Isomorphisms of non noetherian down-up algebras

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Abstract

We solve the isomorphism problem for non noetherian down-up algebras $A(\alpha, 0, \gamma)$ by lifting isomorphisms between some of their non commutative quotients. The quotients we consider are either quantum polynomial algebras in two variables for $\gamma = 0$ or quantum versions of the Weyl algebra A_1 for non zero γ. In particular we obtain that no other down-up algebra is isomorphic to the monomial algebra $A(0, 0, 0)$. We prove in the second part of the article that this is the only monomial algebra within the family of down-up algebras. Our method uses homological invariants that determine the shape of the possible quivers and we apply the abelianization functor to complete the proof.

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1 Introduction

Let $\Bbbk$ be a fixed field of characteristic 0. Given parameters $(\alpha, \beta, \gamma) \in \Bbbk^3$, the associated *down-up algebra* $A(\alpha, \beta, \gamma)$, first defined in [2], is the quotient of the free associative algebra $\Bbbk\langle d, u\rangle$ by the ideal generated by the relations

$$d^2u - (\alpha dud + \beta ud^2 + \gamma d),$$
$$du^2 - (\alpha udu + \beta u^2d + \gamma u).$$

There are several well-known examples of down-up algebras such as $A(2, -1, 0)$, isomorphic to the enveloping algebra of the Heisenberg-Lie algebra of dimension 3, and, for $\gamma \neq 0$, $A(2, -1, \gamma)$, isomorphic to the enveloping algebra of $\mathfrak{sl}(2, \Bbbk)$.

A down-up algebra has a PBW basis given by

$$\{u^i(du)^jd^k : i, j, k \geq 0\}.$$

Note that $A(\alpha, \beta, \gamma)$ can be regarded as a $\mathbb{Z}$-graded algebra where the degrees of u and d are respectively 1 and -1. The field $\Bbbk$ is the trivial module over $A(\alpha, \beta, \gamma)$, with d and u acting as 0.

E. Kirkman, I. Musson and D. Passman proved in [7] that $A(\alpha, \beta, \gamma)$ is noetherian if and only it is a domain, if and only if $\beta \neq 0$.

The isomorphism problem for down-up algebras was posed in [2] where the authors considered algebras $A(\alpha, \beta, \gamma)$ of four different types and proved, by studying one dimensional modules, that algebras of different types are not isomorphic. They considered the following types,

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(a) $\gamma = 0, \alpha + \beta = 1$,

(b) $\gamma = 0, \alpha + \beta \neq 1$,

(c) $\gamma \neq 0, \alpha + \beta = 1$,

(d) $\gamma \neq 0, \alpha + \beta \neq 1$.

As a consequence, we can restrict the question of whether two down-up algebras are isomorphic to each of the four types. In [3] the authors solved the isomorphism problem for noetherian down-up algebras of types (a), (b) and (c) for every algebraically closed field $\Bbbk$, and also for noetherian algebras of type (d) when in addition $\mathsf{char}(\Bbbk) = 0$. More precisely, they proved the following result.

Theorem ([3]). *Let $A = A(\alpha, \beta, \gamma)$ and $A' = A(\alpha', \beta', \gamma')$ be noetherian down-up algebras. Then A is isomorphic to A' if and only if*

1. *$\gamma = \lambda\gamma'$ for some $\lambda \in \Bbbk^\times$, and*
2. *either $\alpha' = \alpha$, $\beta' = \beta$ or $\alpha' = -\alpha\beta^{-1}$, $\beta' = \beta^{-1}$.*

Their solution focuses mainly on the possible commutative quotients of down-up algebras. In contrast with this, there are very well studied non commutative algebras that appear as quotients of non noetherian down-up algebras, for example, when $\alpha \in \Bbbk^\times$, the quantum plane $\Bbbk_\alpha[x, y]$ and the quantum Weyl algebra A^1_α,

$$\Bbbk_\alpha[x, y] := \Bbbk\langle x, y\rangle / (yx - \alpha xy)\,, \qquad A^1_\alpha := \Bbbk\langle x, y\rangle / (yx - \alpha xy - 1)\,.$$

In this article we describe isomorphisms amongst non noetherian down-up algebras by using these quotients. Our main result is the following.

Theorem 1.1. *Let $\Bbbk$ be an algebraically closed field and let $\alpha, \alpha', \gamma, \gamma' \in \Bbbk$. The algebras $A(\alpha, 0, \gamma)$ and $A(\alpha', 0, \gamma')$ are isomorphic if and only if*

1. *$\gamma = \lambda\gamma'$, for some $\lambda \in \Bbbk^\times$, and*
2. *$\alpha' = \alpha$.*

We obtain in particular that no other down-up algebra is isomorphic to $A(0, 0, 0)$. In Section 3 we prove

Theorem 1.2. *The algebra $A(\alpha, \beta, \gamma)$ is monomial if and only if $(\alpha, \beta, \gamma) = (0, 0, 0)$.*

So, the only monomial algebra in the family of down-up algebras is the evident one. Our starting point is the fact that noetherian down-up algebras cannot be monomial since they are a domain of global dimension 3 [7]. The situation can be related to 3-dimensional Sklyanin algebras. In both cases, an algebra A is noetherian if and only if it is a domain. For Sklyanin algebras, these conditions are equivalent to A being monomial [9]. This is not the case for down-up algebras.

