SYMMETRY BREAKING FOR AN ELLIPTIC EQUATION INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. We study the symmetry breaking phenomenon for an elliptic equation involving the fractional Laplacian in a large ball. Our main tool is an extension of the Strauss radial lemma involving the fractional Laplacian, which might be of independent interest; and from which we derive compact embedding theorems for a Sobolev-type space of radial functions with power weights.

1. Introduction

Recently, elliptic problems involving the fractional Laplacian operator $(-\Delta)^s$ (see section 2 for its definition) have received a great deal of attention. Due to the rotation invariance of the fractional Laplacian it makes sense to ask if the solutions of equations of the form

$$(-\Delta)^s u = f(|x|, u) \tag{1}$$

in a radially symmetric domain (for instance, with homogeneous Dirichlet conditions) are necessarily radial or not.

In this work, we investigate the problem

$$(-\Delta)^s u + |x|^a |u|^{q-2} u = |x|^b |u|^{p-2} u \tag{2}$$

with $a, b > 0$, and u in the natural energy space for this problem, $H^s_{q,a}(\mathbb{R}^n)$ (see section 2 for the precise definitions), and analogous problems in a ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ (for $q = 2$).

Our aim is to extend to the fractional Laplacian setting, the results on existence of radial solutions and symmetry breaking obtained by P. Sintzoff in [28] on this equation (2) for the case $s = 1$ (that of the usual Laplacian). Namely, our main result on symmetry breaking reads as follows:

Theorem 1.1. *Let $n \geq 2$, $1/2 < s < 1$, $2 < p < 2^* = \frac{2n}{n-2s}$, $0 < a < n$ and $b > \frac{ap}{2}$. If in addition,*

$$a(p - 2 - 2ps) + 4bs < 2s(p - 2)(n - 1), \tag{3}$$

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then for every $R > 0$ large enough, problem

$$(4) \qquad \begin{cases} (-\Delta)^s u + |x|^a u = |x|^b u^{p-1} & \text{in } B_R \\ u > 0 \text{ a.e. in } B_R, \quad u \equiv 0 \text{ in } \mathbb{R}^n - B_R \end{cases}$$

has a nontrivial radial weak solution and a nonradial one (in the natural energy space $H^s_{q,a,0,rad}(B_R)$ for this problem, see section 2) .

This result extends theorem 3.1 in [28] to the fractional Laplacian setting. We follow the proof there, which is based in the comparison of the energy levels for the radial and non-radial ground states (minimizers of the associated energy functional). However, some technical difficulties arise from the non-local character of the fractional Laplacian. For instance, we shall require a technical lemma from [24] in order to perform the arguments based on cut-off functions. Moreover, we shall need to prove a version of Strauss inequality adapted to this problem (see the discussion below).

There is a large literature on the subject of radial symmetry of solutions of elliptic equations, starting from the classical result of B. Gidas, W.M. Ni and L. Nirenberg, [18] on the radial symmetry of positive solutions of (1) in a ball, for the usual Laplacian ($s = 1$). Hence, we cannot attempt to give a complete list of references here. In that work, the authors employed the moving plane method of A. D. Alexandrov and J. Serrin. We recall that this kind of result typically requires f to be decreasing as a function of $|x|$ (see theorem $1'$ of [18]). For the fractional Laplacian, the radial symmetry of positive solutions of some equations like (1) has been investigated by using the moving plane method in [14] and [13] (in both cases for f independent of x).

On the other hand, when f is not decreasing as a function of $|x|$, a symmetry breaking phenomenon may occur: equations of the form (1) may admit positive non-radial solutions. A case that has been extensively studied is that of the Henón equation

$$-\Delta u = |x|^b u^{p-1} \quad b > 0$$

in a ball ([30], [29]). Hence it is reasonable to expect equation (2) to exhibit a similar behavior. Other related results on symmetry breaking for nonlinear elliptic equations include [6] (for a singular elliptic equation related to the Caffarelli-Kohn-Nirenberg inequalities), [15] and [21] (where the symmetry breaking phenomenon is investigated for the minimizer of the trace inequality), and [19] where the results of [28] were extended to equations involving the p-Laplacian.

There is also an increasing literature on Schrödinger equations with the fractional Laplacian ([26], [26], [13], [16], [11] among other works). However, the question of symmetry breaking for the ground states for (4) (or related equations involving the fractional Laplacian) seems not to have been studied before.

As usual, when studying problems in $\mathbb{R}^n$ by variational methods, it is essential to get some compactness. It is well known that the Sobolev embedding

$$H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)\ 2 \leq p \leq 2^*_s = \frac{ns}{n-2s}$$

is not compact, due to the translation invariance of the norms of those spaces. However, starting by the pioneering work of W. Strauss [32], it is known that one can get compactness (for $2 < p < 2^*$ by restricting the problem to the subspace of radial functions. More precisely, W. Strauss proved in [32] that the following inequality holds for all *radially symmetric* functions in $H^1(\mathbb{R}^n)$.

$$\|u\| \leq C\,|x|^{-(n-1)/2}\,\|u\|_{H^1(\mathbb{R}^n)} \qquad n \geq 2, |x| \geq 1$$

This inequality means that radially symmetric functions in $H^1(\mathbb{R}^n)$ necessarily have a certain decay rate at infinity, and implies the required compactness in [32]. Strauss result was later generalized in many directions, see [8] for a survey of related results and elementary proofs of some of them within the framework of potential spaces.

In this work, our main tool will be a version of the Strauss inequality for the energy space $H^s_{q,a}(\mathbb{R}^n)$ of our problem (theorem 3.1) which might be of independent interest. The proof is completely different from the analogous result for the case of a local operator ($s = 1$) in [28], which was based on a simple integration by parts argument. Instead, we use some ideas from harmonic analysis, namely: we split the function into a high and a low frequency part. As a Strauss-type inequality implies the continuity of radial functions in that functional space outside the origin, we cannot expect this type of result to hold unless $s > 1/2$, and that is why this restriction appears in theorem 1.1.

This paper is organized as follows: In section 2 we collect the basic definitions and notations that we use. In section, 3 we state and prove the generalization of Strauss inequality for the space $H^s_{q,a}(\mathbb{R}^n)$. In section 4, we show that the same type of estimates allows us to prove the Hölder continuity of the functions in that space, outside the origin. This will be useful to us in order to be able to use the Arzela-Ascoli theorem for the compactness arguments. In section 5, we derive from our Strauss-type inequality, a compact embedding result for radial functions in $H^s_{q,a}(\mathbb{R}^n)$ into L^p spaces with power weights. As a corollary, we obtain some existence results for radial solutions of (2) in the whole space $\mathbb{R}^n$. Finally, in section (6) we prove theorem 1.1.

2. Basic definitions and notations

In this section, we define the functional spaces that we are going to work with, and recall some basic facts about the fractional Laplacian.

