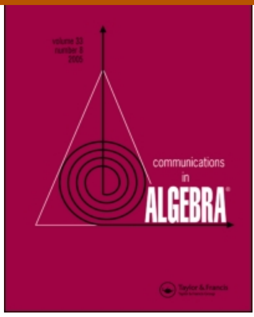

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Simple modules of the quantum double of the Nichols algebra of unidentified diagonal type $\mathfrak{ufo}(7)$

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ABSTRACT
The finite-dimensional simple modules over the Drinfeld double of the bosonization of the Nichols algebra $\mathfrak{ufo}(7)$ are classified.

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1. Introduction

1.1. Motivations and context

The purpose of this paper is to compute explicitly all simple finite-dimensional modules of the Hopf algebra $\mathcal{U}$ introduced by generators and relations in Definition 1.1. In short, $\mathcal{U} \simeq D(H)$ arises as the Drinfeld double of $H = \mathcal{B}(V)\#\Bbbk\Lambda$, where Λ is an abelian group, V is a braided vector space of diagonal type of dimension 2 with Dynkin diagram (1) (realized as a Yetter-Drinfeld module over Λ) and $\mathcal{B}(V)$ denotes its Nichols algebra.

The general context where our results fit is the following. Let W be a braided vector space of diagonal type and assume that its Nichols algebra $\mathcal{B}(W)$ is finite-dimensional; see [2] for an introduction to Nichols algebras and [3] for a survey on Nichols algebras of diagonal type. We recall that finite-dimensional Nichols algebras of diagonal type were classified in [13]. It is useful to organize the classification in four classes:

- Standard type [8], including Cartan type [7].
- Super type [5].
- (Super) modular type [3].
- Unidentified type [9].

Let Γ be an an abelian group such that W is realized as a Yetter-Drinfeld module over it and let U be the Drinfeld double of $\mathcal{B}(W)\#\Bbbk\Gamma$. The representation theory of such Drinfeld doubles U or slight variations thereof was treated in many papers, among them [1, 6, 14, 15, 17–19]. Indeed, the first two articles deal with the representation theory of the finite quantum groups or Frobenius-Lusztig kernels (that roughly arise from W of Cartan type), while in the others some general results are established. Presently we know that the simple U-modules are parametrized by highest weights but we ignore the character formulas and the dimensions in general, except for Frobenius-Lusztig kernels under appropriate conditions.

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Back to the particular V, the goal of working out this example, establishing the dimensions of all simple $\mathcal{U}$-modules, is to gain experience for further developments. The algebra $\mathcal{U}$ is small enough to allow an approach by elementary computations. Arguing as in [6, Theorem 3.7], see also [14, Proposition 5.6], it is possible to prove that $\mathcal{U}$ is a quasi-triangular Hopf algebra, even a ribbon one by the criterion in [16, Theorem 3], what makes it susceptible of applications. If Λ is finite, then the simple $\mathcal{U}$-modules are just the simple Yetter-Drinfeld H-modules; therefore the classification here might have applications to the study of basic Hopf algebras. Also, in the organization in classes mentioned above, $\mathcal{B}(V)$ is the smallest Nichols algebra of unidentified type; in the terminology from [3], V is of type $\mathfrak{ufo}(7)$. Indeed, $\dim \mathcal{B}(V) < \infty$ by [13, Table 1, row 7]; more precisely, cf. (13),

$$\dim \mathcal{B}(V) = 2^4 3^2 = 144.$$

By [9], a consequence of [10, 11], we know that $\mathcal{B}(V)$ has a presentation by generators E_1, E_2 and relations (5) below. Thus $\mathcal{B}(V)$ is manageable yet does not arise from any Lie algebra, what makes it attractive.

There is another reason to address the representation theory of $\mathcal{U}$. A finite-dimensional Nichols algebra of diagonal type admits both a distinguished pre-Nichols algebra [12] and a distinguished post-Nichols algebra [4]; the representation theories of the corresponding Drinfeld doubles seem to be very rich. However our $\mathcal{B}(V)$ coincides with its distinguished pre-Nichols and post-Nichols algebras, being therefore of singular interest (the only other Nichols algebra with this feature has diagram $\overset{\omega}{\circ} \overset{-\omega}{\text{------}} \overset{-1}{\circ}$, $\omega \in \mathbb{G}_3'$, which is of standard type B_2). This peculiar behaviour appeals to the consideration of V.

1.2. The algebra $\mathcal{U}$

We now introduce formally $\mathcal{U}$. Let us begin with some notation.

If $k, \ell \in \mathbb{N}_0$, then we denote $\mathbb{I}_{k,\ell} = \{n \in \mathbb{N}_0 : k \leq n \leq \ell\}$; also $\mathbb{I}_\ell := \mathbb{I}_{1,\ell}$. Let $\Bbbk$ be an algebraically closed field of characteristic zero and $\Bbbk^\times = \Bbbk - 0$. Let $\mathbb{G}_{12}$ be the group of 12-roots of unity in $\Bbbk$, and let $\mathbb{G}_{12}'$ be the subset of primitive roots of order 12.

To define $\mathcal{U}$, we need some data:

- A matrix $\mathbf{q} = (q_{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix} \zeta^4 & q_{12} \\ q_{21} & -1 \end{pmatrix} \in \Bbbk^{2\times 2}$ such that $q_{12}q_{21} = \zeta^{11}$; that is, its associated generalized Dynkin diagram is given by

$$\underset{1}{\overset{\zeta^4}{\circ}} \overset{\zeta^{11}}{\text{------}} \underset{2}{\overset{-1}{\circ}} . \tag{1}$$

- An abelian group Λ whose group of characters is denoted by $\widehat{\Lambda}$. We set $\Gamma = \Lambda \times \widehat{\Lambda}$.
- $g_1, g_2 \in \Lambda, \sigma_1, \sigma_2 \in \widehat{\Lambda}$ such that $\begin{pmatrix} \sigma_1(g_1) & \sigma_2(g_1) \\ \sigma_1(g_2) & \sigma_2(g_2) \end{pmatrix} = \begin{pmatrix} \zeta^4 & q_{12} \\ q_{21} & -1 \end{pmatrix}$.

Starting from these data, we consider vector spaces V and W with bases v_i, respectively w_i, $i \in \mathbb{I}_2$ and define an action and a Γ-grading on V and W by

$$g \cdot v_i = \sigma_i(g) v_i, \qquad \sigma \cdot v_i = \sigma(g_i) v_i, \qquad g \cdot w_i = \sigma_i^{-1}(g) w_i, \qquad \sigma \cdot w_i = \sigma(g_i^{-1}) w_i; \tag{2}$$

$$\deg v_i = g_i, \qquad \deg w_i = \sigma_i, \qquad g \in \Lambda, \sigma \in \widehat{\Lambda}, \qquad i \in \mathbb{I}_2. \tag{3}$$

Then $V \oplus W$ is a Yetter-Drinfeld module over $\Bbbk\Gamma$ and $T(V \oplus W)$ is a braided Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. In particular, V is a braided vector space of diagonal type $\mathfrak{ufo}(7)$, as said.

It is convenient to start with the auxiliary Hopf algebra $\overline{\mathcal{U}} = T(V \oplus W)\#\Bbbk\Gamma$; in particular, $T(V \oplus W)$ and $\Bbbk\Gamma$ are subalgebras of $\overline{\mathcal{U}}$ and

$$gv_i = \sigma_i(g) v_i, \qquad \sigma v_i = \sigma(g_i) v_i \sigma, \qquad g w_i = \sigma_i^{-1}(g) w_i g, \qquad \sigma w_i = \sigma(g_i^{-1}) w_i \sigma,$$

$g \in \Lambda, \sigma \in \widehat{\Lambda}, i \in \mathbb{I}_2$. To stress the similarity with quantum groups, we denote in $\overline{\mathcal{U}}$ or any quotient thereof, as in [6, 14, 15],

$$E_i = v_i, \qquad F_i = w_i\sigma_i^{-1}, \qquad i \in \mathbb{I}_2. \tag{4}$$

Thus

$$gE_i = \sigma_i(g)E_ig, \qquad \sigma E_i = \sigma(g_i)E_i\sigma, \qquad gF_i = \sigma_i^{-1}(g)F_ig, \qquad \sigma F_i = \sigma(g_i^{-1})F_i\sigma.$$

We also need the notation of the so-called root vectors, needed for the relations and for the PBW-basis:

$$\begin{aligned} E_{12} &= E_1E_2 - q_{12}E_2E_1, & E_{112} &= E_1E_{12} - q_{12}\zeta^4E_{12}E_1, & E_{11212} &= E_{112}E_{12} - q_{12}\zeta E_{12}E_{112},\\ F_{12} &= F_1F_2 - q_{21}F_2F_1, & F_{112} &= F_1F_{12} - q_{21}\zeta^4F_{12}F_1, & F_{11212} &= F_{112}F_{12} - q_{21}\zeta F_{12}F_{112}. \end{aligned}$$

We are now ready to define $\mathcal{U}$.

Definition 1.1. The algebra $\mathcal{U}$ is the quotient of $\overline{\mathcal{U}}$ by the ideal generated by

$$E_1^2 = 0, \qquad E_2^2 = 0, \qquad E_{11212}E_{12} = \zeta^{10}q_{12}E_{12}E_{11212}, \tag{5}$$

$$F_1^2 = 0, \qquad F_2^2 = 0, \qquad F_{11212}F_{12} = \zeta^4q_{21}F_{12}F_{11212}, \tag{6}$$

$$E_kF_i - F_iE_k = \delta_{ki}(g_i - \sigma_i^{-1}). \tag{7}$$

The algebra $\mathcal{U}$ is a Hopf algebra with coproduct given by

$$\Delta(E_i) = E_i \otimes 1 + g_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes \sigma_i^{-1} + 1 \otimes F_i, \qquad \Delta(g) = g \otimes g, \qquad g \in \Gamma.$$

Let $\mathcal{U}^-$ (respectively $\mathcal{U}^+$) be the subalgebra of $\mathcal{U}$ generated by F_1, F_2 (respectively E_1, E_2). The following facts are not difficult to prove and can be derived from general results in the literature cited above:

- $\mathcal{U}$ has a triangular decomposition $\mathcal{U} \simeq \mathcal{U}^+ \otimes \Bbbk\Gamma \otimes \mathcal{U}^-$, given by the multiplication map.
- $\mathcal{U}^+ \simeq \mathcal{B}(V)$; in what follows we identify these two algebras.
- $\mathcal{U}, \mathcal{U}^+$ and $\mathcal{U}^-$ admit a $\mathbb{Z}^2$-graduation $\mathcal{U} = \oplus_{\beta\in\mathbb{Z}^2}\mathcal{U}_\beta$ such that $\deg E_i = \alpha_i = -\deg F_i$, $i \in \mathbb{I}_2$, and $\deg x = 0$ for $x \in \Gamma$.
  Here $(\alpha_i)_{i\in\mathbb{I}_2}$ is the canonical basis of $\mathbb{Z}^2$.

1.3. Verma modules

We recall succinctly the description of the simple modules in terms of highest weights.

Let ${}_{\mathcal{U}}\mathcal{M}$ be the category of left $\mathcal{U}$-modules and let $\operatorname{Irr}\mathcal{U}$ be the set of isomorphism classes of finite-dimensional simple $\mathcal{U}$-modules. If $M \in {}_{\mathcal{U}}\mathcal{M}$ and $\lambda \in \widehat{\Gamma}$, then

$$M^\lambda = \{m \in M : g \cdot m = \lambda(g)m \ \forall g \in \Gamma\}$$

is the space of weight vectors with weight λ; if $M = \oplus_{\lambda\in\widehat{\Gamma}}M^\lambda$, then we say that M is diagonalizable.

Let $\lambda \in \widehat{\Gamma}$. We denote by $\Bbbk_\lambda$ the $\Bbbk\Gamma \otimes \mathcal{U}^-$-module defined by $\lambda \otimes \varepsilon$ (the counit). The *Verma module* $M(\lambda)$ associated to λ is the induced module

$$M(\lambda) = \operatorname{Ind}_{\Bbbk\Gamma\otimes\mathcal{U}^-}^{\mathcal{U}}\Bbbk_\lambda \simeq \mathcal{U}/\big(\mathcal{U}F_1 + \mathcal{U}F_2 + \sum_{g\in\Gamma}\mathcal{U}(g - \lambda(g))\big). \tag{8}$$

Let v_λ be the residue class of 1 in $M(\lambda)$; then we have an isomorphism of $\mathcal{U}^+$-modules

$$\mathcal{U}^+ \simeq M(\lambda), \qquad 1 \longmapsto v_\lambda.$$

Hence $\dim M(\lambda) = \dim \mathcal{B}(V) = 144$. Thus the PBW-basis of $\mathcal{U}^+ \simeq \mathcal{B}(V)$ becomes via this isomorphism a basis of $M(\lambda)$.

The $\mathbb{Z}^2$-grading on $\mathcal{U}^+ \simeq \mathcal{B}(V)$ induces a $\mathbb{Z}^2$-grading on $M(\lambda)$ such that

$$M(\lambda)_\beta = \mathcal{U}_\beta \cdot v_\lambda, \qquad \beta \in \mathbb{Z}^2.$$

Thus

$$M(\lambda)_0 = \Bbbk v_\lambda, \qquad \mathcal{U}_\beta \cdot M(\lambda)_\gamma \subset M(\lambda)_{\beta+\gamma}, \qquad \beta, \gamma \in \mathbb{Z}^2.$$

The family of $\mathcal{U}$-submodules of $M(\lambda)$ contained in $\sum_{\beta \neq 0} M(\lambda)_\beta$ has a unique maximal element $N(\lambda)$. We set

$$L(\lambda) = M(\lambda)/N(\lambda).$$

Since $\mathcal{U}$ satisfies the conditions on [19, Section 2], [19, Corollary 2.6] implies that

$$\textit{The map} \qquad \lambda \mapsto L(\lambda) \qquad \textit{provides a bijective correspondence} \qquad \widehat{\Gamma} \simeq \operatorname{Irr} \mathcal{U}. \tag{9}$$

Alternatively we see that $L(\lambda)$ is simple arguing as in [18, Theorem 1]; then [18, Theorem 3] gives (9). Notice that $L(\lambda)$ inherits the grading from $M(\lambda)$. Also, it follows that every simple $M \in {}_{\mathcal{U}}\mathcal{M}$ is diagonalizable.

Lowest weight modules of weight λ are defined as usual; $M(\lambda)$ covers every lowest weight module of weight λ, that in turn covers $L(\lambda)$. Highest weight modules are defined similarly.

