

CONTINUITY RESULTS WITH RESPECT TO DOMAIN PERTURBATION FOR THE FRACTIONAL p -LAPLACIAN

CARLA BARONCINI, JULIÁN FERNÁNDEZ BONDER AND JUAN F. SPEDALETTI

ABSTRACT. In this paper we give sufficient conditions on the approximating domains in order to obtain the continuity of solutions for the fractional p -laplacian. These conditions are given in terms of the fractional capacity of the approximating domains.

1. INTRODUCTION.

In recent years, there has been an increasing amount of attention on nonlocal problems due to some interesting new applications that these operators have shown to possess, such as some models for physics [6, 8, 11, 15, 18, 23], finances [1, 16, 20], fluid dynamics [3], ecology [14, 17, 19] and image processing [12].

In particular, the so-called (s, p) -laplacian operator have been extensively studied and up to date is almost impossible to give an exhaustive list of references. See for instance [5, 4] and references therein.

The (s, p) -laplace operator is defined as

$$(-\Delta_p)^s u(x) := 2 \text{ p.v. } \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{n+sp}} dy,$$

up to some normalization constant. The term p.v. stands for *principal value*.

It is easy to see that this operator is bounded between the fractional order Sobolev space $W^{s,p}(\mathbb{R}^n)$ and its dual $W^{-s,p'}(\mathbb{R}^n)$. Moreover, for any $u \in W^{s,p}(\mathbb{R}^n)$, $(-\Delta_p)^s u$ defines a distribution as

$$\langle (-\Delta_p)^s u, \phi \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y))}{|x - y|^{n+sp}} dx dy,$$

for every $\phi \in C_c^\infty(\mathbb{R}^n)$. In fact this equality holds for any $\phi \in W^{s,p}(\mathbb{R}^n)$. See next section for precise definitions of the Sobolev spaces $W^{s,p}(\mathbb{R}^n)$.

Another elementary fact is that given $f \in L^{p'}(\mathbb{R}^n)$ (or more generally $f \in W^{-s,p'}(\mathbb{R}^n)$) and a bounded open set $\Omega \subset \mathbb{R}^n$ there exists a unique $u \in W_0^{s,p}(\Omega) = \{v \in W^{s,p}(\mathbb{R}^n) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$ that verifies

$$(1.1) \quad \begin{cases} (-\Delta_p)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where the equality is understood in the sense of distributions.

We denote this function by u_Ω^f .

2010 *Mathematics Subject Classification.* 35B30, 35J60.

Key words and phrases. Fractional p -laplacian, domain perturbation, fractional capacity.

The question that we address in this paper is then the following. Assume that we have a sequence of domains $\{\Omega_k\}_{k \in \mathbb{N}}$ such that $\Omega_k \rightarrow \Omega$ in a suitable defined notion of convergence of sets. Is it then true that $u_{\Omega_k}^f \rightarrow u_{\Omega}^f$ in some sense? Or more generally, give necessary and/or sufficient conditions for the above statement to hold true.

When the (s, p) -laplacian is replaced by the classical p -laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ (recall that for $p = 2$ this operator becomes the classical Laplace operator), this problem was studied in [21]. In that article, the author gives additional conditions in terms of the capacity of the symmetric differences of the domains in order to obtain a positive answer, and the famous counterexample of Cioranescu and Murat [2] says that one cannot expect a positive answer without any further assumptions.

In the fractional setting, recently [10] extended the counterexample of Cioranescu-Murat to the (s, p) -laplacian so, as in the classical setting, one cannot expect a positive answer in full generality.

Therefore, our purpose in this work is to find some capacity conditions on the symmetric difference $\Omega_k \Delta \Omega$ in order to have convergence of the solutions $u_{\Omega_k}^f \rightarrow u_{\Omega}^f$.

Though it will be desirable to find *geometric conditions* that assures the continuity of solutions (c.f. [21]) we are not able to obtain those conditions in the fractional case, mostly due to the lack of a Wiener type regularity result on the solutions of (1.1). However the capacity-type conditions that we find is suitable in many applications. For instance, in numerical computations of (1.1) one usually approximate the domain Ω by a sequence of polygonal domains Ω_k and if Ω is convex these approximations verify our hypotheses.

