used to solve the equilibrium equation efficiently using multiple cores.

### 2.3. Optimization

We want to optimize an objective function $f$ by changing the shape and topology of the structure. Therefore, the objective can be anything as long as it is a function of the shape and topology. Furthermore, there are two ways to change the shape and topology. The first is to change the position $\boldsymbol{p}_{n}$ of a design node $n$, the other is to change the material $m_{e}$ of a design element $e$. A node is a design node $n$ if it is

- on the surface of the structure,
- not supported,
- not subjected to any external forces and
- not part of a fixed domain (see Section 2.5).

Furthermore, a tetrahedral element is a design element $e$ if it is

- solid,
- not adjacent to a control point subjected to external forces and
- not part of a fixed domain (see Section 2.5).

For the test cases presented here, we seek to find the tructure which is as stiff as possible. Consequently, the objective function is compliance

$$
\begin{equation*}
f(\boldsymbol{m}, \boldsymbol{p})=\boldsymbol{u}^{T} \boldsymbol{K}(\boldsymbol{m}, \boldsymbol{p}) \boldsymbol{u} \tag{5}
\end{equation*}
$$

Note that since this objective is a function of the displacements $\boldsymbol{u}$, we need to solve Equation 4 to evaluate it. The reason for choosing to minimize compliance and not for example maximal Von Mises stress is that the compliance function is smooth. This is a significant advantage for the optimization algorithm. However, we plan to minimize the maximal Von Mises stress using the same method in the future.
It is often desirable to constrain the optimization. In some test examples, we choose to limit the amount of
not appear in this surface by applying the constraint:

$$
\begin{align*}
g_{3}(\boldsymbol{m}, \boldsymbol{p}) & =\frac{1}{N_{\in O}} \sum_{n \in O} \max \left(d_{n}(\boldsymbol{m}, \boldsymbol{p})-D^{*}, 0\right)^{2} \\
& +\frac{1}{N_{\notin O}} \sum_{n \notin O} \max \left(T^{*}-t_{n}(\boldsymbol{m}, \boldsymbol{p}), 0\right)^{2} \tag{8}
\end{align*}
$$

Here, $D^{*}$ is the maximal change from the original surface and $T^{*}$ is the minimum thickness of the shell of the structure. Note that $g_{3}$ is $C^{1}$ continuous and thereby differentiable.

### 2.3.1. Continuous optimization

The first part of the optimization procedure is to locally perturb the surface of the structure such that it iteratively gets closer to optimum. This part of the optimization procedure consists of calculating an improved


Figure 6: Illustrates the destination $\boldsymbol{p}_{n}\left(x_{n}\right)$ of node $n$ as a function of the design variable $x_{n}$. Furthermore, $\boldsymbol{p}_{n}$ is the current position and $\boldsymbol{n}_{n}$ is the normal.

The relation between the current position $p_{n}$, the optimized position $\boldsymbol{p}_{n}^{*}$ and the optimized design variable $x_{n}^{*}$ for a design node $n$ is

$$
\begin{equation*}
\boldsymbol{p}_{n}^{*}=\boldsymbol{p}_{n}\left(x_{n}^{*}\right)=\boldsymbol{p}_{n}+x_{n}^{*} \boldsymbol{n}_{n} \tag{9}
\end{equation*}
$$

${ }_{421}$ To estimate $\boldsymbol{x}^{*}=\left[\ldots, x_{n}^{*}, \ldots\right]^{T}$, a smooth non-linear op${ }_{422}$ timization problem is solved:

$$
\begin{align*}
\boldsymbol{x}^{*}=\underset{\boldsymbol{x}}{\arg \min } & : f(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x}))=\boldsymbol{u}^{T} \boldsymbol{K}(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x})) \boldsymbol{u} \\
\text { subject to } & : g_{i}(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x})) \leq 0, i=1,2,3  \tag{10}\\
& : \boldsymbol{K}(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x})) \boldsymbol{u}=\boldsymbol{f}(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x})) \\
& : \boldsymbol{x}^{\min } \leq \boldsymbol{x} \leq \boldsymbol{x}^{\max }
\end{align*}
$$