1.4. Main result

In our main theorem, we give the dimension of $L(\lambda)$ for each $\lambda \in \widehat{\Gamma}$, in terms of certain equalities arising from the Shapovalov determinant [15] satisfied by

$$\lambda_i = \lambda(g_i \sigma_i), \qquad i \in \mathbb{I}_2.$$

Indeed, the Shapovalov determinant in the context of this paper is

$$\begin{aligned} \text{Ш} = {} & (\zeta^4\lambda_1^{-1} - \zeta^4)(\zeta^4\lambda_1^{-1} - \zeta^8)(\zeta^2\lambda_1^{-2}\lambda_2^{-1} - \zeta^8)(\zeta^2\lambda_1^{-2}\lambda_2^{-1} - \zeta^4)(\lambda_1^{-3}\lambda_2^{-2} + 1) \\ & \times (\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} - \zeta^9)(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} + 1)(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} - \zeta^3)(\lambda_2^{-1} - 1). \end{aligned} \tag{10}$$

Then $\text{Ш} = 0$ if and only if one of the factors in (10) vanishes. Let

$$\mathtt{S}_1 = \{1, \zeta^8\}, \qquad \mathtt{S}_2 = \{-1, \zeta^{10}\}, \qquad \mathtt{S}_3 = \{\zeta, \zeta^4, \zeta^7\}. \tag{11}$$

The equalities alluded above can be packed as the conditions:

$$\lambda_1 \overset{?}{\in} \mathtt{S}_1, \qquad \lambda_1^2\lambda_2 \overset{?}{\in} \mathtt{S}_2, \qquad \lambda_1^3\lambda_2^2 \overset{?}{=} -1, \qquad \lambda_1\lambda_2 \overset{?}{\in} \mathtt{S}_3, \qquad \lambda_2 \overset{?}{=} 1. \tag{12}$$

To organize the information, we consider 47 subsets of $\widehat{\Gamma}$, organized in classes $\mathcal{C}_j$ according to the quantity j of conditions in (12) satisfied. The class $\mathcal{C}_0$ contains just one family:

$$\mathfrak{I}_1 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\ \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathtt{S}_3,\ \lambda_2 \neq 1\};$$

Here is the class $\mathcal{C}_1$:

$$\begin{aligned} \mathfrak{I}_2 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathtt{S}_3,\ \lambda_2 \neq 1\} \\ &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 \notin \{1, \zeta, \zeta^4, \zeta^7, \zeta^3, \zeta^9, -1, \zeta^{10}\}\}; \\ \mathfrak{I}_3 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathtt{S}_3,\ \lambda_2 \neq 1\} \\ &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 \notin \{\pm 1, \zeta^2, \zeta^3, \zeta^5, \zeta^8, \zeta^9, \zeta^{11}\}\}; \end{aligned}$$

$$\begin{aligned}
\mathfrak{I}_4 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 = -1,\, \lambda_1^3\lambda_2^2 \neq -1,\, \lambda_1\lambda_2 \notin \mathtt{S}_3,\, \lambda_2 \neq 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = -1,\, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\};\\
\mathfrak{I}_5 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 = \zeta^{10},\, \lambda_1^3\lambda_2^2 \neq -1,\, \lambda_1\lambda_2 \notin \mathtt{S}_3,\, \lambda_2 \neq 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = \zeta^{10},\, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\};\\
\mathfrak{I}_6 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\, \lambda_1^3\lambda_2^2 = -1,\, \lambda_1\lambda_2 \notin \mathtt{S}_3,\, \lambda_2 \neq 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^3\lambda_2^2 = -1,\, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\};\\
\mathfrak{I}_7 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\, \lambda_1^3\lambda_2^2 \neq -1,\, \lambda_1\lambda_2 = \zeta,\, \lambda_2 \neq 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta,\, \lambda_1 \notin \{1, \zeta^8, \zeta, \zeta^4, \zeta^9\}\};\\
\mathfrak{I}_8 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\, \lambda_1^3\lambda_2^2 \neq -1,\, \lambda_1\lambda_2 = \zeta^4,\, \lambda_2 \neq 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^4,\, \lambda_1 \notin \{1, \zeta^8, \zeta^4, \zeta^2, -1, \zeta^{10}\}\};\\
\mathfrak{I}_9 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\, \lambda_1^3\lambda_2^2 \neq -1,\, \lambda_1\lambda_2 = \zeta^7,\, \lambda_2 \neq 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^7,\, \lambda_1 \notin \{1, \zeta^8, \zeta^7, \zeta^4, \zeta^{11}\}\};\\
\mathfrak{I}_{10} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\, \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\, \lambda_1^3\lambda_2^2 \neq -1,\, \lambda_1\lambda_2 \notin \mathtt{S}_3,\, \lambda_2 = 1\}\\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{G}_{12},\, \lambda_2 = 1\};
\end{aligned}$$

All the 37 remaining subsets belong to class $\mathcal{C}_2$:

$$\begin{aligned}
&\mathfrak{I}_{11} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta\}, && \mathfrak{I}_{12} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^4\},\\
&\mathfrak{I}_{13} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^7\}, && \mathfrak{I}_{14} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^3\},\\
&\mathfrak{I}_{15} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^9\}, && \mathfrak{I}_{16} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = -1\},\\
&\mathfrak{I}_{17} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^{10}\}, && \mathfrak{I}_{18} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^5\},\\
&\mathfrak{I}_{19} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^8\}, && \mathfrak{I}_{20} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^{11}\},\\
&\mathfrak{I}_{21} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^3\}, && \mathfrak{I}_{22} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^9\},\\
&\mathfrak{I}_{23} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^2\}, && \mathfrak{I}_{24} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = -1\},\\
&\mathfrak{I}_{25} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = \zeta^8\}, && \mathfrak{I}_{26} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = \zeta^8\},\\
&\mathfrak{I}_{27} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^9\}, && \mathfrak{I}_{28} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = \zeta^4\},\\
&\mathfrak{I}_{29} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = -1\}, && \mathfrak{I}_{30} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = \zeta^2\},\\
&\mathfrak{I}_{31} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = \zeta^{10}\}, && \mathfrak{I}_{32} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = -1\},\\
&\mathfrak{I}_{33} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = -1\}, && \mathfrak{I}_{34} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^3\},\\
&\mathfrak{I}_{35} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = \zeta^4\}, && \\
&\mathfrak{I}_{36} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta, \lambda_2 = 1\}, && \mathfrak{I}_{37} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = 1\},\\
&\mathfrak{I}_{38} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = 1\}, && \mathfrak{I}_{39} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = 1\},\\
&\mathfrak{I}_{40} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = 1\}, && \mathfrak{I}_{41} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = 1\},\\
&\mathfrak{I}_{42} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^7, \lambda_2 = 1\}, && \mathfrak{I}_{43} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = 1\},\\
&\mathfrak{I}_{44} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = 1\}, && \mathfrak{I}_{45} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = 1\},\\
&\mathfrak{I}_{46} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = 1\}, && \mathfrak{I}_{47} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = 1\}.
\end{aligned}$$

Main Theorem. *The dimension and the maximal degree of $L(\lambda)$ depend on λ_i, $i \in \mathbb{I}_2$, and appear in Table 1.*

The paper is organized as follows. We collect some general information about $\mathcal{U}$ and the Verma modules in Section 2, where we also deal with $\mathfrak{I}_1$. The proof of the Main Theorem for the families in the class 1, resp. 2, is given in Section 3, respectively 4.

If $M \in \mathcal{U}$, then we write $N \leq M$ to express that N is a submodule of M.

Table 1. Dimensions and highest degrees of simple modules.

Family	dim $L(\lambda)$	max. degree	$L(\lambda)^{\varphi}$
$\mathfrak{I}_1$	144	(12, 8)	$\mathfrak{I}_1$
$\mathfrak{I}_2$	48	(10, 8)	$\mathfrak{I}_2$
$\mathfrak{I}_3$	96	(11, 8)	$\mathfrak{I}_3$
$\mathfrak{I}_4$	48	(8, 6)	$\mathfrak{I}_4$
$\mathfrak{I}_5$	96	(10, 7)	$\mathfrak{I}_5$
$\mathfrak{I}_6$	72	(9, 6)	$\mathfrak{I}_6$
$\mathfrak{I}_7$	36	(9, 5)	$\mathfrak{I}_7$
$\mathfrak{I}_8$	72	(10, 6)	$\mathfrak{I}_8$
$\mathfrak{I}_9$	108	(11, 7)	$\mathfrak{I}_9$
$\mathfrak{I}_{10}$	72	(12, 7)	$\mathfrak{I}_{10}$
$\mathfrak{I}_{11}$	11	(5, 4)	$\mathfrak{I}_{12}$
$\mathfrak{I}_{12}$	11	(5, 4)	$\mathfrak{I}_{11}$
$\mathfrak{I}_{13}$	23	(7, 5)	$\mathfrak{I}_{44}$
$\mathfrak{I}_{14}$	25	(7, 5)	$\mathfrak{I}_{28}$
$\mathfrak{I}_{15}$	37	(9, 6)	$\mathfrak{I}_{41}$
$\mathfrak{I}_{16}$	37	(8, 6)	$\mathfrak{I}_{30}$
$\mathfrak{I}_{17}$	47	(10, 7)	$\mathfrak{I}_{46}$
$\mathfrak{I}_{18}$	11	(5, 3)	$\mathfrak{I}_{38}$
$\mathfrak{I}_{19}$	35	(8, 5)	$\mathfrak{I}_{40}$
$\mathfrak{I}_{20}$	71	(11, 7)	$\mathfrak{I}_{42}$
$\mathfrak{I}_{21}$	61	(9, 6)	$\mathfrak{I}_{32}$
$\mathfrak{I}_{22}$	49	(9, 6)	$\mathfrak{I}_{45}$
$\mathfrak{I}_{23}$	47	(8, 6)	$\mathfrak{I}_{29}$
$\mathfrak{I}_{24}$	85	(10, 7)	$\mathfrak{I}_{35}$
$\mathfrak{I}_{25}$	37	(8, 5)	$\mathfrak{I}_{37}$
$\mathfrak{I}_{26}$	25	(8, 5)	$\mathfrak{I}_{43}$
$\mathfrak{I}_{27}$	35	(9, 5)	$\mathfrak{I}_{36}$
$\mathfrak{I}_{28}$	25	(7, 5)	$\mathfrak{I}_{14}$
$\mathfrak{I}_{29}$	47	(8, 6)	$\mathfrak{I}_{23}$
$\mathfrak{I}_{30}$	37	(8, 6)	$\mathfrak{I}_{16}$
$\mathfrak{I}_{31}$	61	(10, 6)	$\mathfrak{I}_{39}$
$\mathfrak{I}_{32}$	61	(9, 6)	$\mathfrak{I}_{21}$
$\mathfrak{I}_{33}$	71	(9, 6)	$\mathfrak{I}_{34}$
$\mathfrak{I}_{34}$	71	(9, 6)	$\mathfrak{I}_{33}$
$\mathfrak{I}_{35}$	85	(10, 7)	$\mathfrak{I}_{24}$
$\mathfrak{I}_{36}$	35	(9, 5)	$\mathfrak{I}_{27}$
$\mathfrak{I}_{37}$	37	(8, 5)	$\mathfrak{I}_{25}$
$\mathfrak{I}_{38}$	11	(5, 3)	$\mathfrak{I}_{18}$
$\mathfrak{I}_{39}$	61	(10, 6)	$\mathfrak{I}_{31}$
$\mathfrak{I}_{40}$	35	(8, 5)	$\mathfrak{I}_{19}$
$\mathfrak{I}_{41}$	37	(9, 6)	$\mathfrak{I}_{15}$
$\mathfrak{I}_{42}$	71	(11, 7)	$\mathfrak{I}_{20}$
$\mathfrak{I}_{43}$	25	(8, 5)	$\mathfrak{I}_{26}$
$\mathfrak{I}_{44}$	23	(7, 5)	$\mathfrak{I}_{13}$
$\mathfrak{I}_{45}$	49	(9, 6)	$\mathfrak{I}_{22}$
$\mathfrak{I}_{46}$	47	(10, 7)	$\mathfrak{I}_{17}$
$\mathfrak{I}_{47}$	1	(0, 0)	$\mathfrak{I}_{47}$

2. Preliminaries

2.1. The algebra $\mathcal{U}$

The Nichols algebra $\mathcal{B}(V)$ has a PBW-basis given by

$$\left\{E_2^{a_2}E_{12}^{a_{12}}E_{11212}^{a_{11212}}E_{112}^{a_{112}}E_1^{a_1} \mid \quad a_2, a_{11212} \in \mathbb{I}_{0,1}; \quad a_{12} \in \mathbb{I}_{0,3}; \quad a_{112}, a_1 \in \mathbb{I}_{0,2}\right\}. \tag{13}$$

See [9]. We obtain a new PBW-basis by reordering the PBW-generators:

$$\left\{E_1^{a_1}E_{112}^{a_{112}}E_{11212}^{a_{11212}}E_{12}^{a_{12}}E_2^{a_2} \mid \quad a_2, a_{11212} \in \mathbb{I}_{0,1}; \quad a_{12} \in \mathbb{I}_{0,3}; \quad a_{112}, a_1 \in \mathbb{I}_{0,2}\right\}. \tag{14}$$

Thus the set of positive roots of $\mathcal{B}(V)$ (the degrees of the generators of the PBW-basis) is

$$\Delta_+^V = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

By [11, Theorem 4.9], we have

$$E_{112}^3 = E_{11212}^2 = E_{12}^4 = 0. \tag{15}$$

From the defining relations (5), we can deduce that the following are valid in $\mathcal{B}(V)$:

$$\begin{aligned}
E_1E_{112} &= q_{12}\zeta^8 E_{112}E_1,\\
E_{112}E_2 &= -q_{12}^2 E_2E_{112} + q_{12}\zeta^8 E_{12}^2,\\
E_1E_{11212} &= q_{12}^2 E_{11212}E_1 + q_{12}\zeta^7(1+\zeta)E_{112}^2\\
E_1E_{12}^2 &= E_{11212} + q_{12}\zeta(1+\zeta^3)E_{12}E_{112} + q_{12}^2\zeta^8 E_{12}^2E_1\\
E_1E_{12}^3 &= q_{12}\zeta^{10}E_{12}E_{11212} + q_{12}^2\zeta^5 E_{12}^2E_{112} + q_{12}^3E_{12}^3E_1,\\
E_1^2E_2 &= E_{112} + q_{12}^2\zeta^2E_{12}E_1 + q_{12}^2E_2E_1^2,\\
E_1^2E_{12} &= -q_{12}^2E_{112}E_1 + q_{12}^2\zeta^8E_{12}E_1^2,\\
E_{112}E_{12}^2 &= -q_{12}\zeta^4(1+\zeta^3)E_{12}E_{11212} + q_{12}^2\zeta^2E_{12}^2E_{112}\\
E_{112}E_{12}^3 &= q_{12}^2\zeta^{11}E_{12}^2E_{11212} + q_{12}^3\zeta^3E_{12}^3E_{112},\\
E_{11212}E_{12} &= q_{12}\zeta^{10}E_{12}E_{11212},\\
E_{112}E_{11212} &= q_{12}\zeta^9E_{11212}E_{112},\\
E_{11212}E_2 &= q_{12}^3E_2E_{11212} + q_{12}^2\zeta^2(1+\zeta)E_{12}^3,\\
E_{12}E_2 &= -q_{12}E_2E_{12}.
\end{aligned}$$

