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The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information

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ABSTRACT
In this paper, we shall present some novel Dice similarity measures of hesitant fuzzy linguistic term sets and the generalized Dice similarity measures of hesitant fuzzy linguistic term sets and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Then, we propose the generalized Dice similarity measures-based multiple attribute decision making models with hesitant fuzzy linguistic term sets. Finally, a practical example concerning the evaluation of the quality of movies is given to illustrate the applicability and advantage of the proposed generalized Dice similarity measures.

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Multiple attribute decision making; generalized dice similarity measures; dice similarity measures; hesitant fuzzy linguistic term sets; asymmetric measures; projection measures; quality of movies

Introduction
Multiple attribute decision making problems under linguistic information processing environment is an interesting research topic having received more and more attention during the last several years (Beg & Rashid, 2015; Dutta & Guha, 2015; Herrera, Herrera-Viedma 2000a-b; Herrera, Martínez 2001a-b; Herrera, Martínez, & Sánchez, 2005; Hu, Rao, Zheng, & Huang, 2015; Martínez-López, Rodríguez, R, & Herrera, 2015; Rao, Zheng, Wang, & Xiao, 2016; Wu et al. 2015; Zhang & Chu, 2009; Zhang & Liu, 2010). Herrera and Martínez (2001a) show 2-tuple linguistic information processing manner can effectively avoid the loss and distortion of information. Herrera, Herrera-Viedma (2000a) developed 2-tuple arithmetic average (TAA) operator, 2-tuple weighted average (TWA) operator, 2-tuple ordered weighted average (TOWA) operator and extended 2-tuple weighted average (ET-WA) operator. Herrera-Viedma, Martínez, Mata, and Chiclana (2005) proposed the consensus support system with multi-granular linguistic information. Herrera et al. (2005) presented the group decision making model for managing non-homogeneous information. Herrera, Herrera-Viedma, and Martínez (2008) developed the fuzzy linguistic model...

The similarity measure is one of the important and useful tools for degree of similarity between objects (Hung, 2012; Hung & Yang, 2004; Hung & Yang, 2007; Li, Olson, & Zheng, 2007; Liao, Xu, & Zeng, 2014; Liao & Xu, 2015; Liu, 2005; Rajarajeswari & Uma, 2013; Shi & Ye, 2013; Singh, 2014; Su, Xu, Liu, & Liu, 2015; Szmidt, 2014; Szmidt & Kacprzyk, 2000; Tian, 2013; Wei et al., 2017a, 2018a; Wei & Gao, 2018; Wei & Wei, 2018; Xu & Xia, 2010; Ye, 2011, 2016b, 2017). Functions expressing the degree of similarity of items or sets are used in physical anthropology, automatic classification, ecology, psychology, citation analysis, information retrieval, patterns recognition and numerical taxonomy (Ye, 2012a). The degree of similarity or dissimilarity between the objects under study plays an important role. In vector space, especially the Jaccard, Dice, and cosine similarity measures (Dice, 1945; Jaccard, 1901; Salton & McGill, 1987) are often used in information retrieval, citation analysis, and automatic classification. Therefore, Ye (2012a) proposed the Jaccard, Dice, and cosine similarity measures between trapezoidal intuitionistic fuzzy numbers (TIFNs) and applied them to group decision-making problems. Ye (2012b) proposed the
multi-criteria decision making models by using the Dice similarity measure between expected intervals of trapezoidal fuzzy numbers. Ye (2012c) investigated the multicriteria decision-making method by using the Dice similarity measure based on the reduct intuitionistic fuzzy sets of IVIFSs. Ye (2014) developed the Dice measures for simplified neutrosophic sets. Ye (2016a) proposed the generalized Dice measures for multiple attribute decision making under simplified neutrosophic environments.

However, these Dice similarity measures do not deal with the similarity measures for hesitant fuzzy linguistic term sets (HFLTSs) directly. Therefore, it is necessary to extend the Dice measure to HFLTSs to handle patterns recognition, citation analysis, information retrieval and multiple attribute decision making problems to satisfy the requirements of decision makers’ preference and flexible decision making. In order to do so, the main purposes of this paper are: 1) to propose two forms of the Dice measures of HFLTSs, 2) to present the generalized Dice measures of HFLTSs, and 3) to develop the generalized Dice measures multiple attribute decision making (MADM) methods of HFLTSs. In the MADM process, the main advantage of the proposed methods is more general and more flexible than existed MADM methods with HFLTSs to satisfy the practical requirements.

In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to HFLTSs. In Section 3, we shall propose some Dice similarity measure and some weighted Dice similarity measure between HFLTSs. In Section 4, we propose the generalized Dice similarity measures-based MADM models with HFLTSs in Section 5, an illustrative example is given to demonstrate the efficiency of the similarity measures for concerning the evaluation of the quality of movies. Section 6 concludes the paper with some remarks.