${ }_{423}$ Here, $\boldsymbol{x}^{\min }=\left[\ldots, x_{n}^{\min }, \ldots\right]^{T}$ and $\boldsymbol{x}^{\max }=\left[\ldots, x_{n}^{\max }, \ldots\right]^{T}$ 424 are move limits on the design variables $\boldsymbol{x}$. Generally, ${ }_{425} \boldsymbol{x}^{\text {min }}$ and $\boldsymbol{x}^{\max }$ are chosen such that the design nodes ${ }_{426}$ will not create degenerate tetrahedra during the opti${ }_{427}$ mization. Consequently, the new shape can only be a ${ }_{428}$ small perturbation from the current shape and Equation ${ }_{429} 10$ will be solved many times. Furthermore, the move

$$
\begin{align*}
& \frac{\partial f(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x}))}{\partial x_{n}}=  \tag{11}\\
& -\boldsymbol{u}^{T} \frac{\partial \boldsymbol{K}(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x}))}{\partial x_{n}} \boldsymbol{u}+2 \boldsymbol{u}^{T} \frac{\partial \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x}))}{\partial x_{n}}
\end{align*}
$$

446

## 450 2.3.2. Discrete optimization

In addition to changing the shape by moving the de2 sign nodes, a discrete optimization step is performed where the materials $\boldsymbol{m}$ are changed and the positions $\boldsymbol{p}$ are not. The step has two purposes; introducing holes 5 inside the structure and increasing the convergence rate 56 of the continuous optimization. The optimization problem can be written as

$$
\begin{align*}
\boldsymbol{m}^{*}=\underset{\boldsymbol{m}}{\arg \min } & : f(\boldsymbol{m}, \boldsymbol{p})=\boldsymbol{u}^{T} \boldsymbol{K}(\boldsymbol{m}, \boldsymbol{p}) \boldsymbol{u} \\
\text { subject to } & : g_{i}(\boldsymbol{m}, \boldsymbol{p}) \leq 0, i=1,2,3  \tag{12}\\
& : \boldsymbol{K}(\boldsymbol{m}, \boldsymbol{p}) \boldsymbol{u}=\boldsymbol{f}(\boldsymbol{m}, \boldsymbol{p}) \\
& : m_{e} \in\{\text { void, solid }\}
\end{align*}
$$

${ }_{8}$ Note that the set of possible materials is limited to void and solid. However, it is possible to extend this approach to handle multiple materials. Furthermore, we choose that only solid elements are design elements. Consequently, this step only removes material from the structure. If it removes material near the surface, this will speed up shape changes. On the other hand, if it removes material inside the structure, a hole is created.

466
The discrete optimization problem in Equation 12 is ${ }_{467}$ NP-hard. However, since this optimization problem is ${ }_{468}$ combined with a continuous optimization, it is not nec${ }_{469}$ essary to solve it to optimality. Consequently, this step 470 will seek to improve the objective while trying to satisfy ${ }_{471}$ the constraints by relabeling tetrahedra. The relabeling 472 will be based on discrete derivatives, i.e. the change in ${ }_{473}$ objective or constraints when changing the material in ${ }_{474}$ element $e$ from solid to void:

$$
\begin{gather*}
\Delta_{e} f(\boldsymbol{m}, \boldsymbol{p})=f\left(\boldsymbol{m}_{e}^{v}, \boldsymbol{p}\right)-f(\boldsymbol{m}, \boldsymbol{p})  \tag{13}\\
\Delta_{e} g_{i}(\boldsymbol{m}, \boldsymbol{p})=g_{i}\left(\boldsymbol{m}_{e}^{v}, \boldsymbol{p}\right)-g_{i}(\boldsymbol{m}, \boldsymbol{p}), i=1,2,3 \tag{14}
\end{gather*}
$$

475 Here, $\boldsymbol{m}_{e}^{v}$ equals $\boldsymbol{m}$ where $m_{e}$ is void instead of solid. 476 However, computing these discrete derivatives for com${ }_{477}$ pliance is inefficient since the equilibrium equations 478 then have to be evaluated once for each solid tetrahe${ }_{479}$ dron. Instead, we will use an approximation based on ${ }_{480}$ the theory of topological derivatives [4][5][45][6]. The ${ }_{481}$ topological derivative corresponds to the influence on 482 the objective function of introducing an infinitesimal ${ }_{483}$ hole in element $e$. For compliance, the discrete deriva${ }_{484}$ tive can therefore be approximated by

$$
\begin{equation*}
\Delta_{e} f(\boldsymbol{m}, \boldsymbol{p}) \approx 3 \boldsymbol{u}^{T} \boldsymbol{K}_{e}(\boldsymbol{m}, \boldsymbol{p}) \boldsymbol{u}-\frac{2 V_{e}(\boldsymbol{p})}{N_{\in e}} \sum_{c \in e} \boldsymbol{u}_{c}^{T} \boldsymbol{g} \tag{15}
\end{equation*}
$$