Our proof uses homological invariants that determine the possible shapes of the quiver. We think that these methods may be useful for other families of algebras.

2 Isomorphisms of non noetherian down-up algebras

The purpose of this section is to prove Theorem 1.1. Let $\Bbbk$ be an algebraically closed field. Note that the condition $\gamma = \lambda\gamma'$ for $\lambda \in \Bbbk^\times$ is equivalent to the condition of γ and γ' being both zero or both non zero. We already know from [2] that if $\gamma \neq 0$, then $A(\alpha, 0, \gamma)$ is isomorphic to $A(\alpha, 0, 1)$. This is done by rescaling d by γd. Also, observe that $A(\alpha, 0, 0)$ is not isomorphic to $A(\alpha', 0, 1)$ for all $\alpha, \alpha' \in \Bbbk$, since they belong to different types. Gathering all this information, we deduce that Theorem 1.1 is equivalent to the following two propositions:

Proposition 2.1. *Let $\alpha, \alpha' \in \Bbbk$. The algebras $A(\alpha, 0, 0)$ and $A(\alpha', 0, 0)$ are isomorphic if and only if $\alpha = \alpha'$.*

Proposition 2.2. *Let $\alpha, \alpha' \in \Bbbk$. The algebras $A(\alpha, 0, 1)$ and $A(\alpha', 0, 1)$ are isomorphic if and only if $\alpha = \alpha'$.*

We will thus prove both of them in order to obtain our result.

Lemma 2.3. *Let $\alpha \in \Bbbk^\times$, $\gamma \in \Bbbk$ and let $A = A(\alpha, 0, \gamma)$ be a down-up algebra. Denote $\omega := du - \alpha ud - \gamma$. The algebra $A/\langle\omega\rangle$ is isomorphic to $\Bbbk_\alpha[x, y]$ if $\gamma = 0$ and it is isomorphic to A^1_α if $\gamma \neq 0$. Moreover, in case $\gamma = 0$ or $\gamma = 1$, the isomorphism maps the class of d to y and the class of u to x.*

Proof. The algebra $A(\alpha, 0, \gamma)$ is the quotient of the free algebra generated by the variables d and u subject to the relations $d^2u - \alpha dud - \gamma d = 0$ and $du^2 - \alpha udu - \gamma u = 0$. Denote by Ω the element $du - \alpha ud - \gamma$ in the free algebra. The projection of Ω onto $A(\alpha, 0, \gamma)$ is ω. The defining relations of A are $d\Omega = 0$ and $\Omega u = 0$. Therefore, the algebra $A/\langle\omega\rangle$ is isomorphic to the algebra freely generated by letters d, u subject to the relation $\Omega = 0$. If $\gamma = 0$, then this is exactly the definition of $\Bbbk_\alpha[x, y]$. If $\gamma \neq 0$, then $\omega = \gamma^{-1}((\gamma d)u - \alpha u(\gamma d) - 1)$, and so $A/\langle\omega\rangle$ is the quantum Weyl algebra generated by x and y, with $y = \gamma d$ and $x = u$. □

In [8] the authors describe all isomorphisms and automorphisms for quantum Weyl algebras A^1_α for $\alpha \in \Bbbk^\times$ not a root of unity. In [10] this result is generalized to the family of *quantum generalized Weyl algebras*, including the quantum plane and the quantum Weyl algebra for all values of $\alpha \in \Bbbk^\times$. We recall some of their results in the cases relevant to us.

Theorem 2.4 ([8],[10])**.** *Let $\alpha, \alpha' \in \Bbbk \setminus \{0, 1\}$.*

i) The algebras $\Bbbk_\alpha[x, y]$ and $\Bbbk_{\alpha'}[x, y]$ are isomorphic if and only if $\alpha' \in \{\alpha, \alpha^{-1}\}$. Moreover, if $\varphi : \Bbbk_\alpha[x, y] \to \Bbbk_{\alpha^{-1}}[x, y]$ is an isomorphism and $\alpha \neq -1$, then there exist $\lambda, \mu \in \Bbbk^\times$ such that $\varphi(x) = \lambda y$ and $\varphi(y) = \mu x$.

ii) The algebras A^1_α and $A^1_{\alpha'}$ are isomorphic if and only if $\alpha' \in \{\alpha, \alpha^{-1}\}$. If $\alpha \neq -1$, then every isomorphism $\eta : A^1_\alpha \to A^1_{\alpha^{-1}}$ is of the form $\eta(x) = \lambda y$ and $\eta(y) = -\lambda^{-1}\alpha^{-1}x$, for some $\lambda \in \Bbbk^\times$.

