Aspiration-Based Choice*

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Abstract

Numerous studies and experiments suggest that aspirations for desired but perhaps unavailable alternatives influence decisions. A common finding is that an unavailable aspiration steers agents to choose similar available alternatives. We propose and axiomatically characterize a choice theory consistent with this aspirational effect. Similarity is modeled using a subjective metric derived from choice data. This model offers implications for consumer welfare and its distribution between rich and poor when firms compete for aspirational agents, and a novel rationale for sales.

Keywords: Choice; Similarity; Aspirations; Consumer Welfare

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1 Introduction

There is widespread evidence that decision makers do not always behave as utility maximizers. While violations of rationality occur, they do not appear to be random but often follow systematic patterns. To better understand these patterns, the conditions under which they occur and their economic implications, a growing literature has proposed choice models that accommodate departures from rationality. In this paper, we introduce a choice model that focuses on the effect of unavailable alternatives.

The question of how unavailable alternatives influence decision making has received scant attention in the decision theory literature, even though numerous studies and experiments suggest that they have an important effect. Some examples are: (1) The high price (and hence unavailability) of luxury brands leads consumers to purchase similar and cheaper counterfeit products. (2) Individuals may attempt to "keep up with the Joneses" and mimic the choices of their neighbors, even though (or perhaps because) those alternatives are not feasible for them. (3) A manager's consumption-leisure decision is unavailable to her subordinates due to wage differences, but can affect how they trade off between the two. (4) An employer may change her ranking of job applicants after interviewing a "superstar" who is clearly out of reach. (5) Past consumption can create habits that influence current consumption, especially when past consumption levels are no longer attainable. (6) Many advertised products are intentionally unavailable (such as limited editions or vaporware) in order to influence consumer choice among available alternatives.[1]

A number of experiments have found that the presence of a *phantom* alternative, which is desired and unavailable, leads agents to choose a similar available alternative. Farquhar and Pratkanis (1992, 1993) were the first to conduct such experiments. Highhouse (1996) and Pettibone and Wedell (2000) respectively found this *aspirational effect* to be equal in magnitude to the well-known attraction and compromise effects.

[1] Vaporware refers to advertised software or hardware that is not available to buy.

The above examples and experiments present a basic violation of rationality because alternatives that are unavailable (and therefore irrelevant) affect decisions. However, two systemic patterns appear and they are not accounted for by existing models of bounded rationality: the unavailable alternatives influencing these decisions are desired by the decision maker, and when a highly desired alternative is unavailable, decision makers tend to choose *similar* available alternatives. The first part of this paper proposes an axiomatic choice model that captures the effect of unavailable alternatives. The second part studies the economic implications.

A choice problem is a pair of sets: a set of observed alternatives and a subset of those which are also available for choice. The observed alternatives which are not in this subset are unavailable. For each choice problem, an agent chooses a subset of the available alternatives, but her choice may depend upon the unavailable alternatives.

We axiomatically characterize a choice procedure defined by two primitives: a linear order over alternatives and an endogenous metric. A decision maker focuses on the maximal element she observes according to the linear order, which we call her *aspiration*. She selects the closest available alternative to this aspiration where distances are measured using her metric. This procedure lines up with basic intuition as well as the experimental findings that an unavailable aspiration steers the agent to the most similar available option.

Three straightforward axioms characterize this procedure. First, in the absence of unavailable alternatives, choices are rational. Second, the agent behaves rationally across choice problems with the same observed alternatives. Third, the agent's most desired alternatives (aspirations) are revealed by her choices when every observed alternative is also available. We assume that the agent chooses the same alternatives from choice problems with the same available alternatives and the same aspirations.

A key feature of the model is that agents' choices reveal their similarity judgements. This reflects the view that similarity (like utility) is subjective[2]

[2]After all, two diamonds may appear indistinguishable to one decision maker, but couldn't be further apart to another.

and we prove that the metric that an agent uses to measure similarity is identified from the choice data, provided that the axioms are satisfied. The proof is intricate because a metric must satisfy additional properties (reflexivity, symmetry and the triangle inequality) that a utility function need not.

The notion that an aspiration towards a higher goal may influence behavior is widespread across the social sciences. Notable examples from the economics literature include Veblen (1899), Duesenberry (1949), Pollak (1976), Hopkins and Kornienko (2004), and Ray (2006). More recently, Ray and Robson (2012) and Genicot and Ray (2017) study risk-taking and consumption-savings decisions of agents whose utility functions have inflection points at certain aspirational levels.[3]

Our choice model is the first revealed preference theory of aspirations. It resembles some of those above in that outcomes are assessed relative to an aspiration, but aspirations are for unavailable alternatives rather than for payoff or wealth thresholds. The main advantages of the revealed preference approach is that the choice axioms can be empirically tested and the model applies to many choice settings (including those with non-monetary outcomes and non-Euclidean spaces).

Existing reference-dependent choice models do not account for aspirations. For example, a status quo alternative is what the decision maker already has (Samuelson and Zeckhauser 1988), the reference point in Kőszegi and Rabin (2006, 2007) is what she expects, and an aspiration is what she wants. Each reference point will have a different effect on choice. Status quos may rule out alternatives from consideration (Masatlioglu and Ok 2005, 2014). In Kőszegi and Rabin, an agent suffers loss-aversion relative to her expectations determined in a personal equilibrium. On the other hand, aspirations steer choices to similar options. We will illustrate these differences in Section 2.1.

[3]Satisficing thresholds that influence behavior (but not payoffs) are used in reinforcement learning models (Borgers and Sarin 2000), repeated games (Karandikar, Mookherjee, Ray, and Vega-Redondo 1998), and in network choice models (Bendor, Diermeier, and Ting 2016). Outside of economics, aspirations are used to study a wide range of topics including poverty traps (Appadurai 2004), occupational choice (Correll 2004), educational attainment (Kao and Tienda 1998), general happiness (Stutzer 2004) and voter turnout (Bendor, Diermeier, and Ting 2003).

What is important to emphasize now is that the differences are not only theoretical, but have consequences in economic settings. The second part of the paper draws implications of similarity-based decision making in two standard economic settings.