The fractional Laplacian $(-\Delta)^s$ can be defined for functions $u \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class) and $0 < s < 1$ by means of the Fourier transform as

$$\widehat{(-\Delta)^s u}(\omega) = |\omega|^{2s}\, \widehat{u}(\omega). \tag{5}$$

Alternatively, we may define the fractional Laplacian by a hypersingular integral

$$(-\Delta)^s u(x) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

where

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}$$

is a normalization constant. It is also possible to define the fractional Laplacian by means of an extension problem (see [5]).

For $0 < s < 1$, we consider the Gagliardo seminorm,

$$[u]_{H^s} = \|(-\Delta)^{s/2} u\|_{L^2} = \left(\frac{C(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2} \tag{6}$$

which, for $u \in \mathcal{S}(\mathbb{R}^n)$, can be also expressed in terms of the Fourier transform

$$[u]_{H^s} = \left(\int_{\mathbb{R}^n} |\omega|^{2s} |\widehat{u}(\omega)|^2\, d\omega \right)^{1/2}.$$

We refer to [10] for more details.

We denote by $L^q_a(\mathbb{R}^n)$ the weighted Lebesgue space with the norm as

$$\|u\|_{L^q_a} = \left(\int_{\mathbb{R}^n} |u(x)|^q\, |x|^a\, dx \right)^{1/q}.$$

Then, we define the space $H^s_{q,a}(\mathbb{R}^n)$, which will be the natural energy space for problem (2), as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^s_{q,a}} = \left([u]^2_{H_s} + \|u\|^2_{L^q_a} \right)^{1/2}. \tag{7}$$

Moreover, we denote by $H^s_{q,a,rad}(\mathbb{R}^n)$ the subspace of radial functions in $H^s_{q,a}(\mathbb{R}^n)$.

In a similar way, given a domain $\Omega \subset \mathbb{R}^n$ we denote by $H^s_{q,a,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^s_{q,a}(\mathbb{R}^n)$. And by $H^s_{q,a,0,rad}(\Omega)$ its subspace consisting of radial functions, when Ω is radially symmetric.

When $q = 2$ and $a = 0$, we have that $H^s_{q,a}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ (the usual fractional Sobolev space, sometimes also denoted by $W^{s,2}(\mathbb{R}^n)$, see [10]).

We recall that we have the following *fractional Sobolev inequality* (see for instance [10], theorem 6.5):

Theorem 2.1. *Let $0 < s < \min(\frac{n}{2}, 1)$ and define the Sobolev critical exponent*

$$2_s^* = 2^*(n, s) = \frac{2n}{n - 2s}. \tag{8}$$

Then, there exists a constant $C = C(n, s)$ such that

$$\|u\|_{L^{2_s^*}} \leq C\ [u]_{H^s} \quad \textit{for all} \ \ u \in H^s(\mathbb{R}^n).$$

Corollary 2.2. *Let $a \geq 0$ and $0 < s < \min(\frac{n}{2}, 1)$. Then*

$$H_{2,a}^s(\mathbb{R}^n) \subset H^s(\mathbb{R}^n).$$

Proof. It suffices to show that

$$\|u\|_{L^2} \leq C[u]_{H^s} \quad \ \forall\, u \in C_0^\infty(\mathbb{R}^n). \tag{9}$$

However,

$$\int_{|x|>1} u^2\, dx \leq \int_{|x|>1} |x|^a u^2\, dx$$

and

$$\int_{|x|\leq 1} u^2\, dx \leq C \left(\int_{|x|\leq 1} u^{2^*}\, dx \right)^{2/2^*} \leq C[u]_{H^s}.$$

Inequality (9) follows. □

Remark 2.3. *It is desirable to have a more concrete characterization of the functional space $H_{q,a,0}^s(\Omega)$. For this propose, we consider the fractional homogeneous Solev space*

$$\dot{H}^s(\mathbb{R}^n) = \{u \in L^{2_s^*}(\mathbb{R}^n) : [u]_{H^s} < \infty\}.$$

Then it is well known that $C_0^\infty(\mathbb{R}^n)$ is dense in $\dot{H}^s(\mathbb{R}^n)$ (This for instance the case $a = 0$ of theorem 1.1 in [12]*, which is established by the standard procedure of truncation and regularization by convolution with a standard molifier). Then,*

$$H_{q,a}^s(\mathbb{R}^n) = \dot{H}^s(\mathbb{R}^n) \cap L_a^q(\mathbb{R}^n) \ \textit{if} \ -n < a < n(q-1)$$

(We need the restrictions on a so that $|x|^a \in A_q$, the Muckenhoupt class of weights, and the regularization by convolution with a standard molifier works, see for example lemma 1.5 in [20]*). Likewise, if Ω is a ball*

$$H_{q,a}^s(\Omega) = \{u \in \dot{H}^s(\mathbb{R}^n) \cap L_a^q(\mathbb{R}^n) : u = 0 \ a.e \ \textit{in} \ \Omega^c\ \} \quad \textit{for} \ -n < a < n(q-1).$$

For some other similar density results for fractional Sobolev spaces in domains see [17].

Since we are going to use variational methods, it is natural to interpret our solutions in a weak sense:

Definition 2.4. *We say that $u \in H^s_{q,a,0}(\Omega)$ is a weak solution of*

$$\begin{cases} (-\Delta)^s u + |x|^a |u|^{q-2} u = |x|^b u^{p-2} u \quad \text{in } \Omega \\ u \equiv 0 \text{ in } \mathbb{R}^n - \Omega \end{cases}$$

provided that

$$C(n,s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))}{|x-y|^{n+2s}} \cdot (\varphi(x)-\varphi(y)) \, dx \, dy + \int_\Omega |x|^a |u|^{q-2} u \, \varphi \, dx$$
$$= \int_\Omega |x|^b |u|^{p-2} u \, \varphi \, dx$$

holds for any test function $\varphi \in C_0^\infty(\Omega)$.

Remark 2.5. *It is important to notice that different notions of fractional Laplacian in domains $\Omega \subset \mathbb{R}^n$, with different interpretations of the Dirichlet boundary conditions, have been defined in the literature, which should not be confused.*

i) *We may consider equations with the standard fractional Laplacian on $\mathbb{R}^n$ and require that the solutions vanish outside Ω. The associated* Dirilect form *is*

$$\mathcal{E}_{\mathbb{R}^n}(u,v) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y)) \cdot (v(x)-v(y))}{|x-y|^{n+2s}} \, dx \, dy.$$

In probabilistic terms, the associated stochastic process is the standard symmetric α-stable Lévy process (with $\alpha = 2s$), killed upon living Ω. This is the operator that we consider in definition 2.4 and theorem 1.1, and it has been also used for instance in [22] *(in even a more general form, called there the fractional p-Laplacian).*

ii) *A second option is to consider the so-called regional fractional Laplacian, with corresponds to the Dirichlet form*

$$\mathcal{E}_{\Omega}(u,v) = \frac{C(n,s)}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y)) \cdot (v(x)-v(y))}{|x-y|^{n+2s}} \, dx \, dy.$$

The associated stochastic process is the censored stable process, for which jumps outside Ω are completely forbidden [1].

iii) *Another approach is to consider the fractional powers of the Laplacian in Ω with Dirichlet conditions, defined from the spectral decomposition. This coincides with the operator obtained from the Caffarelli-Silvestre extension on a cylinder based in Ω (see* [2]*). The associated stochastic process is the subordinate killed Brownian motion studied in* [34].