Preliminaries

Let \( S = \{ s_i | i = -\tau, ..., -1, 0, 1, ..., \tau \} \) be a linguistic term set with odd cardinality. Any label, \( s_i \), represents a possible value for a linguistic variable, and it should satisfy the following characteristics (Herrera, Martínez & Sánchez, 2005; Herrera & Martínez, 2000a, 200b; 2001a-b; Xu, 2004a, 2006):

1. The set is ordered: \( s_i > s_j \), if \( i > j \); (2) Max operator: \( \max(s_i, s_j) = s_i \), if \( s_i \geq s_j \); (3) Min operator: \( \min(s_i, s_j) = s_i \), if \( s_i \leq s_j \). For example, \( S \) can be defined as

\[
S = \{ s_{-3} = \text{extremely poor}, s_{-2} = \text{very poor}, s_{-1} = \text{poor}, s_0 = \text{medium}, s_1 = \text{good}, s_2 = \text{very good}, s_3 = \text{extremely good} \}
\]

Hesitant fuzzy sets, which permit the membership degree of an element to a reference set represented by several possible values, is a powerful structure in reflecting a decision maker’s hesitance.

**Definition 1** (Torra, 2010). Given a fixed set \( X \), then a hesitant fuzzy sets (HFSs) on \( X \) is in terms of a function that when applied to \( X \) returns a subset of \([0, 1]\). A hesitant fuzzy set (HFS) can be expressed the HFS by the mathematical symbol:
\[ H_E = \{ (x, h_E(x)) | x \in X \}, \]  

where \( h_E(x) \) is a set of some values in \([0, 1]\), denoting the possible membership degree of the element \( x \in X \) to the set \( E \). For convenience, Xia and Xu (2011) call \( h = h_E(x) \) a hesitant fuzzy element (HFE) and \( H_E \) the set of all HFEs.

Similar to the situations of HFSs where a decision maker may hesitate between several possible values as the membership degree when evaluating an alternative, in a qualitative circumstance, a decision maker may hesitate between several terms to assess a linguistic variable. Hence, motivated by the idea of HFSs, Rodríguez, Martínez, and Herrera (2012) introduced the hesitant fuzzy linguistic term set (HFLTS), whose envelope is an uncertain linguistic variable (Xu et al., 2014).

**Definition 2** (Rodríguez et al., 2012). Let \( S = \{ s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau \} \) be a linguistic term set, a hesitant fuzzy linguistic term sets (HFLTSs), \( H_S \), is an ordered finite subset of the consecutive linguistic terms of \( S \).

Let \( S = \{ s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau \} \) be a linguistic term set. The HFLTS \( H_S \) for a linguistic variable \( \nu \in S \) can then be represented mathematically as \( H_S(\nu) \). For the convenience of statement, we call \( \eta = \{ H_S(\nu) | \nu \in S \} \) a set of HFLTSs. The aim of introducing HFLTS is to improve the elicitation of linguistic information, mainly when decision makers hesitate between several values in assessing linguistic variables. Linguistic information, which is more similar to the decision makers’ expressions, is semantically represented by HFLTS and generated by a context-free grammar (Rodríguez et al., 2012).

**Definition 3** (Rodríguez et al., 2012). For three HFLTSs \( H_S, H^1_S \) and \( H^2_S \), the following operations are defined:

1. Lower bound: \( h_S^\downarrow = \min(s_i) = s_j, s_i \in h, \text{and } s_i \geq s_j, \forall i \);
2. Upper bound: \( h_S^\uparrow = \max(s_i) = s_j, s_i \in h, \text{and } s_i \leq s_j, \forall i \);
3. Complement operation: \( H_S^c = S - H_S = \{ s_i | s_i \in S \text{ and } s_i \notin H_S \} \).
4. Union operation: \( H_S^1 \cup H_S^2 = \{ s_i | s_i \in H_S^1 \text{ or } s_i \in H_S^2 \} \).
5. Intersection operation: \( H_S^1 \cap H_S^2 = \{ s_i | s_i \in H_S^1 \text{ and } s_i \in H_S^2 \} \).

**Some dice similarity measure for hesitant fuzzy linguistic information**

The Dice similarity measure can’t induce this undefined situation when one vector is zero, which overcomes the disadvantage of the cosine similarity measure (Dice, 1945). Therefore, the concept of the Dice similarity measure is introduced in the section.

**Definition 4** (Dice, 1945). Let \( X = (x_1, x_2, \ldots, x_n) \) and \( Y = (y_1, y_2, \ldots, y_n) \) be two vectors of length \( n \) where all the coordinates are positive real numbers. Then the Dice similarity measure is defined as follows:

\[
D(X, Y) = \frac{2X \cdot Y}{\|X\|_2^2 + \|Y\|_2^2} = \frac{2 \sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} (x_j)^2 + \sum_{j=1}^{n} (y_j)^2} \]  

\( (2) \)
where $X \cdot Y = \sum_{j=1}^{n} x_j y_j$ is called the inner product of the vector $X$ and $Y$ and $\|X\|_2 = \sqrt{\sum_{j=1}^{n} (x_j)^2}$ and $\|Y\|_2 = \sqrt{\sum_{j=1}^{n} (y_j)^2}$ are the Euclidean norms of $X$ and $Y$ (also called the $L_2$ norms).