Organization of the paper. After this introduction, in section 2 we revise the definitions and results on fractional order Sobolev spaces and on fractional capacities that are needed in the paper. Then, in section 3, we prove our main result (Theorem 3.6).

2. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^n$ be an open, connected set. For $0 < s < 1 < p < \infty$, we consider the fractional order Sobolev space $W^{s,p}(\Omega)$ defined as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p} + s}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{s,p;\Omega} = \left(\int_{\Omega} |u|^p dx + \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}.$$

The term

$$[u]_{W^{s,p}(\Omega)}^p = [u]_{s,p;\Omega}^p = \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy$$

is called the *Gagliardo seminorm* of u . We refer the interested reader to [5] for a throughout introduction to these spaces.

When $\Omega = \mathbb{R}^n$, we omit it in the notation, i.e.

$$\|u\|_{s,p;\mathbb{R}^n} = \|u\|_{s,p} \text{ and } [u]_{s,p;\mathbb{R}^n} = [u]_{s,p}.$$

In order to consider Dirichlet boundary conditions, it is customary to define the spaces

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Let us observe that $W_0^{s,p}(\Omega)$ is a closed subset of $W^{s,p}(\mathbb{R}^n)$. Therefore it has the same properties as a functional space, that is $(W_0^{s,p}(\Omega), \|\cdot\|_{s,p})$ is a separable, uniformly convex and reflexive Banach space.

An alternative definition for $W_0^{s,p}(\Omega)$ is to consider the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{s,p}$. If Ω is Lipschitz, both definitions are known to coincide (see [5]).

2.1. Elementary properties. We will now present some well-known properties of the norm that will be useful for our results. We state the results without proof for future references.

Proposition 2.1 (Poincaré Inequality). *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure. Then, there exists a positive constant $c = c(s, p, n, |\Omega|) > 0$ such that*

$$\|u\|_p \leq c[u]_{s,p} \text{ for every } u \in W_0^{s,p}(\Omega).$$

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure. Then, $[\cdot]_{s,p}$ and $\|\cdot\|_{s,p}$ define equivalent norms in $W_0^{s,p}(\Omega)$.*

We will now define a notion of convergence of domains that will be essential for our next results.

Definition 2.3 (Hausdorff complementary topology.). Let $D \subset \mathbb{R}^n$ be compact. Given $K_1, K_2 \subset D$ compact sets, we define the Hausdorff distance d_H as

$$d_H(K_1, K_2) := \max \left\{ \sup_{x \in K_1} \inf_{y \in K_2} \|x - y\|, \sup_{x \in K_2} \inf_{y \in K_1} \|x - y\| \right\}.$$

Now, let $\Omega_1, \Omega_2 \subset D$ be open sets, we define the Hausdorff complementary distance d^H as

$$d^H(\Omega_1, \Omega_2) := d_H(D \setminus \Omega_1, D \setminus \Omega_2).$$

Finally, we say that $\{\Omega_k\}_{k \in \mathbb{N}}$ converges to Ω in the sense of the Hausdorff complementary topology, denoted by $\Omega_k \xrightarrow{H} \Omega$, if $d^H(\Omega_k, \Omega) \rightarrow 0$.

We will use the notation

$$\mathcal{A}(D) := \{\Omega \subset D : \text{open}\}$$

and therefore this space has a natural structure of a metric space with metric d^H .

Remark 2.4. Is a well known fact that the space $(\mathcal{A}(D), d^H)$ is a compact metric space when D is compact.

For the proof of the following proposition, we refer to the book [13].

Proposition 2.5. *If $\Omega_k \xrightarrow{H} \Omega$, then for every $\phi \in C_c^\infty(\Omega)$ there is an integer k_0 such that $\phi \in C_c^\infty(\Omega_k)$ for $k \geq k_0$.*

2.2. Fractional Capacity. In this subsection, we recall some definitions of the (s, p) –capacity and relative capacity that can be found, for instance, in [22].

For a detailed analysis of the (s, p) –capacity, we refer to the above mentioned article [22].

We start with the definition of the (s, p) –capacity and the relative (s, p) –fractional capacity.