We don't know if the analogue of theorem 1.1 holds for the fractional Laplacian in Ω in the sense of ii) or iii).

We shall need the following strong minimum principle for weak supersolutions, which can be found in [3]. We start by a definition:

Definition 2.6. *We say that $u \in H_0^{s,2}(\Omega)$ is a weak supersolution of*

$$(-\Delta)^s u = 0 \text{ in } \Omega \quad u \equiv 0 \text{ in } \Omega - \mathbb{R}^n \tag{10}$$

if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))}{|x-y|^{n+2s}} \, (\varphi(x) - \varphi(y)) \, dx \, dy \geq 0$$

for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$.

Then we may state the result:

Theorem 2.7 ([3], theorem A.1, case $p = 2$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, which is connected. Let $s \in (0,1)$ and $u \in H_0^s(\Omega)$ be a weak supersolution of* (10) *in the sense of the previous definition such that $u \geq 0$ in Ω. Let us suppose that*

$$u \not\equiv 0 \text{ in } \Omega.$$

Then $u > 0$ almost everywhere in Ω .

We shall need also the following elementary calculus lemma, whose proof is straightforward.

Lemma 2.8. *Consider the function $f : (0,\infty) \to \mathbb{R}$*

$$f(\lambda) = C_1 \, \lambda_1^e + C_2 \, \lambda^{-e_2}$$

where $C_1, C_2, e_1, e_2 > 0$. Then f archives its minimum at the point

$$\lambda_0 = \left(\frac{C_2 \, e_2}{C_1 \, e_1} \right)^{1/(e_1+e_2)}$$

and

$$f(\lambda_0) = C_1^{e_2/(e_1+e_2)} \, C_2^{e_1/(e_1+e_2)} \, k(e_1, e_2)$$

where $k(e_1, e_2)$ depends only on the exponents k_1 and k_2.

3. A generalization of Strauss Radial Lemma

In this section, we shall prove a version of Strauss radial lemma for the space $H^s_{q,a,rad}(\mathbb{R}^n)$, generalizing lemma 2.2 in [28].

Theorem 3.1. *Assume that $s > \frac{1}{2}$, and that $-(n-1) \leq a < n(q-1)$. Define the exponents*

$$\theta = \theta(s,q) = \frac{2}{2sq + 2 - q} \quad (0 < \theta < 1)$$

and

$$\sigma = \theta \, \frac{n-1}{2} + (1-\theta) \, \frac{n-1+a}{q} \; = \; \frac{2\,as + 2\,ns - a - 2\,s}{2\,qs - q + 2}.$$

For any radial function $u \in H^s_{q,a,rad}(\mathbb{R}^n)$, we have that

$$|u(x)| \leq C(n,s,q,a) \, |x|^{-\sigma} \, [u]_{H^s}^{\theta} \, \|u\|_{L^q_a}^{1-\theta}. \tag{11}$$

As a consequence, any function $u \in H^s_{q,a,rad}(\mathbb{R}^n)$ is equal a.e. to a continuous function in $\mathbb{R}^n - \{0\}$ and we have that

$$|u(x)| \leq C(n,s,q,a)\ |x|^{-\sigma}\ \|u\|_{H^s_{q,a}}. \tag{12}$$

For the proof we need the following lemmas.

Lemma 3.2 ([31],theorem 3.3 of chapter IV)**.** *Let $u \in L^1_{rad}(\mathbb{R}^n)$ be a radial function, $u(x) = u_0(|x|)$. Then its Fourier transform $\widehat{u}$ is also radial, and it is given by*

$$\widehat{u}(\omega) = (2\pi)^{n/2}|\omega|^{-\nu} \int_0^\infty u_0(r)\ J_\nu(r|\omega|)\ r^{n/2}\ dr$$

where $\nu = \frac{n}{2} - 1$ and J_ν denotes the Bessel function of order ν.

Remark 3.3. *If u is a radial function, in particular is even ($u(-x) = u(x)$). It follows than the inverse Fourier transform of u, coincides with the Fourier transform. In other words, for radial functions $u \in L^1_{rad}(R^n)$, we can write Fourier inversion formula as*

$$u(x) = (2\pi)^{n/2}|x|^{-\nu} \int_0^\infty (\widehat{u})_0(r)\ J_\nu(r|x|)\ r^{n/2}\ dr \tag{13}$$

Lemma 3.4 (Assymptotics of Bessel functions, [31], lemma 3.1 of chapter IV)**.** *If $\lambda > -1/2$, then*

$$|J_\lambda(r)| \leq C\ r^{-1/2}. \tag{14}$$

Lemma 3.5. *Let $\gamma > n - \frac{1}{p}$ and consider the operator*

$$S_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{(1+|x-y|^2)^{\gamma/2}}\ dy.$$

Then if $-\frac{n-1}{p} < \alpha < \frac{n}{p'}$, there exists $C > 0$ such that for any radial function f,

$$|S_\gamma f(x)| \leq C|x|^{-(n-1)/p-\alpha}\ \||x|^\alpha f\|_{L^p}.$$

Proof. This is a special case $\beta = -\frac{n-1}{p} - \alpha$, $q = \infty$ and $\tilde{p} = \tilde{q} = p$ of lemma 2.3 in [9]. □

Now we proceed to the proof of theorem 3.1.

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$. We choose a radial function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ in the ball $B(0,1)$. Moreover, we consider $\phi = \widehat{\psi}$ which will be a radial function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, and we use it to split u into low and high frequency parts, as

$$u(x) = h(x) + l(x)$$

where $l(x) = (u * \phi_t)(x)$ with $\phi_t(x) = t^{-n}\phi(x/t)$. We call l the low frequency part of u, since its Fourier transform

$$\widehat{l}(\omega) = \widehat{u}(\omega)\psi(t\omega)$$

is supported in the ball $B(0, 1/t)$. In a similar way,

$$\widehat{h}(\omega) = \widehat{u}(\omega)(1 - \psi(t\omega)) \tag{15}$$

is supported in the region $|\omega| \geq 1/t$.