485 The first part of the optimization strategy is to im${ }_{486}$ prove the objective function while decreasing or satis${ }_{487}$ fying all constraints. A constraint $i$ is decreased if

$$
\begin{equation*}
\Delta_{e} g_{i}(\boldsymbol{m}, \boldsymbol{p}) \leq 0 \tag{16}
\end{equation*}
$$

${ }_{488}$ and satisfied if

$$
\begin{equation*}
g_{i}(\boldsymbol{m}, \boldsymbol{p})+\Delta_{e} g_{i}(\boldsymbol{m}, \boldsymbol{p}) \leq 0 \tag{17}
\end{equation*}
$$

489 Hence, a design element $e$ is relabeled from solid to void 490 if either of equations 16 and 17 are satisfied for all con491 straints and

$$
\begin{equation*}
\Delta_{e} f(\boldsymbol{m}, \boldsymbol{p})<0 \tag{18}
\end{equation*}
$$

The second part of the optimization is to try to im${ }_{493}$ prove constraints which are not satisfied. Therefore, if 494 constraint $i$ is not satisfied, i.e. $g_{i}(\boldsymbol{m}, \boldsymbol{p})>0$, we will 495 try to find an optimal design element $e^{*}$ to relabel from ${ }_{496}$ solid to void. Noting that $\Delta_{e} f(\boldsymbol{m}, \boldsymbol{p}) \geq 0$, the optimal ${ }_{497}$ design element $e^{*}$ is found by solving

$$
\begin{equation*}
e^{*}=\underset{e}{\arg \min }-\frac{\Delta_{e} f(\boldsymbol{m}, \boldsymbol{p})}{\Delta_{e} g_{i}(\boldsymbol{m}, \boldsymbol{p})} \tag{19}
\end{equation*}
$$

498 where all arguments $e$ satisfy

$$
\begin{equation*}
\Delta_{e} g_{i}(\boldsymbol{m}, \boldsymbol{p})<0 \tag{20}
\end{equation*}
$$

499 and either Equation 16 or 17 for all constraints. Design ${ }_{500}$ element $e^{*}$ is then relabeled from solid to void. This ${ }_{501}$ process is repeated as long as constraint $i$ is not satisfied 502 an

## 503 2.4. Disconnected material

The continuous and discrete optimization steps can very well result in material which is disconnected from the main structure. These parts do not contribute to the objective. Furthermore, since void elements are eliminated from the finite element analysis, disconnected material will result in the equilibrium equations not having a unique solution. Consequently, disconnected material is removed by performing a connected component analysis and making every component, except for the largest, void.

### 2.5. Initialization

To initialize the optimization, the user has to specify boundary conditions, an objective function, constraints and an initial structure.
The boundary conditions are the supports and external forces applied to the surface of the structure as described in Section 2.2. Furthermore, the boundaries of the design domain (the domain where material can reside) have to be specified. Finally, it is possible to specify fixed domains (areas that are either always solid or always void). The fixed void areas are implemented as not being a part of the design domain. However, the fixed solid domains are enforced by assigning a different label to the tetrahedra inside these domains. Consequently, an invisible surface exists between the fixed and non-fixed solid domains. The shape of this surface should not be changed in any way. However, we still want the DSC method to improve the mesh quality and control the level of detail at this surface. Consequently, the DSC method is modified such that only mesh operations which do not change the surface are performed at the surface between fixed and non-fixed domains.
In all of the example problems presented here, the objective is to minimize compliance since it is often desirable to produce as stiff a structure as possible. However, choosing another objective is as simple as changing the objective function and calculating the shape and topological derivatives of the new function. For example, the same approach has been used for balancing of 3D models [46]. Furthermore, different problems require different constraints. In this paper, we present several different global constraints to illustrate their effect on
hedra discretize parts far away from the surface. Furthermore, the main computational power should be used ${ }_{591}$ to achieve a fine resolution near the optimum. When

## 562 Step 2: Continuous optimization

Solves the optimization problem in Equation 10 using the gradient-based optimization algorithm MMA (Section 2.3.1). MMA hereby estimates the optimal values of the design variables $\boldsymbol{x}^{*}=$ $\left[\ldots, x_{n}^{*}, \ldots\right]^{T}$. Then, each design node $n$ is moved from position $\boldsymbol{p}_{n}$ to $\boldsymbol{p}_{n}^{*}=\boldsymbol{p}_{n}+x_{n}^{*} \quad \boldsymbol{n}_{n}$ using the DSC method as described in Section 2.1. Finally, disconnected material is removed.