The Dice similarity measure takes value in the interval $[0, 1]$. However, it is undefined if $x_j = y_j = 0 (j = 1, 2, ..., n)$. In this case, let the Dice measure value be zero when $x_j = y_j = 0 (j = 1, 2, ..., n)$. 

**Dice similarity measure for hesitant fuzzy linguistic information**

In this section, we shall propose some Dice similarity measure and some weighted Dice similarity measure for hesitant fuzzy linguistic information.

**Definition 5.** Let $S = \{s_t|t = -\tau, ..., -1, 0, 1, ..., \tau\}$ be a linguistic term set. For two HFLTSs $H^1_S = \{<x_j, h^1_S(x_j)>|x_j \in X\}$ and $H^2_S = \{<x_j, h^2_S(x_j)>|x_j \in X\}$ with $h^k_S(x_j) = \{s^l_{h_j}(x_j)|s^l_{h_j}(x_j) \in S, l = 1, 2, ..., L, j = 1, 2, ..., n\}$, $k = 1, 2$, a Dice similarity measure between HFLTSs $H^1_S$ and $H^2_S$ is proposed as follows:

$$D^1(H^1_S, H^2_S) = \frac{1}{n} \sum_{j=1}^{n} \frac{2 \left( \sum_{l=1}^{L_1} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)}{\left( \sum_{l=1}^{L_1} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)^2 + \left( \sum_{l=1}^{L_2} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)^2} \quad (3)$$

The Dice similarity measure between HFLTSs $H^1_S$ and $H^2_S$ also satisfies the following properties:

1. $0 \leq D^1(H^1_S, H^2_S) \leq 1$;
2. $D^1(H^1_S, H^1_S) = D^1(H^2_S, H^1_S)$;
3. $D^1(H^1_S, H^2_S) = 1$, if $H^1_S = H^2_S$, i.e. $s^l_{h_j}(x_j) = s^l_{h_j}(x_j), j = 1, 2, ..., n$.

**Proof.**

1. Let us consider the $j$th item of the summation in Eq.(3).

$$D^1(H^1_S(x_j), H^2_S(x_j)) = \frac{2 \left( \sum_{l=1}^{L_1} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)}{\left( \sum_{l=1}^{L_1} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)^2 + \left( \sum_{l=1}^{L_2} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)^2}$$

It is obvious that $D^1(H^1_S(x_j), H^2_S(x_j)) \geq 0$, and

$$\left( \sum_{l=1}^{L_1} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)^2 + \left( \sum_{l=1}^{L_2} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)^2 \geq 2 \left( \sum_{l=1}^{L_1} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^l_{h_j}(x_j)|}{2\tau+1} \right)$$

according to the inequality $a^2 + b^2 \geq 2ab$. Thus, $0 \leq D^1(H^1_S(x_j), H^2_S(x_j)) \leq 1$. 


From Eq.(3), the summation of \( n \) terms is \( 0 \leq D^1(H^1_S, H^2_S) \leq 1 \).

1. It is obvious that the proposition is true.
2. When \( H^1_S = H^2_S \), there are \( s_{\delta j} (x_j) = s_{\delta i} (x_j) \), for \( j = 1, 2, \ldots, n \). So, there is

\[
D^1(H^1_S, H^2_S) = \frac{1}{n} \sum_{j=1}^{n} \frac{2 \left( \sum_{i=1}^{L_1} \frac{|\delta_i^1 (x_j)|}{2 \tau + 1} \cdot \sum_{i=1}^{L_2} \frac{|\delta_i^2 (x_j)|}{2 \tau + 1} \right)^2}{\left( \sum_{i=1}^{L_1} \frac{|\delta_i^1 (x_j)|}{2 \tau + 1} \right)^2 + \left( \sum_{i=1}^{L_2} \frac{|\delta_i^2 (x_j)|}{2 \tau + 1} \right)^2}.
\]

Therefore, we have finished the proofs.

If we consider the weights \( x_j \), a weighted Dice similarity measure between HFLTSs \( H^1_S \) and \( H^2_S \) is proposed as follows:

\[
WD^1(H^1_S, H^2_S) = \sum_{j=1}^{n} \omega_j \frac{2 \left( \sum_{i=1}^{L_1} \frac{|\delta_i^1 (x_j)|}{2 \tau + 1} \cdot \sum_{i=1}^{L_2} \frac{|\delta_i^2 (x_j)|}{2 \tau + 1} \right)^2}{\left( \sum_{i=1}^{L_1} \frac{|\delta_i^1 (x_j)|}{2 \tau + 1} \right)^2 + \left( \sum_{i=1}^{L_2} \frac{|\delta_i^2 (x_j)|}{2 \tau + 1} \right)^2} \tag{4}
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( x_j (j = 1, 2, \ldots, n) \), with \( w_j \in [0, 1], i = 1, 2, \ldots, n, \sum_{j=1}^{n} w_j = 1 \). In particular, if \( \omega = (1/n, 1/n, \ldots, 1/n)^T \), then the weighted Dice similarity measure reduces to Dice similarity measure. That is to say, if we take \( \omega_j = \frac{1}{n}, j = 1, 2 \ldots, n \), then there is \( WD^1(H^1_S, H^2_S) = D^1(H^1_S, H^2_S) \).