For the high frequency part, h we use an estimate on the frequency side. Indeed, writing the Fourier inversion formula (13) for h, and using the bound (14) for the Bessel functions, we get that:

$$|h(x)| \leq C_n \, |x|^{-(n-1)/2} \int_0^\infty |(\widehat{h})_0(r)| \, r^{(n-1)/2} \, dr. \tag{16}$$

Hence, using the Cauchy-Schwarz inequality and (15), we see that

$$\begin{aligned}
|h(x)| &\leq C_n \, |x|^{-(n-1)/2} \left(\int_0^\infty |(\widehat{u})_0(r)|^2 \, r^{2s} r^{n-1} \, dr \right)^{1/2} \left(\int_0^\infty |1 - \psi_0(tr)|^2 \, r^{-2s} \, dr \right)^{1/2} \\
&\leq C_n \, |x|^{-(n-1)/2} \left(\int_{\mathbb{R}^n} |\widehat{u}(\omega)|^2 |\omega|^{2s} \, d\omega \right)^{1/2} \left(\int_{1/t}^\infty |1 - \psi_0(tr)|^2 \, r^{-2s} \, dr \right)^{1/2} \\
&\leq C_n \, |x|^{-(n-1)/2} \, t^{s-1/2} \, [u]_{H^s} \, \left(\int_1^\infty |1 - \psi_0(z)|^2 \, z^{-2s} \, dz \right)^{1/2} \\
&\leq C \, |x|^{-(n-1)/2} \, t^{s-1/2} \, [u]_{H^s}
\end{aligned}$$

as the last integral is finite, since $s > 1/2$ by hypothesis.

Now we consider the low frequency part, and we use an estimate on the space side. We fix $\gamma > (n - 1 + a)/q$. Then, since $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exist $C > 0$ such that

$$|\phi(x)| \leq C(1 + |x|^2)^{-\gamma/2}.$$

Then we estimate l as follows,

$$
\begin{aligned}
|l(x)| &\leq |u(x)| * |\phi_t(x)| \leq C \int_{\mathbb{R}^n} \frac{1}{t^n} \frac{|u(y)|}{\left(1+\left|\frac{x-y}{t}\right|\right)^{\gamma/2}} \, dy \\
&\leq C \int_{\mathbb{R}^n} \frac{|u(tz)|}{\left(1+\left|\frac{x}{t}-z\right|^2 \, dz\right)^{\gamma/2}} \, dz \quad (y = tz) \\
&\leq C S_\gamma(u_t)\left(\frac{x}{t}\right) \quad \text{where } u_t(z) = u(tz) \\
&\leq C \, \left|\frac{x}{t}\right|^{-(n-1+a)/q} \left(\int_{\mathbb{R}^n} |u(tz)|^q \, |z|^q \, dz\right)^{1/q} \quad \text{by lemma 3.5 (with } p = q, \alpha = a/q) \\
&\leq C \, t^{(n-1+a)/q} \, |x|^{-(n-1+a)/q} \left(\int_{\mathbb{R}^n} |u(y)|^q \, \left|\frac{y}{t}\right|^q \, \frac{dy}{t^n}\right)^{1/q} \\
&\leq C \, t^{-1/q} \, |x|^{-(n-1+a)/q} \, \|u\|_{L^q_a}.
\end{aligned}
$$

Therefore, collecting our estimates, we have that

$$
\begin{aligned}
|u(x)| &\leq |h(x)| + |l(x)| \\
&\leq C \, \left[\, |x|^{-(n-1)/2} \, t^{s-1/2} \, [u]_{H^s} + \, |x|^{-(n-1+a)/q} \, t^{-1/q} \, \|u\|_{L^q_a} \right].
\end{aligned}
$$

We choose t in order to minimize the right hand side of this expression, using lemma 2.8, with

$$
\begin{gathered}
e_1 = s - 1/2, \\
e_2 = 1/q, \\
C_1 = |x|^{-(n-1)/2} \, [u]_{H^s}, \\
C_2 = |x|^{-(n-1+a)/q} \, \|u\|_{L^q_a}.
\end{gathered}
$$

We obtain (11), and (12) follows immediately.

Now let $u \in H^s_{q,a,rad}(\mathbb{R}^n)$ and consider a sequence of radial functions (u_k) in $C_0^\infty(\mathbb{R}^n)$ that $u_k \to u$ in $H^s_{q,a,rad}(\mathbb{R}^n)$. Then,

$$
|u_k(x) - u_j(x)| \leq C \, |x|^{-\sigma} \, \|u_k - u_j\|_{H^s_{q,a}}
$$

It follows that (u_n) converges uniformly on compact subsets of $\mathbb{R}^n - \{0\}$ to a continuous function, which can be taken as a continuous representative of u, and which satisfies inequality (11). □

Remark 3.6. *When $q = 2$ and $a = 0$, we get the Strauss inequality for the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ (for $s > \frac{1}{2}$)*

$$
|u(x)| \leq C(n,s) \, |x|^{-(n-1)/2} \, [u]_{H^s}^{\frac{1}{2s}} \, \|u\|_{L^2}^{1-\frac{1}{2s}} \tag{17}
$$

and as a consequence

$$
|u(x)| \leq C(n,s) \, |x|^{-(n-1)/2} \, \|u\|_{H^s(\mathbb{R}^n)}. \tag{18}
$$

Indeed in this particular case, the above proof can be simplified. We can take $\psi(x)$ as the characteristic function of the unit ball (which is not C^∞, but *we don't need it for this alternative proof), and estimate the low frequency part l on the Fourier side, by using lemma 3.2, lemma 3.4 and Plancherel theorem (as we did before for h).*

$$\begin{aligned}
|l(x)| &\leq C_n\, |x|^{-(n-1)/2} \int_0^\infty |(\widehat{l})_0(r)|\, r^{(n-1)/2}\, dr \\
&\leq C_n\, |x|^{-(n-1)/2} \int_0^{1/t} |(\widehat{u})_0(r)|\, r^{(n-1)/2}\, dr \\
&\leq C_n\, |x|^{-(n-1)/2}\, t^{-1/2} \left(\int_0^{1/t} |(\widehat{u})_0(r)|^2\, r^{n-1}\, dr \right)^{1/2} \\
&\leq C_n\, |x|^{-(n-1)/2}\, t^{-1/2} \|\widehat{u}\|_{L^2} \\
&\leq C_n\, |x|^{-(n-1)/2}\, t^{-1/2} \|u\|_{L^2}
\end{aligned}$$

and we have arrived to the same estimate for l as before, but without using lemma 3.5. This is essentially the argument that we have used in [8] *to derive* (17)*. A technique similar to the ours, has been used by Y. Cho and T. Ozawa* [4] *to derive various related Sobolev inequalities with symmetries. However, this simpler approach does not work in the general case of theorem 3.1 (as Plancherel theorem is not available for the weighted L^q-norm).*

The inequality (18) *is a particular case of the results of W. Sickel and L. Skrzypczak* [33]*, who proved a version of Strauss lemma for Besov and Tribel-Lizorkin spaces, since $H^s(\mathbb{R}^n)$ coincides with the Tribel-Lizorkin space $F^s_{p,2}(\mathbb{R}^n)$. However, the above proof is much simpler than the proof in* [33]*, which is based on an atomic decomposition.*

Remark 3.7. *When $s = 1$, $H^s_{a,q}(\mathbb{R}^n)$ coincides with the space X considered by Sintzoff* [28]*, and as said before, our result (partially) extends lemma 2.2 in* [28] *to the fractional case $s > 1/2$.*

However, Sintzoff's proof (which is based like the original proof in [32] *in an integration by parts argument), does not have the restriction $a < (n-1)q$ (that comes from the use of lemma 3.5). We conjecture that this restriction is not necessary for theorem 3.1 to hold, but we have not been able to remove it.*

4. Hölder continuity estimates

In this section, we show that the same kind of estimates in the proof of theorem 3.1, can be used to obtain local Hölder continuity of the functions in $H^s_{q,a,rad}(\mathbb{R}^n)$. This fact will be useful later for proving compactness of the embedding of this space into weighted L^p spaces.