Proof of Proposition 2.1. Let α and α' be elements of $\Bbbk$ and suppose there exists an isomorphism of $\Bbbk$-algebras $\varphi : A(\alpha, 0, 0) \to A(\alpha', 0, 0)$. Denote $A := A(\alpha, 0, 0)$, $A' := A(\alpha', 0, 0)$ and let d' and u' be the usual generators of A'.

Suppose $\alpha, \alpha' \in \Bbbk \setminus \{0, 1\}$ and $\alpha \neq \alpha'$. Let $\omega := du - \alpha ud$ and $\omega' := d'u' - \alpha' u'd'$. By Lemma 2.3, we can identify $A/\langle\omega\rangle$ with $\Bbbk_\alpha[x, y]$, where the canonical projection $\pi : A \to \Bbbk_\alpha[x, y]$ sends d to y and u to x, and similarly for $A'/\langle\omega'\rangle$ and $\Bbbk_{\alpha'}[x, y]$; here we denote by π' the canonical projection. Define $\psi_1 = \pi' \circ \varphi$. The equalities $d\omega = 0$ and $\omega u = 0$ hold in A, so

$$\psi_1(d)\psi_1(\omega) = 0 = \psi_1(\omega)\psi_1(u).$$

The algebra $\Bbbk_{\alpha'}[x, y]$ is a non commutative domain generated by $\psi_1(d)$ and $\psi_1(u)$. Thus, $\psi_1(d)$ and $\psi_1(u)$ are not zero and from the above equations we deduce $\psi_1(\omega) = 0$. This implies that there exists an algebra map $\overline{\psi}_1 : \Bbbk_\alpha[x, y] \to \Bbbk_{\alpha'}[x, y]$ such that $\psi_1 = \overline{\psi}_1 \circ \pi$. In the other direction we obtain that $\psi_2 := \pi \circ \varphi^{-1}$ factors as $\psi_2 = \overline{\psi}_2 \circ \pi'$. Since $\overline{\psi}_1 \circ \overline{\psi}_2 \circ \pi' = \pi'$ and $\overline{\psi}_2 \circ \overline{\psi}_1 \circ \pi = \pi$, we deduce $\overline{\psi}_1$ is an isomorphism. The situation is illustrated by the following commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \Bbbk_\alpha[x,y] & \underset{\overline{\psi}_2}{\overset{\overline{\psi}_1}{\rightleftarrows}} & \Bbbk_{\alpha'}[x,y] \end{array}$$

By Theorem 2.4 and our assumption that $\alpha \neq \alpha'$, we obtain $\alpha' = \alpha^{-1}$. Theorem 2.4 also says that there exist $\lambda, \mu \in \Bbbk^\times$ and $z_1, z_2 \in \langle \omega' \rangle$ such that $\varphi(u) = \lambda d' + z_1$ and $\varphi(d) = \mu u' + z_2$. Note that A' is graded considering the generators d' and u' in degree 1. Since $\deg(\omega') = 2$, it follows that z_1 and z_2 are either 0 or sums of homogeneous elements of degree at least 2 with respect to this grading. On the other hand $d^2u - \alpha dud$ is 0, and so

$$0 = \varphi(d)^2\varphi(u) - \alpha\varphi(d)\varphi(u)\varphi(d).$$

In particular the degree 3 component of the right hand side of this equality, that is $(u')^2d' - \alpha u'd'u'$, must be 0. But the set $\{(u')^i(d'u')^j(d')^l : i, j, l \in \mathbb{N}_0\}$ is a $\Bbbk$-basis of A', and this is a contradiction.

In case $\alpha = 0$, an argument similar to the above one shows that there is an epimorphism $\psi : A \to \Bbbk_{\alpha'}[x, y]$. As a consequence the elements $\psi(d)$ and $\psi(u)$ generate $\Bbbk_{\alpha'}[x, y]$. If $\alpha' \neq 0$, then the algebra $\Bbbk_{\alpha'}[x, y]$ is a domain and it is not commutative, thus it cannot be generated by one element. From the equality $0 = d^2u$ we obtain that $0 = \psi(d^2u) = \psi(d)^2\psi(u)$, implying $\psi(d) = 0$ or $\psi(u) = 0$. This is a contradiction and so $\alpha' = 0$.

If $\alpha = 1$, then A belongs to type (a) and so does A'. This implies $\alpha' = 1$, concluding the proof of the proposition. □

Now we turn our attention to Proposition 2.2. Let $A = A(\alpha, 0, 1)$ for $\alpha \in \Bbbk$. Recall that $\omega := du - \alpha ud - 1$. Using Lemma 2.2 in [11], the set $\{u^i\omega^jd^l : i, j, l \geq 0\}$ is a $\Bbbk$-basis of A.