Obviously, the weighted Dice similarity measure of between two HFLTSs \( H^1_S \) and \( H^2_S \) also satisfies the following properties:

1. \( 0 \leq WD^1(H^1_S, H^2_S) \leq 1 \);
2. \( WD^1(H^1_S, H^2_S) = WD^1(H^2_S, H^1_S) \);
3. \( WD^1(H^1_S, H^2_S) = 1 \), if \( H^1_S = H^2_S \), i.e. \( s_{\delta j} (x_j) = s_{\delta i} (x_j) \), \( j = 1, 2, \ldots, n \).

Similar to the previous proof method, we can prove the above three properties.
Another form of the dice similarity measure for hesitant fuzzy linguistic information

In this section, we shall develop another form of Dice similarity measure for hesitant fuzzy linguistic information, which is defined as follows:

**Definition 6.** Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau \}$ be a linguistic term set. For two HFLTs $H^1_S = \{(x_j, h^1_S(x_j)) | x_j \in X \}$ and $H^2_S = \{(x_j, h^2_S(x_j)) | x_j \in X \}$ with $h^2_S(x_j) = \{s^j_{\alpha_i}(x_j) | s^j_{\alpha_i}(x_j) \in S, \alpha_i = 1, 2, \ldots, L_i, k = 1, 2, \ldots, n \}$, a Dice similarity measure between HFLTs $H^1_S$ and $H^2_S$ is proposed as follows:

$$D^2(H^1_S, H^2_S) = \frac{\sum_{j=1}^{n} 2 \left( \sum_{l=1}^{L_1} |\delta^1_l(x_j)| \cdot \sum_{l=1}^{L_2} |\delta^2_l(x_j)| \right)}{\sum_{j=1}^{n} \left( \left( \sum_{l=1}^{L_1} |\delta^1_l(x_j)|^2 \right) + \left( \sum_{l=1}^{L_2} |\delta^2_l(x_j)|^2 \right) \right)}$$

(5)

The Dice similarity measure between HFLTs $H^1_S$ and $H^2_S$ also satisfies the following properties:

1. $0 \leq D^2(H^1_S, H^2_S) \leq 1$;
2. $D^2(H^1_S, H^1_S) = D^2(H^2_S, H^1_S)$;
3. $D^2(H^1_S, H^1_S) = 1$, if $H^1_S = H^2_S$, i.e. $s^j_{\alpha_i}(x_j) = s^j_{\alpha_i}(x_j)$, $j = 1, 2, \ldots, n$.

Similar to the previous proof method, we can prove the above three properties.

If we consider the weights $x_j$, a weighted Dice similarity measure between HFLTs $H^1_S$ and $H^2_S$ is proposed as follows:

$$WD^2(H^1_S, H^2_S) = \frac{\sum_{j=1}^{n} 2\omega_j \left( \sum_{l=1}^{L_1} |\delta^1_l(x_j)| \cdot \sum_{l=1}^{L_2} |\delta^2_l(x_j)| \right)}{\sum_{j=1}^{n} \omega_j \left( \left( \sum_{l=1}^{L_1} |\delta^1_l(x_j)|^2 \right) + \left( \sum_{l=1}^{L_2} |\delta^2_l(x_j)|^2 \right) \right)}$$

(6)

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j (j = 1, 2, \ldots, n)$, with $w_j \in [0, 1], i = 1, 2, \ldots, n, \sum_{j=1}^{n} w_j = 1$. In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the weighted Dice similarity measure reduces to Dice similarity measure. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1, 2 \cdots, n$, then there is $WD^2(\phi, \varphi) = D^2(\phi, \varphi)$.

Obviously, the weighted Dice similarity measure of between two HFLTs $H^1_S$ and $H^2_S$ also satisfies the following properties:

1. $0 \leq WD^2(H^1_S, H^2_S) \leq 1$;
2. $WD^2(H^1_S, H^1_S) = WD^2(H^2_S, H^1_S)$;
3. $WD^2(H^1_S, H^1_S) = 1$, if $H^1_S = H^2_S$, i.e. $s^j_{\alpha_i}(x_j) = s^j_{\alpha_i}(x_j)$, $j = 1, 2, \ldots, n$. 

The generalized dice similarity measure for hesitant fuzzy linguistic information

In this section, we develop the generalized Dice similarity measure between two HFLTSs $H^1_S$ and $H^2_S$. As the generalization of the Dice similarity measure, the generalized Dice similarity measures between two HFLTSs $H^1_S$ and $H^2_S$ are defined below.