Theorem 4.1. *Let $\Omega_\varepsilon = \{x \in \mathbb{R}^n : |x| \geq \varepsilon\}$. Then the continuous representative of a function $u \in H^s_{q,a,rad}(\mathbb{R}^n)$ is Hölder continuous in Ω_ε, and* *moreover there exists a constant $C_\varepsilon > 0$ such that*

$$|u(x_1) - u(x_2)| \leq C_\varepsilon \, |x_1 - x_2|^\alpha \, \|u\|_{H^s_{q,a}}(\mathbb{R}^n) \tag{19}$$

with

$$\alpha = \frac{s - 1/2}{s + 1/q - 1/2}. \tag{20}$$

Proof. We use the same decomposition $u = h + l$ as in the proof of theorem 3.1. For the low frequency part l, we know hat $\nabla l(x) = u * \nabla(\phi_t)(x)$. Hence,

$$|\nabla l(x)| \leq C \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} |u(y)| \, \left| \nabla \phi \left(\frac{x-y}{t} \right) \right| \, dy.$$

Now all the partial derivatives $\frac{\partial \phi}{\partial x_i}$ are also in $\mathcal{S}(\mathbb{R}^n)$. Hence estimating this integral using lemma 3.5 as before, we get

$$|\nabla l(x)| \leq C \, t^{-1/q-1} \, |x|^{-(n-1+a)/q} \, \|u\|_{L^q_a}.$$

Recalling that l is radial, and using the mean value theorem, we get

$$|l_0(\rho_1) - l_0(\rho_2)| \leq C_\varepsilon \, t^{-1/q-1} \, |\rho_1 - \rho_2| \, \|u\|_{L^q_a} \quad \text{for all } \rho_1, \rho_2 \geq \varepsilon.$$

Now, for the high frequency part, using the estimates in the proof of theorem 3.1, we have that

$$|h_0(\rho_1) - h_0(\rho_2)| \leq |h(\rho_1)| + |h(\rho_2)| \leq C_\varepsilon \, t^{s-1/2} \, [u]_{H^s}$$

Collecting our estimates, we find that

$$|u_0(\rho_1) - u_0(\rho_2)| \leq C_\varepsilon \left[t^{s-1/2} \, [u]_{H^s} + t^{-1/q-1} \, |\rho_1 - \rho_2| \, \|u\|_{L^q_a} \right]$$

Choosing t to minimize the right hand side according to lemma 2.8 as before, gives

$$|u_0(\rho_1) - u_0(\rho_2)| \leq C_\varepsilon |\rho_1 - \rho_2|^\alpha \, [u]^\theta_{H^s} \| u|x|^{a/q} \|^{1-\theta}_{L^q}$$

with α given by (20), and the same θ as before. We conclude that

$$|u_0(\rho_1) - u_0(\rho_2)| \leq \; C_\varepsilon \, H^s_{q,a}(\Omega) \subset |\rho_1 - \rho_2|^\alpha \|u\|_{H^s_{q,a}}$$

and as a consequence, we obtain (19). □

5. Emebedding theorems

In this section, we prove some embedding theorems for spaces of radial functions. We remark that the power weight $|x|^b$ produces the presence of a shifted Sobolev critical exponent 2^*_b in all of them.

The following lemma is a special case of the result of [7], that we state here in our present notation for convenience of the reader, and generalizes a result due to Rother [25] (that corresponds to the case $s = 1$).

Lemma 5.1. *Let* $0 < s < \frac{n}{2}$, $c > -2s$, $c(1-2s) \leq 2s(n-1)$, $2 \leq q = 2_c^* := \frac{2(n+c)}{n-2s}$. *Then we have that*

$$\left(\int_{\mathbb{R}^n} |x|^c |u|^q \right)^{1/q} \leq C\ [u]_{H^s}$$

for any radial function $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Take $\gamma = n - s$, $p = 2$, $\alpha = 0$, $\beta = -c/q$ in the result of [7] (theorem 1.2), and recall that $T_{n-s}u = C(-\Delta)^{-s/2}u$, where T_γ denotes the fractional integral

$$T_\gamma u(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^\gamma}\, dy.$$

□

We start by proving a Gagliardo-Nirenberg type inequality.

Lemma 5.2. *Assume that* $\max(q, 2) < p < p < 2_b^* = \frac{2(n+b)}{n-2s}$, $a < n(q-1)$ *and*

$$a\,(p-2-2ps) + b\,(2qs-q+2) < 2s\,(p-q)\,(n-1) \tag{21}$$

Then, there exist $\eta = \eta(n, s, p, q, a, b)$ *with* $0 < \eta < 1$ *such that for any radial function* $u \in C_0^\infty(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |x|^b |u|^p\ dx \right)^{1/p} \leq C\ [u]_{H^s}^\eta\ \left(\int_{\mathbb{R}^n} |x|^a |u|^q\ dx \right)^{(1-\eta)/q}. \tag{22}$$

Proof. We devide the integral in $|x| \leq \lambda$, and $|x| \geq \lambda$. We define c by $p = \frac{2(n+c)}{n-2s}$. The hypothesis $p < 2_b^*$ gives that $c < b$. Moreover $c > -2s$ since $p > 2$. Then lemma 5.1, gives that

$$\int_{|x| \leq \lambda} |x|^b |u|^p\ dx \leq C\lambda^{b-c}\ [u]_{H^s}^p.$$

On the other hand, using theorem 3.1, we have that

$$\begin{aligned}
\int_{|x|>\lambda} |x|^b\ |u|^p\ dx &= \int_{|x|>\lambda} |x|^a\ |u|^q\ |x|^{b-a}\ |u|^{p-q} dx \\
&\leq C \int_{|x|>\lambda} |x|^a\ |u|^q\ |x|^{b-a-\sigma(p-q)}\ [u]_{H^s}^{\theta(p-q)} \|u\|_{L_a^q}^{(1-\theta)(p-q)}\ dx \\
&\leq C\lambda^{b-a-\sigma(p-q)}\ [u]_{H^s}^{\theta(p-q)}\ \|u\|_{L_a^q}^{(1-\theta)(p-q)+q}
\end{aligned}$$

with

$$\sigma = \frac{2\,as + 2\,ns - a - 2\,s}{2\,qs - q + 2}, \quad \theta = \frac{2}{2sq + 2 - q} \quad (0 < \theta < 1)$$

provided that $b - a < \sigma(p-q)$ which is equivalent to (21).