The first setting is a competitive market with profit-maximizing firms and aspirational buyers. A "red" firm and a "blue" firm both sell a high- and low-quality good to a continuum of wealth-constrained buyers. Firms engage in price competition. Buyers that cannot afford the high-quality goods will aspire to the choices of richer buyers who can. Thus, firms have an added incentive to attract these richer buyers in order to create aspirations for its goods. In the unique equilibrium, firms lose money on their aspirational good and cross-subsidize these losses with profits from the low-quality market. We find that welfare may increase or decrease in comparison to the rational benchmark, but the distribution of welfare always changes in favor of richer buyers.

The second is a standard consumption setting where the agent may aspire to previously consumed bundles. We focus on her response to price and income shocks. For example, following a decrease in income, previously consumed bundles are no longer affordable but will have an impact as aspirations. On the other hand, following an increase in income, previously consumed bundles remain affordable and therefore will not affect current consumption. Turning to price changes leads to a rationale for sales: a low enough sale price will increase the post-sale demand.

The rest of the paper is organized as follows: Section 2 presents the model, the main representation result, and a discussion of the related experimental and theoretical literature. Section 3 studies some economic implications of our model. Section 4 concludes. Two appendices contain all proofs and extensions of the main theorem.

2 The Choice Model

We begin with some preliminary definitions. A compact metric space X (finite or infinite) is the *grand set* of alternatives. Let $\mathcal{X}$ denote the set of all nonempty closed subsets of X. $\mathcal{X}$ is endowed with the Hausdorff metric and convergence with respect to this metric is denoted by $\xrightarrow{H}$. A *linear order* is a reflexive, complete, transitive and anti-symmetric binary relation.

A choice problem is a pair of sets (S, Y) where $S, Y \in \mathcal{X}$ and $S \subseteq Y$. The set of all choice problems is denoted by $\mathcal{C}(X)$ and a choice correspondence is a map $C : \mathcal{C}(X) \to \mathcal{X}$ such that $C(S, Y) \subseteq S$ for all $(S, Y) \in \mathcal{C}(X)$. When an agent faces a choice problem (S, Y), she observes the *potential* set Y, but chooses from the *feasible* set S. For an alternative to be feasible, it must be observed and hence $S \subseteq Y$.

Unavailable alternatives constitute part of the choice data. There are various settings, including the examples mentioned in the introduction, where this is the case. For instance, when a high school student applies to colleges or a job seeker applies to jobs, both the schools or jobs applied to (the potential set) and the schools or job offers (the feasible set) are externally observable. A worker making consumption-leisure tradeoffs usually observes the decisions of her co-workers and managers which may be unavailable due to wage differences. Firm-level data on wages, hours worked, and organizational hierarchy is available. Online consumer searches are tracked and many alternatives are unavailable due to financial or capacity constraints (e.g. a fully booked hotel or a sold-out flight). There are advertised products which are intentionally unavailable (luxury products, limited editions and vaporware).

We take an axiomatic approach and characterize a choice model that allows for such unavailable alternatives to influence choice. At the same time, we aim to keep the model as close as possible to the rational one. The next two axioms limit the domains in which rationality violations can occur. The first requires that agents behave rationally when all the observed alternatives are also available. The second assumes rational behavior when a potential set is held fixed.

Axiom 2.1. *(WARP Without Unavailability) For any $S, Y \in \mathcal{X}$ such that $S \subseteq Y$,*

$$C(S, S) = C(Y, Y) \cap S, \text{ provided that } C(Y, Y) \cap S \neq \emptyset.$$

Axiom 2.2. *(WARP Given Potential Set) For any* $(S, Y), (T, Y) \in \mathcal{C}(X)$ *such that* $T \subseteq S$*,*

$$C(T, Y) = C(S, Y) \cap T, \text{ provided that } C(S, Y) \cap T \neq \emptyset.$$

Given the above two axioms, violations of rationality are due to unavailable alternatives and can only occur when potential sets vary. These restrictions allow for the experimental evidence that choice reversals may occur when unavailable alternatives are introduced.

In the previously discussed examples and experiments, the unavailable alternatives that influence choice are desired by the agent. We call these alternatives "aspirations" and identify them by the choices made in ideal situations without feasibility restrictions. Therefore, choices from problems of the form (S, S) reveal an agent's aspirations. The following axiom states that if two potential sets generate the same aspirations, then they will influence choices in the same way.

Axiom 2.3. *(Independence of Irrelevant Unavailable Alternatives) For any* (S, Z)*,*$(S, Y) \in \mathcal{C}(X)$*,*

$$C(S, Z) = C(S, Y), \text{ provided that } C(Z, Z) = C(Y, Y).$$

A choice correspondence is called *aspirational* if it satisfies Axioms 2.1-2.3.

The next two axioms are technical. The first guarantees a unique aspiration in any given problem. The second is a standard continuity axiom, which ensures the existence of maximums and trivially holds when the grand set is finite. Appendix B considers relaxations of these two axioms.

Axiom 2.4. *(Single Aspiration Point)* $|C(Y, Y)| = 1$ *for all* $Y \in \mathcal{X}$*.*

Axiom 2.5. *(Continuity) For any* $(S_n, Y_n), (S, Y) \in \mathcal{C}(X)$*,* $n = 1, 2, \ldots$ *and* $x_n \in S_n$ *for all* n *such that* $S_n \xrightarrow{H} S$*,* $Y_n \xrightarrow{H} Y$ *and* $x_n \to x$*,*

$$\text{if } x_n \in C(S_n, Y_n) \text{ for every } n, \text{ then } x \in C(S, Y).$$

The above axioms yield our main representation result:

Theorem 2.1. *C satisfies Axioms 2.1-2.5 if and only if there exists*

1. *a continuous linear order $\succeq$ and*
2. *a continuous metric $d : X^2 \to \mathbb{R}_+$*

such that

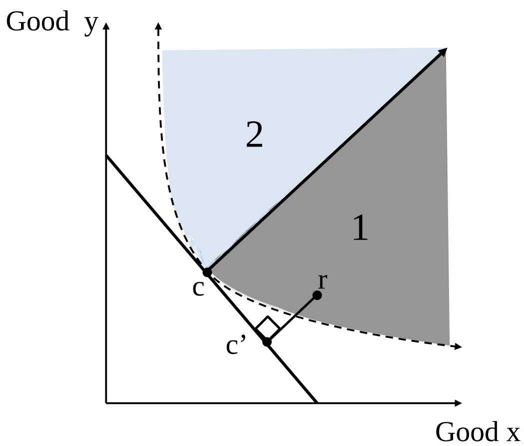
$$C(S,Y) = \operatorname*{argmin}_{s \in S} d(s, a(Y)) \quad \text{for all } (S,Y) \in \mathcal{C}(X),$$

where $a(Y)$ is the $\succeq$-maximum element of Y.