Lemma 2.5. *The set $\{u^i\omega^jd^l : i, l \geq 0$ and $j \geq 1\}$ is a $\Bbbk$-linear basis of the two sided ideal $\langle\omega\rangle$, and, for each $n \in \mathbb{N}$, the set $\{u^i\omega^jd^l : i, l \geq 0$ and $j \geq n\}$ is a $\Bbbk$-linear basis of $\langle\omega\rangle^n$.*

Proof. Every element of the form $u^i\omega^jd^l$ with $j \geq 1$ belongs to $\langle\omega\rangle$, so it only remains to prove that $\langle\omega\rangle$ is contained in the $\Bbbk$-vector space with basis the set $\{u^i\omega^jd^l : i, l \geq 0 \text{ and } j \geq 1\}$. Given $z \in \langle\omega\rangle$, write $z = \sum_{i,j,l} \lambda_{i,j,l}u^i\omega^jd^l$ with $i, j, l \geq 0$ and $\lambda_{i,j,l} \in \Bbbk$. By Lemma 2.3 we can identify $A/\langle\omega\rangle$ with A^1_α, and the canonical projection $\pi : A \to A^1_\alpha$ sends u to x and d to y. The set $\{x^iy^l : i, l \geq 0\}$ is a basis of A^1_α. From the equalities $\sum_{i,l} \lambda_{i,0,l}x^iy^l = \pi(z) = 0$, we deduce $\lambda_{i,0,l} = 0$ for all $i, l \geq 0$.

Taking into account the description we now have of $\langle\omega\rangle$, we see that the elements of $\langle\omega\rangle^2$ are linear combinations of monomials of type $u^i\omega^jd^lu^{i'}\omega^{j'}d^{l'}$, with $j, j' \geq 1$. Similarly, the elements of $\langle\omega\rangle^n$ are linear combinations of n-fold products of the same type. Therefore, to prove the second claim, it is sufficient to show that for every $r, s \geq 0$ there exist $\lambda_i \in \Bbbk$ such that $\omega d^ru^s\omega = \sum_{i\geq 2} \lambda_i\omega^i$. Indeed, there exist $\lambda_{i,j,l} \in \Bbbk$ such that

$$d^ru^s = \sum_{i,j,l\geq 0} \lambda_{i,j,l}u^i\omega^jd^l.$$

So

$$\omega d^ru^s\omega = \sum_{i,j,l\geq 0} \lambda_{i,j,l}\omega u^i\omega^jd^l\omega = \sum_{j\geq 0} \lambda_{0,j,0}\omega^{j+2}.$$

The last equality follows from $d\omega = 0$ and $\omega u = 0$. □

Corollary 2.6. *The set $\{[u^i\omega d^l] : i,l \geq 0\}$, where $[p]$ denotes the class of an element p in $\langle\omega\rangle/\langle\omega\rangle^2$, is a $\Bbbk$-linear basis of the A-bimodule $\langle\omega\rangle/\langle\omega\rangle^2$. Moreover, in case $\alpha \neq 1$, the following equalities hold*

$$
\begin{aligned}
[u^i\omega d^l u] &= \frac{\alpha^l - 1}{\alpha - 1}[u^i\omega d^{l-1}], \\
[du^i\omega d^l] &= \frac{\alpha^i - 1}{\alpha - 1}[u^{i-1}\omega d^l],
\end{aligned}
$$

where the terms on the right are considered to be zero for $l = 0$ or $i = 0$.

Proof. The first claim is a direct consequence of Lemma 2.5. To prove the first formula, we fix $i \geq 0$ and proceed by induction on l, the case $l = 0$ being trivial from the equalities $\omega u = 0 = d\omega$. On the other hand, since $\omega^2 = \omega(du - \alpha ud - 1) = \omega du - \omega$, we obtain that $\omega du = \omega^2 + \omega$. Similarly $du\omega = \omega^2 + \omega$. Therefore, $[u^i\omega du] = [u^i\omega]$. Now, for $l \geq 2$

$$
\begin{aligned}
[u^i\omega d^l u] = [u^i\omega d^{l-2}(\alpha dud + d)] &= \alpha[u^i\omega d^{l-1}ud] + [u^i\omega d^{l-1}] \\
&= \alpha\frac{\alpha^{l-1} - 1}{\alpha - 1}[u^i\omega d^{l-2}d] + [u^i\omega d^{l-1}] \\
&= \frac{\alpha^l - 1}{\alpha - 1}[u^i\omega d^{l-1}].
\end{aligned}
$$

The second formula can be proved analogously. □

Proof of Proposition 2.2. Let $\alpha, \alpha' \in \Bbbk$. Denote $A := A(\alpha, 0, 1)$, $A' := A(\alpha', 0, 1)$. Let d', u' be the generators of A'. Suppose there exists an isomorphism of $\Bbbk$-algebras $\varphi : A \to A'$. Recall that $\omega = du - \alpha ud - 1$ and $\omega' = d'u' - \alpha' u'd' - 1$.