Definition 2.6. Let $E \subset \mathbb{R}^n$ be an arbitrary set. We define the (s, p) –fractional capacity of the set E as

$$(2.1) \quad \text{cap}_{s,p}(E) := \inf\{[u]_{s,p}^p : u \in C_c^\infty(\mathbb{R}^n), u \geq 1 \text{ in } E\}$$

Given $\Omega \subset \mathbb{R}^n$ an open and bounded set and $E \subset \Omega$, we can define the capacity of the set E relative to the set Ω as follows.

Definition 2.7.

$$\text{cap}_{s,p}(E; \Omega) = \inf\{[u]_{s,p}^p : u \in W_0^{s,p}(\Omega), u \geq 1 \text{ in an open neighborhood of } E\}.$$

Remark 2.8. It is an immediate consequence of the above definitions that $\text{cap}_{s,p}(E) \leq \text{cap}_{s,p}(E; \Omega)$.

Now, when we deal with pointwise properties of Sobolev functions we must change the concept of *almost everywhere* for *quasi everywhere*. The following definition expresses such idea.

Definition 2.9. We say that a property is valid (s, p) –quasi everywhere if it is valid except in a set of null (s, p) –capacity. We note this fact writing (s, p) –q.e.

Definition 2.10. Let $D \subset \mathbb{R}^n$ be an open and bounded set, we say that $\Omega \subset D$ is (s, p) –quasi open if there is a decreasing sequence $\{\omega_k\}_{k \in \mathbb{N}}$ of open sets such that $\text{cap}_{s,p}(\omega_k, D) \rightarrow 0$ and $\Omega \cup \omega_k$ is an open set for each $k \in \mathbb{N}$.

Remark 2.11. Recall that the (s, p) –capacity is homogeneous of degree $n - sp$, that is

$$\text{cap}_{s,p}(\lambda E) = \lambda^{n-sp} \text{cap}_{s,p}(E).$$

See [22]. Therefore, it follows that

$$\text{cap}_{s,p}(E) \leq C\mathcal{H}^{n-sp}(E),$$

where \mathcal{H}^α is the α –dimensional Hausdorff measure (see the proof of Theorem 2, page 151 in [9] in the case $s = 1$, here is analogous).

From this inequality it follows that any set of Hausdorff dimension $d < n - sp$ is a set of null (s, p) –capacity.

Definition 2.12. A function $u : \Omega \rightarrow \mathbb{R}$ is called an (s, p) –quasi continuous function if for every $\varepsilon > 0$, there is an open set U such that $\text{cap}_{s,p}(U, \Omega) < \varepsilon$ and $u|_{\Omega \setminus U}$ is continuous.

The next results, which proofs can be found in [22] will be needed in the course of the proof of the main result of this paper.

Theorem 2.13 (Theorem 3.7 in [22]). *For each $u \in W^{s,p}(\mathbb{R}^n)$ there exists a (s, p) –quasicontinuous function $v \in W^{s,p}(\mathbb{R}^n)$ such that $u = v$ a.e. in \mathbb{R}^n .*

Remark 2.14. It is easy to see that two (s, p) -quasicontinuous representatives of a given function $u \in W^{s,p}(\mathbb{R}^n)$ can only differ in a set of zero (s, p) -capacity. Therefore, the unique (s, p) -quasicontinuous representative (defined (s, p) -q.e.) of u will be denoted by \tilde{u} .

Proposition 2.15 (Lemma 3.8 in [22]). *Let $0 < s < 1 < p < \infty$. and let $\{v_k\}_{k \in \mathbb{N}} \subset W^{s,p}(\mathbb{R}^n)$ be such that $v_k \rightarrow v$ in $W^{s,p}(\mathbb{R}^n)$ for some $v \in W^{s,p}(\mathbb{R}^n)$. Then there is a subsequence $\{v_{k_j}\}_{j \in \mathbb{N}} \subset \{v_k\}_{k \in \mathbb{N}}$ such that $\tilde{v}_{k_j} \rightarrow \tilde{v}$ (s, p) -q.e.*

Theorem 2.16 (Theorem 4.5 in [22]). *Let $D \subset \mathbb{R}^n$ be an open set and $\Omega \subset D$ an open subset. Then,*

$$u \in W_0^{s,p}(\Omega) \Leftrightarrow u \in W_0^{s,p}(D) \text{ and } \tilde{u} = 0 \text{ } (s, p)\text{-q.e. in } D \setminus \Omega.$$

3. CONTINUITY OF THE PROBLEMS WITH RESPECT TO VARIABLE DOMAINS

Throughout this section we consider $0 < s < 1 < p < \infty$ fixed.