In the above procedure, from the choice problem (S,Y) a decision maker chooses the closest available alternative to her aspiration $a(Y)$. Her aspiration is the maximum element (according to the linear order $\succeq$) that she observes and closeness is measured using her metric d. This metric is derived endogenously and need not coincide with the metric that the grand set is originally endowed with. Generally, agents making different choices will be represented with different metrics. Notice that if an aspiration is available $a(Y) \in S$, then it is uniquely chosen $C(S,Y) = a(Y)$, because an alternative is always closest to itself.

Identification: The two primitives, d and $\succ$, are uniquely identified from choice data up to order-preserving transformations. That is, $\succ$ is uniquely identified by observing the agent's choices from problems of the form $C(S,S)$. The metric d cannot be uniquely identified because it is cardinal and choice is ordinal, but the distance order is unique. That is, if $a \succ x, y$, then whether $d(x,a) < d(y,a)$ or $d(y,a) < d(x,a)$ is identified by observing $C(\{x,y\},\{x,y,a\})$.

Example - Choice from a Budget Set: To illustrate the model, we now present a standard consumption example which will serve as a running example throughout the paper. An agent chooses a pair (x,y) from the budget set $S = \{(x,y) : px + y = I\}$ where x is a quantity of a nondurable good, p is the price of the good, y is the expenditure on all other goods, and I is income.



- If the aspiration is in Region 1, she will choose a point on the budget line below c.
- If the aspiration is in Region 2, she will choose a point on the budget line above c.
- Otherwise, she will choose c.

Figure 1: The Effect of an Aspiration on Budget Consumption

Let point c in Figure 1 denote the choice of a rational agent with continuous, strictly monotonic and strictly convex preferences. An aspirational agent with the same underlying preferences will only aspire to unavailable alternatives in Regions 1 or 2. Furthermore, if she measures distances using the Euclidean metric, then her choices will be affected as in Figure 1. That is, an aspiration in Region 1 influences the agent to purchase more of good x than the rational agent and vice-versa for an aspiration in Region 2.

Remark 1: In this framework, for an alternative to be choosable it must be observable, and therefore $S \subseteq Y$. An equivalent formulation would drop the subset restriction and take aspirations to be drawn from $S \cup Y$. In this formulation, if S and Y do not intersect, then Y is the set of unavailable alternatives. Subject to relabeling, the main representation theorem holds.

In our model, S and Y are externally observable and so constitute the choice data. A natural extension of our model would be to a setting with less rich choice data, such as when only S is externally observable. Unfortunately, this leads to an "anything goes" result. Consider, for example, choice from a budget set where the agent uses the Euclidean metric. Then, the choice $z = C(S)$ implies that the agent's aspiration lies northeast of z on the line perpendicular to S. That is, any choice from S can be rationalized by an appropriately chosen aspiration. If the external observer is aware of the metric, then she can make inferences about aspirations: the choice from a budget set pins down a line on which the aspiration must lie. But even with this additional knowledge, the anything goes result still applies.

Remark 2: A rational agent who maximizes a single continuous utility function over all available alternatives satisfies the above axioms and therefore, the rational model is nested. Representing this agent's choices through distance minimization is straightforward as the order $\succ$ is uniquely identified and the distance metric must be ordinally identified with the utility distance, $d(a,b) = |U(a) - U(b)|$. However, when agents are irrational, potential sets influence choices and such a trivial representation does not exist.

Remark 3: The rational model is strictly nested in ours and our model is strictly nested in the more general class of reference-dependent utility maximization models. To see this nesting, the reference point in our model is $r = \max(Y, \succ)$ and we can define a reference-dependent utility function U as $U(s,t) = -d(s,t)$, so $C(S,Y) = \text{argmax}_{s \in S} U(s,r)$. To see that the nesting is strict, if $U : X^2 \to \mathbb{R}$ is a reference-dependent utility function and $C(S,Y) = \max_{s \in S} U(s, \max(Y, \succ))$, then there may be no distance-minimization representation because Axiom 2.3 may not hold.[4] Furthermore, for a grand set of n alternatives, a reference-dependent utility model has n^2 degrees of freedom whereas our model only has $n(n-1)/2$ because it is based upon distances. Thus, the distance-based model is better identified from choice data.

Remark 4: Theorem 2.1 represents choices as $C(S) = \underset{s \in S}{\text{argmin}}\ d(s, \max(Y, \succsim))$ where d is a metric. A metric satisfies four properties: non-negativity, reflexivity, symmetry, and the triangle inequality. Of these properties, non-negativity and reflexivity are fundamental for the model, but the symmetry and triangle inequality properties do not restrict the model. To see this, take a choice correspondence defined as $C'(S) = \text{argmin}_{s \in S} \phi(s, \max(Y, \succsim))$ where ϕ is non-negative and reflexive. Axioms 2.1 and 2.4 rely upon ϕ being non-negative and reflexive. Axioms 2.2 and 2.3 follow from the use of a minimization procedure and do not rely upon any properties of ϕ. As C' satisfies all of our axioms,

[4]Suppose $U(a,z) > U(y,z) > U(b,z) > U(z,z)$, $U(a,y) > U(b,y) > U(y,y)$ and $a \prec b \prec y \prec z$. Then $C(abyz, abyz) = C(aby, aby) = a$, but $C(by, aby) = b, C(by, abyz) = y$.

then there is an equivalent representation using a metric d. Thus, the essential features of the minimizing function are that it is non-negative and reflexive, but the proof demonstrates that we can always find a metric d which additionally satisfies symmetry and the triangle inequality.

Remark 5: According to our procedure, if an aspiration is available, then it must be chosen. There are interesting cases where an available alternative which we wouldn't choose may still affect choice. For example, suppose that an agent faces the option of eating at a Michelin restaurant and emptying his bank account. Even if the agent does not choose this option, he may still be influenced by its presence. One way to extend our model in the spirit of Kalai and Smorodinsky (1975) is that the presence of the available alternative (Michelin Star, empty bank account) can bring to mind an unavailable aspiration (Michelin Star, bank account) which in turn influences the agent's choice.