**Definition 7.** Let $\phi = \{ (\gamma_1, x_1), (\gamma_2, x_2), \ldots, (\gamma_n, x_n) \}$ and $\varphi = \{ (\eta_1, \beta_1), (\eta_2, \beta_2), \ldots, (\eta_n, \beta_n) \}$ be two groups of HFLTSs, a generalized Dice similarity measure between two HFLTSs $H^1_S$ and $H^2_S$ is proposed as follows:

\[
GD^1(H^1_S, H^2_S) = \frac{1}{n} \sum_{j=1}^{n} \frac{\left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)} {\lambda \left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2}
\]

(7)

\[
GD^2(H^1_S, H^2_S) = \frac{1}{n} \sum_{j=1}^{n} \frac{\left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)} {\lambda \sum_{j=1}^{n} \left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + (1 - \lambda) \sum_{j=1}^{n} \left( \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2}
\]

(8)

where $\lambda$ is a positive parameter for $0 \leq \lambda \leq 1$.

Then, the generalized Dice similarity measure includes some special cases by altering the parameter value $\lambda$.

If $\lambda = 0.5$, the two generalized Dice similarity measures (7) and (8) reduced to Dice similarity measures (3) and (5):

\[
GD^1(H^1_S, H^2_S) = \frac{1}{n} \sum_{j=1}^{n} \frac{\left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)} {0.5 \left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + 0.5 \left( \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2}
\]

(9)

\[
GD^2(H^1_S, H^2_S) = \frac{1}{n} \sum_{j=1}^{n} \frac{\left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)} {2 \left( \sum_{l=1}^{L^1_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + \left( \sum_{l=1}^{L^2_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2}
\]
Asymmetric similarity measures, respectively:

\[
GD^2(H_1^k, H_2^k) = \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2
\]

\[
\lambda \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + (1 - \lambda) \sum_{j=1}^{n} \left( \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2
\]

\[
= \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + \sum_{j=1}^{n} \left( \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2
\]

If \( \lambda = 0, 1 \), the two generalized Dice similarity measures reduced to the following asymmetric similarity measures, respectively:

\[
GD^1(H_1^k, H_2^k) = \frac{1}{n} \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)
\]

\[
\lambda \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)
\]

\[
0 \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \right)^2 + (1 - 0) \left( \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^1_l(x_j)|}{2\tau + 1} \cdot \sum_{l=1}^{L_2} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)
\]

\[
\frac{1}{n} \sum_{j=1}^{n} \left( \sum_{l=1}^{L_1} \frac{|\delta^2_l(x_j)|}{2\tau + 1} \right)^2, \text{for } \lambda = 0.
\]
\[
\frac{1}{n} \sum_{j=1}^{n} \frac{GD^1(H_1^1, H_2^1)}{\lambda \left( \sum_{l=1}^{L_1} |\delta_l^1(x_j)|^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} |\delta_l^2(x_j)|^2 \right) \right)}
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} \frac{GD^2(H_1^1, H_2^1)}{\lambda \left( \sum_{l=1}^{L_1} |\delta_l^1(x_j)|^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} |\delta_l^2(x_j)|^2 \right) \right)}, \text{ for } \lambda = 1.
\]
\[
\frac{1}{n} \sum_{j=1}^{n} \frac{GD^1(H_1^2, H_2^2)}{\left( \sum_{l=1}^{L_1} |\delta_l^1(x_j)|^2 + \sum_{l=1}^{L_2} |\delta_l^2(x_j)|^2 \right)^2}
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} \frac{GD^2(H_1^2, H_2^2)}{\left( \sum_{l=1}^{L_1} |\delta_l^1(x_j)|^2 + \sum_{l=1}^{L_2} |\delta_l^2(x_j)|^2 \right)^2}, \text{ for } \lambda = 0.
\]
\[
GD^2(H^1_S, H^2_S) = \frac{\sum_{j=1}^{n} \left( \frac{\delta^1_j(x_j)}{2\tau + 1} \cdot \frac{\delta^2_j(x_j)}{2\tau + 1} \right)}{\lambda \sum_{j=1}^{n} \left( \frac{\delta^1_j(x_j)}{2\tau + 1} \right)^2 + (1 - \lambda) \sum_{j=1}^{n} \left( \frac{\delta^2_j(x_j)}{2\tau + 1} \right)^2}
\]

\[
= \frac{\sum_{j=1}^{n} \left( \frac{\delta^1_j(x_j)}{2\tau + 1} \cdot \frac{\delta^2_j(x_j)}{2\tau + 1} \right)}{\sum_{j=1}^{n} \left( \frac{\delta^1_j(x_j)}{2\tau + 1} \right)^2 + \sum_{j=1}^{n} \left( \frac{\delta^2_j(x_j)}{2\tau + 1} \right)^2}, \quad \text{for } \lambda = 1.
\]

From above analysis, it can be seen that the above four asymmetric similarity measures are the extension of the relative projection measure of the HFLTSs.