The following equalities hold by direct computation from (5) and the previous ones:

$$\begin{aligned}
F_1E_{12} &= E_{12}F_1 + q_{12}(\zeta-1)E_2\sigma_1^{-1},\\
F_1E_{112} &= E_{112}F_1 + q_{12}\zeta^8(1+\zeta^3)E_{12}\sigma_1^{-1},\\
F_1E_{11212} &= E_{11212}F_1 + q_{12}^2(\zeta^5-1)E_{12}^2\sigma_1^{-1},\\
F_1E_{112}^2 &= E_{112}^2F_1 - q_{12}(1+\zeta^3)(E_{11212}\sigma_1^{-1} + \zeta^4E_{112}E_{12}\sigma_1^{-1}),\\
F_1E_{12}^2 &= E_{12}^2F_1 + q_{12}^2(3)_{\zeta^5}E_2E_{12}\sigma_1^{-1},\\
F_1E_{12}^3 &= E_{12}^2F_1 + q_{12}^3\zeta^3(\zeta-1)E_2E_{12}^2\sigma_1^{-1},\\
F_2E_{12} &= E_{12}F_2 + (\zeta^{11}-1)E_1g_2,\\
F_2E_{112} &= E_{112}F_2 - (3)_{\zeta^7}E_1^2g_2,\\
F_2E_{11212} &= E_{11212}F_2 - E_{112}E_1g_2,\\
F_2E_{12}^2 &= E_{12}^2F_2 + q_{21}(1+\zeta^5)E_{112}g_2 - (3)_{\zeta^7}E_{12}E_1g_2,\\
F_2E_{112}^2 &= E_{112}^2F_2 + (3)_{\zeta^7}\zeta^4E_{112}E_1^2g_2,\\
F_2E_{12}^3 &= E_{12}^3F_2 + \zeta^8(1-\zeta)(E_{12}^2E_1g_2 - q_{21}\zeta^3E_{12}E_{112}g_2 + q_{21}^2\zeta^3E_{11212}g_2),\\
F_{11212}E_{11212} &= E_{11212}F_{11212} + \sigma_1^{-3}\sigma_2^{-2} - g_{11212},\\
F_{12}E_2 &= E_2F_{12} + (1-\zeta^{11})F_1\sigma_2^{-1},\\
F_{12}E_{12} &= E_{12}F_{12} + \sigma_1^{-1}\sigma_2^{-1} - g_1g_2,\\
F_{12}E_{112} &= E_{112}F_{12} + \zeta^3(3)_{\zeta^7}E_1g_1g_2,\\
F_{12}E_{112}^2 &= E_{112}^2F_{12} + \zeta^{11}(3)_{\zeta^7}E_{112}E_1g_1g_2,\\
F_{12}E_1 &= E_1F_{12} + q_{21}(1-\zeta)F_2g_1,\\
F_{12}E_{11212} &= E_{11212}F_{12} + \zeta^{11}E_{112}g_1g_2,\\
F_{112}E_{112} &= E_{112}F_{112} + \sigma_1^{-2}\sigma_2^{-1} - g_1^2g_2,\\
F_{112}E_2 &= E_2F_{112} + (\zeta-1)F_1^2\sigma_2^{-1}.
\end{aligned}$$

2.2. Verma modules

We shall use the notation for q-factorial numbers: for each $q \in \Bbbk^{\times}$,

$$(n)_q = 1 + q + \ldots + q^{n-1}, \qquad (n)_q! = (1)_q(2)_q \cdots (n)_q, \qquad n \in \mathbb{N}.$$

We shall investigate the lattice of submodules of a Verma module. We record the following standard fact for future use.

Remark 2.1. Let $v \in M(\lambda)_\alpha$ be such that $F_i \cdot v = 0$ for $i \in \mathbb{I}_2$. By the triangular decomposition of $\mathcal{U}$, $\mathcal{U} \cdot v = \mathcal{U}^+ \cdot v$. In particular, if $\alpha \neq 0$, then $\mathcal{U} \cdot v \cap \Bbbk v_\lambda = 0$.

We consider two families in $M(\lambda)$, corresponding to PBW-bases (13) and (14). We set

$$\widetilde{m}_{a,b,c,d,e} := E_2^a E_{12}^b E_{11212}^c E_{112}^d E_1^e \cdot v_\lambda, \qquad \widetilde{n}_{a,b,c,d,e} := E_1^e E_{112}^d E_{11212}^c E_{12}^b E_2^a \cdot v_\lambda$$

for $a, b, c, d, e \in \mathbb{Z}$. Clearly, $v_\lambda = \widetilde{m}_{0,0,0,0,0} = \widetilde{n}_{0,0,0,0,0}$ and

$$\widetilde{m}_{a,b,c,d,e} \neq 0 \iff a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2} \iff \widetilde{n}_{a,b,c,d,e} \neq 0.$$

We denote by $\langle S \rangle$ the subspace generated by a subset S of a vector space. Let

$$\begin{aligned} W_1(\lambda) &= \langle \widetilde{m}_{a,b,c,d,e} \mid a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{1,2} \rangle, \\ W_2(\lambda) &= \langle \widetilde{m}_{a,b,c,d,2} \mid a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2} \rangle, \\ W(\lambda) &= \langle \widetilde{n}_{1,b,c,d,e} \mid c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2} \rangle. \end{aligned}$$

By a direct computation, we can prove:

Lemma 2.2.
(a) $F_2 \cdot W_i(\lambda) \subseteq W_i(\lambda)$, $i \in \mathbb{I}_2$,
(b) $F_1 \cdot \widetilde{m}_{a,b,c,d,i} \in \lambda(\sigma_1^{-1})(i)_{\zeta^4}(\zeta^{(i-1)8} - \lambda_1)\widetilde{m}_{a,b,c,d,i-1} + W_i(\lambda)$, $i \in \mathbb{I}_2$,
(c) $F_1 \cdot W(\lambda) \subseteq W(\lambda)$,
(d) $F_2 \cdot \widetilde{n}_{1,b,c,d,e} \in \lambda(\sigma_2^{-1})(1 - \lambda_2)\widetilde{n}_{0,b,c,d,e} + W(\lambda)$.

In consequence,

- $W_1(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda_1 = 1$;
- $W_2(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda_1 = \zeta^8$;
- $W(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda_2 = 1$. □

We denote by $m_{a,b,c,d,e}$, $n_{a,b,c,d,e}$ the classes of $\widetilde{m}_{a,b,c,d,e}$, $\widetilde{n}_{a,b,c,d,e}$ in $L(\lambda)$. We order lexicographically the set of all $m_{a,b,c,d,e}$:

$$m_{a,b,c,d,e} < m_{a',b',c',d',e'} \iff a < a', \text{ or } a = a', b < b', \text{ or } \cdots. \tag{16}$$

2.3. Simple modules

Let $\varphi : \mathcal{U} \to \mathcal{U}$ be the algebra automorphism such that

$$\varphi(K_i) = K_i^{-1}, \qquad \varphi(L_i) = L_i^{-1}, \qquad \varphi(E_i) = F_i L_i^{-1}, \qquad \varphi(F_i) = K_i^{-1} E_i,$$

$i \in \mathbb{I}_2$, cf. [14, Proposition 4.9]; this resembles the Chevalley involution. If M is a $\mathcal{U}$-module, then we denote by M^φ the $\mathcal{U}$-module with $M^\varphi = M$ as vector space and action given by $a \triangleright v = \varphi(a) \cdot v$, $v \in V$, $a \in \mathcal{U}$. If $v \in M$ has weight λ (with respect the action of Γ), then $v \in M^\varphi$ has weight λ^{-1}. The functor $M \mapsto M^\varphi$ preserves simple objects and sends lowest weight modules to highest weight modules, and vice versa. The following result is standard.

Lemma 2.3. *The subspace $X(\lambda) := \{x \in L(\lambda) : E_i x = 0 \text{ for all } i\}$ of $L(\lambda)$ is one-dimensional and there exists $\mu \in \widehat{\Gamma}$ such that $X(\lambda) \overset{(1)}{=} L(\lambda)_\mu$, $L(\lambda)^\varphi \overset{(2)}{\simeq} L(\mu^{-1})$.*

Proof. $X(\lambda) \neq 0$ because there exists $\beta \in \mathbb{N}_0^2$ maximal such that $L(\lambda)_\beta \neq 0$. Since $X(\lambda)$ is Γ-stable, there exists a weight vector $0 \neq x \in X(\lambda)$ with weight $\mu \in \widehat{\Gamma}$. Thus $\mathcal{U}^- x = \mathcal{U} x = L(\lambda)$ and (1) follows. Also $L(\lambda)^\varphi = (\mathcal{U}^- x)^\varphi \twoheadrightarrow L(\mu^{-1})$ implying (2). □

Lemma 2.4. *Let $M \in {}_{\mathcal{U}}\mathcal{M}$ a highest weight module of highest weight μ and $0 \neq v \in M^\mu$. If $m_{a,b,c,d,e} \neq 0$ in $L(\mu^{-1})$ then $z := F_2^a F_{12}^b F_{11212}^c F_{112}^d F_1^e v \neq 0$.*

There is an analogue statement for $n_{a,b,c,d,e}$.

Proof. Indeed M^φ is lowest weight of lowest weight μ^{-1}, hence $M^\varphi \twoheadrightarrow L(\mu^{-1})$; up to a non-zero scalar, $z \mapsto m_{a,b,c,d,e} \neq 0$, hence $z \neq 0$. □

2.4. A relative of $\mathfrak{u}_q(\mathfrak{sl}_2)$

We consider for a moment the algebra $\mathcal{V}$ constructed as $\mathcal{U}$ above but starting from a braided vector space of dimension 1, with braiding given by $q = \sigma(g) \in \mathbb{G}'_N$, $g \in \Lambda$, $\sigma \in \widehat{\Lambda}$. The algebra $\mathcal{V}$ is close to $\mathfrak{u}_q(\mathfrak{sl}_2)$ and has a presentation by generators $h \in \Lambda$, $\tau \in \widehat{\Lambda}$, E, F with relations

$$E^N = F^N = 0, \qquad hE = \sigma(h)Eh, \qquad \tau E = \tau(g)E\tau,$$
$$EF - FE = g - \sigma^{-1}, \qquad hF = \sigma^{-1}(h)Fh, \qquad \tau F = \tau(g^{-1})F\tau,$$

and $h\tau = \tau h$ for $h \in \Lambda$, $\tau \in \widehat{\Lambda}$, and the relations defining Λ, $\widehat{\Lambda}$. Thus

$$E^jF - FE^j = (j)_q E^{j-1}(g - q^{1-j}\sigma^{-1}), \qquad j \in \mathbb{N}. \tag{17}$$

Let $\lambda \in \widehat{\Gamma}$. Let $L(\lambda)$ be lowest weight $\mathcal{V}$-module of lowest weight λ defined in the same usual way. The same argument as for $\mathfrak{u}_q(\mathfrak{sl}_2)$ gives the following.

Lemma 2.5.
(a) *If there exists $j \in \mathbb{I}_{N-1}$ such that $\lambda(g\sigma) = q^{1-j}$, then* $\dim L(\lambda) = j$.
(b) *If $\lambda(g\sigma) \notin \{q^h | h \in \mathbb{I}_{0,N-2}\}$, then* $\dim L(\lambda) = N$.
(c) *$L(\lambda)$ has a basis $v_0, \dots, v_{\dim L(\lambda)-1}$ such that for all i,*

$$Ev_i = v_{i+1}, \qquad Fv_i = (i)_q(q^{1-i}\lambda(\sigma_1^{-1}) - \lambda(g_1))v_{i-1}, \qquad h\tau v_i = \lambda(h\tau)\sigma^i(h)\tau(g^i)v_i. \tag{18}$$

(d) *Let M be a lowest weight $\mathcal{V}$-module with lowest weight $\lambda \in \widehat{\Gamma}$. If $0 \neq v \in M^\lambda$, then $v, Ev, \dots, E^{n-1}v$ are linearly independent, where*
 (1) *either $n = j$ if $\lambda(g\sigma) = q^{1-j}$ for some (unique) $j \in \mathbb{I}_{N-1}$,*
 (2) *or else $n = N - 1$ if $\lambda(g\sigma) \notin \{q^h | h \in \mathbb{I}_{0,N-2}\}$.*
 Moreover $F^iE^iv = a_i v$ for some $a_i \in \Bbbk^\times$ when $i \in \mathbb{I}_{0,n-1}$. □

2.5. The class $\mathcal{C}_0$

The first family is easy to deal with.

Lemma 2.6. *If $\lambda \in \mathfrak{I}_1$, then $M(\lambda)$ is simple.*

Proof. By [15, 5.16] that says: if Ш $\neq 0$, then $M(\lambda)$ is simple. □

3. Simple $\mathcal{U}$-modules in class $\mathcal{C}_1$

Here we deal with the class of families satisfying exactly one of the conditions in (12). Recall that $\Gamma = \Lambda \times \widehat{\Lambda}$; we introduce $\chi_i \in \widehat{\Gamma}$ by

$$\chi_i(g, \sigma) = \sigma_i(g)\sigma(g_i), \qquad i \in \mathbb{I}_2.$$

For simplicity, we introduce the following notation:

$$g_{12} = g_1 g_2, \qquad g_{112} = g_1^2 g_2, \qquad g_{11212} = g_1^3 g_2^2,$$
$$\sigma_{12} = \sigma_1\sigma_2, \qquad \sigma_{112} = \sigma_1^2\sigma_2, \qquad \sigma_{11212} = \sigma_1^3\sigma_2^2.$$

We outline the method to compute $L(\lambda)$, $\lambda \in \mathfrak{I}_j$, $j \in \mathbb{I}_{2,10}$.