We collect our estimates

$$\int_{\mathbb{R}^n} |x|^b \; |u|^p \; dx \leq C \left[\lambda^{b-c} \; [u]^p_{H^s} + \lambda^{b-a-\sigma(p-q)} \; [u]^{\theta(p-q)}_{H^s} \; \|u\|^{(1-\theta)(p-q)+q}_{L^q_a} \right]$$

and optimize for λ using lemma 2.8, with

$$e_1 = b - c$$

$$e_2 = \sigma(p-q) - (b-a)$$

$$C_1 = [u]^p_{H^s}$$

$$C_2 = [u]^{\theta(p-q)}_{H^s} \; \|u\|^{(1-\theta)(p-q)+q}_{L^q_a}.$$

Hence we get (22) with

$$\eta = \frac{e_2}{e_1 + e_2} + \theta(1 - q/p) \left(\frac{e_1}{e_1 + e_2} \right).$$

□

Theorem 5.3. *If* $1 < q < p$, $2 < p < 2^*_b = \frac{2(n+b)}{n-2s}$, $a < n(q-1)$ *and*

$$a\,(p - 2 - 2ps) + b\,(2qs - q + 2) < 2s\,(p-q)\,(n-1) \tag{23}$$

then we have a continuous and compact embedding

$$H^s_{q,a,rad}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, |x|^b \; dx).$$

Proof. The continuity of the embedding follows from lemma 5.2. To prove the compactness, we shall follow the argument employed by Sintzoff in [28], with some modifications.

Since $H^s_{q,a}(\mathbb{R}^n)$ is reflexive, it is enough to see that for every given sequence (u_n) that converges weakly to 0 in $H^s_{q,a,rad}(\mathbb{R}^n)$, we have that $\|u_n\|_{L^p_b} \to 0$.

Since (u_n) is weakly convergent, (u_n) is bounded in $H^s_{q,a}(\mathbb{R}^n)$,

$$\|u_n\|_{H^s_{q,a}} \leq M.$$

Given $\varepsilon > 0$, we divide the domain $\mathbb{R}^n$ in three parts:

$$\|u_n\|^p_{L^p_b} = \int_{|x| \leq \lambda} |x|^b |u_n|^p \; dx + \int_{|x| \geq \frac{1}{\lambda}} |x|^b |u_n|^p \; dx + \int_{\lambda \leq |x| \leq \frac{1}{\lambda}} |x|^b |u_n|^p \; dx \tag{24}$$

where $\lambda = \lambda(\varepsilon)$ will be chosen later.

First term: As before, we define c by $p = \frac{2(n+c)}{n-2s}$, where $c < b$. Then by lemma 5.1, we deduce that

$$\int_{|x| \leq \lambda} |x|^b \; |u_n|^p \; dx \leq C \lambda^{b-c} [u_n]^p_{H^s} \leq C \lambda^{b-c} \; M^p < \frac{\varepsilon}{3} \text{ for } \lambda \geq \lambda_0(\varepsilon).$$

Second term: Again, we use theorem 3.1 in this part, recalling that $b-a<\sigma(p-q)$ is equivalent to (23):

$$\begin{aligned}\int_{|x|\geq\frac{1}{\lambda}}|x|^b|u_n|^p\,dx &= \int_{|x|>\lambda}|x|^a\,|u_n|^q\,|x|^{b-a}\,|u_n|^{p-q}dx\\ &\leq C\int_{|x|>\lambda}|x|^a\,|u|^q\,|x|^{b-a-\sigma(p-q)}[u_n]_{H^s}^{\theta(p-q)}\|u_n\|_{L^q_a}^{(1-\theta)(p-q)}\,dx\\ &\leq C\lambda^{b-a-\sigma(q-p)}\,M^{p-q}\\ &< \frac{\varepsilon}{3}\text{ for }\lambda\geq\lambda_1(\varepsilon).\end{aligned}$$

Third term: Finally, we fix $\lambda=\max(\lambda_0(\varepsilon),\lambda_1(\varepsilon))$.

Consider then, any subsequence (u_{n_k}) of (u_n). From theorems 3.1 and 4.1, (u_n) is equibounded and equicontinuous in $A_\lambda=\{x\in\mathbb{R}^n:\lambda\leq|x|\leq\frac{1}{\lambda}\}$. From the Arzela-Ascoli theorem, u_{n_k} admits a subsequence $u_{n_{k_j}}$ such that $u_{n_{k_j}}$ converges uniformly in A_λ.

Therefore, the whole sequence (u_n) converges uniformly to 0 in Ω_ε, and hence

$$\int_{A_\varepsilon}|x|^b|u_n|^p\,dx<\frac{\varepsilon}{3}\text{ if }n\geq n_0(\varepsilon)$$

and hence $\|u_n\|<\varepsilon$ if $n\geq n_0$.

Therefore $u_n\to 0$ in $L^p_b(\mathbb{R}^n)$, and we conclude the proof. □

Using this theorem with $q=2$, the Lagrange multiplier rule and the symmetric criticality principle [23] ($H^s_{2,a,rad}(\mathbb{R}^n)$ is the invariant subespace of $H^s_{2,a}(\mathbb{R}^n)$ under the action of the orthogonal group $O(n)$, which is a compact Lie group), we immediately get:

Corollary 5.4. *Let $2<p<2^*_b=\frac{2(n+b)}{n-2s}$, $a<n$ and*

$$a(p-2-2ps)+4bs<2s(p-2)(n-1).$$

Let us consider the minimization problem

$$m\,=\,\inf\left\{[u]^2_{H^s}+\int_{\mathbb{R}^n}|x|^a u^2\,dx\;:\;u\in H^s_{2,a,rad}(\mathbb{R}^n),\int_{\mathbb{R}^n}|x|^b(u^+)^p=1\right\}$$

Then $m=m(a,b,p,s)>0$ and is achieved. Hence, problem

$$(-\Delta)^s u+|x|^a u=|x|^b u^{p-1}\quad u\in H^s_{2,a,rad}(\mathbb{R}^n)$$

admits a non-negative radial solution.