In many situations, the weight of the elements \(x_j \in X\) should be taken into account. For example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. Thus, we further propose the following two weighted generalized Dice similarity measures for HFLTSs, respectively, as follows:

\[
WGD^1(H^1_S, H^2_S) = \sum_{j=1}^{n} \omega_j \frac{\left( \frac{\delta^1_j(x_j)}{2\tau + 1} \cdot \frac{\delta^2_j(x_j)}{2\tau + 1} \right)}{\lambda \left( \sum_{j=1}^{n} \frac{\delta^1_j(x_j)}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{j=1}^{n} \frac{\delta^2_j(x_j)}{2\tau + 1} \right)^2}
\]

\[
WGD^2(H^1_S, H^2_S) = \sum_{j=1}^{n} \omega_j^2 \frac{\left( \sum_{j=1}^{n} \frac{\delta^1_j(x_j)}{2\tau + 1} \cdot \sum_{j=1}^{n} \frac{\delta^2_j(x_j)}{2\tau + 1} \right)^2}{\lambda \sum_{j=1}^{n} \left( \sum_{j=1}^{n} \frac{\delta^1_j(x_j)}{2\tau + 1} \right)^2 + (1 - \lambda) \sum_{j=1}^{n} \left( \sum_{j=1}^{n} \frac{\delta^2_j(x_j)}{2\tau + 1} \right)^2}
\]

where \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T\) is the weight vector of \(x_j (j = 1, 2, \ldots, n)\), with \(\omega_j \in [0, 1], j = 1, 2, \ldots, n, \sum_{j=1}^{n} \omega_j = 1\). In particular, if \(\omega = (1/n, 1/n, \ldots, 1/n)^T\), then the weighted generalized Dice similarity measures reduce to generalized Dice similarity measures. That is to say, if we take \(\omega_j = \frac{1}{n^2}, j = 1, 2 \cdots, n\), then there is \(WGD^k_{\text{asy}}(\phi, \varphi) = GD^k_{\text{asy}}(\phi, \varphi) (k = 1, 2)\).
Then, the weighted generalized Dice similarity measure includes some special cases by altering the parameter value \( \lambda \).

If \( \lambda = 0.5 \), the two weighted generalized Dice similarity measures (15) and (16) reduced to weighted Dice similarity measures (4) and (6):

\[
WGD^1(H^1_3, H^2_3) = \sum_{j=1}^{n} o_j \left( \frac{\lambda}{\left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2} + (1 - \lambda) \left( \sum_{i=1}^{L_2} |\delta^2_i(x_j)| / (2\tau + 1) \right)^2 \right)
\]

\[
= \sum_{j=1}^{n} o_j \left( \frac{L_1 \left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2}{\left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2} \right)
\]

\[
WGD^2(H^1_3, H^2_3) = \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2 + \left( \sum_{i=1}^{L_2} |\delta^2_i(x_j)| / (2\tau + 1) \right)^2
\]

\[
= \lambda \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2 + (1 - \lambda) \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_2} |\delta^2_i(x_j)| / (2\tau + 1) \right)^2
\]

\[
= 0.5 \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2 + (1 - 0.5) \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_2} |\delta^2_i(x_j)| / (2\tau + 1) \right)^2
\]

\[
= \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_1} |\delta^1_i(x_j)| / (2\tau + 1) \right)^2 + \sum_{j=1}^{n} o_j^2 \left( \sum_{i=1}^{L_2} |\delta^2_i(x_j)| / (2\tau + 1) \right)^2
\]
If $\lambda = 0, 1$, the two weighted generalized Dice similarity measures reduced to the following asymmetric weighted similarity measures, respectively:

$$
WGD^1(H^1_S, H^2_S) = \sum_{j=1}^{n} \omega_j \frac{\left( \sum_{l=1}^{L_1} \left| \delta_l^1(x_j) \right| \cdot \sum_{l=1}^{L_2} \left| \delta_l^2(x_j) \right| \right)}{\left( \sum_{l=1}^{L_1} \frac{\left| \delta_l^1(x_j) \right|}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} \frac{\left| \delta_l^2(x_j) \right|}{2\tau + 1} \right)^2}.
$$

(19)

$$
\lambda \left( \sum_{l=1}^{L_1} \frac{\left| \delta_l^1(x_j) \right|}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} \frac{\left| \delta_l^2(x_j) \right|}{2\tau + 1} \right)^2
$$

$$
= \sum_{j=1}^{n} \omega_j \frac{\left( \sum_{l=1}^{L_1} \left| \delta_l^1(x_j) \right| \cdot \sum_{l=1}^{L_2} \left| \delta_l^2(x_j) \right| \right)}{\left( \sum_{l=1}^{L_1} \frac{\left| \delta_l^1(x_j) \right|}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} \frac{\left| \delta_l^2(x_j) \right|}{2\tau + 1} \right)^2}, \text{ for } \lambda = 0.
$$