Remark 6: The representation relies upon the single aspiration axiom as each alternative is assessed relative to a unique aspiration. Relaxing this assumption raises questions regarding how to generalize the above procedure, as it is not immediately obvious how an agent should assess a considered alternative relative to a set of aspirations. For example, she may select the alternative that is closest to any of the aspirations or she may consider the average distance of an alternative to each of her aspirations.

In the appendix we axiomatically characterize an aggregation method: For each considered alternative, the agent aggregates the distance to the aspiration point into a single score and selects the minimum score alternative. This aggregator agrees with the distance when an agent faces a single aspiration and thus agrees with the above representation. Several different aggregators are plausible, such as using the minimum distance, maximum distance, or the sum of distances faced. It is possible to focus on a specific aggregator and refine the representation by imposing additional technical restrictions which characterize it. This is left for future work.

Remark 7: The conjunction of Axioms 3.4 and 3.5 has implications for the space X. Nishimura and Ok (2014) show that the existence of a continuous choice function on a compact metric space implies that it is of topological dimension at most 1. This dimensionality restriction has no impact when X is a finite set, but has consequences when X is a larger path-connected space. In Appendix B, we relax these axioms so that this dimensionality restriction no longer applies. Note that Section 4 considers several economic settings and the choice domains therein satisfy this dimensionality restriction.

2.1 Related Literature

As mentioned in the introduction, the present paper is the first to axiomatically study the effect of unavailable alternatives on choice. Aspiration based choice also differs from existing reference dependent choice models in two important ways: how the reference point is determined and how it affects choices. We will use the previous example of choice from budget sets (depicted in Figure 1) to illustrate the behavioral differences between our model and others in the literature. When appropriate, we consider extensions of these choice models to incorporate unavailable alternatives. Throughout this example, we assume that the agent's metric is the standard Euclidean one.

Figure 2(i) depicts the aspirational effect when choices are made from a budget set and the agent also observes the unavailable alternative r. The agent aspires to r because it is the maximal observed alternative and it leads her to purchase more x. Figure 2(ii) depicts the regions in which an unavailable aspiration will influence her choice. Any reference point that lies outside the shaded regions is inferior to c, and hence has no influence on choice. In Figures 2(ii) and throughout this section, a reference point in Region 1 influences the agent to consume more x and a reference point in Region 2 influences her towards more y.

The most closely related papers to ours are the axiomatic works of Masatlioglu and Ok (2005, 2014) and Ok, Ortoleva, and Riella (2015) on status quo bias. The former two papers take a choice problem to be a set of available alternatives plus a status quo alternative and the latter paper endogenizes the selection of the status quo. In all three papers, the status quo alternative is available for choice and it may rule out other alternatives from consideration,

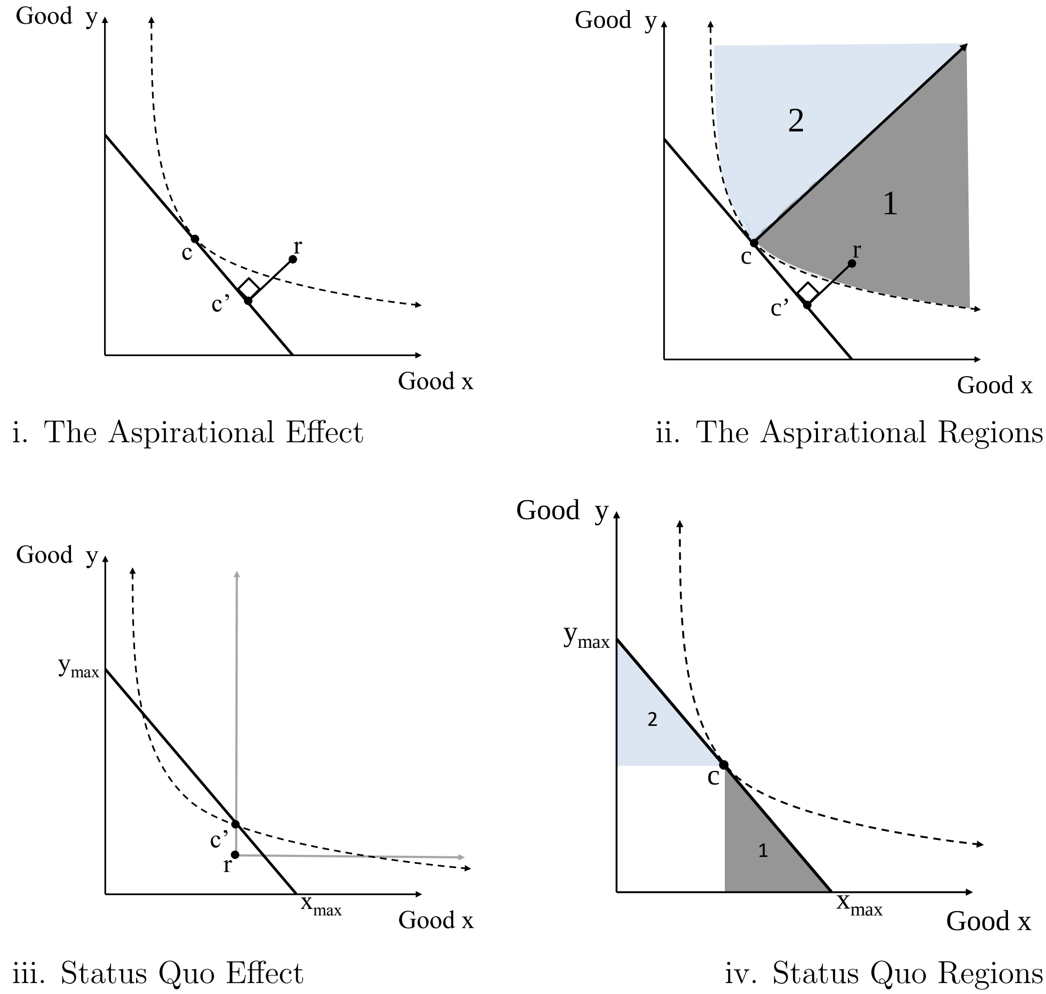


Figure 2: Aspirations and Status Quos

but it does not change the ranking of considered alternatives. In contrast, aspirations are desired unavailable alternatives and they may change the ranking of the available alternatives by similarity. Notice that when all the alternatives are available, the aspirational agent behaves rationally, unlike the status quo bias models.