(a) As (exactly) one of the factors of the Shapovalov determinant III vanishes, there exists $\beta \neq 0$ and $w \in M(\lambda)_\beta - 0$, such that $F_i w = 0$, $i \in \mathbb{I}_2$, see Remarks 3.5, 3.8, 3.11, 3.14, 3.17, 3.20, or Lemma 2.2. Thus $\mathcal{U}w$ is a proper submodule.

(b) Assume we are dealing with $\mathfrak{I}_j$, $j \in \mathbb{I}_{2,6}$. Write $w = \sum \mathrm{p}_{a,b,c,d,e}\, \widetilde{m}_{a,b,c,d,e}$. Then there exist a, b, c, d, e such that $\mathrm{p}_{a,b,c,d,e} \neq 0$ and exactly four of the integers $a, \dots, e$ are zero. The same holds for $j \in \mathbb{I}_{7,10}$ exchanging $\widetilde{m}_{a,b,c,d,e}$ by $\widetilde{n}_{a,b,c,d,e}$. From here we describe a basis B_j of the quotient $L'(\lambda)$ of $M(\lambda)$ by $\mathcal{U}w$, $j \in \mathbb{I}_{2,10}$.

(c) Let v be the element of maximal degree of $L'(\lambda)$. A short computation shows that v belongs to every submodule of $L'(\lambda)$. Because of the inequalities defining $\mathfrak{I}_j$, there exists $F \in \mathcal{U}$ such that $Fv = v_\lambda$. Hence $L'(\lambda)$ is simple.

We work out the details for $\mathfrak{I}_2$, with shorter expositions for the other families in $\mathcal{C}_1$.

3.1. The family $\mathfrak{I}_2$

Recall that

$$\mathfrak{I}_2 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 \notin \{1, \zeta, \zeta^4, \zeta^7, \zeta^3, \zeta^9, -1, \zeta^{10}\}\}.$$

Lemma 3.1. *If $\lambda \in \mathfrak{I}_2$, then $\dim L(\lambda) = 48$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_2 = \{m_{a,b,c,d,0} : a, c \in \mathbb{I}_{0,1},\ b \in \mathbb{I}_{0,3},\ d \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = \widetilde{m}_{0,0,0,0,1}$; then $F_i w = 0$, $i \in \mathbb{I}_2$, hence $\mathcal{U}^+ w = W_1(\lambda) \leq M(\lambda)$ is proper by Lemma 2.2. Let $L'(\lambda) = M(\lambda)/\mathcal{U}^+ w$. Let $\widehat{m}_{a,b,c,d,0}$ be the class of $\widetilde{m}_{a,b,c,d,0}$ in $L'(\lambda)$. Then

$$\widehat{\mathrm{B}}_2 = \{\widehat{m}_{a,b,c,d,0} : a, c \in \mathbb{I}_{0,1},\ b \in \mathbb{I}_{0,3},\ d \in \mathbb{I}_{0,2}\}$$

is a basis of $L'(\lambda)$, ordered by (16). Thus, it is enough to show that $L'(\lambda)$ is simple. Let $0 \neq W \leq L'(\lambda)$ and pick $u \in W - 0$. Fix $\widehat{m}_{a,b,c,d,0} \in \widehat{\mathrm{B}}_2$ minimal among those whose coefficient in u is non-zero. Then

$$E_{112}^{2-d} E_{11212}^{1-c} E_{12}^{3-b} E_2^{1-a} u \in \mathbb{k}^\times \widehat{m}_{1,3,1,2,0} \implies \widehat{m}_{1,3,1,2,0} \in W.$$

By abuse of notation, we denote by v_λ its class in $L'(\lambda)$. We claim that

$$F_2 F_{12}^3 F_{11212} F_{112}^2 \widehat{m}_{1,3,1,2,0} \in \mathbb{k}^\times v_\lambda; \tag{19}$$

this implies that $v_\lambda \in W$, so $L'(\lambda)$ is simple.

To prove (19), we first consider the subalgebra $\mathcal{V}_1 = \mathbb{k}\langle g, \sigma, E_{112}, F_{112}\rangle$ of $\mathcal{U}$; clearly $\mathcal{V}_1 \simeq \mathcal{V}$ from §2.4. Then

$$F_{112}\widehat{m}_{1,3,1,0,0} = 0, \qquad g_{112}\sigma_{112}\widehat{m}_{1,3,1,0,0} = -\lambda_2 \widehat{m}_{1,3,1,0,0}, \qquad E_{112}^2 \widehat{m}_{1,3,1,0,0} = \sigma_{112}^2(g_{12}^{-6})\widehat{m}_{1,3,1,2,0}.$$

By Lemma 2.5, we conclude that

$$F_{112}^2 \widehat{m}_{1,3,1,2,0} \in \mathbb{k}^\times \widehat{m}_{1,3,1,0,0} \implies \widehat{m}_{1,3,1,0,0} \in W.$$

We next consider $\mathcal{V}_2 = \Bbbk\langle g, \sigma, E_{11212}, F_{11212}\rangle \hookrightarrow \mathcal{U}$; again, $\mathcal{V}_2 \simeq \mathcal{V}$. Then

$$F_{11212}\widehat{m}_{1,3,0,0,0} = 0, \qquad g_{11212}\sigma_{11212}\widehat{m}_{1,3,0,0,0} = -\lambda_2^2\widehat{m}_{1,3,0,0,0},$$
$$E_{11212}\widehat{m}_{1,3,0,0,0} = \sigma_{11212}(g_1^{-3}g_2^{-4})\widehat{m}_{1,3,1,0,0},$$
$$\overset{\text{Lemma 2.5}}{\Longrightarrow} F_{11212}\widehat{m}_{1,3,1,0,0} \in \Bbbk^\times\widehat{m}_{1,3,0,0,0} \implies \widehat{m}_{1,3,0,0,0} \in W.$$

Once again, we consider $\mathcal{V}_3 = \Bbbk\langle g, \sigma, E_{12}, F_{12}\rangle \hookrightarrow \mathcal{U}$; thus $\mathcal{V}_3 \simeq \mathcal{V}$ from §2.4. Then

$$F_{12}\widehat{m}_{1,0,0,0,0} = 0, \qquad g_{12}\sigma_{12}\widehat{m}_{1,0,0,0,0} = \lambda_2\zeta^{11}\widehat{m}_{1,0,0,0,0}, \qquad E_{12}^3\widehat{m}_{1,0,0,0,0} = \sigma_{12}^3(g_2^{-1})\widehat{m}_{1,3,0,0,0}$$
$$\overset{\text{Lemma 2.5}}{\Longrightarrow} F_{12}^3\widehat{m}_{1,3,0,0,0} \in \Bbbk^\times\widehat{m}_{1,0,0,0,0} \implies \widehat{m}_{1,0,0,0,0} \in W.$$

Now $F_2\widehat{m}_{1,0,0,0,0} = \lambda(\sigma_2)^{-1}(\lambda_2 - 1)v_\lambda \neq 0$, and (19) follows. □

Corollary 3.2. *If $\lambda \in \mathfrak{I}_2$, then $N(\lambda) \simeq L(\chi_1\lambda)$ and $\chi_1\lambda \in \mathfrak{I}_3$.*

Proof. By the proof of the Lemma, $N(\lambda)$ is of lowest weight $\chi_1\lambda$ and $\dim N(\lambda) = 96$. It is easy to see that $\chi_1\lambda \in \mathfrak{I}_3$; hence $\dim L(\chi_1\lambda) = 96$ by Lemma 3.3 and the claim follows. □

3.2. The family $\mathfrak{I}_3$

Recall that

$$\mathfrak{I}_3 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 \notin \{\pm 1, \zeta^2, \zeta^3, \zeta^5, \zeta^8, \zeta^9, \zeta^{11}\}\}.$$

Lemma 3.3. *If $\lambda \in \mathfrak{I}_3$, then $\dim L(\lambda) = 96$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_3 = \{m_{a,b,c,d,e} | a, c \in \mathbb{I}_{0,1},\ b \in \mathbb{I}_{0,3},\ d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\}.$$

Proof. Let $w = \widetilde{m}_{0,0,0,0,2}$ and $L'(\lambda) = M(\lambda)/\mathcal{U}^+w$. We identify B_3 with a basis of $L'(\lambda)$. Now $F_2F_{12}^3F_{11212}F_{112}^2F_1m_{1,3,1,2,1} \in \Bbbk^\times v_\lambda$, hence $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.4. *If $\lambda \in \mathfrak{I}_3$, then $N(\lambda) \simeq L(\chi_1^2\lambda)$ and $\chi_1^2\lambda \in \mathfrak{I}_2$.* □

3.3. The family $\mathfrak{I}_4$

Recall that

$$\mathfrak{I}_4 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = -1,\ \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}.$$

We start by a Remark that will be useful elsewhere.

Remark 3.5. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1^2\lambda_2 = -1$, then $w = F_1^2E_{112}E_1^2v_\lambda \in M(\lambda)$ satisfies

$$F_1w = F_2w = 0. \tag{20}$$

Proof. By a direct computation,

$$F_{112}E_{112}E_1^2v_\lambda = \lambda(\sigma_1^{-2}\sigma_2^{-1})q_{21}^2\zeta^4(\lambda_1^2\lambda_2 + 1)E_1^2v_\lambda.$$

As $M(\lambda)_{4\alpha_1} = M(\lambda)_{3\alpha_1} = 0$, we have that $F_2E_{112}E_1^2v_\lambda = F_1E_{112}E_1^2v_\lambda = 0$, so

$$0 = F_{112}E_{112}E_1^2v_\lambda = \zeta^8q_{12}^2F_2F_1^2E_{112}E_1^2v_\lambda.$$

This shows that $F_2w = 0$; on the other hand, $F_1w = F_1^3(E_{112}E_1^2v_\lambda) = 0$, since $F_1^3 = 0$. □

Lemma 3.6. *If $\lambda \in \mathfrak{I}_4$, then* $\dim L(\lambda) = 48$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_4 = \{m_{a,b,c,0,e} : a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_1^2 E_{112} E_1^2 v_\lambda$. By Remark 3.5, $\mathcal{U}w$ is a proper submodule. We identify B_4 with a basis of $L'(\lambda) := M(\lambda)/\mathcal{U}w$. We check that there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.7. *If $\lambda \in \mathfrak{I}_4$, then $N(\lambda) \simeq L(\chi_1^2 \chi_2 \lambda)$ and $\chi_1^2 \chi_2 \lambda \in \mathfrak{I}_5$.*

3.4. The family $\mathfrak{I}_5$

Recall that

$$\mathfrak{I}_5 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2 \lambda_2 = \zeta^{10}, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}.$$

Here is another Remark that will be useful later, proved as Remark 3.5.

Remark 3.8. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1^2 \lambda_2 = \zeta^{10}$, then $w = F_1^2 E_{112}^2 E_1^2 v_\lambda \in M(\lambda)$ satisfies (20).

Lemma 3.9. *If $\lambda \in \mathfrak{I}_5$, then* $\dim L(\lambda) = 96$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_5 = \{m_{a,b,c,d,e} | a, c, d \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_1^2 E_{112}^2 E_1^2 v_\lambda$. By Remark 3.8, $\mathcal{U}w$ is a proper submodule. We identify B_5 with a basis of $L'(\lambda) := M(\lambda)/\mathcal{U}w$. We check that there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,1,2} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.10. *If $\lambda \in \mathfrak{I}_5$, then $N(\lambda) \simeq L(\chi_1^4 \chi_2^2 \lambda)$ and $\chi_1^4 \chi_2^2 \lambda \in \mathfrak{I}_4$.*

3.5. The family $\mathfrak{I}_6$

Recall that

$$\mathfrak{I}_6 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1^3 \lambda_2^2 = -1, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}.$$

Still another Remark useful elsewhere, with an analogous proof as above.

Remark 3.11. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1^3 \lambda_2^2 = -1$, then $w = F_1^2 F_{112}^2 E_{11212} E_{112}^2 E_1^2 v_\lambda$ satisfies (20).

Lemma 3.12. *If $\lambda \in \mathfrak{I}_6$, then* $\dim L(\lambda) = 72$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_6 = \{m_{a,b,0,d,e} | a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let w be as in Remark 3.11; then $\mathcal{U}w$ is proper. Again B_6 is identified with a basis of $L'(\lambda) = M(\lambda)/\mathcal{U}w$; since there is $F \in \mathcal{U}$ such that $Fm_{1,3,0,2,2} = v_\lambda$, $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.13. *If $\lambda \in \mathfrak{I}_6$, then $N(\lambda) \simeq L(\chi_1^3 \chi_2^2 \lambda)$ and $\chi_1^3 \chi_2^2 \lambda \in \mathfrak{I}_6$.*

3.6. The family $\mathfrak{I}_7$

Recall that

$$\mathfrak{I}_7 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta,\ \lambda_1 \notin \{1, \zeta^8, \zeta, \zeta^4, \zeta^9\}\}.$$

Again we start by a useful remark.

Remark 3.14. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1\lambda_2 = \zeta$, then $w = F_2E_2E_{12}v_\lambda \in M(\lambda)$ satisfies (20).

Lemma 3.15. *If $\lambda \in \mathfrak{I}_7$, then $\dim L(\lambda) = 36$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_7 = \{n_{a,0,c,d,e} | a, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_2E_2E_{12}v_\lambda$. By Remark 3.14, $\mathcal{U}w \subsetneq M(\lambda)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$, so B_7 is a basis of $L'(\lambda)$. There exists $F \in \mathcal{U}$ such that $Fn_{1,0,1,2,2} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.16. *If $\lambda \in \mathfrak{I}_7$, then $N(\lambda) \simeq L(\chi_1\chi_2\lambda)$ and $\chi_1\chi_2\lambda \in \mathfrak{I}_9$.*

3.7. The family $\mathfrak{I}_8$

Recall that

$$\mathfrak{I}_8 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^4,\ \lambda_1 \notin \{1, \zeta^8, \zeta^4, \zeta^2, -1, \zeta^{10}\}\}.$$

Remark 3.17. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1\lambda_2 = \zeta^4$, then $w = F_2E_2E_{12}^2v_\lambda \in M(\lambda)$ satisfies (20).