Let $D \subset \mathbb{R}^n$ be a bounded, open set and let $\Omega \subset D$ be an open set. The Dirichlet problem for the (s, p) -laplacian consists of finding $u \in W_0^{s,p}(\Omega)$ such that

$$(3.1) \quad \begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c := \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $f \in W^{-s,p'}(D) := [W_0^{s,p}(D)]'$.

In its weak formulation, this problem consists of finding $u \in W_0^{s,p}(\Omega)$ such that

$$\langle (-\Delta)_p^s u, v \rangle = \langle f, v \rangle \text{ for every } v \in W_0^{s,p}(\Omega).$$

That is, for every $v \in W_0^{s,p}(\Omega)$, the following equality holds

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} dx dy = \langle f, v \rangle.$$

Lemma 3.1. *Let $f \in W^{-s,p'}(D)$ and $\Omega \in \mathcal{A}(D)$. Then there exists a unique $u \in W_0^{s,p}(\Omega)$, which we will denote u_Ω^f , solution of (3.1).*

Proof. It is enough to consider $\mathfrak{F}: W_0^{s,p}(\Omega) \rightarrow \mathbb{R}$ defined by $\mathfrak{F}(v) := \frac{1}{p}[v]_{s,p}^p - \langle f, v \rangle$ and observe that u is solution of (3.1) if and only if u is a minimizer for \mathfrak{F} .

By Poincaré inequality, Proposition 2.1, a minimizing sequence is bounded and hence it contains a weakly convergent subsequence. By the sequentially weak lower semicontinuity of \mathfrak{F} (recall that is strongly continuous and convex), it follows that this weak limit is a minimizer.

Since \mathfrak{F} has a unique minimizer (observe that \mathfrak{F} is strictly convex), this completes the proof. \square

Now we observe that these solutions u_Ω^f are bounded independently of Ω .

Lemma 3.2. *There is a constant $C = C(\|f\|_{-s,p'}, s, p, n, |D|)$ such that $\|u_\Omega^f\|_{s,p} \leq C$ for every $\Omega \in \mathcal{A}(D)$.*

Proof. Let us observe that

$$[u_\Omega^f]_{s,p}^p = \langle (-\Delta)_p^s u_\Omega^f, u_\Omega^f \rangle = \langle f, u_\Omega^f \rangle \leq \|f\|_{-s,p'} \|u_\Omega^f\|_{s,p}.$$

Combining this inequality with Theorem 2.1, there exists a constant $C > 0$ such that

$$\|u_\Omega^f\|_{s,p}^p \leq C \|f\|_{-s,p'} \|u_\Omega^f\|_{s,p},$$

from where the conclusion of the lemma follows. \square

As an immediate corollary, we have the following result.

Corollary 3.3. *Let $\{\Omega_k\}_{k \in \mathbb{N}} \subset \mathcal{A}(D)$. Then, $\{u_{\Omega_k}^f\}_{k \in \mathbb{N}}$ is bounded in $W_0^{s,p}(D)$ and, therefore, there exists $u^* \in W_0^{s,p}(D)$ and a subsequence $\{u_{\Omega_{k_j}}^f\}_{j \in \mathbb{N}} \subset \{u_{\Omega_k}^f\}_{k \in \mathbb{N}}$ such that $u_{\Omega_{k_j}}^f \rightharpoonup u^*$ weakly in $W_0^{s,p}(D)$.*

The next result is a first step in proving the continuity result.

Theorem 3.4. *Let $\{\Omega_k\}_{k \in \mathbb{N}} \subset \mathcal{A}(D)$ and $\Omega \in \mathcal{A}(D)$ be such that $\Omega_k \xrightarrow{H} \Omega$. Assume that $u_{\Omega_k}^f \rightharpoonup u^*$ weakly in $W_0^{s,p}(D)$ for some $u^* \in W_0^{s,p}(D)$ when $k \rightarrow \infty$. Then*

$$(-\Delta)_p^s u^* = f \text{ in } \Omega,$$

in the sense of distributions. That is

$$(3.2) \quad \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u^*(x) - u^*(y)|^{p-2} (u^*(x) - u^*(y)) (\phi(x) - \phi(y))}{|x - y|^{n+sp}} dx dy = \langle f, \phi \rangle,$$

for every $\phi \in C_c^\infty(\Omega)$.