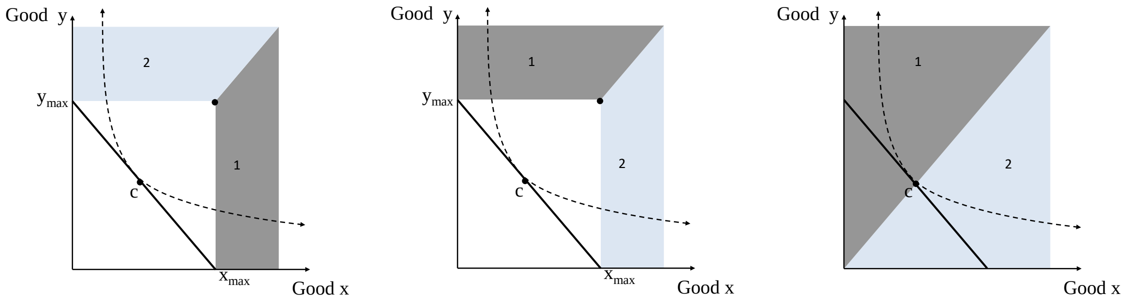
One common method by which the status quo rules out alternatives is dominance. That is, only alternatives which dominate the status quo are considered. An unavailable status quo which lies above the budget line is inappropriate here because it would rule out all available alternatives. Figure

2(iii) depicts the effect of an available status quo r. Since the agent only considers alternatives which dominate r, she selects c' instead of c. Figure 2(iv) depicts the regions where a status quo may change choice. If r lies outside the shaded region, then the agent still chooses c, her most preferred alternative. Otherwise, if r is in Region 1 (Region 2) will influence her to choose more x (y resp.).

In the canonical reference dependent preferences model of Tversky and Kahneman (1991), the agent assesses each alternative relative to the reference point with loss aversion, and therefore *any* reference point will influence choice. Since their value function is S-shaped for gains and losses, reference points which dominate (in each dimension) all available alternatives lead to concave indifference curves, and so the agent chooses a corner. The corner selected is as in Figure 3.2. For other reference points, the effect will depend on the specific functional form. In Kőszegi and Rabin (2006, 2007), the reference point is endogenously selected in a personal equilibrium and also influences choice through loss-aversion. In particular, the reference point is undominated and therefore lies on the budget frontier in the budget set example.

Our model is also related to the recent literature on context-dependent choice (Kőszegi and Szeidl 2013, Bushong, Rabin, and Schwartzstein 2015, Bordalo, Gennaioli, and Shleifer 2013). In these models, alternatives are multi-dimensional and an agent assigns a weight to each dimension which is determined by the choice set. Formally, for a bundle x, an agent's utility function is $U(x) = \sum_i \omega_i x_i$ where ω_i depends upon the choice set and perhaps on x_i as well.

In the models of Kőszegi and Szeidl (2013) (KS) and Bushong, Rabin, and Schwartzstein (2015) (BRS), the weights that an agent assigns to each good depends upon its observed range. An unavailable alternative can influence choice only if it changes this range. Therefore, when an agent chooses from a budget set, unavailable alternatives in the white region of Figures 3(i) & 3(ii) do not influence an agent's choice. According to KS, good x is given more weight when the x-range is larger than the y-range. Therefore, an unavailable alternative in Region 1 will increase the consumption of x because the x-range



i. KS Regions ii. BRS Regions iii. Saliency Regions

Figure 3: Context Dependent Choice

is now relatively larger (Figure 3(i)). Similarly, an unavailable alternative in Region 2 will increase the consumption of y. According to BRS, good x is given *less* weight when the x-range is larger and thus the influence regions are reversed. That is, in Figure 3(ii), an unavailable alternative in Region 1 will increase the consumption of x because the x-range is now relatively smaller. Of these two models, KS is closer to ours whereas BRS makes the opposite predictions.

In salience theory (Bordalo, Gennaioli, and Shleifer 2013), for a bundle (x, y), the weight assigned to good x depends upon a comparison between the ratios x/y and $\bar{x}/\bar{y}$ where $\bar{x}$ is the average x in the choice set (similarly for $\bar{y}$). Thus, for a symmetric finite budget set, a decoy below the 45° line will increase the ratio of the average bundle and it follows[5] that the decoy will distort the weights in favor of good y, the opposite of the aspiration effect.

To sum up the differences, the above models do not possess the two key features of aspirations – that only desired unavailable alternatives influence choice and that they influence choice through similarity. Our model is axiomatic and distances are endogenously derived from choice data. Thus, the model applies to a variety of settings. Of the models depicted above, only the status quo bias models are axiomatic (Masatlioglu and Raymond (2016) further axiomatize the Kőszegi and Rabin models).

Rubinstein and Zhou (1999) axiomatize a bargaining solution which selects

[5]From their assumptions of ordering and diminishing sensitivity.

from a utility possibility set the point closest to an external reference. The bargaining setting is specific and their solution resembles our choice procedure except that the metric and reference point are exogenous. This model and the context-dependent choice models above require clear measures of distance and therefore are applied to monetary outcomes, probabilities, or vectors in $\mathbb{R}^N$. The current model may prove useful for extending distance-based theories to settings without such clear-cut distances.

Finally, the choice framework that we develop consists of choice problems with frames, as in Rubinstein and Salant (2008) and Bernheim and Rangel (2009). The procedure that we characterize can be thought of as taking place in two stages, and thus tangentially relates to other two-stage choice procedures, such as: triggered rationality (Rubinstein and Salant 2006), sequential rationality (Manzini and Mariotti 2007), limited attention (Masatlioglu, Nakajima, and Ozbay 2012) and the warm glow effect (Cherepanov, Feddersen, and Sandroni 2013). The current model differs significantly from these models with respect to both framework and procedure.

2.2 Experimental Evidence

Farquhar and Pratkanis (1992, 1993) were the first to experimentally examine how a desired unavailable alternatives affect choice. They let subjects chose between two alternatives T and R as in Figure 4, and found that the addition of a phantom P increased the proportion of subjects that chose T (the closest alternative to P), as predicted by our model.[6]

Highhouse (1996) compares this *aspiration effect* to the well-known *attraction effect* that introducing a decoy D which is dominated by exactly one alternative increases the choice share of that dominating alternative (Huber, Payne, and Puto 1982).[7] His experiment varies which alternative is targeted (T or R in Figure 4) and how it is targeted: either with a dominating phantom

[6] In our notation, an aspirational effect occurs when $C(S,S) = R$ and $C(S,Y) = T$ where $S = \{R,T\}$, $Y = \{P,R,T\}$, $P \succ R \succ T$ and $d(T,P) < d(R,P)$.