Proof. Analogous to Remark 3.5. □

Lemma 3.18. *If $\lambda \in \mathfrak{I}_8$, then $\dim L(\lambda) = 72$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_8 = \{n_{a,b,c,d,e} | a, b, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_2E_2E_{12}^2v_\lambda$. By Remark 3.17, $\mathcal{U}w \subsetneq M(\lambda)$. Now B_8 identifies with a basis of $L'(\lambda) := M(\lambda)/\mathcal{U}w$. Since there is $F \in \mathcal{U}$ such that $Fn_{1,1,1,2,2} = v_\lambda$, $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.19. *If $\lambda \in \mathfrak{I}_8$, then $N(\lambda) \simeq L(\chi_1^2\chi_2^2\lambda)$ and $\chi_1^2\chi_2^2\lambda \in \mathfrak{I}_8$.*

3.8. The family $\mathfrak{I}_9$

Recall that

$$\mathfrak{I}_9 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^7,\ \lambda_1 \notin \{1, \zeta^8, \zeta^7, \zeta^4, \zeta^{11}\}\}.$$

Remark 3.20. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1\lambda_2 = \zeta^7$, then $w = F_2E_2E_{12}^3v_\lambda \in M(\lambda)$ satisfies (20).

Proof. Analogous to Remark 3.5. □

Lemma 3.21. *If $\lambda \in \mathfrak{I}_9$, then $\dim L(\lambda) = 108$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_9 = \{n_{a,b,c,d,e} | a, c \in \mathbb{I}_{0,1}, b, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_2E_2E_{12}^3v_\lambda$. By Remark 3.20, $\mathcal{U}w \subsetneq M(\lambda)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$, so B_9 is a basis of $L'(\lambda)$. Since there exists $F \in \mathcal{U}$ such that $Fn_{1,2,1,2,2} = v_\lambda$, $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.22. *If $\lambda \in \mathfrak{I}_9$, then $N(\lambda) \simeq L(\chi_1^3\chi_2^3\lambda)$ and $\chi_1^3\chi_2^3\lambda \in \mathfrak{I}_7$.*

3.9. The family $\mathfrak{I}_{10}$

Recall that

$$\mathfrak{I}_{10} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{G}_{12},\ \lambda_2 = 1\}.$$

Lemma 3.23. *If $\lambda \in \mathfrak{I}_{10}$, then $\dim L(\lambda) = 72$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{10} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = \widetilde{n}_{1,0,0,0,0}$ and $L'(\lambda) = M(\lambda)/\mathcal{U}^+w$. We identify B_{10} with a basis of $L'(\lambda)$. Now $F_1^2F_{112}^2F_{11212}F_{12}^3n_{0,3,1,2,2} \in \Bbbk^\times v_\lambda$, hence $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.24. *If $\lambda \in \mathfrak{I}_{10}$, then $N(\lambda) \simeq L(\chi_2\lambda)$ and $\chi_2\lambda \in \mathfrak{I}_{10}$.*

4. Simple $\mathcal{U}$-modules in class $\mathcal{C}_2$

We start by the method to compute $L(\lambda)$, $\lambda \in \mathfrak{I}_j$, $j \in \mathbb{I}_{11,47}$. We illustrate by considering $\mathfrak{I}_{11}$, which is small enough to allow complete details; and $\mathfrak{I}_{13}$, with less explicit yet complete enough arguments. Then we give the main features of the proofs for the other families in $\mathcal{C}_2$. Here are the steps of the method:

(1) We identify easily a proper submodule $W = \mathcal{U}w_1$ of $M(\lambda)$ as follows:
- ◇ if $j \in \mathbb{I}_{11,17}$, then $w_1 = \widetilde{m}_{0,0,0,0,1}$, so $W = W_1(\lambda)$, see Lemma 2.2;
- ◇ if $j \in \mathbb{I}_{18,24}$, then $w_1 = \widetilde{m}_{0,0,0,0,2}$, so $W = W_2(\lambda)$, again by Lemma 2.2;
- ◇ if $j \in \mathbb{I}_{25,35}$, then w_1 is as in one of the Remarks 3.5, 3.8, 3.14, 3.17, 3.20;
- ◇ if $j \in \mathbb{I}_{36,47}$, then $w_1 = \widetilde{n}_{1,0,0,0,0}$, so $W = W(\lambda)$ by Lemma 2.2.

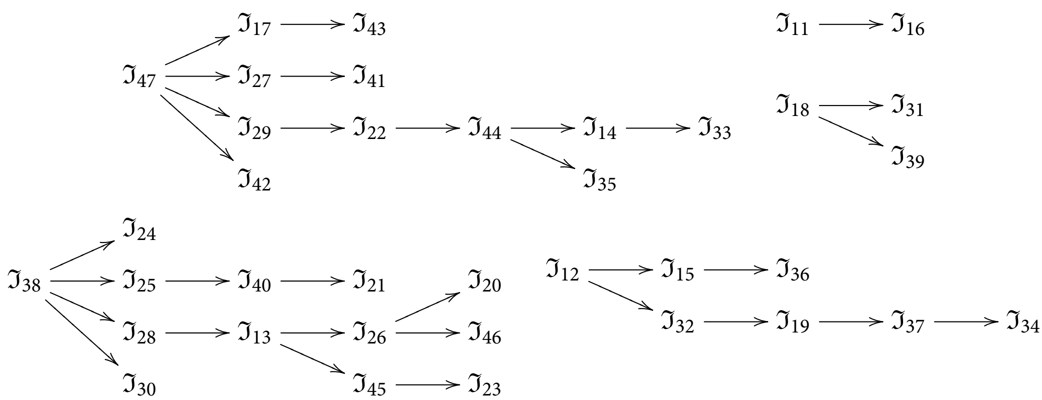
A basis of $M(\lambda)/W$ is obtained by restriction of the height of a specific PBW generator. Below we denote by w_2 an element of $M(\lambda)$ or its class modulo W, indistinctly.

(2) Next we show that there exists $\beta \neq 0$ and $w_2 \in (M(\lambda)/W)_\beta - 0$, such that $F_iw_2 = 0$, $i \in \mathbb{I}_2$; for this, we either apply one of Remarks 3.5, 3.8, 3.11, 3.14, 3.17, 3.20, or else proceed by direct computation. Hence $\mathcal{U}w_2$ is a proper submodule of $M(\lambda)/W$.

(3) Let $L'(\lambda) = M(\lambda)/(W + \mathcal{U}w_2)$. We consider a suitable set B_j inside the image of the PBW-basis in $L'(\lambda)$ that spans $L'(\lambda)$. To prove that B_j is linearly independent, we apply one of the following procedures:
   (a) For $j \in \mathbb{J} = \{11, 12, 18, 38\}$, the elements of B_j are homogeneous of different degrees.
   (b) Assume that $j \notin \mathbb{J}$. Then $\mathcal{U}w_2 \leq M(\lambda)/W$ projects onto the simple module $L(\nu)$, where ν is the weight of w_2. Also, let $u \in M(\lambda)/W$ be the element of maximal degree; then $(\mathcal{U}u)^\varphi$ projects onto a simple $L(\mu)$. Let $\mathfrak{I}_k$ and $\mathfrak{I}_\ell$ be the families containing ν and μ, respectively. At this point, we observe that we are proceeding recursively, so that we already know the simple modules in $\mathfrak{I}_k$ and $\mathfrak{I}_\ell$. With this information on hand, we check that $\mathcal{U}u = \mathcal{U}w_2 \simeq L(\nu)$. This isomorphism provides a basis of $\mathcal{U}w_2$; we conclude that there is a linear complement of $\mathcal{U}w_2$ with a basis $\widetilde{\mathrm{B}}_j$ projecting onto B_j; thus B_j is a basis of $L'(\lambda)$.

(4) Finally we prove that $L'(\lambda)$ is simple. Let v be the element of maximal degree of $L'(\lambda)$. A short computation shows that v belongs to every submodule of $L'(\lambda)$. Applying Lemma 2.5 (or by direct computation when we have a table for the action), there exists $F \in \mathcal{U}$ such that $Fv = v_\lambda$. Hence $L'(\lambda)$ is simple.

As said, we proceed recursively, but with respect to an ad hoc partial ordering of the families in $\mathcal{C}_2$. In the quiver below, we describe this ordering; $\mathfrak{J}_{11} \longrightarrow \mathfrak{J}_{16}$ means that knowledge on $\mathfrak{J}_{11}$ is used for $\mathfrak{J}_{16}$. As we see, there is no vicious circle.



4.1. The family $\mathfrak{J}_{11}$

Recall that $\mathfrak{J}_{11} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = \zeta\}$.

Lemma 4.1. *If $\lambda \in \mathfrak{J}_{11}$, then $\dim L(\lambda) = 11$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{11} = \{m_{a,b,0,d,0} | a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{0,2}\} - \{m_{1,1,0,0,0}\}.$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ is described in Table 2.

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = \widetilde{m}_{1,1,0,0,0}$; hence $F_i w_1 = 0$, $i \in \mathbb{I}_2$,

$$F_1 \widetilde{m}_{1,1,0,0,0} = 0, \qquad F_2 \widetilde{m}_{1,1,0,0,0} = (\zeta^{11} - 1)\lambda(g_2)\widetilde{m}_{1,0,0,0,1} \in W_1(\lambda) = \mathcal{U} w_1.$$

Table 2. Simple modules for $\lambda \in \mathfrak{J}_{11}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	0	$v_{0,1}$	0	0
$v_{0,1}$	$v_{1,1}$	0	0	$(\zeta^{11} - 1)v_{0,0}$
$v_{1,1}$	$v_{2,1}$	0	$q_{12}(\zeta - 1)v_{0,1}$	0
$v_{2,1}$	0	$v_{2,2}$	$q_{12}\zeta^8(1 + \zeta^3)v_{1,1}$	0
$v_{2,2}$	$v_{3,2}$	0	0	$q_{21}^2(1 - \zeta)v_{2,1}$
$v_{3,2}$	$v_{4,2}$	$v_{3,3}$	$q_{12}^2(\zeta^2 - 1)v_{2,2}$	0
$v_{4,2}$	0	$v_{4,3}$	$2q_{12}^2(\zeta^2 - 1)v_{3,2}$	0
$v_{3,3}$	$q_{12}\frac{\zeta^8(\zeta^3-1)}{2}v_{4,3}$	0	0	$q_{21}^3(\zeta^2 - 1)v_{3,2}$
$v_{4,3}$	$v_{5,3}$	0	$2q_{12}^2(\zeta^2 - 1)v_{3,3}$	$q_{21}^4(\zeta^3 - 1)v_{4,2}$
$v_{5,3}$	0	$v_{5,4}$	$q_{12}^3\zeta^8(1 - \zeta^{11})v_{4,3}$	0
$v_{5,4}$	0	0	0	$q_{21}^5(\zeta^{11} + 1)v_{5,3}$

Thus $\mathcal{U}w_1 + \mathcal{U}w_2$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$ is simple. Consider the following elements of $L'(\lambda)$:

$$\begin{aligned} v_{0,0} &= \widetilde{m}_{0,0,0,0,0}, & v_{0,1} &= \widetilde{m}_{1,0,0,0,0}, & v_{1,1} &= \widetilde{m}_{0,1,0,0,0}, & v_{2,1} &= \widetilde{m}_{0,0,0,1,0}, \\ v_{2,2} &= \widetilde{m}_{1,0,0,1,0}, & v_{3,2} &= \widetilde{m}_{0,1,0,1,0}, & v_{4,2} &= \widetilde{m}_{0,0,0,2,0}, & v_{3,3} &= \widetilde{m}_{1,1,0,1,0}, \\ v_{4,3} &= \widetilde{m}_{1,0,0,2,0}, & v_{5,3} &= \widetilde{m}_{0,1,0,2,0}, & v_{5,4} &= \widetilde{m}_{1,1,0,2,0}. \end{aligned}$$

Notice that $v_{i,j} \in L'(\lambda)_{i\alpha_1 + j\alpha_2}$. The action of E_i, F_i on these vectors is given in Table 2, and we check that $L'(\lambda)$ is spanned by the $v_{i,j}$'s by direct computation.

For each $v_{i,j}$ there exists $E_{i,j} \in \mathcal{U}^+_{(5-i)\alpha_1+(4-j)\alpha_2}$ such that $E_{i,j}v_{i,j} = v_{5,4}$; also, there exists $F_{5,4} \in \mathcal{U}^-_{-5\alpha_1-4\alpha_2}$ such that $F_{5,4}v_{5,4} = v_\lambda$. This implies that the $v_{i,j}$'s are $\neq 0$; hence they are linearly independent, since they have different degrees, and B_{11} is identified with a basis of $L'(\lambda)$.

Let now $0 \neq U \leq L'(\lambda)$ and pick $v \in U - 0$. Expressing v in the basis B_{11}, we see that there exists $E \in \mathcal{U}^+$ such that $Ev = v_{5,4}$. But $\mathcal{U}v_{5,4} = L'(\lambda)$. Hence $L'(\lambda)$ is simple. □

Remark 4.2. If $\lambda \in \mathfrak{I}_{11}$, then $N(\lambda)/W_1(\lambda) \simeq L(\chi_1\chi_2^2\lambda)$, with $\chi_1\chi_2^2\lambda \in \mathfrak{I}_{41}$ has dimension 37. Now $W_1(\lambda)$ is a lowest weight module of lowest weight $\chi_1\lambda \in \mathfrak{I}_{43}$; since $\dim L(\chi_1\lambda) = 25$ by Lemma 4.34, the kernel of $W_1(\lambda) \twoheadrightarrow L(\chi_1\lambda)$ is a submodule of dimension 71.

4.2. *The family $\mathfrak{I}_{12}$*

Recall that $\mathfrak{I}_{12} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^4\}$.

Lemma 4.3. *If $\lambda \in \mathfrak{I}_{12}$, then $\dim L(\lambda) = 11$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{12} = \{m_{a,b,0,d,0} : a, b, d \in \mathbb{I}_{0,1}\} \cup \{m_{0,1,1,0,0}, m_{1,0,1,1,0}, m_{0,0,1,1,0}\}.$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ is described in Table 3.