Proof. We denote $u_k = u_{\Omega_k}^f$, and also denote

$$\xi_k(x, y) = \frac{|u_k(x) - u_k(y)|^{p-2} (u_k(x) - u_k(y))}{|x - y|^{\frac{n+sp}{p'}}}.$$

Then, $\xi_k \in L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$\|\xi_k\|_{L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)}^{p'} = [u_k]_{s,p}^p.$$

Therefore, from Lemma 3.2, we get that $\{\xi_k\}_{k \in \mathbb{N}}$ is bounded in $L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$. So, up to some subsequence, there exists a function $\xi \in L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\xi_k \rightharpoonup \xi \text{ weakly in } L^{p'}(\mathbb{R}^n \times \mathbb{R}^n).$$

Therefore,

$$(3.3) \quad \begin{aligned} \lim_{k \rightarrow \infty} \langle (-\Delta_p)^s u_k, \phi \rangle &= \lim_{k \rightarrow \infty} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \xi_k(x, y) \frac{\phi(x) - \phi(y)}{|x - y|^{\frac{n+sp}{p}}} dx dy \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \xi(x, y) \frac{\phi(x) - \phi(y)}{|x - y|^{\frac{n+sp}{p}}} dx dy, \end{aligned}$$

for all $\phi \in W^{s,p}(\mathbb{R}^n)$. In particular, (3.3) holds for every $\phi \in C_c^\infty(\Omega)$. Moreover, by the compactness of the immersion $W_0^{s,p}(D) \subset L^p(D)$ (see [5]), since $u_k \rightharpoonup u^*$ weakly in $W_0^{s,p}(D)$ we can conclude that $u_k \rightarrow u^*$ a.e. in \mathbb{R}^n , then

$$\xi_k(x, y) \rightarrow \frac{|u^*(x) - u^*(y)|^{p-2} (u^*(x) - u^*(y))}{|x - y|^{\frac{n+sp}{p'}}},$$

a.e. in $\mathbb{R}^n \times \mathbb{R}^n$, from where it follows that

$$(3.4) \quad \xi(x, y) = \frac{|u^*(x) - u^*(y)|^{p-2}(u^*(x) - u^*(y))}{|x - y|^{\frac{n+sp}{p}}}.$$

Finally, observe that if $\phi \in C_c^\infty(\Omega)$ then $\phi \in C_c^\infty(\Omega_k)$ for every k sufficiently large (Proposition 2.5). Therefore, from (3.3) we conclude that

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \xi(x, y) \frac{\phi(x) - \phi(y)}{|x - y|^{\frac{n}{p} + s}} dx dy = \langle f, \phi \rangle.$$

The proof is then completed by combining this last equality with (3.4). \square

Remark 3.5. In order to show that $u^* = u_\Omega^f$, what remains is to show that $u^* = 0$ on Ω^c . This is the hard part and is where some geometric hypotheses on the nature of the convergence of the domains needs to be made.

Theorem 3.6. *Assume that the hypotheses of Theorem 3.4 are satisfied. If, in addition,*

$$(3.5) \quad \text{cap}_{s,p}(\Omega_k \setminus \Omega, D) \rightarrow 0,$$

then $u_{\Omega_k}^f \rightharpoonup u_\Omega^f$ weakly in $W_0^{s,p}(D)$.

Proof. As before, we denote $u_k = u_{\Omega_k}^f$. By Corollary 3.3, $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $W_0^{s,p}(D)$ and therefore we can assume that $u_k \rightharpoonup u^*$ weakly in $W_0^{s,p}(D)$.

By Theorem 3.4 the proof will be finished if we can prove that $u^* \in W_0^{s,p}(\Omega)$, and by Theorem 2.16, it is enough to prove that $\tilde{u}^* = 0$ (s, p) -q.e. in Ω^c .