[7] Aspirations do not offer an explanation for the attraction effect as the addition of inferior alternatives do not influence choice. For recent theoretical explanations of the attraction effect, see Natenzon (2018) and Tserenjigmid (2017).

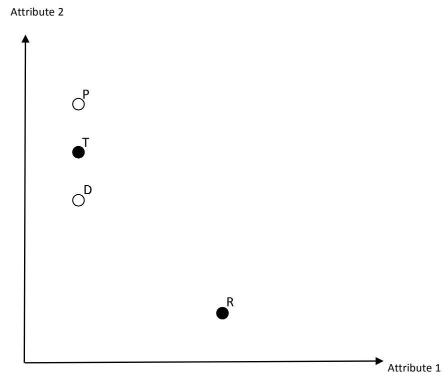


Figure 4: Two Choice Alternatives, a Phantom, and a Decoy

P (the aspiration effect), or with a dominated decoy D (the attraction effect). He finds that targeting an alternative (with either a decoy or a phantom) increases the frequency with which it is chosen and he finds no statistically significant difference in the magnitude of the two effects.

Pettibone and Wedell (2000) similarly compare the aspiration effect to the compromise effect. In their treatments, the unavailable alternative either dominates the target alternative or is placed so that the target becomes a compromise alternative. Like Highhouse (1996), they find that both treatments significantly increase the proportion of agents who choose the target T and there is no statistically significant difference in the magnitude of the two effects.

Recently, Guney and Richter (2015) find additional evidence of the aspirational effect in a deterministic bundle choice setting. Soltani, De Martino, and Camerer (2012) study the impact of unavailable alternatives on choices over lotteries. They find evidence of the attraction effect, but do not find evidence for the compromise and aspiration effects. In the previously discussed experiments mentioned, the alternatives were deterministic. Choice behavior is often different in lottery settings and further experimental research is needed to elucidate the specific factors which are relevant for the aspiration effect.[8]

[8]The strength of the endowment effect in a deterministic setting may differ from that of a lottery setting, e.g., Kahneman, Knetsch, and Thaler (1991), Camerer, Chapman, Dean, Ortoleva, and Snowberg (2017), and Isoni, Loomes, and Sugden (2011).

3 Economic Implications

This section studies equilibrium implications of our choice model in two standard economic environments. The first is a market where firms compete for aspirational buyers. The second is a standard consumption setting where agents aspire to previously consumed bundles.

3.1 Competition over aspirational buyers

In this subsection, we study a competitive market with profit-maximizing firms and aspirational buyers. Each firm sells goods of two different quality levels $q_H > q_L$ with costs $1 > c_H > c_L > 0$ and $q_L - c_L > 0$. Some firms can differentiate themselves by brand and others cannot. There are at least two firms of each type. For expositional clarity, we consider two branded firms "red" and "blue" and refer to the generic firms as "colorless". Firms engage in price competition.

There is a continuum of buyers with wealth w distributed uniformly on $[0, 1]$, each of whom cannot pay more than their wealth. Buyers are aspirational and each observes the choices of all others. Therefore, a buyer's feasible set consists of the goods that she can afford and her potential set also includes the goods chosen by others. The aspirational ranking is determined by $u = q - p$ and each buyer breaks ties randomly. Similarity is lexicographic first by color and then by utility (that is, two goods of the same color are always closer to each other than to a third good of a different color). A buyer who aspires to a red good will always purchase a red good if one is affordable and if not, will purchase a maximal utility good amongst those that are affordable.

An *aspirational market* is the tuple $< q_H, q_L, c_H, c_L >$ where $q_H > q_L$, $1 > c_H > c_L > 0$, and $q_L - c_L > 0$.

As a benchmark, consider rational buyers with the utility function $u = q - p$. In equilibrium, Bertrand-competition drives prices down to marginal costs and firms break even. If the high-quality good is more efficient, $q_H - c_H > q_L - c_L$, then both goods are sold at marginal costs, c_H and c_L. If the low-quality good is more efficient, then only this good is sold at c_L.

However, when buyers are aspirational, both goods being sold at marginal cost is not an equilibrium. The reason is that buyers who cannot afford the high-quality good will aspire to it, and therefore the red firm can profitably increase the price of its low-quality good without losing all of these buyers. In fact, the firm can do even better by slightly decreasing the price of its high-quality good to capture the entire high-quality market and consequently monopolize the low-quality market. The following proposition characterizes the equilibrium prices when buyers are aspirational.

Proposition 3.1. *In an aspirational market, generically there is a unique pure strategy NE. There exists a number $\Delta > 0$ (which depends only upon the costs) such that*

1. *If $q_H - q_L > \Delta$, the red and blue firms charge the same prices $p_L^*,\ p_H^*$ where $c_L < p_L^* < p_H^* < c_H$, the uncolored firms sell only the low-quality good at price c_L, and all firms break even.*

2. *If $q_H - q_L < \Delta$, then only the low-quality good is sold and the price is c_L.*

Furthermore, $c_H - c_L > \Delta$.

We use the term generically because in the measure 0 case where $q_H - q_L = \Delta$, there may be multiple NE and ties. To understand the proposition, consider two cases. First, when the high-quality good is more efficient, then case 1 holds because $q_H - q_L > c_H - c_L > \Delta$. In equilibrium, both goods are sold and prices are jointly determined by:

(i) Monopoly pricing of the low-quality good $p_L = (p_H + c_L)/2$.
(ii) Zero-profit condition $(1 - p_H)(p_H - c_H) + (p_H - p_L)(p_L - c_L) = 0$.