If $\alpha = 1$, then A belongs to type (a), and so does A', hence $\alpha' = 1$. Now suppose $\alpha = 0$ and $\alpha' \neq 0$. By Lemma 2.3 the algebra $A'/\langle\omega'\rangle$ is isomorphic to $A^1_{\alpha'}$ and, if π' denotes the canonical projection, then $\pi'(u') = x$ and $\pi'(d') = y$. Let $\psi = \pi' \circ \varphi$. Since $d\omega = \omega u = 0$, we have $\psi(d)\psi(\omega) = \psi(\omega)\psi(u) = 0$. Note that $\psi(d)$ and $\psi(u)$ generate $A^1_{\alpha'}$, and therefore they cannot belong to $\Bbbk$; in particular they cannot be zero. We deduce $0 = \psi(\omega) = \psi(d)\psi(u) - 1$. The algebra $A^1_{\alpha'}$ has a filtration whose associated graded algebra $\mathsf{Gr}(A^1_{\alpha'})$ is $\Bbbk_{\alpha'}[x,y]$. The equality $\psi(d)\psi(u) = 1$ implies that $\mathsf{Gr}(A^1_{\alpha'})$ is not a domain, which is a contradiction since $\alpha' \neq 0$.

Suppose $\alpha, \alpha' \in \Bbbk \setminus \{0,1\}$ and $\alpha \neq \alpha'$. By the same arguments as in the proof of Proposition 2.1, the map $\psi := \pi' \circ \varphi : A \to A^1_{\alpha'}$ induces an isomorphism of $\Bbbk$-algebras $\overline{\psi} : A^1_\alpha \to A^1_{\alpha'}$. Theorem 2.4 implies $\alpha' = \alpha$ or $\alpha' = \alpha^{-1}$. Since we are assuming $\alpha \neq \alpha'$, we deduce $\alpha' = \alpha^{-1}$. Again, by Theorem 2.4 we obtain that there exist $\lambda \in \Bbbk^\times$ and $z_1, z_2 \in \langle\omega\rangle$ such that

$$
\begin{aligned}
\varphi^{-1}(d') &= -\lambda^{-1}\alpha u + z_1, \\
\varphi^{-1}(u') &= \lambda d + z_2.
\end{aligned}
$$

By rescaling the variables d, u, we may assume $\lambda = 1$. The equality $(d')^2u' - \alpha^{-1}d'u'd' - d' = 0$ implies

$$
\begin{aligned}
0 &= \varphi^{-1}(d')^2\varphi^{-1}(u') - \alpha^{-1}\varphi^{-1}(d')\varphi^{-1}(u')\varphi^{-1}(d') - \varphi^{-1}(d') \\
&= \alpha^2u^2d - \alpha udu + \alpha u + \alpha^2u^2z_2 - \alpha uz_1d - \alpha z_1ud \\
&\quad + udz_1 - \alpha uz_2u + z_1du - z_1 + z \\
&= -\alpha u\omega + \alpha(\alpha u^2z_2 - uz_1d) + (udz_1 - \alpha uz_2u) + z_1\omega + z \in \langle\omega\rangle,
\end{aligned}
$$

where z denotes the sum of all terms in which at least two factors z_1 or z_2 are involved. Note that $z_1\omega$ and z belong to $\langle\omega\rangle^2$. Taking classes modulo $\langle\omega\rangle^2$,

$$\alpha[u\omega] + \alpha([uz_1d] - \alpha[u^2z_2]) = [udz_1] - \alpha[uz_2u].$$

Write now $z_1 = \sum_{i,l\geq 0, j\geq 1} \lambda_{i,j,l}u^i\omega^j d^l$ and $z_2 = \sum_{i,l\geq 0,j\geq 1} \mu_{i,j,l}u^i\omega^j d^l$. Using the formulas of Corollary 2.6 we obtain

$$\alpha[u\omega] + \sum_{i,l\geq 0} \alpha(\lambda_{i,1,l}[u^{i+1}\omega d^{l+1}] - \alpha\mu_{i,1,l}[u^{i+2}\omega d^l]) =$$
$$= \sum_{i\geq 1, l\geq 0} \lambda_{i,1,l}\frac{\alpha^i - 1}{\alpha - 1}[u^i\omega d^l] - \sum_{i\geq 0, l\geq 1} \alpha\mu_{i,1,l}\frac{\alpha^l - 1}{\alpha - 1}[u^{i+1}\omega d^{l-1}].$$

By Corollary 2.6, the set $\{[u^i\omega d^l] : i, l \geq 0\}$ is a $\Bbbk$-linear basis of $\langle\omega\rangle/\langle\omega\rangle^2$. For $m \geq 0$, define $\Lambda_m := \lambda_{m+1,1,m} - \alpha\mu_{m,1,m+1}$. Looking at the coefficient corresponding to the term $[u^{m+1}\omega d^m]$ in the last equation for each $m \geq 0$, we deduce

$$\alpha = \Lambda_0,$$
$$\alpha\Lambda_{m-1} = \frac{\alpha^{m+1} - 1}{\alpha - 1}\Lambda_m, \text{ for } m \geq 1.$$