Moreover, using symmetric versions of the Mountain pass theorem as in [28], applied to the functional

$$\Phi(u)=\frac{1}{2}\,[u]^2_{H^s}+\frac{1}{q}\,\|u\|^q_{L^q_a}-\frac{1}{p}\,\|u\|^p_{L^p_a}\quad u\in H^s_{q,a,rad}(\mathbb{R}^n)$$

we can also prove the following result:

Corollary 5.5. *Let $n \geq 3$, $q > 1$, $s > n/2$, $\max(2,q) < p < 2_b^* = \frac{2(n+b)}{n-2s}$ and*

$$a\,(p-2-2ps) + b\,(2qs-q+2) < 2s\,(p-q)\,(n-1)$$

Then the problem

$$(-\Delta)^s u + |x|^a |u|^{q-2} u = |x|^b u^{p-1} \quad u \in H^s_{2,a,rad}(\mathbb{R}^n)$$

admits a non negative radial solution, and infinitely many radial solutions u_k such that $\Phi(u_k) \to +\infty$

The proofs of these corollaries are exactly as in [28]. Hence, they are omitted.

6. Proof of the Symmetry Breaking Result

In this section, we prove theorem 1.1, on symmetry breaking for problem (4) in a large ball. For the proof, we need the following lemma which allows us to carry the arguments involving cut-off functions in the setting of the fractional Laplacian.

Lemma 6.1. *Let $0 < s < \min(1, \frac{n}{2})$ and $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 1$ on $(-\infty, 1/2)$, $0 \leq \eta \leq 1$ and $sup(\eta) \subset (-\infty, 1]$. Set*

$$\eta_R(x) = \eta(|x|/R)$$

Then, if $u \in H^s(\mathbb{R}^n)$, then

$$\eta_R \cdot u \to u \quad \text{in } H^s(\mathbb{R}^n) \text{ as } R \to \infty,$$

Moreover if $u \in H^s_{q,a}(\mathbb{R}^n)$ then

$$\eta_R \cdot u \to u \quad \text{in } H^s_{q,a}(\mathbb{R}^n) \text{ as } R \to \infty.$$

Proof. A simple proof of the first assertion is given in [24], which is based on a bounded convergence argument. The second assertion follows immediately, likewise (recall remark 2.3). □

Now we are ready to give the proof of the symmetry breaking result.

Proof. Following the idea of [28], we consider the minimization problems:

$$M(R) = \inf\left\{ [u]^2_{H^s} + \int_{B_R} |x|^a u^2 \, dx \ : \ u \in H^s_{2,a,0}(B_R), \int_{B_R} |x|^b (u^+)^p = 1 \right\}$$

$$m(R) = \inf\left\{ [u]^2_{H^s} + \int_{B_R} |x|^a u^2 \, dx \ : \ u \in H^s_{2,a,0,rad}(B_R), \int_{B_R} |x|^b (u^+)^p = 1 \right\}$$

and the corresponding minimization problems for $\mathbb{R}^n$

$$M(\infty) = \inf\left\{ [u]^2_{H^s} + \int_{\mathbb{R}^n} |x|^a u^2 \, dx \ : \ u \in H^s_{2,a}(\mathbb{R}^n), \int_{\mathbb{R}^n} |x|^b (u^+)^p = 1 \right\}$$

$$m(\infty) = \inf \left\{ [u]_{H^s}^2 + \int_{\mathbb{R}^n} |x|^a u^2 \, dx \ : \ u \in H^s_{2,a,rad}(\mathbb{R}^n), \int_{\mathbb{R}^n} |x|^b (u^+)^p = 1 \right\}$$

where the Gagliardo $[\cdot]_{H^s}$ is allways taken in $\mathbb{R}^n$, i.e. is given by (6) in both cases.

The main idea of the proof is to compare the minimum energy levels for those problems.

First we consider the minimization problems $m(R)$ and $M(R)$, for the ball B_R. We note that we have a continuous embedding

$$H^s_{2,a,0}(B_R) \hookrightarrow H^s_0(B_R)$$

hence $m(R)$ and $M(R)$ are well defined. Moreover since $p < 2^*$, by Rellich theorem, the embedding $H^s_{2,a,0}(B_R) \hookrightarrow L^p(B_R)$ is compact. It follows that $m(R)$ and $M(R)$ are achieved, and are positive for every $R > 0$. Moreover, since for every $u \in H^s_{2,a,0}(B_R)$, u^+ is in the same space (for checking that, the characterizations in remark 2.3 are useful) and we have that

$$[u^+]_{H^s} \leq [u]_{H^s},$$

we see that they have non-negative minimizer, which by the Lagrange multiplier rule (and the symmetric criticality principle in [23] in the case of $m(R)$) are weak solutions of in the sense of definition 2.4 of

$$\begin{cases} (-\Delta)^s u + |x|^a u = \lambda |x|^b u^{p-1} & \text{in } B_R \\ u \geq 0 \text{ a.e. in } B_R, \quad u \equiv 0 \text{ in } \mathbb{R}^n - B_R \end{cases} \tag{25}$$

for some λ. Using u itself as a test function, we see that $\lambda = M(R) > 0$ (respectively $\lambda = m(R) > 0$), so since $p \neq 2$, by replacing u by some positive multiple, we can get solutions of (25) with $\lambda = 1$. Moreover, by the strong minimum principle (theorem 2.7) those minimizers are strictly positive almost everywhere in B_R.

Next we consider the minimization problems $m(\infty)$ and $M(\infty)$ for the whole space $\mathbb{R}^n$. In a similar way as before, it follows from theorem 5.3 (with $q = 2$), that $m(\infty)$ is achieved and is positive. We remark that the condition $b - a < \sigma(n, s, 2, a)(p - 2)$ in that theorem is equivalent to (3).

However, we claim that $M(\infty) = 0$. To see that, choose $u \in C_0^\infty(B_1)$, $u \not\equiv 0$, $u \geq 0$ and consider the translated functions

$$u_t(x) = u(x - tv)$$

for a fixed unitary vector $v \in \mathbb{R}^n$. Then,

$$M(\infty) \leq \frac{[u_t]_{H_s}^2 + \int_{\mathbb{R}^n} |x|^a u_t^2 \, dx}{\left(\int_{\mathbb{R}^n} |x|^b u_t^p \, dx \right)^{2/p}}. \tag{26}$$

We have the following estimates

$$[u_t]_{H^s} = [u]_{H^s},$$

$$\int_{\mathbb{R}^n} |x|^a u_t^2 \, dx = \int_{B_1} |y+tv|^a u^2(y) \, dy \leq (1+t)^a \int_{B_1} u^2(y) \, dy,$$

$$\int_{\mathbb{R}^n} |x|^b u_t^p \, dx = \int_{B_1} |y+tv|^b u^p(y) \, dy \geq (t-1)^b \int_{B_1} u^p(y) \, dy.$$

Hence, since $2b > ap$, we see that $M(\infty) = 0$ by letting $t \to +\infty$ in (26).