Consider $\tilde{\Omega}_j = \bigcup_{k \geq j} \Omega_k$ and $E = \bigcap_{j \geq 1} \tilde{\Omega}_j$.

Since $u_k \rightharpoonup u^*$ in $W_0^{s,p}(D)$, by Mazur's Lemma (see for instance [7]), there is a sequence $v_j = \sum_{k=j}^{N_j} a_k^j u_k$ such that $a_k^j \geq 0$, $\sum_{k=j}^{N_j} a_k^j = 1$ and $v_j \rightarrow u^*$ strongly in $W_0^{s,p}(D)$.

Since $u_k \in W_0^{s,p}(\Omega_k)$, by Theorem 2.16, $\tilde{u}_k = 0$ (s, p) -q.e. in Ω_k^c . Therefore, $\tilde{v}_j = \sum_{k=j}^{N_j} a_k^j \tilde{u}_k = 0$ (s, p) -q.e. in $\bigcap_{k=j}^{N_j} \Omega_k^c \supset \tilde{\Omega}_j^c$ for every $j \in \mathbb{N}$.

Then, $\tilde{v}_j = 0$ (s, p) -q.e. in $\tilde{\Omega}_j^c$ for every $j \in \mathbb{N}$ and, since $\tilde{\Omega}_j^c \subset \tilde{\Omega}_{j+1}^c$, we conclude that $\tilde{v}_j = 0$ (s, p) -q.e. $\tilde{\Omega}_i^c$ for every $i \leq j$.

On the other hand, since $v_j \rightarrow u^*$ strongly in $W_0^{s,p}(D)$, by Proposition 2.15, $\tilde{v}_{j_k} \rightarrow \tilde{u}^*$ (s, p) -q.e. Then we conclude that $\tilde{u}^* = 0$ (s, p) -q.e. in E^c .

In order to finish the proof of the theorem, we show that the capacity condition (3.5) implies that $\Omega^c \subset E^c$ up to some set of zero (s, p) -capacity.

In fact, since $\text{cap}_{s,p}(\Omega_k \setminus \Omega) \rightarrow 0$, passing to a subsequence, if necessary, we can assume that $\text{cap}_{s,p}(\Omega_k \setminus \Omega) \leq \frac{1}{2^k}$. Therefore,

$$\text{cap}_{s,p}(\tilde{\Omega}_j \setminus \Omega) = \text{cap}_{s,p}(\bigcup_{k \geq j} \Omega_k \setminus \Omega) \leq \sum_{k \geq j} \text{cap}_{s,p}(\Omega_k \setminus \Omega) \leq \sum_{k \geq j} \frac{1}{2^k} = \frac{1}{2^{j-1}}.$$

Recall now that $E \subset \tilde{\Omega}_j$ for every $j \in \mathbb{N}$, then we have that

$$\text{cap}_{s,p}(E \setminus \Omega) \leq \text{cap}_{s,p}(\tilde{\Omega}_j \setminus \Omega) \leq \frac{1}{2^{j-1}} \text{ for every } j \in \mathbb{N}.$$

Taking the limit $j \rightarrow \infty$, we have that $\text{cap}_{s,p}(E \setminus \Omega) = \text{cap}_{s,p}(\Omega^c \setminus E^c) = 0$ and the proof is finished. \square

As a simple corollary, we can show that the convergence of the solutions in Theorem 3.6 is actually strong.

Corollary 3.7. *Under the assumptions of Theorem 3.6 we have that $u_{\Omega_m}^f \rightarrow u_{\Omega}^f$ strongly in $W_0^{s,p}(D)$.*

Proof. The proof is simple. Just observe that from the weak convergence $u_{\Omega_m}^f \rightharpoonup u_{\Omega}^f$ given by Theorem 3.6, we get

$$[u_{\Omega_m}^f]_{s,p}^p = \langle f, u_{\Omega_m}^f \rangle \rightarrow \langle f, u_{\Omega}^f \rangle = [u_{\Omega}^f]_{s,p}^p.$$

Since $W_0^{s,p}(D)$ is a uniformly convex Banach space, the result follows. \square

ACKNOWLEDGEMENTS

This paper was partially supported by grants UBACyT 20020130100283BA, CONICET PIP 11220150100032CO and ANPCyT PICT 2012-0153.