The fact that $\Lambda_0 = \alpha \neq 0$ implies, by an inductive argument, that $\Lambda_m \neq 0$ for all $m \in \mathbb{N}$. As a consequence, either $\lambda_{m+1,1,m} \neq 0$ for infinitely many values of $m \in \mathbb{N}$, or $\mu_{m,1,m+1} \neq 0$ for infinitely many values of $m \in \mathbb{N}$. This is a contradiction that comes from the assumption $\alpha \neq \alpha'$. □

3 Monomial down-up algebras

An algebra is monomial if it is isomorphic to an algebra of the form $\Bbbk Q/I$, where Q is a quiver with a finite number of vertices and I is a two-sided ideal in $\Bbbk Q$ generated by paths of length at least 2. The algebra $A(0, 0, 0)$ is monomial and no other down-up algebra is isomorphic to it. However, other monomial down-up algebras may exist. In this section we prove Theorem 1.2. Before doing it we prove a series of preparatory lemmas.

We will make use of the abelianization functor defined on $\Bbbk$-algebras as

$$A \mapsto A^{\mathsf{ab}} := A/J_A,$$

where J_A is the two sided ideal in A generated by the set $\{xy - yx : x, y \in A\}$. The canonical projection $\pi_A : A \to A^{\mathsf{ab}}$ is a natural transformation from the identity to the abelianization functor.

In order to state the next lemma we need some previous definitions. Given a quiver Q with a finite number of vertices and $e, e' \in Q_0$, define $eQ_1e' := \{\alpha \in Q_1 : \mathsf{t}(\alpha) = e, \mathsf{s}(\alpha) = e'\}$, where t and s are the usual target and source maps. Also, denote by B_e the $\Bbbk$-algebra $\Bbbk[X_\alpha : \alpha \in eQ_1e]$. That is, B_e is the polynomial algebra in variables indexed by the elements of the set eQ_1e. In case $eQ_1e = \emptyset$ we set $B_e = \Bbbk$. If I is a two-sided ideal in $\Bbbk Q$ generated by paths of length at least 2, define I_e to be the ideal in B_e generated by the set

$$\bigcup_{n\geq 2}\{X_{\alpha_n}\cdots X_{\alpha_1} : \alpha_n\cdots\alpha_1 \in I, \alpha_i \in eQ_1e\}.$$

Lemma 3.1. *Let Q be a quiver with a finite number of vertices and I a two-sided ideal in $\Bbbk Q$ generated by paths of length at least 2. There is an isomorphism of $\Bbbk$-algebras*

$$\left(\frac{\Bbbk Q}{I}\right)^{\mathsf{ab}} \cong \bigoplus_{e\in Q_0} \frac{B_e}{I_e}.$$

Proof. The classes $\overline{e}$ in $\Bbbk Q^{\mathsf{ab}}$ of the vertices e in Q_0 are a complete set of central orthogonal idempotents and $\overline{e}(\Bbbk Q^{\mathsf{ab}})\overline{e}$ is isomorphic to $(e\Bbbk Qe)^{\mathsf{ab}}$, and thus isomorphic to B_e. As a consequence, there is an isomorphism $\theta : \Bbbk Q^{\mathsf{ab}} \to \bigoplus_{e\in Q_0} B_e$ such that $\theta(\pi(\alpha)) = X_\alpha$ for all $\alpha \in ekQe$ and $e \in Q_0$. Let g be the map $f^{\mathsf{ab}} \circ \theta^{-1}$. The commutativity of the diagram

$$\begin{array}{ccccc}
\Bbbk Q & \xrightarrow{\pi} & \Bbbk Q^{\mathsf{ab}} & \xrightarrow{\theta} & \bigoplus_{e\in Q_0} B_e \\
\big\downarrow f & & \big\downarrow f^{\mathsf{ab}} & \swarrow g & \\
\Bbbk Q/I & \longrightarrow & (\Bbbk Q/I)^{\mathsf{ab}} & &
\end{array}$$

implies that g is surjective and that its kernel is $(\theta \circ \pi)(I + J_{\Bbbk Q}) = (\theta \circ \pi)(I) = \oplus_{e\in Q_0} I_e$, and the lemma follows.

□

The following lemma is a well known result for finite dimensional algebras replacing Tor by Ext. Here we give the proof for completeness.

Lemma 3.2. *Let $B = \Bbbk Q/I$ be a monomial algebra. For each $e \in Q_0$ denote by T_e the simple B-module corresponding to e. If $e, e' \in Q_0$, then $\#eQ_1e' = \dim_\Bbbk \operatorname{Tor}_1^B(T_e, T_{e'})$. Moreover, if Q has only one vertex e, then the dimension of $\operatorname{Tor}_1^B(T_e, T_e)$ is*

$$\sup\{\dim_\Bbbk \operatorname{Tor}_1^B(T_1, T_2) : T_1, T_2 \textit{ are one dimensional } B\textit{-modules}\}.$$

Proof. Let $e, e' \in Q_0$. The first terms of the minimal projective bimodule resolution of B are

$$\cdots \to B \otimes_E \Bbbk Q_1 \otimes_E B \to B \otimes_E B \to B \to 0,$$

where $E = \Bbbk Q_0$.