These two steps make up one time step and are iterated until the changes on the surface from consecutive time steps are small.

Problems can arise if a volume or perimeter con5 straint is applied. The optimization will seek to obey the constraint before taking the objective into account. $\rightarrow$ This can lead to undesired removal of material from 8 places where it is necessary. Our solution to this problem is to gradually lower the constraint such that $V^{*}(t)=$ $\max \left(\alpha^{t}, V^{*}\right)$ and $A^{*}(t)=\max \left(\beta^{t}, A^{*}\right)$ where $t$ is the time step and $0<\alpha<1$ and $0<\beta<1$ are constants.

### 2.7. Efficiency

Efficiency is essential when performing topology optimization in 3D. A major piece of the puzzle to make 5 this approach more efficient than standard fixed grid methods is to take advantage of the mesh adaptivity inherent to the DSC method. Consequently, the surface is represented by a fine discretization whereas large tetra-
the design. The effect can be quite drastic and consequently the constraints are as important as the objective.
Finally, the initial model is a triangle mesh. Consequently, any surface mesh can be used as a starting point for the optimization without any conversions. In this paper, we choose to initialize the optimization by triangle meshes of existing models and by generated meshes that fill the entire design domain.

### 2.6. Method summary

## The method consists of two steps:

## Step 1: Discrete optimization

Improves the objective as well as unsatisfied constraints by relabeling elements from solid to void based on their topological derivatives as described in Section 2.3.2. Then, removes disconnected material.
the optimization is initialized by a 3D model, the optimum is assumed to be close. However, that is probably not the case when the optimization is initialized by filling the design domain with material. Consequently, in these cases, we slowly lower the discretization parameter $\delta$ by multiplying it by 0.99 at each time step. The detail control, described in Section 2.1, will then increase the mesh complexity. Note that this strategy is especially effective since the method only calculates on solid elements. However, solving the equilibrium equations is still the most time-consuming part. Consequently, we utilize multiple threads on the CPU to speed up these computations. Also, computing the gradients of the compliance function and assembling the global stiffness matrix $K$ and force vector $F$ are parallelized.

## 3. Results

The proposed method can be used in the fabrication design process in areas such as construction, manufacturing and design. In this section, we will illustrate this statement by solving problems within each of these fields. The results are generated on a laptop with a 2.4 GHz quad-core Intel Core i7 processor and 8 GB of 1333 MHz DDR3 RAM. Parameters and performance measures are depicted in Table 1. Furthermore, the objective of all examples is to minimize compliance subject to constraints as depicted in Table 1.

The raw surface triangle meshes of the optimized structures, i.e. the output as it looks from the optimization method, are visualised using Blender. No post processing like subdivision and smoothing has been utilized to improve the appearance. Furthermore, when material has been removed from inside a structure, the internal cavities are visualized by making the structure transparent. In addition to the optimized result, we will in some cases visualize the strain energy density (SED) at the surface of the final model. The SED depicts how much strain an element at the surface is subjected to. Here, the jet colormap is used, where blue and red depict low and high SED respectively. Furthermore, the SEDs are scaled between the minimum and maximum SED of the initial structure. Consequently, this visualizes how the stiffness has changed as a consequence of the optimization. In the same cases, we will also visualize the difference from the original model by a grayscale colormap. Here, gray means no change, darker means it has moved in the negative normal direction and lighter that it has moved in the normal direction. The distance is scaled by the largest change.