We claim that

$$\lim_{R \to +\infty} M(R) = M(\infty) \ , \tag{27}$$

$$\lim_{R \to +\infty} m(R) = m(\infty) \ . \tag{28}$$

It is clear that $M(R) \geq M(\infty)$, $m(R) \geq m(\infty)$. On the other hand, let us prove (27): given $\varepsilon > 0$, choose $u \in H^s_{2,a}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |x|^b (u^+)^p = 1$$

and

$$E(u) := [u]^2_{H^s} + \int_{\mathbb{R}^n} |x|^a u^2 \, dx < M(R) + \varepsilon.$$

Consider the cut-off functions η_R constructed in lemma 6.1. Then, from that lemma,

$$E(\eta_R \cdot u) \to E(u) \text{ as } R \to +\infty.$$

Since $\eta_R \cdot u \in H^s_0(B_R)$, $M(R) \leq M(\eta_R u)$. We conclude that

$$M(R) \leq M(\infty) + \varepsilon \text{ for } R \geq R_0(\varepsilon).$$

This proves (27). The proof of (28) is similar.

It follows that if R is large enough, $M(R) < m(R)$, and hence for R large enough, the positive minimizers that we have found for $m(R)$ and $M(R)$ are different. Hence, problem (4) admits both a radial positive solution, and a non-radial one.

□

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References

[1] K. Bogdan, K. Burzy and Z.Q. Chen. *Censored Stable Process*. Probab. Theory Related Fields 127 (2003), no. 1, 89–152.

[2] C. Brändle, E. Colorado, and A. De Pablo. *A concave-vonvex elliptic problem involving the fractional Laplacian*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics - Volume 143 - Issue 01 - February (2013), 39–71

[3] L. Brasco and G. Franzina *Convexity properties of Dirichlet integral and Picone-type inequalities*. Preprint arXiv:1403.0280

[4] Y. Cho and T. Ozawa. *Sobolev inequalities with symmetry*. Commun. Contemp. Math. 11 (2009), no. 3, 355—365.

[5] L. Caffarelli and L. Silvestre *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245–1260.

[6] F. Catrina and Z. Wang *On the Caffarelli-Khon-Nirenberg Inequalities*. Communications on Pure and Applied Mathematics. Volume 54, Issue 2, 229—258, February (2001)

[7] I. Drelichman and R. G. Durán. *On weighted inequalities for fractional integrals of radial functions. P. L. De Nápoli*, Illinois Journal of Mathematics. Volume 55, Number 2 (2011), 575–587.

[8] P. L. De Nápoli and I. Drelichman. *Elementary proofs of embedding theorems for potential spaces of radial functions*. Preprint 2014, arXiv:1404.7468.

[9] P. D'Ancona and R. Luca. *Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities with angular integrability*. J. Math. Anal. Appl. 388 (2012), no. 2, 1061–1079.

[10] E. Di Nezza, G. Palatucci and E. Valdinoci *Hitchhiker's guide to the fractional Sobolev spaces.* Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[11] S, Dipierro, G. Palatucci and E. Valdinoci *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian* Le Matematiche (Catania), Vol. 68 (2013), no. 1.

[12] S. Dipierro and E. Valdinoci, *A density property for fractional weighted Sobolev spaces.* arXiv preprint arXiv:1501.04918. (2015)

[13] P. Felmer, A. Quaas and J. Tan *Positive solutions of Nonlinear Schrödinger equations with the Fractional Laplacian*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics. Volume 142, Issue 06, December (2012), pp 1237-1262

[14] P. Felmer and Y. Wang. *Radial symmetry of positive solutions involving the fractional Laplacian* Commun. Contemp. Math. 16, 1350023 (2014).

[15] J. Fernández Bonder, E. J. Lami Dozo and J. D. Rossi. *Symmetry properties for the extremals of the Sobolev trace embedding*. Annales de l'Institut Henri Poincare. Analyse Non Lineaire. Vol. 21 (6), 795–805, (2004).

[16] R. L. Frank, E. Lenzmann and L. Silvestre *Uniqueness of radial solutions for the fractional Laplacian* Preprint arxiv arXiv:1302.2652.

[17] A. Fiscella, R. Servadei and E. Valdinoci. *Density properties for fractional Sobolev spaces*. Ann. Acad. Sci. Fenn. Math, 40(1), 235–253. (2015)

[18] B. Gidas, W.M. Ni and L. Nirenberg, *Symmetry and related propreties via the maximum principle*, Comm. Math. Phys., 68 (1979), 209-243.

[19] H-M He and J-Q Chen. *On the existence of Solutions to a class of p-Laplace Elliptic Equations.*Volume 10 (2009), Issue 2, Article 59, 8 pp.

[20] T. Kilpeläinen. *Weighted Sobolev spaces and capacity*. Ann. Acad. Sci. Fenn. Ser. AI Math, 19(1), 95-113. (1994)

[21] E. J. Lami Dozo and O. Torné *Symmetry and symmetry breaking for minimizers in the trace inequality*. Commun. Contemp. Math. 07, 727 (2005).

[22] E. Lindgren and P. Lindqvist. *Fractional eigenvalues* Calc. Var. (2014) 49:795–826.

[23] R. S. Palais. *The principle of Symmetric Criticality*. Comm. Math. Phys. 69 (1979), pp. 19-30.

[24] J. F. Raman *On Some properties of fractional Sobolev Spaces* Annals of University of Craiova, Math. Comp. Sci. Ser. Volume 30(1), 2003, Pages 1–2

[25] W. Rother. *Some existence theorems for the equation* $-\Delta u + K(x)u^p = 0$. Comm. Partial Diff. Eq. 15 (1990), pp. 1461-1473.

[26] S. Secchi. *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^n$. J. Math. Phys. 54 (2013), no. 3, 031501 , 17 pp.

[27] S. Secchi, *On fractional Schrödinger equations in RN without the Ambrosetti-Rabinowitz condition*. Preprint arXiv:1210.0755

[28] P. Sintzoff, *Symmetry and singularities for some semilinear elliptic problems*. Ph.D. Thesis Université Catolique de Louvain, (2005). The results have been published in *Symmetry of solutions of a semilinear elliptic equation with unbounded coefficients* Differential Integral Equations Volume 16, Number 7 (2003), 769–786.

[29] E. Serra *Non radial positive solutions for the Hénon equation with critical growth* Calc. Var. Partial Differential Equations, 23 (2005), pp. 301—326

[30] D. Smets, J. Su, M. Willem *Non-radial ground states for the Hénon equation*. Comm. Contemp. Math., 4 (2002), pp. 467–480

[31] E.M. Stein , G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, 1971.

[32] W. A. Strauss. Existence of Solitary Waves in Higher Dimensions. Comm. Math. Phys. 55 (1977), 149-162.

[33] W. Sickel and L. Skrzypczak. *Radial subspaces of Besov and Lizorkin-Triebel classes: Extended strauss lemma and compactness of embeddings*. J. Fourier Anal. Appl. Volume 6, issue 6, (2000), pp. 639–662.

[34] R. Song and Z. Vondracek. *Potential Theory of Subordinate Killed Brownian Motion in a Domain*. Probab. Theory Related Fields 125 (2003), no. 4, 578–592.

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