Applying the functor $T_e \otimes_B (-) \otimes_B T_{e'}$ to this resolution we obtain the following complex

$$\cdots \to T_e \otimes_\Bbbk \Bbbk eQ_1e' \otimes_\Bbbk T_{e'} \to T_e \otimes_\Bbbk T_{e'} \to 0,$$

whose homology is isomorphic to $\operatorname{Tor}_\bullet^B(T_e, T_{e'})$. The minimality of the resolution implies that every arrow in the above complex is zero. As a consequence, $\operatorname{Tor}_1^B(T_e, T_{e'}) \cong T_e \otimes_\Bbbk \Bbbk eQ_1e' \otimes_\Bbbk T_{e'} \cong e\Bbbk Q_1e'$, from where we deduce that the dimension of $\operatorname{Tor}_1^B(T_e, T_{e'})$ is $\#eQ_1e'$. As for the second assertion, the same argument shows that if T_1, T_2 are one dimensional B-modules, then the homology of the complex

$$\cdots \to T_1 \otimes_\Bbbk \Bbbk Q_1 \otimes_\Bbbk T_2 \to T_1 \otimes_\Bbbk T_2 \to 0,$$

is isomorphic to $\operatorname{Tor}_\bullet^B(T_1, T_2)$. It follows that $\dim_\Bbbk \operatorname{Tor}_1^B(T_1, T_2) \leq \dim_\Bbbk(\Bbbk Q_1) = \#Q_1 = \dim_\Bbbk \operatorname{Tor}_1^B(T_e, T_e)$, where e is the only vertex of Q. □

Lemma 3.3. *Let $A = A(\alpha, 0, \gamma)$ and let T_1, T_2 be one dimensional A-modules.*

(i) If $\gamma \neq 0$ and $\alpha = 1$, then $\dim_\Bbbk \operatorname{Tor}_1^A(T_1, T_2) = 0$.

(ii) If $\gamma \neq 0$ and $\alpha \neq 1$, then $\dim_\Bbbk \operatorname{Tor}_1^A(T_1, T_2) \leq 1$.

(iii) If $\gamma = 0$, then $\dim_{\Bbbk} \operatorname{Tor}_1^A(T_1, T_2) \leq 2$ *and* $\dim_{\Bbbk} \operatorname{Tor}_1^A(\Bbbk, \Bbbk) = 2$*. Moreover, if $\alpha \neq 1$ and $T_1 \neq \Bbbk$, then* $\dim_{\Bbbk} \operatorname{Tor}_1^A(T_1, T_1) = 1$.

Proof. Let T_1, T_2 be one dimensional A-modules with bases $\{v_1\}$ and $\{v_2\}$, respectively. Let $\delta_1, \delta_2, \mu_1, \mu_2 \in \Bbbk$ be such that $d \cdot v_i = \delta_i v_i$ and $u \cdot v_i = \mu_i v_i$ for $i = 1, 2$. From the equalities $d^2u - \alpha dud - \gamma d = 0 = du^2 - \alpha udu - \gamma u$ we deduce

$$\begin{aligned} &\delta_i((1-\alpha)\delta_i\mu_i - \gamma) = 0, \\ &\mu_i((1-\alpha)\delta_i\mu_i - \gamma) = 0, \end{aligned} \tag{3.1}$$

for $i = 1, 2$. Consider the following resolution of A as A-bimodule [4]

$$0 \to A \otimes_{\Bbbk} \Omega \otimes_{\Bbbk} A \xrightarrow{d_3} A \otimes_{\Bbbk} R \otimes_{\Bbbk} A \xrightarrow{d_2} A \otimes_{\Bbbk} V \otimes_{\Bbbk} A \xrightarrow{d_1} A \otimes_{\Bbbk} A \to 0,$$

where V, R and Ω are the subspaces of the free algebra $\Bbbk\langle d, u\rangle$ spanned, respectively, by the sets $\{d, u\}$, $\{d^2u, du^2\}$ and $\{d^2u^2\}$. The differentials are

$$\begin{aligned} d_1(1 \otimes v \otimes 1) &= v \otimes 1 - 1 \otimes v, \qquad \text{for all } v \in V, \\ d_2(1 \otimes d^2u \otimes 1) &= 1 \otimes d \otimes du + d \otimes d \otimes u + d^2 \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes d \otimes ud + d \otimes u \otimes d + du \otimes d \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes d^2 + u \otimes d \otimes d + ud \otimes d \otimes 1) \\ &\quad - \gamma \otimes d \otimes 1, \\ d_2(1 \otimes du^2 \otimes 1) &= 1 \otimes d \otimes u^2 + d \otimes u \otimes u + du \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes u \otimes du + u \otimes d \otimes u + ud \otimes u \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes ud + u \otimes u \otimes d + u^2 \otimes d \otimes 1) \\ &\quad - \gamma \otimes u \otimes 1, \end{aligned}$$

and

$$\begin{aligned} d_3(1 \otimes d^2u^2 \otimes 1) &= d \otimes du^2 \otimes 1 + \beta \otimes du^2 \otimes d \\ &\qquad - 1 \otimes d^2u \otimes u - \beta u \otimes d^2u \otimes 1. \end{aligned} \tag{3.2}$$