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = F_2E_2E_{12}^2v_\lambda$; then $F_iw_j = 0$ for $i,j \in \mathbb{I}_2$, so $\mathcal{U}w + W_1(\lambda)$ is a proper submodule of $M(\lambda)$. Let $L'(\lambda) := M(\lambda)/\mathcal{U}w + W_1(\lambda)$. We label the elements of B_{12} as follows:

$$\begin{aligned} v_{0,0} &= m_{0,0,0,0,0}, & v_{0,1} &= m_{1,0,0,0,0}, & v_{1,1} &= m_{0,1,0,0,0}, & v_{2,1} &= m_{0,0,0,1,0}, \\ v_{2,2} &= m_{1,0,0,1,0}, & v_{1,2} &= m_{1,1,0,0,0}, & v_{3,2} &= m_{0,1,0,1,0}, & v_{3,3} &= m_{1,1,0,1,0}, \\ v_{4,3} &= m_{0,1,1,0,0}, & v_{5,3} &= m_{0,0,1,1,0}, & v_{5,4} &= m_{1,0,1,1,0}. \end{aligned}$$

The action of E_i, F_i on these vectors is given in Table and B_{12} is a basis of $L'(\lambda)$. Looking at the table, there exists $F \in \mathcal{U}^-$ such that $Fm_{1,0,1,1,0} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Table 3. Simple modules for $\lambda \in \mathfrak{I}_{12}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	0	$v_{0,1}$	0	0
$v_{0,1}$	$v_{1,1}$	0	0	$(\zeta^{10}+1)v_{0,0}$
$v_{1,1}$	$v_{2,1}$	$v_{1,2}$	$q_{12}(\zeta-1)v_{0,1}$	0
$v_{2,1}$	0	$v_{2,2}$	$q_{12}\zeta^8(1+\zeta^3)v_{1,1}$	0
$v_{1,2}$	$\zeta^{11}(1+\zeta^3)q_{12}v_{2,2}$	0	0	$q_{21}(1+\zeta^3)\zeta^4v_{1,1}$
$v_{2,2}$	$v_{3,2}$	0	$q_{12}(\zeta^3+1)\zeta^8v_{1,2}$	$-q_{2,1}^2v_{2,1}$
$v_{3,2}$	0	$v_{3,3}$	$q_{12}^2\zeta^{10}v_{2,2}$	0
$v_{3,3}$	0	0	0	$q_{21}^3\zeta^3(1-\zeta)v_{3,2}$
$v_{4,3}$	$\zeta^9q_{12}v_{5,3}$	0	$q_{12}^4\zeta(3)_{\zeta^{11}}v_{3,3}$	0
$v_{5,3}$	0	$v_{5,4}$	$-q_{12}^2(1+\zeta^3)v_{4,3}$	0
$v_{5,4}$	0	0	0	$q_{21}^5(1-\zeta)\zeta^4v_{5,3}$

4.3. The family $\mathfrak{I}_{13}$

Recall that $\mathfrak{I}_{13} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^7\}$.

Lemma 4.4. *If $\lambda \in \mathfrak{I}_{13}$, then $\dim L(\lambda) = 23$. A basis of $L(\lambda)$ is given by*

$$B_{13} = \{m_{a,b,0,d,0} | b \in \mathbb{I}_{0,2}\} \cup \{m_{a,0,1,0,0}, m_{0,3,0,d,0}, m_{1,3,0,1,0} | a \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = F_2E_2E_{12}^3 v_\lambda$. Then $W_1(\lambda) = \mathcal{U}w_1$ by Lemma 2.2, and $F_1w_2 = F_2w_2 = 0$ by Remark 4.22, so $\mathcal{U}w_1 + \mathcal{U}w_2 \lneq M(\lambda)$. We claim that $L'(\lambda) := M(\lambda)/(\mathcal{U}w_1 + \mathcal{U}w_2)$ is simple and B_{13} is a basis of $L'(\lambda)$.

Let $M = M(\lambda)/W_1(\lambda)$ and $u = m_{1,3,1,2,0} \in M$. Notice that $E_{112}^2 E_{11212} E_2 w_2 = -q_{12}^{18}\, u$, so $u \in \mathcal{U}w_2$. On the other hand, $E_i u = 0$, $i \in \mathbb{I}_2$, $g_1\sigma_1 u = u$ and $g_2\sigma_2 u = \zeta^9 u$, so $(\mathcal{U}u)^\varphi$ projects over a simple module $L(\mu)$ with $\mu \in \mathfrak{I}_{14}$, see Lemma 2.3; in particular there exists $F' \in \mathcal{U}_{-7\alpha_1-5\alpha_2}$ such that $F'u \neq 0$. As $\mathcal{U}u \subseteq \mathcal{U}w_2$ and $\mathcal{U}w_2$ is a lowest weight module,

$$F'u \in (\mathcal{U}u)_{3\alpha_1+3\alpha_2} \subseteq (\mathcal{U}w_2)_{3\alpha_1+3\alpha_2} = \Bbbk w.$$

Hence we may assume that $F'u = w_2$, and $\mathcal{U}u = \mathcal{U}w_2$.

Also $g_1\sigma_1 w_2 = \zeta^9 w_2$, $g_2\sigma_2 w_2 = \zeta^4 w_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\nu)$ with $\nu \in \mathfrak{I}_{28}$. For any $v \in M$, $v \neq 0$, there exists $E \in \mathcal{U}$ such that $Ev = u$. Thus we conclude that $\mathcal{U}w_2 \simeq L(\nu)$, and then $\dim L'(\lambda) = 48 - 25 = 23$ by Lemma 4.19.

Applying Lemma 2.5, there exists $F \in \mathcal{U}^-$ such that $Fm_{0,3,0,2,0} = v_\lambda$. Note that

$$E_2 m_{0,3,0,2,0} = m_{1,3,0,2,0} = 0$$

since $0 = E_{12}m_{0,3,1,0,0}$ and $\Bbbk m_{1,2,1,1,0} = \Bbbk m_{1,3,0,2,0}$. Also $E_1 m_{0,3,0,2,0} = 0$ because it is a scalar multiple of $m_{0,1,1,2,0}$, which is 0. Using this fact and previous relations, we are able to prove that B_{13} spans $L'(\lambda)$, but as B_{13} has 23 elements, it is a basis.

Let $0 \neq W \leq L'(\lambda)$, $w \in W - 0$. Arguing as before, there exists $E \in \mathcal{U}^+$ such that $Ew = m_{0,3,0,2,0}$, so $m_{0,3,0,2,0} \in W$, but then $v_\lambda \in W$, so $L'(\lambda)$ is simple. □

4.4. The family $\mathfrak{I}_{14}$

Recall that $\mathfrak{I}_{14} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^3\}$.

Lemma 4.5. *If $\lambda \in \mathfrak{I}_{14}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$B_{14} = \{m_{a,b,0,d,0} \mid a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \cup \{m_{0,0,1,0,0}, m_{0,0,1,2,0}\} - \{m_{1,3,0,2,0}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = (1+\zeta^3)\widetilde{m}_{1,0,1,0,0} + q_{12}\zeta^3(1+\zeta)\widetilde{m}_{1,1,0,1,0}$. Then $W_1(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$ by direct computation.

Let $M = M(\lambda)/W_1(\lambda)$, $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_1(\lambda)$ and $u = m_{1,3,1,2,0} \in M$. Then $(\mathcal{U}u)^\varphi$ projects over $L(\mu)$ for some $\mu \in \mathfrak{I}_{13}$. Also, $\mathcal{U}w_2$ projects over $L(\nu)$ for some $\nu \in \mathfrak{I}_{44}$. Hence $\mathcal{U}u = \mathcal{U}w_2$, and moreover $\mathcal{U}w_2$ is simple, so $\dim L'(\lambda) = 48 - 25 = 23$ by Lemma 4.35. By direct computation $L'(\lambda)$ is spanned by B_{14}, so B_{14} is a basis of $L'(\lambda)$.

Moreover there exists $F \in \mathcal{U}^-$ such that $Fm_{1,0,1,2,0} = v_\lambda$, so $L'(\lambda)$ is simple. □

4.5. The family $\mathfrak{I}_{15}$

Recall that $\mathfrak{I}_{15} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^9\}$.

Lemma 4.6. *If $\lambda \in \mathfrak{I}_{15}$, then* $\dim L(\lambda) = 37$. *A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned} \mathrm{B}_{15} = &\{m_{a,b,c,d,0} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \\ &- \{m_{a,b,1,d,0} | a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{2,3}, d \in \mathbb{I}_{0,2}, (a, b, d) \neq (0, 2, 2)\}. \end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $u = \widetilde{m}_{1,3,1,2,0}$, $w_2 = F_2F_{12}F_{112}^2u$. Then $W_1(\lambda) = \mathcal{U}w_1$.

Let $M = M(\lambda)/W_1(\lambda)$, so $E_1u = E_2u = 0$ in M, and $(\mathcal{U}u)^{\varphi} \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{11}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{12}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 37$ by Lemma 4.3, and B_{15} is a basis of $L'(\lambda)$. There exists F such that $Fm_{0,2,1,2,0} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.6. The family $\mathfrak{I}_{16}$

Recall that $\mathfrak{I}_{16} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = -1\}$.

Lemma 4.7. *If $\lambda \in \mathfrak{I}_{16}$, then* $\dim L(\lambda) = 37$. *A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned} \mathrm{B}_{16} = &\{m_{a,b,c,d,0} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \\ &- \big(\{m_{a,3,c,d,0} | a, c \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\} \cup \{m_{1,2,1,2,0}, m_{0,2,1,2,0}, m_{1,2,0,2,0}\}\big). \end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $u = \widetilde{m}_{1,3,1,2,0}$, $w_2 = F_2F_{11212}F_{112}u$. Then $W_1(\lambda) = \mathcal{U}w_1$.

Let $M = M(\lambda)/W_1(\lambda)$, so $E_1u = E_2u = 0$ in M', and $(\mathcal{U}u)^{\varphi} \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{12}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{11}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 37$ by Lemma 4.1, and B_{16} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,1,1,2,0} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.7. The family $\mathfrak{I}_{17}$

Recall that $\mathfrak{I}_{17} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = \zeta^{10}\}$.

Lemma 4.8. *If $\lambda \in \mathfrak{I}_{17}$, then* $\dim L(\lambda) = 47$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{17} = \{m_{a,b,c,d,0} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}, (a, b, c, d) \neq (1, 3, 1, 2)\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = \widetilde{m}_{1,3,1,2,0}$. Then $W_1(\lambda) = \mathcal{U}w_1$, and $F_iw = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{47}$. Let $M = M(\lambda)/W_1(\lambda)$, hence $\mathcal{U}w_2 \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 47$ by Lemma 4.38, and B_{17} is a basis of $L'(\lambda)$. There exists F such that $Fm_{0,3,1,2,0} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.8. The family $\mathfrak{I}_{18}$

Recall that $\mathfrak{I}_{18} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^5\}$.

Lemma 4.9. *If $\lambda \in \mathfrak{I}_{18}$, then* $\dim L(\lambda) = 11$. *A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned} \mathrm{B}_{18} = &\{m_{a,b,1,0,1} | a, b \in \mathbb{I}_{0,1}\} \cup \{m_{0,b,0,0,e} | e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}\} \cup \{m_{1,0,0,0,0}\} \\ &- \{m_{1,1,1,0,1}, m_{3,0,0,0,1}\}. \end{aligned}$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ is described in Table 4.

Table 4. Simple modules for $\lambda \in \mathfrak{I}_{18}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(\sigma_1)F_1 \cdot w$	$\lambda(g_2)^{-1}F_2 \cdot w$
$v_{0,0}$	$v_{1,0}$	$v_{0,1}$	0	0
$v_{1,0}$	0	$q_{21}\zeta^9(4)_\zeta v_{1,1}$	$(1+\zeta^2)v_{0,0}$	0
$v_{0,1}$	$\zeta^8(4)_\zeta v_{1,1}$	0	0	$(\zeta^7-1)v_{0,0}$
$v_{1,1}$	$\frac{q_{12}\zeta^4(4)_{\zeta^7}}{3}v_{2,1}$	0	$q_{12}(\zeta-1)v_{0,1}$	$(\zeta^{11}-1)v_{1,0}$
$v_{2,1}$	0	$q_{21}^2\zeta^{10}(4)_\zeta v_{2,2}$	$(1-\zeta^4)v_{1,1}$	0
$v_{2,2}$	$(1-\zeta^4)v_{3,2}$	0	0	$\frac{-(1+\zeta^2)(3)_{\zeta^7}}{3}v_{2,1}$
$v_{3,2}$	$v_{4,2}$	$q_{12}\zeta^{10}(4)_\zeta v_{3,3}$	$\zeta^{10}(4)_\zeta v_{2,2}$	0
$v_{4,2}$	0	$v_{4,3}$	$q_{12}^2\zeta(\zeta+1)v_{3,2}$	0
$v_{3,3}$	$\frac{q_{12}^4\zeta^7(4)_\zeta}{3}v_{4,3}$	0	0	$\frac{\zeta^8-1}{3}v_{3,2}$
$v_{4,3}$	$v_{5,3}$	0	$q_{12}^3(\zeta^{11}+1)(4)_\zeta^2 v_{3,3}$	$q_{21}^4(\zeta^{11}-1)v_{4,2}$
$v_{5,3}$	0	0	$q_{12}^3\zeta^4 v_{4,3}$	0

Proof. $W_2(\lambda) \le M(\lambda)$ by Lemma 2.2 and $w := F_2E_2E_{12}$ satisfies $F_1w = F_2w = 0$ by Remark 3.14. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$. We fix the following notation for B_{18}:

$$v_{0,0} = m_{0,0,0,0,0},\quad v_{1,0} = m_{0,0,0,0,1},\quad v_{0,1} = m_{1,0,0,0,0},\quad v_{1,1} = m_{0,1,0,0,0},$$
$$v_{2,1} = m_{0,1,0,0,1},\quad v_{2,2} = m_{0,2,0,0,0},\quad v_{3,2} = m_{0,2,0,0,1},\quad v_{4,2} = m_{0,0,1,0,1},$$
$$v_{3,3} = m_{0,3,0,0,0},\quad v_{4,3} = m_{1,0,1,0,1},\quad v_{5,3} = m_{0,1,1,0,1}.$$

We check that $L'(\lambda)$ is spanned by B_{18}. From Table 4 there exist $E_{i,j} \in \mathcal{U}^+_{(5-i)\alpha_1+(3-j)\alpha_2}$, $F_{5,3} \in \mathcal{U}^-_{-5\alpha_1-3\alpha_2}$ such that $E_{i,j}v_{i,j} = v_{5,3}$, $F_{5,3}v_{5,3} = v_\lambda$. Thus $L'(\lambda)$ is simple. □

4.9. The family $\mathfrak{I}_{19}$

Recall that $\mathfrak{I}_{19} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^8\}$.

Lemma 4.10. *If $\lambda \in \mathfrak{I}_{19}$, then* $\dim L(\lambda) = 35$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{19} = \{m_{0,b,0,d,e} | b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\} \cup \{m_{1,b,0,0,e} |\ b, e \in \mathbb{I}_{0,1}\} \cup \{m_{0,b,1,0,0} |\ b \in \mathbb{I}_{1,3}\}$$
$$\cup \{m_{1,b,0,0,1} \mid b \in \mathbb{I}_{2,3}\} \cup \{m_{1,0,0,1,1}, m_{0,0,1,1,0}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $w_2 = F_2E_2E_{12}^2v_\lambda$. Then $W_2(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{m}_{1,3,1,2,1}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{32}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, so $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 61 = 35$ by Lemma 4.23, and B_{19} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.10. The family $\mathfrak{I}_{20}$

Recall that $\mathfrak{I}_{20} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^{11}\}$.