| Problem | $\delta$ | $V^{*}(\alpha)$ | $A^{*}(\beta)$ | $D^{*}$ | $T^{*}$ | $f^{*} / f^{0}$ | Surface | Complex | Running time |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | mm | $\% V^{0}(-)$ | $\% A^{0}(-)$ | $\% \delta$ | $\% \delta$ | - | \# faces | \# elements | minutes (\#) |
| Bridge | 423 | $20(0.96)$ | $30(0.98)$ | - | - | $304 \%$ | 9883 | 29836 | $68(70)$ |
| Statue | 50 | $50(0.95)$ | - | 15 | 100 | $27 \%$ | 35868 | 66314 | $275(20)$ |
| Dinosaur | 1.4 | - | - | 15 | 100 | $46 \%$ | 6876 | 15071 | $11(5)$ |
| Armadillo | 2.8 | - | - | 15 | 100 | $13 \%$ | 9872 | 15819 | $60(50)$ |
| Table 1 | 42 | $15(0.96)$ | $30(0.98)$ | - | - | $2671 \%$ | 5492 | 11761 | $16(100)$ |
| Table 2 | 62 | $15(0.96)$ | $35(0.98)$ | - | - | $964 \%$ | 3543 | 5521 | $13(60)$ |
| Table 3 | 42 | $15(0.96)$ | $30(0.98)$ | - | - | $5929 \%$ | 5374 | 11759 | $20(100)$ |
| Chair 1 | 21 | $12.5(0.96)$ | $25(0.98)$ | - | - | $1199 \%$ | 4413 | 7929 | $15(100)$ |
| Chair 2 | 21 | $12.5(0.96)$ | $30(0.98)$ | - | - | $625 \%$ | 5527 | 9026 | $18(100)$ |
| Chair 3 | 27 | $12.5(0.96)$ | $30(0.98)$ | - | - | $927 \%$ | 3382 | 4927 | $8(75)$ |
| Support | 655 | $20(0.96)$ | $20(0.98)$ | - | - | $17 \%$ | 15064 | 27120 | $109(100)$ |

Table 1: Method parameters and performance measures for all example problems. The displayed values are the values as they appear after the optimization. The $V^{*}$ and $A^{*}$ values are stated in percent of the initial volume $V^{0}$ and surface area $A^{0}$ respectively whereas $D^{*}$ and $T^{*}$ are stated in percent of the discretization parameter $\delta$. Furthermore, $f^{0}$ and $f^{*}$ are initial and final compliance respectively. Finally, the \# in the right-most column is the number of time steps.


Figure 7: Topology optimized cow statue which show that the method can optimize stiffness while saving material. ${ }_{650}$ as depicted in Figure 1. The surface of the bridge is ${ }_{651}$ fixed and subjected to a distributed load pushing down${ }_{652}$ wards ( 100 MPa ). The result and optimization process ${ }_{653}$ are also depicted in Figure 1. The result shows that com${ }_{654}$ pliance has increased to $304 \%$ of the initial value during ${ }_{655}$ the optimization process. However, the optimized struc${ }_{656}$ ture only uses $20 \%$ of the material used by the initial 657 658

Next, a 4 m-long concrete statue is initialized by a 659 3D model of a cow (source: Aim@Shape). The statue
is solid concrete, only subjected to gravitational forces and supported underneath all of its hoofs. The change in SED, shape changes and the optimized cow statue are depicted in Figure 7. This example shows that our method extends previous methods by being able to initialize an optimization by a 3D model (represented by a triangle mesh) without any conversion and, furthermore, remain close to this shape. Also, since the statue is subjected to gravitational forces only, compliance is improved at the same time as the amount of material is reduced.

### 3.2. Manufacturing

An important application of our method is as a tool to improve the stiffness of a given shape. Assume, we are given a 3D shape that is to be fabricated. The problem is to change the exterior shape as little as possible while using a minimum amount of material and ensuring that the fabricated object will be able to support itself and moreover withstand specified external loads. Further-


Figure 8: Toy models optimized to improve both stiffness and balance while remaining close to the initial shape.

708 When humans design a given 3D object, the main 709 concerns are often to satisfy aesthetic and functional 710 requirements. Topology optimization is not concerned 711 with aesthetics but it satisfies functional requirements. However, topology-optimized shapes exhibit an organic ${ }_{3}$ and sparse feeling that is often visually pleasing. There${ }_{74}$ fore, such a tool is useful as part of a design workflow 5 [47]. Furthermore, the method can be used to generate 6 significantly different designs by slight changes to the input. This is significantly simpler for a designer than remodeling a surface.