(20)

$$
WGD^1(H^1_S, H^2_S) = \sum_{j=1}^{n} \omega_j \frac{\left( \sum_{l=1}^{L_1} \left| \delta_l^1(x_j) \right| \cdot \sum_{l=1}^{L_2} \left| \delta_l^2(x_j) \right| \right)}{\left( \sum_{l=1}^{L_1} \frac{\left| \delta_l^1(x_j) \right|}{2\tau + 1} \right)^2 + \left( \sum_{l=1}^{L_2} \frac{\left| \delta_l^2(x_j) \right|}{2\tau + 1} \right)^2}.
$$

$$
\lambda \left( \sum_{l=1}^{L_1} \frac{\left| \delta_l^1(x_j) \right|}{2\tau + 1} \right)^2 + (1 - \lambda) \left( \sum_{l=1}^{L_2} \frac{\left| \delta_l^2(x_j) \right|}{2\tau + 1} \right)^2
$$

$$
= \sum_{j=1}^{n} \omega_j \frac{\left( \sum_{l=1}^{L_1} \left| \delta_l^1(x_j) \right| \cdot \sum_{l=1}^{L_2} \left| \delta_l^2(x_j) \right| \right)}{\left( \sum_{l=1}^{L_1} \frac{\left| \delta_l^1(x_j) \right|}{2\tau + 1} \right)^2 + \left( \sum_{l=1}^{L_2} \frac{\left| \delta_l^2(x_j) \right|}{2\tau + 1} \right)^2}, \text{ for } \lambda = 1.
$$
The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information

In this section, we shall extend the generalized Dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information. Let $A =$
\{A_1, A_2, \ldots, A_m\} be a discrete set of alternatives, and \(G = \{G_1, G_2, \ldots, G_n\}\) be the set of attributes, \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)\) is the weighting vector of the attributes \(G_j(j = 1, 2, \ldots, n)\), where \(\omega_j \in [0, 1]\), \(\sum_{j=1}^{n} \omega_j = 1\). Suppose that \(H = (H_{ij}^g)_{m \times n}\) is the hesitant fuzzy linguistic decision matrix, where \(H_{ij}^g = \cup_{s \in H_{ij}} \{s_{ij}^g| l = 1, 2, \ldots, H_{ij}^g\}\) \((i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)\) is the hesitant fuzzy linguistic values, which take the form of HFLTSs, given by the decision maker for the alternative \(A_i \in A\) with respect to the attribute \(G_j \in G\).

Then, in the following, we shall develop an algorithm to utilize the generalized Dice similarity measures to solve the multiple attribute decision making with hesitant fuzzy linguistic information.

**Step 1.** Defining the hesitant fuzzy linguistic positive ideal solution (HFLPIS) \(A^+\) as

\[ A^+ = (H_{S1}^{H^+}, H_{S2}^{H^+}, L, H_{Sn}^{H^+}) \]  

where

\[
H_{Sj}^{H^+} = \begin{cases} 
\max_{i=1, \ldots, m} H_{ij}^g = \max_{i=1, \ldots, m} \{s_{ij}^g\}, & \text{for benefit attribute } G_j \ \\
\min_{i=1, \ldots, m} H_{ij}^g = \min_{i=1, \ldots, m} \{s_{ij}^g\}, & \text{for cost attribute } G_j
\end{cases}, \quad j = 1, 2, L, n. \tag{24}
\]

Note that the hesitant fuzzy linguistic positive ideal solution \(A^+\) is linguistic term sets. Hence, they certainly can be taken as special HFLTSs with only one linguistic term in each HFLTS.

**Step 2.** Calculating the weighted generalized Dice similarity measures between \(A_i (i = 1, 2, \ldots, m)\) and \(A^+\) as follows:

\[
WGD^I(A_i, A^+) = \sum_{j=1}^{n} \omega_j \left( \frac{\max_{i=1, \ldots, m} \{s_{ij}^g\}}{H_{ij}^g} \right) \left( \frac{1}{2\tau + 1} \right)^2 (1 - \lambda) \left( \frac{\max_{i=1, \ldots, m} \{s_{ij}^g\}}{H_{ij}^g} \right) \left( \frac{1}{2\tau + 1} \right)^2
\]  

or


\[
WGD^2(A_i, A^+) = \frac{\sum_{j=1}^{n} \omega_j^2 \left( \frac{\max_{i=1, \ldots, m} \left\{ \delta_{ij}^l \right\}}{2\tau + 1} \right)}{\lambda \sum_{j=1}^{n} \omega_j^2 \left( \frac{\sum_{l=1}^{2\tau+1} \delta_{ij}^l}{2\tau + 1} \right)^2 + (1 - \lambda) \sum_{j=1}^{n} \omega_j^2 \left( \frac{\max_{i=1, \ldots, m} \left\{ \delta_{ij}^l \right\}}{2\tau + 1} \right)^2}
\]

(26)

**Step 3.** Rank all the alternatives \(A_i (i = 1, 2, \ldots, m)\) and select the best one(s) in accordance with the weighted generalized Dice similarity measures \(WGD^1(A_i, A^+) (WGD^2(A_i, A^+)) \) \(i = 1, 2, \ldots, m\). If any alternative has the highest \(WGD^1(A_i, A^+) (WGD^2(A_i, A^+))\) value, then, it is the most important alternative.