Lemma 4.11. *If $\lambda \in \mathfrak{I}_{20}$, then* $\dim L(\lambda) = 71$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{20} = \{m_{a,b,c,d,e} | a, c, e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\}$$
$$- \Big(\{m_{1,b,1,d,e} | b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}, (b,d,e) \neq (2,2,1)\} \cup \{m_{1,0,0,2,1}, m_{1,3,0,0,0}\}\Big).$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $w_2 = F_2E_2E_{12}^3 v_\lambda$. Then $W_2(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{m}_{1,3,1,2,1}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{26}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, so $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 25 = 71$ by Lemma 4.17 and B_{20} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.11. The family $\mathfrak{I}_{21}$

Recall that $\mathfrak{I}_{21} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^3\}$.

Lemma 4.12. *If $\lambda \in \mathfrak{I}_{21}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{21} = \{m_{a,b,c,d,e} \mid a,b,c,e \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{0,2}\} \cup \{m_{a,2,c,0,e} \mid a,c,e \in \mathbb{I}_{0,1}\}$$
$$\cup \{m_{1,3,0,0,e} \mid e \in \mathbb{I}_{0,1}\} \cup \{m_{0,3,1,0,1}, m_{1,3,1,0,1}, m_{0,2,0,1,0}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $u = \widetilde{m}_{1,3,1,2,1}$, $w_2 = F_1F_{11212}F_{12}u$. Then $W_2(\lambda) = \mathcal{U}w_1$.

Let $M' = M(\lambda)/W_2(\lambda)$, so $E_1u = E_2u = 0$ in M', and $(\mathcal{U}u)^\varphi \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{19}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{40}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_2(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 61$ by Lemma 4.31, and B_{21} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,1,1,2,1} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.12. The family $\mathfrak{I}_{22}$

Recall that $\mathfrak{I}_{22} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^9\}$.

Lemma 4.13. *If $\lambda \in \mathfrak{I}_{22}$, then $\dim L(\lambda) = 49$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{22} = \{m_{a,b,c,d,e} \mid a,c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d,e \in \mathbb{I}_{0,1}\}$$
$$- \{m_{a,b',1,0,0}, m_{1,3,1,1,1}, m_{a,b,1,1,0} \mid a \in \mathbb{I}_{0,1}, b' \in \mathbb{I}_{0,3}, b \in \mathbb{I}_{1,3}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $w_2 = F_1^2E_{112}^2E_1v_\lambda$. Then $W_2(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{m}_{1,3,1,2,1}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{29}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, so $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 47 = 49$ by Lemma 4.20, and B_{22} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.13. The family $\mathfrak{I}_{23}$

Recall that $\mathfrak{I}_{23} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^2\}$.

Lemma 4.14. *If $\lambda \in \mathfrak{I}_{23}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{23} = \Big(\{m_{a,b,0,d,e} \mid a,e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \cup \{m_{a,b,1,0,0} \mid a,b \in \mathbb{I}_{0,1}\}$$
$$\cup \{m_{0,2,1,0,0}, m_{1,3,1,0,0}\}\Big) - \Big(\{m_{1,b,0,1,e} \mid b \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\} \cup \{m_{0,2,0,2,0}\}\Big).$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $u = \widetilde{m}_{1,3,1,2,1}$, $w_2 = F_{12}^3F_{11212}F_{112}F_1u$. Then $W_2(\lambda) = \mathcal{U}w_1$.

Let $M' = M(\lambda)/W_2(\lambda)$, so $E_1u = E_2u = 0$ in M', and $(\mathcal{U}u)^\varphi \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{22}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{45}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 47$ by Lemma 4.36, and B_{23} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,3,0,2,1} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.14. The family $\mathfrak{I}_{24}$

Recall that $\mathfrak{I}_{24} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = -1\}$.

Lemma 4.15. *If $\lambda \in \mathfrak{I}_{24}$, then $\dim L(\lambda) = 85$. A basis of $L(\lambda)$ is given by*

$$B_{24} = \{m_{a,b,c,d,e} | a, c, e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} - \left(\{m_{a,3,c,2,e}, m_{1,3,c,1,1} | a, c, e \in \mathbb{I}_{0,1}\} \cup \{m_{0,3,1,1,1}\}\right).$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $u = \widetilde{m}_{1,3,1,2,1}$, $w_2 = F_{12}F_{11212}F_1u$. Then $W_2(\lambda) = \mathcal{U}w_1$.

Let $M' = M(\lambda)/W_2(\lambda)$, so $E_1u = E_2u = 0$ in M', and $(\mathcal{U}u)^\varphi \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{18}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{38}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 85$ by Lemma 4.29, and B_{24} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,2,1,2,1} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.15. The family $\mathfrak{I}_{25}$

Recall that $\mathfrak{I}_{25} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = \zeta^8\}$.

Lemma 4.16. *If $\lambda \in \mathfrak{I}_{25}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$B_{25} = B'_{25} - \left(\{m_{0,3,0,0,e} \mid e \in \mathbb{I}_{0,1}\} \cup \{m_{1,3,c,0,e}, m_{1,2,1,0,e} \mid c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\}\right), \text{ where}$$

$$B'_{25} = \{m_{a,b,c,0,e} \mid a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_1^2E_{112}E_1^2v_\lambda$. By Remark 3.5, $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$, so B'_{25} is a basis of M'. Notice that $w_2 = E_2E_{12}^3v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{38}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,0,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,0,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 11 = 37$ by Lemma 4.29 and B_{25} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.16. The family $\mathfrak{I}_{26}$

Recall that $\mathfrak{I}_{26} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = \zeta^8\}$.

Lemma 4.17. *If $\lambda \in \mathfrak{I}_{26}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$B_{26} = \{m_{0,b,c,0,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\} \cup \{m_{1,0,0,0,0}, m_{1,0,0,0,2}\} - \{m_{0,3,1,0,0}\}.$$

Proof. Let $w_1 = F_1^2E_{112}E_1^2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{25} as in Lemma 4.17 is a basis of M'. Notice that $w_2 = F_2E_2E_{12}v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{13}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,0,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = w$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,0,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 23 = 25$ by Lemma 4.4, and B_{26} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.17. The family $\mathfrak{I}_{27}$

Recall that $\mathfrak{I}_{27} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^9\}$.

Lemma 4.18. *If $\lambda \in \mathfrak{I}_{27}$, then $\dim L(\lambda) = 35$. A basis of $L(\lambda)$ is given by*

$$B_{27} = B'_{27} - \{n_{0,0,1,2,2}\}, \qquad \text{where} \qquad B'_{27} = \{n_{a,0,c,d,e} \mid a, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_2E_{12}E_2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{27} is a basis of M'. Notice that $w_2 = E_{11212}E_{112}^2E_1^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{47}$; as also $E_1w_2 = E_2w_2 = 0$, we have that $\mathcal{U}w_2 \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 36 - 1 = 35$ by Lemma 4.38, and B_{27} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.18. The family $\mathfrak{I}_{28}$

Recall that $\mathfrak{I}_{28} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = \zeta^4\}$.

Lemma 4.19. *If $\lambda \in \mathfrak{I}_{28}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$B_{28} = B'_{27} - \Big(\{n_{0,0,1,1,e}, n_{0,0,c,2,e} | c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} \cup \{n_{1,0,1,2,e} \mid e \in \mathbb{I}_{1,2}\}\Big).$$

Proof. Let $w_1 = F_2E_{12}E_2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{27} is a basis of M'. Notice that $w_2 = F_1^2E_1^2E_{112}^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{38}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,0,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,0,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,0,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 36 - 11 = 25$ by Lemma 4.29, and B_{28} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.19. The family $\mathfrak{I}_{29}$

Recall that $\mathfrak{I}_{29} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = -1\}$.

Lemma 4.20. *If $\lambda \in \mathfrak{I}_{29}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is given by*

$$B_{29} = B'_{29} - \{m_{1,3,1,0,0}\}, \qquad \text{where} \qquad B'_{29} = \{m_{a,b,c,0,e} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_1^2E_{112}E_1^2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{29} is a basis of M'. Notice that $w_2 = E_2E_{12}^3E_{11212}v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{47}$; as also $E_1w_2 = E_2w_2 = 0$, we have that $\mathcal{U}w_2 \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 1 = 47$ by Lemma 4.38, and B_{29} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.20. The family $\mathfrak{I}_{30}$

Recall that $\mathfrak{I}_{30} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = \zeta^2\}$.

Lemma 4.21. *If $\lambda \in \mathfrak{I}_{30}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$B_{30} = B'_{29} - \{m_{1,b,c,0,e} \mid c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{2,3}, e \in \mathbb{I}_{0,2}, (b, c, e) \neq (3, 1, 2)\}.$$

Proof. Let $w_1 = F_1^2E_{112}E_1^2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{29} is a basis of M'. Notice that $w_2 = E_2E_{12}^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{38}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,0,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = w_2$, and then

$\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,0,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 11 = 37$ by Lemma 4.29, and B_{30} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.21. The family $\mathfrak{I}_{31}$

Recall that $\mathfrak{I}_{31} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1,\ \lambda_2 = \zeta^{10}\}$.

Lemma 4.22. *If $\lambda \in \mathfrak{I}_{31}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{31} = \mathrm{B}'_{31} - \big(\{n_{0,0,0,2,e} \mid e \in \mathbb{I}_{0,1}\} \cup \{n_{0,0,1,1,e}, n_{0,0,1,2,e}, n_{0,1,1,2,e} | e \in \mathbb{I}_{0,2}\}\big), \quad \textit{where}$$
$$\mathrm{B}'_{31} = \{n_{a,b,c,d,e} | a,b,c \in \mathbb{I}_{0,1}, d,e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_2E_2E_{12}^2 v_\lambda$. By Remark 3.17, $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$, so B'_{31} is a basis of M'. Notice that

$$w_2 = n_{0,0,0,2,1} + \tfrac{q_{21}}{3}\zeta(1+\zeta^3)(1+\zeta^2)\big(n_{0,0,1,0,2} + \zeta^4 n_{0,1,0,1,2}\big)$$

satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{18}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = n_{1,1,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fn_{1,1,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}n_{1,1,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 72 - 11 = 61$ by Lemma 4.9, and B_{31} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.22. The family $\mathfrak{I}_{32}$

Recall that $\mathfrak{I}_{32} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10},\ \lambda_2 = -1\}$.

Lemma 4.23. *If $\lambda \in \mathfrak{I}_{32}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{32} = \mathrm{B}'_{31} - \Big(\{n_{a,b,1,d,2} \mid a,b \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\} \cup \{n_{0,0,1,0,2}, n_{1,0,1,0,2}, n_{1,0,0,2,2}\}\Big).$$

Proof. Let $w_1 = F_2E_2E_{12}^2 v_\lambda$. By Remark 3.17, $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$, so B'_{31} is a basis of M'. Moreover $u = n_{1,1,1,2,2} \in V_{10\alpha_1+6\alpha_2}$ satisfies that $E_1u = E_2u = 0$, $g_1\sigma_1 u = u$, $g_2\sigma_2 u = \zeta^8 u$, so $(\mathcal{U}w)^\varphi \twoheadrightarrow L(\nu)$, $\nu \in \mathfrak{I}_{12}$. Also $\mathcal{U}u$ is a proper submodule. Set $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}u$. By Lemma 4.3,

$$61 = \dim L(\lambda) \leq \dim L'(\lambda) = \dim W - \dim \mathcal{U}w \leq \dim W - \dim L(\nu) = 61,$$

so $L(\lambda) = L'(\lambda)$ and $\mathcal{U}w \simeq L(\nu)^\varphi$. In particular $w_2 := F_2F_{11212}F_{112}u \neq 0$, $F_iw_2 = 0$ and $\mathcal{U}w_2 = \mathcal{U}u$. Moreover B_{32} is a basis of $L(\lambda)$. □

4.23. The family $\mathfrak{I}_{33}$

Recall that $\mathfrak{I}_{33} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2,\ \lambda_2 = -1\}$.

Lemma 4.24. *If $\lambda \in \mathfrak{I}_{33}$, then $\dim L(\lambda) = 71$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{33} = \{m_{a,b,c,d,e} \mid a,c,d \in \mathbb{I}_{0,1}, b,e \in \mathbb{I}_{0,2}\} \cup \{m_{1,3,0,0,0}\} - \{m_{0,0,1,0,0}, m_{1,2,0,1,2}\}.$$

Proof. Let $w_1 = F_1^2E_{112}^2E_1^2 v_\lambda$. By Remark 3.8, $F_1w_1 = F_2w_1 = 0$. By a direct computation, $\mathcal{U}w_1 \simeq L(\mu)$, with $\mu \in \mathfrak{I}_{23}$, and $\mathrm{B}' = \{m_{a,b,c,d,e} \mid d \neq 2\} \cup \{m_{0,0,0,2,2}\}$ is a basis of $W' = M(\lambda)/\mathcal{U}w_1$. Now $\mathcal{U}m_{0,0,0,2,2} = \Bbbk m_{0,0,0,2,2}$ in W', so $\mathrm{B} = \{m_{a,b,c,d,e} \mid d \neq 2\}$ is a basis of $M' = W'/\Bbbk m_{0,0,0,2,2}$.

Let $w_2 = F_1^2F_{112}^2E_{11212}E_{112}^2E_1^2 v_\lambda$. By Remark 3.11, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, and $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$, with $\mu \in \mathfrak{I}_{14}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,1,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,1,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,1,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}m_{0,0,0,2,2} + \mathcal{U}w_2$, so $\dim L'(\lambda) = 96 - 25 = 71$ by Lemma 4.5, and B_{33} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.24. The family $\mathfrak{I}_{34}$

Recall that $\mathfrak{I}_{34} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^3\}$.

Lemma 4.25. *If $\lambda \in \mathfrak{I}_{34}$, then* $\dim L(\lambda) = 71$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{34} = \{n_{a,b,c,d,e} | a, c, d \in \mathbb{I}_{0,1}, b, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,0,0,2,e} \mid e \in \mathbb{I}_{0,2}\} - \big(\{n_{0,0,1,0,e} | e \in \mathbb{I}_{0,2}\} \cup \{n_{0,1,1,1,0}\}\big).$$

Proof. Let $w_1 = F_2E_{12}^3E_2v_\lambda$. By Remark 3.20, $F_1w_1 = F_2w_1 = 0$. By a direct computation, $\mathcal{U}w_1 \simeq L(\mu)$, with $\mu \in \mathfrak{I}_{36}$, and $\mathrm{B}' = \mathrm{B}'_{35} \cup \{n_{1,3,0,0,0}\}$ is a basis of $W' = M(\lambda)/\mathcal{U}w_1$. Now $\mathcal{U}n_{1,3,0,0,0} = \Bbbk n_{1,3,0,0,0}$ in W', so B'_{35} is a basis of $M' = W'/\Bbbk n_{1,3,0,0,0}$.