REFERENCES

- [1] Vedat Akgiray and G. Geoffrey Booth. The stable-law model of stock returns. *Journal of Business & Economic Statistics*, 6(1):51–57, 1988.
- [2] Doina Cioranescu and François Murat. A strange term coming from nowhere [MR0652509 (84e:35039a); MR0670272 (84e:35039b)]. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 45–93. Birkhäuser Boston, Boston, MA, 1997.
- [3] Peter Constantin. Euler equations, Navier-Stokes equations and turbulence. In *Mathematical foundation of turbulent viscous flows*, volume 1871 of *Lecture Notes in Math.*, pages 1–43. Springer, Berlin, 2006.
- [4] Françoise Demengel and Gilbert Demengel. *Functional spaces for the theory of elliptic partial differential equations*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2012. Translated from the 2007 French original by Reinie Ern e.
- [5] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
- [6] Qiang Du, Max Gunzburger, R. B. Lehoucq, and Kun Zhou. Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM Rev.*, 54(4):667–696, 2012.
- [7] Ivar Ekeland and Roger Temam. *Convex analysis and variational problems*. North-Holland Publishing Co., Amsterdam, 1976. Translated from the French, Studies in Mathematics and its Applications, Vol. 1.
- [8] A. Cemal Eringen. *Nonlocal continuum field theories*. Springer-Verlag, New York, 2002.
- [9] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [10] Matteo Focardi. Aperiodic fractional obstacle problems. *Adv. Math.*, 225(6):3502–3544, 2010.
- [11] Giambattista Giacomin and Joel L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Statist. Phys.*, 87(1-2):37–61, 1997.
- [12] Guy Gilboa and Stanley Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.
- [13] Antoine Henrot and Michel Pierre. *Variation et optimisation de formes*, volume 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].
- [14] Nicolas Humphries et al. Environmental context explains lévy and brownian movement patterns of marine predators. *Nature*, 465:1066–1069, 2010.
- [15] Nikolai Laskin. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A*, 268(4-6):298–305, 2000.

- [16] Sergei Levendorski. Pricing of the american put under lévy processes. *International Journal of Theoretical and Applied Finance*, 7(03):303–335, 2004.
- [17] A. Massaccesi and E. Valdinoci. Is a nonlocal diffusion strategy convenient for biological populations in competition? *ArXiv e-prints*, March 2015.
- [18] Ralf Metzler and Joseph Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):77, 2000.
- [19] A. M. Reynolds and C. J. Rhodes. The lvy flight paradigm: random search patterns and mechanisms. *Ecology*, 90(4):877–887, 2009.
- [20] Wim Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives*. Willey Series in Probability and Statistics. Willey, New York, 2003.
- [21] V. Šverák. On optimal shape design. *J. Math. Pures Appl. (9)*, 72(6):537–551, 1993.
- [22] Mahamadi Warma. The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets. *Potential Anal.*, 42(2):499–547, 2015.
- [23] Kun Zhou and Qiang Du. Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions. *SIAM J. Numer. Anal.*, 48(5):1759–1780, 2010.

(C. Baroncini) DEPARTAMENTO DE MATEMÁTICA FCEN - UNIVERSIDAD DE BUENOS AIRES AND IMAS - CONICET. CIUDAD UNIVERSITARIA, PABELLÓN I (C1428EGA) AV. CANTILO 2160. BUENOS AIRES, ARGENTINA.

E-mail address: `cbaronci@dm.uba.ar`

(J. Fernández Bonder) DEPARTAMENTO DE MATEMÁTICA FCEN - UNIVERSIDAD DE BUENOS AIRES AND IMAS - CONICET. CIUDAD UNIVERSITARIA, PABELLÓN I (C1428EGA) AV. CANTILO 2160. BUENOS AIRES, ARGENTINA.

E-mail address: `jfbonder@dm.uba.ar`

URL: `http://mate.dm.uba.ar/~jfbonder`

(J. F. Spedaletti) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD NACIONAL DE SAN LUIS AND IMASL - CONICET. EJÉRCITO DE LOS ANDES 950 (D5700HHW), SAN LUIS, ARGENTINA.

E-mail address: `jfspedaletti@unsl.edu.ar`