The prices $p_L^*,\ p_H^*$ are the unique admissible solution to the above equations. Given these prices, neither firm has a profitable deviation. If a firm were to decrease the price of their high-quality good, then they would monopolize the high-quality market and all agents would aspire to them. In this case, it is optimal for the firm to charge the monopoly price for their low-quality good (i). By charging a lower price for their high-quality good, they make less money per unit on their high-quality goods and less per-unit money on their low-quality goods as well. This would not be profitable by (ii). If a firm were to increase the price of the high-quality good, they would lose all of their high-quality customers and no agent would aspire to their good. Then, for such a firm to have any customers, it cannot charge more than the colorless firms do for the low-quality good and hence this firm cannot make a profit on their low-quality good

The driving force behind the above equilibrium is the demand externality of the richer buyers on the poorer ones: the red and blue firms compete for aspirations by lowering the price of their high-quality good in order to monopolize the low-quality market. This process settles when the losses in the high-quality market are exactly offset by the monopoly profits in the low-quality market.

Buyers with wealth above p_H^* purchase the red or blue high-quality good (regions III & IV, Figure 5). Buyers with lower wealth cannot afford these high-quality goods, but aspire to them. Buyers in region II purchase either the red or blue low-quality goods at the price p_L^* and buyers in region I purchase a colorless low-quality good at the price c_L.

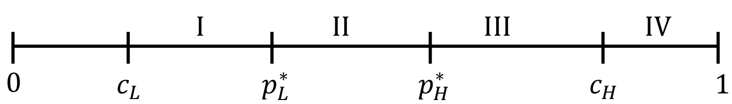


Figure 5: Competition for aspirational buyers

Second, when the low-quality good is more efficient, whether both goods are sold depends upon the quality gap $q_H - q_L$. If only the low-quality good is sold, it must be sold at cost. Notice that the red firm may still introduce a high-quality good to monopolize the low-quality market. To create aspirations, the red firm must price its high-quality good low enough to attract some buyers (incurring losses), and this deviation is profitable when there are sufficiently many buyers in the low-quality market. When the quality gap is small (case 2), any price which attracts buyers to the high-quality market will cannibalize the low-quality market and this deviation is not profitable. But, when the gap is large (case 1), this is a profitable deviation.

We now compare the social welfare of the aspirational and rational equilibria. Notice that they differ only when the quality gap is large. In the aspirational equilibrium, poorer buyers who purchase the red or blue low-quality goods cross-subsidize richer buyers who purchase the red or blue high-quality goods. The utility gains of the richer buyers are larger than the utility losses of the poorer buyers exactly when the high-quality good is more efficient.

Proposition 3.2. *Social welfare is higher in the aspirational equilibrium than in the rational equilibrium if and only if* $q_H - c_H > q_L - c_L$.

In both the rational and aspirational models firms break even and therefore a welfare comparison can focus solely on the change in allocations. When the high-quality good is more efficient, allocations change only for buyers with wealth in region III (Figure 5). These buyers purchase the more-efficient high-quality good instead of the less-efficient low-quality one and welfare increases. Conversely, when the high-quality good is less-efficient, since buyers in regions III and IV switch their purchases to it, welfare decreases.

Thus, overall welfare may increase or decrease, but the welfare distribution always changes to the benefit of richer buyers. Of course, wealthier buyers are better off because they have more options, but we find that competition for aspirations amplifies their advantage, increasing the welfare gap between rich and poor.

There are a number of anecdotal examples of "premium loss leaders", high-quality goods sold beneath costs in order to increase demand for other products. Examples include: high-end low-volume cars produced alongside more affordable and similar options,[9] designer dresses given to celebrities which positively impact their mass market offerings, and restaurant empires anchored by a money-losing highly-rated flagship which is subsidized by profitable cheaper franchises. An article in Fortune[10] explains:

[9]In 2000, 1,371 BMW Z8s were sold while about 40,000 Z3s were sold (BMW's 2000 Annual Report which can be accessed at `https://bib.kuleuven.be/files/ebib/jaarverslagen/BMW_2000.pdf`).

[10]"The Curse of the Michelin-Star Restaurant Rating", *Fortune*, December 11, 2014.

> "For many Michelin star restaurateurs, the restaurant is a loss leader whose fame allows the chef to charge high speaking or private cooking fees; others start lines of premade food or lower priced restaurants. In a recent interview, David Muñoz of DiverXo told me that 2015 would be the first breakeven year in his company's eight-year existence, and then only because of the expansion of his street-food chain, StreetXo."

To sum up, there is a unique equilibrium with the following features: (1) for a good to serve as an aspiration, it need not only be better, but the quality gap must be sufficiently large; (2) firms lose money on their aspirational good and cross-subsidize these losses with profits from the low-quality market; and (3) whether competition for aspirations increases or decreases welfare depends upon the relative efficiency of the goods, but the distribution of welfare always changes in favor of the richer buyers.

Welfare: In markets with sophisticated firms and boundedly rational consumers, it is natural to expect that manipulation of consumers leads to welfare losses, (e.g. Gabaix and Laibson (2006), Spiegler (2006), and Spiegler (2011)). Proposition 3.2 proves that this need not always be the case. A market with aspirational consumers may have higher social welfare (even when judged using the rational utility function) due to equilibrium effects.

Another welfare approach is to compare agents' opportunities. In the rational equilibrium, the available quality-price bundles are $\{(q_L, c_L), (q_H, c_H)\}$ and in the aspirational equilibrium, they are $\{(q_l, c_L), (q_H, c_H), (q_H, p^*_H), (q_L, p^*_L)\}$, where the last two bundles are offered by the branded firms. Thus, respecting agents' autonomy, the aspirational equilibrium Pareto-dominates the rational one because all agents have weakly more opportunities.

It is also possible to consider welfare criteria which explicitly take the agents' aspirations into account. For example, an agent might suffer a utility loss from failed aspirations, that is, when observing other agents consume a desired unavailable good. On the other hand, an agent may have a utility gain from "dressing like a celebrity".

Related Literature: Kamenica (2008) studies buyers whose preferences depend upon a global taste parameter which the firm knows but buyers may not. The firm offers a menu of goods and the buyers make inferences about the parameter through the firm's menu choice. In equilibrium, firms may introduce unprofitable loss leaders to influence the beliefs of uninformed buyers. In another recent paper, Kircher and Postlewaite (2008) consider an infinitely repeated search model where firms vary in unobservable quality and prices are fixed. Since wealthy buyers consume more frequently, they are better informed, and as a result, the poor have an incentive to imitate the rich. In equilibrium, firms offer higher quality to wealthier buyers in order to attract other buyers. In both models, firms may produce superior goods for the sake of influencing less informed buyers. In our model, the high-quality good is subsidized for a different reason, to create aspirations.