Let $w_2 = F_1^2F_{112}^2E_{11212}E_{112}^2E_1^2v_\lambda$. By Remark 3.11, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, and $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$, with $\mu \in \mathfrak{I}_{37}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = n_{1,2,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fn_{1,2,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}n_{1,2,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}n_{1,2,1,2,2} + \mathcal{U}w_2$, so $\dim L'(\lambda) = 108 - 37 = 71$ by Lemma 4.28, and B_{34} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.25. The family $\mathfrak{I}_{35}$

Recall that $\mathfrak{I}_{35} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = \zeta^4\}$.

Lemma 4.26. *If $\lambda \in \mathfrak{I}_{35}$, then* $\dim L(\lambda) = 85$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{35} = \mathrm{B}'_{35} - \Big(\{n_{0,b,c,2,e} | c \in \mathbb{I}_{0,1}, b, e \in \mathbb{I}_{0,2}\} \cup \{n_{1,2,1,2,2}, n_{1,0,0,2,2}, n_{1,0,1,2,e} | e \in \mathbb{I}_{0,2}\}\Big) \text{ where}$$

$$\mathrm{B}'_{35} = \{n_{a,b,c,d,e} \mid a, c \in \mathbb{I}_{0,1}, b, d, e \in \mathbb{I}_{0,2}\}$$

Proof. Let $w_1 = F_2E_2E_{12}^3v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{35} is a basis of M'. Notice that $w_2 = F_1^2E_{112}^2E_1^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{44}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = n_{1,2,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fn_{1,2,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}n_{1,2,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 108 - 23 = 85$ by Lemma 4.35, and B_{35} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.26. The family $\mathfrak{I}_{36}$

Recall that $\mathfrak{I}_{36} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta, \lambda_2 = 1\}$.

Lemma 4.27. *If $\lambda \in \mathfrak{I}_{36}$, then* $\dim L(\lambda) = 35$. *A basis of $L(\lambda)$ is given by* $\mathrm{B}_{36} =$

$$\{n_{0,b,0,d,e}, n_{0,0,1,2,e}, n_{0,0,1,0,e} | b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \{n_{0,1,0,1,e}, n_{0,2,0,2,e}, n_{0,1,0,0,2} | e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = E_1^2E_{12}v_\lambda$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{15}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 37 = 35$ by Lemma 4.6, and B_{36} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.27. The family $\mathfrak{J}_{37}$

Recall that $\mathfrak{J}_{37} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = 1\}$.

Lemma 4.28. *If $\lambda \in \mathfrak{J}_{37}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$B_{37} = \{n_{0,b,0,d,e}, n_{0,0,1,0,0}, n_{0,3,1,0,e} | b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \{n_{0,3,0,2,e} | e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = \widetilde{n}_{0,1,0,1,1} - \zeta \widetilde{n}_{0,2,0,0,2} - \zeta^{10}(1-\zeta)^2 \widetilde{n}_{0,0,1,0,1}$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{J}_{19}$, and there exists $E \in \mathcal{U}$ such that $E w_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 35 = 37$ by Lemma 4.10, and B_{37} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.28. The family $\mathfrak{J}_{38}$

Recall that $\mathfrak{J}_{38} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = 1\}$.

Lemma 4.29. *If $\lambda \in \mathfrak{J}_{38}$, then $\dim L(\lambda) = 11$. A basis of $L(\lambda)$ is given by*

$$B_{38} = \{n_{0,b,c,0,e} | b, c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} - \{n_{0,1,1,0,2}\}.$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ is described in Table 5.

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 E_{112} E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$. We label the elements of B_{38} as follows:

$$\begin{aligned} &v_{0,0} = n_{0,0,0,0,0}, && v_{1,1} = n_{0,1,0,0,0}, && v_{3,2} = n_{0,0,1,0,0}, && v_{4,3} = n_{0,1,1,0,0},\\ &v_{1,0} = n_{0,0,0,0,1}, && v_{2,1} = n_{0,1,0,0,1}, && v_{4,2} = n_{0,0,1,0,1}, && v_{5,3} = n_{0,1,1,0,1},\\ &v_{2,0} = n_{0,0,0,0,2}, && v_{3,1} = n_{0,1,0,0,2}, && v_{5,2} = n_{0,0,1,0,2}. \end{aligned}$$

We check that the action of E_k, F_k on $v_{i,j}$ is given by Table 5 and $L'(\lambda)$ is spanned by B_{38}. Moreover there exists $F \in \mathcal{U}^-$ such that $F v_{5,3} = v_\lambda$, and for each pair (i,j) there is $E_{i,j} \in \mathcal{U}_{(5-i)\alpha_1 + (3-j)\alpha_2}$ such that $E_{i,j} v_{i,j} = v_{5,3}$. Thus $L'(\lambda)$ is simple. □

Table 5. Simple modules for $\lambda \in \mathfrak{J}_{38}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	$v_{1,0}$	0	0	0
$v_{1,0}$	$v_{2,0}$	$\zeta^7 q_{21} v_{1,1}$	$(1-\zeta^3)v_{0,0}$	0
$v_{2,0}$	0	$\zeta^8 q_{21}^2 (1+\zeta^3) v_{2,1}$	$\zeta^7(1+\zeta)v_{1,0}$	0
$v_{1,1}$	$v_{2,1}$	0	0	$(\zeta^{11}-1)v_{1,0}$
$v_{2,1}$	$v_{3,1}$	0	$q_{12}\zeta^8 v_{1,1}$	$(\zeta^{11}-1)v_{2,0}$
$v_{3,1}$	0	$q_{21}^2 \zeta v_{3,2}$	$q_{12}\zeta^2 v_{2,1}$	0
$v_{3,2}$	$v_{4,3}$	0	0	$q_{21}\zeta^{11}(1-\zeta^3)v_{3,1}$
$v_{4,2}$	$v_{5,2}$	$q_{21}^2 \zeta^{10} v_{4,3}$	$q_{12}^2(\zeta^{11}-1)v_{3,2}$	0
$v_{5,2}$	0	$q_{21}^3 (3)_\zeta v_{5,3}$	$q_{12}^2 \zeta^8 (1+\zeta) v_{4,2}$	0
$v_{4,3}$	$v_{5,3}$	0	0	$q_{21}^2 \zeta^{10} (3)_{\zeta^{11}} v_{4,2}$
$v_{5,3}$	0	0	$q_{12}^3 \zeta^8 (1+\zeta^2) v_{4,3}$	$q_{21}^2 \zeta^{10} (3)_{\zeta^{11}} v_{5,2}$

4.29. The family $\mathfrak{I}_{39}$

Recall that $\mathfrak{I}_{39} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = 1\}$.

Lemma 4.30. *If $\lambda \in \mathfrak{I}_{39}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{39} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \Big(\{n_{0,3,c,2,e}, n_{0,2,1,2,e} | c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,2,0,2,e} \mid e \in \mathbb{I}_{1,2}\}\Big).$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $u = \widetilde{n}_{0,3,1,2,2}$, $w_2 = F_1 F_{11212} F_{12}^2 u$. Then $W_2(\lambda) = \mathcal{U} w_1$.

Let $M' = M(\lambda)/W(\lambda)$, so $E_1 u = E_2 u = 0$ in M', and $(\mathcal{U}u)^{\varphi} \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{38}$; thus $w_2 \neq 0$. By direct computation, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U} w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{18}$. From here, $\mathcal{U} w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U} w_2$. Then $\dim L'(\lambda) = 61$ by Lemma 4.9, and B_{39} is a basis of $L'(\lambda)$. There exists F such that $Fu = v_\lambda$, and $L'(\lambda)$ is simple. □

4.30. The family $\mathfrak{I}_{40}$

Recall that $\mathfrak{I}_{40} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = 1\}$.

Lemma 4.31. *If $\lambda \in \mathfrak{I}_{40}$, then $\dim L(\lambda) = 35$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{40} = \{n_{0,b,c,0,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,b,c,1,e} \mid c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,3,0,2,e} \mid e \in \mathbb{I}_{0,1}\} - \{n_{0,3,1,0,e} | e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 E_{112}^2 E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{25}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 37 = 35$ by Lemma 4.16, and B_{40} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.31. The family $\mathfrak{I}_{41}$

Recall that $\mathfrak{I}_{41} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = 1\}$.

Lemma 4.32. *If $\lambda \in \mathfrak{I}_{41}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{41} = \{n_{0,b,c,d,0} | c \in \mathbb{I}_{0,1}, b, d \in \mathbb{I}_{0,2}\} \cup \{n_{0,b,c,d,e} | c, b \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{1,2}\} - \{n_{0,1,c,d,2}, n_{0,0,1,2,2} | c \in \mathbb{I}_{0,1} d \in \mathbb{I}_{1,2}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 F_{112}^2 E_{11212} E_{112}^2 E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{27}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 35 = 37$ by Lemma 4.18, and B_{41} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.32. The family $\mathfrak{I}_{42}$

Recall that $\mathfrak{I}_{42} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^7, \lambda_2 = 1\}$.

Lemma 4.33. *If* $\lambda \in \mathfrak{I}_{42}$, *then* $\dim L(\lambda) = 71$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{42} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}, (b,c,d,e) \neq (3,1,2,2)\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = \widetilde{n}_{0,3,1,2,2}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = E_1w_2 = E_2w_2 = 0$, so $\mathcal{U}w_2 \simeq L(\mu)$ for $\mu \in \mathfrak{I}_{47}$. Let $L'(\lambda) = M(\lambda)/W(\lambda) + \mathcal{U}w_2$, so B_{42} is a basis of $L'(\lambda)$. There exists $F \in \mathcal{U}^-$ such that $Fn_{0,3,1,2,1} = v_\lambda$. If $n_{0,b,c,d,e} \in \mathrm{B}_{42}$, then $E_1^{1-e}E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{3-b}n_{0,b,c,d,e} \in \Bbbk^\times n_{0,3,1,2,1}$, so $L'(\lambda)$ is simple. □

4.33. The family $\mathfrak{I}_{43}$

Recall that $\mathfrak{I}_{43} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = 1\}$.

Lemma 4.34. *If* $\lambda \in \mathfrak{I}_{43}$, *then* $\dim L(\lambda) = 25$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{43} = \{n_{0,b,c,d,e} | c, e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} - \Big(\{n_{0,2,1,2,0}\} \cup \{n_{0,b,c,d,1} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{1,3}, d \in \mathbb{I}_{0,2}\} \cup \{n_{0,3,c,d,0} | c \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\}\Big).$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{17}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 47 = 25$ by Lemma 4.8, and B_{43} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.34. The family $\mathfrak{I}_{44}$

Recall that $\mathfrak{I}_{44} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = 1\}$.

Lemma 4.35. *If* $\lambda \in \mathfrak{I}_{44}$, *then* $\dim L(\lambda) = 23$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{44} = \{n_{0,b,0,d,e} | b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\} \cup \{n_{0,0,0,0,2}\} - \{n_{0,3,0,1,1}, n_{0,3,0,2,1}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = \zeta^4\widetilde{n}_{0,0,0,1,1} + \widetilde{n}_{0,1,0,0,2}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{22}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 49 = 23$ by Lemma 4.13, and B_{44} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.35. The family $\mathfrak{I}_{45}$

Recall that $\mathfrak{I}_{45} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = 1\}$.

Lemma 4.36. *If* $\lambda \in \mathfrak{I}_{45}$, *then* $\dim L(\lambda) = 49$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{45} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \big(\{n_{0,b,c,2,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{1,3}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,0,1,2,e} | e \in \mathbb{I}_{0,2}\} \cup \{n_{0,0,1,0,2}, n_{0,3,1,1,2}\}\big).$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = n_{0,1,0,1,2} - \zeta^{11}(3)_{\zeta^7} n_{0,0,1,0,2}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{13}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 23 = 49$ by Lemma 4.4, and B_{45} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.36. *The family $\mathfrak{I}_{46}$*

Recall that $\mathfrak{I}_{46} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = 1\}$.

Lemma 4.37. *If $\lambda \in \mathfrak{I}_{46}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is*

$$\mathrm{B}_{46} = \{n_{0,b,c,d,e}, | c, d \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,1,0,2,0}, n_{0,3,1,2,0}\} \\ -\{n_{0,1,1,0,2}, n_{0,3,0,0,1}, n_{0,1,1,0,1}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 E_{112}^2 E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{26}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 25 = 47$ by Lemma 4.17, and B_{46} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.37. *The family $\mathfrak{I}_{47}$*

Recall that $\mathfrak{I}_{47} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = 1\}$.

Lemma 4.38. *If $\lambda \in \mathfrak{I}_{47}$, then $\dim L(\lambda) = 1$ and $E_i v_\lambda = 0$, $F_i v_\lambda = 0$, $g\sigma v_\lambda = \lambda(g\sigma) v_\lambda$.*

Proof. Let $N'(\lambda) = W(\lambda) + W_1(\lambda)$. By a direct computation, $N'(\lambda) = \sum_{\beta \neq 0} M(\lambda)_\beta = N(\lambda)$. Therefore $L'(\lambda) = M(\lambda)/N'(\lambda)$ is one-dimensional and simple. □

Example 4.39. Take $\Lambda = \mathbb{Z}_{12} = \langle g_2 \rangle$, $g_1 = g_2^8$ and $\sigma_1, \sigma_2 \in \widehat{\Lambda}$ such that

$$\sigma_1(g_2) = \zeta^{11}, \qquad \sigma_2(g_2) = -1; \qquad \text{hence} \qquad \sigma_1(g_1) = \zeta^4, \qquad \sigma_2(g_1) = 1. \tag{21}$$

Applying the Main Theorem, we see that there is one simple module of dimension one and exactly # different isoclasses of a given dimension as in Table 6:

Table 6. Quantity of simple modules of dimension > 1.

#	dimension	#	dimension	#	dimension	#	dimension
67	144	7	108	10	96	2	85
6	72	4	71	4	61	2	49
10	48	4	47	6	37	7	36
4	35	4	25	2	23	4	11

Note that $\mathfrak{I}_6$ and $\mathfrak{I}_{10}$ are empty.

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