Three plastic tables are modeled by a fixed layer of material at the top of a design domain (1.8 $1.2 \quad 1.2$ $\mathrm{m}^{3}$ ) and a distributed load ( 2 MPa ) pressing down on this layer. Furthermore, three chairs are initialized by filling a $0.6 \quad 0.8 \quad 0.6 \mathrm{~m}^{3}$ design domain. The seat is modeled by a fixed void domain of size $0.4 \quad 0.4 \quad 0.4 \mathrm{~m}^{3}$ and a fixed solid domain underneath which is subjected to a load ( 1 MPa ). Finally, a backrest is modeled by a small fixed solid domain and subjected to a horizontal force ( 0.5 MPa ). The difference between the problems are the position and extent of the supports. All supports are placed at the bottom of the design domain and have the shape depicted in figures $9(\mathrm{a}), 9(\mathrm{~d})$ and $9(\mathrm{~g})$ as seen


Figure 9: Topology optimized tables and chairs which show the design capabilities of the suggested method. The difference between the problems are the supports (illustrated at the left of each row) and possibly the values of parameters. Note that the same illustration is used for both a table and a chair problem, therefore the dimensions of these illustrations are not correct.
from above. The optimized designs are depicted in Fig7з3 ure 9.

Finally, we will use the Qatar National Convention ${ }_{735}$ Center as an example of a real-world architectural de736 737 fa ${ }_{738}$ optimized structure [47]. To model this, we take advan${ }_{739}$ tage of the symmetry and thereby only optimize a quar740 ter of the structure (the symmetry axes are depicted in ${ }_{741}$ Figure $10(\mathrm{~d})$ ). Consequently, the problem is initialized 742 by a $125 \quad 20 \quad 15 \mathrm{~m}^{3}$ cube where the top layer $(1 \mathrm{~m})$ is ${ }_{743}$ fixed and solid. The structure is supported at the bottom 744 in a half circular area (Figure 10(d)) and only subjected 745 to gravity. The result can be seen in Figure 10(e) and, in 746 addition, we illustrate in Figure 10 the effect of chang${ }_{747}$ ing the parameter for the perimeter constraint. Note that 748 the result is not expected to look like the Convention 749 Center since [47] use different boundary conditions and ${ }_{750}$ do not specify material, objective and constraints.

## 4. Conclusion

The presented method is the first to optimize both ${ }_{3}$ the 3D shape and topology of a surface triangle mesh 54 without the use of an implicit representation. This is 5 achieved by embedding the triangle mesh in a simplicial complex and using the Deformable Simplicial Complex method. Consequently, the method accepts a surface triangle mesh as input and outputs another surface triangle mesh which is only different from the input mesh where it has been optimized. Furthermore, as opposed to standard fixed grid methods, our method makes it possible to generate detailed designs within reasonable time on an ordinary laptop.
We have shown that the method automatically generates designs which satisfy some user-defined structural ${ }_{66}$ requirements. However, note that the search space is limited by global constraints and that there is no guarantee that the global optimum is reached. The bridge and the cow statue show that material can be saved where it 70 is expensive or inconvenient while maintaining or improving stiffness. The dinosaur and Armadillo models ${ }_{2}$ show that 3D models automatically can be made stiffer


Figure 10: Topology optimized roof support, optimized using different values for the perimeter constraint. This problem is inspired by the real world problem of supporting the roof of the Qatar National Convention Center. The supports are placed as depicted in Figure 10(d) where also symmetry axes are visualized as black lines.
and more balanced, while retaining the shape. Finally, the tables, chairs and roof support show that functional and, in our opinion, visually pleasing designs can be achieved with little effort from a designer. This is far from an exhaustive list of problems that can be solved using the presented method. As mentioned, topology optimization has been used to solve a wide variety of problems. To solve these or other problems, one only needs to model the boundary conditions and choose the objective, constraints and an initial structure. However, more advanced problems might require additional work. For example implementing additional objective functions and constraints, handling multiple load cases, using an anisotropic material model, handling dynamic problems and taking non-linearity into account.

We have shown that furniture and support structures for buildings can be modeled by specifying a few input parameters. Furthermore, both the input and output models are in the form of a surface triangle mesh. Consequently, this tool has potential to be used for modeling for films, videogames and other offline productions in addition to designing physical structures, especially if performance and user friendliness are improved. To increase performance, one idea is to take full advantage of the parallel nature of the finite element computations by, for example, feeding the computations to the GPU. Furthermore, parallelization of the DSC method would be beneficial. Another idea is to take even further advantage of the mesh adaptivity by lowering the discretization parameter more wisely. To increase the user friendliness, automatic determination of worst-case
loads could be useful to limit the amount of user input. Also, finding an alternative to the perimeter constraint would be desirable since it can limit the optimization and its parameter is unintuitive and difficult to choose. Finally, most designers want to influence the design regularly during the design process. Therefore, a workflow which includes user feedback and post processing is needed.

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[^0]:    ${ }^{1}$ An open-source framework is available at www.github.com/ asny/DSC