Applying the functor $T_1 \otimes_A (-) \otimes_A T_2$ to this resolution of A we obtain the following complex of $\Bbbk$-vector spaces whose homology is isomorphic to $\operatorname{Tor}_\bullet^A(T_1, T_2)$,

$$0 \longrightarrow \Bbbk \xrightarrow{f_2} \Bbbk^2 \xrightarrow{f_1} \Bbbk^2 \xrightarrow{f_0} \Bbbk \longrightarrow 0,$$

where

$$f_0 = \begin{pmatrix} \delta_2 - \delta_1 & \mu_2 - \mu_1 \end{pmatrix} \text{ and}$$

$$f_1 = \begin{pmatrix} (1-\alpha)\delta_1\mu_1 + \delta_2(\mu_1 - \alpha\mu_2) - \gamma & \mu_1(\mu_1 - \alpha\mu_2) \\ \delta_2(\delta_2 - \alpha\delta_1) & (1-\alpha)\delta_2\mu_2 + \mu_1(\delta_2 - \alpha\delta_1) - \gamma \end{pmatrix}.$$

The claims of the lemma follow from these formulas and Equation (3.1). □

We now turn to the proof of Theorem 1.2. Suppose $(\alpha, \beta, \gamma) \neq (0,0,0)$. Denote as usual $A(\alpha, \beta, \gamma)$ by A. Let $B = \Bbbk Q/I$ be a monomial algebra and suppose there exists an isomorphism of $\Bbbk$-algebras $\varphi : A \to B$. Since every down-up algebra has global dimension 3 [6], we deduce that $I \neq 0$ and so B is not a domain, thus we get $\beta = 0$.

Suppose $\gamma = 0$. In this case $\alpha \neq 0$ since we are assuming $(\alpha, \beta, \gamma) \neq (0,0,0)$. Note that

$$A^{\mathsf{ab}} = \Bbbk[d,u]/\langle (1-\alpha)d^2u, (1-\alpha)du^2 \rangle,$$

in particular it is connected and so is B^{ab}. By Lemma 3.1, the quiver Q has only one vertex e. Moreover, by Lemmas 3.2 and 3.3 we deduce

$$2 = \dim_{\Bbbk}(T_e, T_e) = \#Q_1,$$

thus, Q has exactly two arrows a and b. By Lemma 2.3 the quantum plane $\Bbbk_\alpha[x,y]$ is a quotient of A and so there exists an epimorphism $\varphi : B \to \Bbbk_\alpha[x,y]$. Since $\alpha \neq 0$, the quantum plane $\Bbbk_\alpha[x,y]$ is a domain. Given a non zero path p in I, $\varphi(p) = 0$, implying that either $\varphi(a)$ or $\varphi(b)$ is zero. As a consequence, the quantum plane is generated as algebra by one variable, which is a contradiction.

Now suppose $\gamma \neq 0$. If $\alpha = 1$, then $\mathrm{Tor}_1^A(T_1, T_2) = 0$ for every pair of one dimensional A-modules T_1, T_2, from Lemma 3.3. Since $A \cong B$, this is also true for B and one dimensional B-modules. By Lemma 3.2, the quiver Q has no arrows. This is impossible, and so $\alpha \neq 1$. The same lemmas imply in this case that there is at most one arrow between each pair of vertices in Q. In particular, there is at most one element in eQ_1e for every vertex e.

Define $V = \{e \in Q_0 : \#eQ_1e = 1\}$ and for every $e \in V$, denote a_e the unique element in eQ_1e. Since A, and thus B, is of global dimension 3, Bardzell's resolution [1] of B is of finite length. So ${a_e}^n \notin I$ for all $n \in \mathbb{N}$. This implies $B_e = \Bbbk[X]$ and $I_e = 0$ for all $e \in V$, and $B_e = \Bbbk$ for all $e \notin V$. By Lemma 3.1,

$$B^{\mathsf{ab}} \cong \bigoplus_{e \notin V} \Bbbk \oplus \bigoplus_{e \in V} \Bbbk[X].$$

In particular, its group of units is contained in the finite dimensional vector space $\Bbbk^{\#Q_0}$. On the other hand, the fact that the ideals $\langle d, u \rangle$ and $\langle (1-\alpha)du - \gamma) \rangle$ in $\Bbbk[d,u]$ are coprime implies

$$A^{\mathsf{ab}} \cong \Bbbk \oplus \frac{\Bbbk[d,u]}{\langle (1-\alpha)du - \gamma \rangle}.$$

The group of units of this algebra is contained in no finite dimensional space. Since B^{ab} is isomorphic to A^{ab}, this is a contradiction and we conclude the proof of Theorem 1.2.

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