**Step 4.** End.

**Numerical example and comparative analyses**

In this section, we consider a movie recommender system (adapted from Liao et al., 2014) to demonstrate the efficiency of the proposed generalized Dice similarity measures. Suppose that a company intends to give ratings on five movies \(A_i (i = 1, 2, \ldots, 5)\) with respect to four attributes: story (\(G_1\)), acting (\(G_2\)), visuals (\(G_3\)) and direction (\(G_4\)). The weighing vector of these four attributes is: \(\omega = (0.4, 0.2, 0.2, 0.2)\). The ratings provide information about the quality of the movies as well as the taste of the users who give the ratings. Since these criteria are all qualitative, it is convenient and only feasible for the decision makers to express their feelings by using linguistic terms. As pointed out by Miller (1956), most decision makers cannot handle more than nine factors when making their decision. Hence, the company constructs a seven-point linguistic scale to assess the movies, which is

\[
S = \{s_{-3} = \text{extremely poor}, s_{-2} = \text{very poor}, s_{-1} = \text{poor}, s_0 = \text{medium},
\]
\[
s_1 = \text{good}, s_2 = \text{very good}, s_3 = \text{extremely good}\}
\]

The five possible movies \(A_i (i = 1, 2, \ldots, 5)\) are to be evaluated using the linguistic sets \(S\) by the the decision makers under the above four attributes, and construct the hesitant fuzzy linguistic decision matrix as follows \(\tilde{H} = (H_{ij}^{kl})_{5 \times 4}\) in **Table 1**.

In the following, we shall utilize the proposed approach in this paper getting the most desirable movies

**Step 1.** Defining the hesitant fuzzy linguistic positive ideal solution (HFLPIS) \(A^+\) as

\[
A^+ = (\{s_3\}, \{s_3\}, \{s_3\}, \{s_2\})^T
\]

**Step 2.** According to **Equations (25) and (26)** and different values of the parameter \(\lambda\), the weighted generalized Dice measure values between \(A_i (i = 1, 2, 3, 4, 5)\) and \(A^+\) can be obtained, which are shown in **Tables 2 and 3**, respectively.
From the Tables 2 and 3, different ranking orders are shown by taking different values of $\lambda$ and different Dice similarity measures. Then the best movies should belong to $A_3$ or $A_5$ according to the principle of the maximum degree of Dice similarity measures between HFLTSs.

Furthermore, for the special cases of the two generalized Dice measures we obtain the following results:

- When $\lambda = 0$, the two weighted generalized Dice measures are reduced to the weighted projection measures of $A_i (i = 1, 2, 3, 4, 5)$ on $A^+$. Thus, the best movies should belong to $A_3$ according to the principle of the maximum degree of Dice similarity measures between HFLTSs. For this case, we can derive the same best alternative as the method proposed in Ref. Liao et al. (2014). Thus our method is effective.

- When $\lambda = 0.5$, the two weighted generalized Dice measures are reduced to the weighted Dice similarity measures of $A_i (i = 1, 2, 3, 4, 5)$ and $A$. Thus, the best movies should belong to $A_3$ according to the principle of the maximum degree of Dice similarity measures between HFLTSs. For this case, we can derive the same best alternative as the method proposed in Ref. Liao et al. (2014). Thus our method is effective.
When $\lambda = 1$, the two weighted generalized Dice measures are reduced to the weighted projection measures of $A^+$ on $A_i (i = 1, 2, 3, 4, 5)$. Thus, the best movies should belong to $A_5$ according to the principle of the maximum degree of Dice similarity measures between HFLTSs.

Therefore, according to different Dice similarity measures and different values of the parameter $\lambda$, ranking orders may be also different. Thus the proposed multiple attribute decision making methods can be assigned some value of $\lambda$ and some measure to satisfy the requirements of decision makers’ preference and flexible decision making.

Obviously, the multiple attribute decision making methods based on the Dice measures and the projection measures are the special cases of the proposed multiple attribute decision making methods based on generalized Dice measures. Therefore, in the multiple attribute decision making process, the multiple attribute decision making models developed in this paper are more general and more flexible than existing multiple attribute decision making models under a hesitant fuzzy linguistic environment.

**Conclusion and future work**

In this paper, we present some novel Dice similarity measures of HFLTSs and the generalized Dice similarity measures of HFLTSs and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Then, we propose the generalized Dice similarity measures-based multiple attribute decision making models with HFLTSs. Finally, an illustrative example for concerning the evaluation of the quality of movies is given to demonstrate the efficiency of the similarity measures. In the future, the application of the proposed Dice similarity measure of HFLTSs needs to be explored in dynamic and complex decision making, risk analysis and many other fields under an uncertain environment (Gao, 2018; Gao, Lu, Wei, & Wei, 2018; Gao, Wei, & Huang, 2018; Huang & Wei, 2018; Liao, Li, & Lu, 2007; Tang, Wen, & Wei, 2017; Tang & Wei, 2018; Wei & Wei, 2018).

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