3.2 Aspirations for Previously Consumed Bundles

This application analyzes the consumption example of Section 2 where agents may aspire to previously consumed bundles. In period t, an agent chooses a pair (x, y) from a budget set $B_t = \{(x, y) : p_t x + y = I_t\}$ where p_t is the price of the good x at time t, y is the expenditure on all other goods, and I_t is the agent's income at time t.

We are interested in the influence of previous consumption on the choice of an aspirational agent who uses the Euclidean distance. Suppose that income increases. Then across two periods $t = 0, 1$ the budget sets are ordered as $B_0 \subset B_1$. In this case, previous bundles are still affordable and therefore will not affect the decision of an aspirational agent. In contrast, if income decreases, past consumption bundles are no longer affordable and will affect decisions. Thus, an aspirational agent responds rationally to positive income shocks but may respond irrationally to negative income shocks. A similar logic applies to price changes. Figure 6 depicts a price change in the good x. Suppose that both an aspirational and a rational agent choose c' from the larger budget set. Then, following a price increase of good x, the aspirational agent will consume more x than the rational agent.

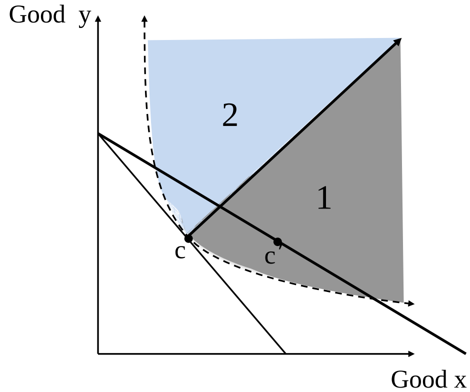


Figure 6: The Influence of Price Changes

We analyze the post-sale demand for a good. A number of empirical studies document a difference between pre-sale and post-sale consumption levels of a good (see, Van Heerde, Leeflang, and Wittink (2000), Hendel and Nevo (2003) and DelVecchio, Henard, and Freling (2006)). During a sale, an agent's budget set expands and her choice will change. After the sale ends, her budget set returns to what it was and an aspirational agent's choice will depend upon her consumption during the sale. This aspirational effect can go either way in that the sale may be beneficial or harmful for the agent's future consumption of the sale good. But, the following proposition proves that if the sales price is sufficiently low, post-sales consumption of the sale good will increase.

Proposition 3.3. *There is always a sale price $p' < p$ such that a temporary sale will increase the post-sale consumption level of the good.*

To determine the direction of the aspirational effect, suppose that an agent with demand $x(p, I)$, $y(p, I)$ undergoes an infinitesimal income or price shock. If income increases or price decreases, then the behavior of the rational agent and aspirational agent coincide. Otherwise, the aspirational agent will consume more x than her rational counterpart precisely when:

$$\text{If incomes decreases: } \left(\frac{I}{p}, -I\right) \cdot \left(\frac{\partial x}{\partial I}, \frac{\partial y}{\partial I}\right) > 0$$
$$\text{If price increases: } \left(\frac{I}{p}, -I\right) \cdot \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}\right) < 0$$

Remarks: First, while the analysis here uses the Euclidean metric, if the agent were to employ a different metric, then the shape of the regions in Figure 6 could be suitably altered. Second, we make the standard Walrasian convexity assumptions which guarantee the existence of a unique aspiration. Third, in habit formation models, an agent's choices may depend upon her past consumption in various forms (for example, see Pollak (1970)), but the aspiration effect has a distinctive one-sided feature: only superior previously consumed bundles may affect choices.

4 Conclusion

This paper introduced a novel choice framework where choices may depend upon observed but unavailable alternatives. There are two important features to any positive choice theory: the axioms which provide necessary and sufficient conditions on individual choice and the economic implications when accounting for equilibrium effects. Our analysis focused on both.

First, we posited three simple conditions on an agent's choice behavior: (1) she behaves rationally across choice problems without unavailable alternatives, (2) she behaves rationally across choice problems with the same observed alternatives, and (3) across choice problems with the same available alternatives and aspirations, she makes the same choices. We showed that these conditions along with single aspirations characterize the aspirational choice procedure: the agent focus on her aspiration and chooses the closest available alternative to it.

Second, we applied the aspirational model and addressed several economic issues including the welfare gains and losses of competition over aspirational buyers; and how previously consumed bundles may influence choice as aspirations.

Part of our contribution is technical. To our knowledge, this is the first axiomatic characterization of a distance-based procedure where the metric is endogenously derived. This "revealed similarity" approach may also prove useful for other models that rely upon exogenously specified distances.

There are several questions that may be of interest for future work. The current model assumes that agents behave rationally when unavailable alternatives are not present. This can be relaxed to allow for other biases. One possibility is an axiomatic theory that simultaneously incorporates both the attraction and aspiration effects. In this vein, an experiment that investigates both effects together, rather than separately as in Highhouse (1996), may shed light on both of them. Another interesting avenue is to extend the first application to study the choices of aspirational agents who may only observe each other through a network. Finally, it may be worth analyzing a dynamic consumption-savings problem where aspirations are drawn from all past consumption bundles.

References

Appadurai, A. (2004): "The Capacity to Aspire: Culture and the Terms of Recognition," in *Culture and Public Action: A Cross-Disciplinary Dialogue on Development Policy*, ed. by V. Rao, and M. Walton. Stanford University Press.

Ben Yaacov, I., A. Berenstein, and S. Ferri (2011): "Reflexive Representability and Stable Metrics," *Mathematische Zeitschrift*, 267(1), 129–138.

Bendor, J., D. Diermeier, and M. Ting (2003): "A Behavioral Model of Turnout," *American Political Science Review*, 97(2), 261–280.

——— (2016): "Inequality, Aspirations, and Social Comparisons," *Political Science Research and Methods*, 4(3), 451–476.

Bernheim, B. D., and A. Rangel (2009): "Beyond Revealed Preference: Choice-Theoretic Foundations for Behavioral Welfare Economics," *The Quarterly Journal of Economics*, 124(1), 51–104.

Bordalo, P., N. Gennaioli, and A. Shleifer (2012a): "Salience and Experimental Tests of the Endowment Effect," *The American Economic Review: Papers and Proceedings*, 102(3), 47–52.

——— (2012b): "Salience Theory of Choice Under Risk," *The Quarterly Journal of Economics*, 127(3), 1243–1285.

——— (2013): "Salience and Consumer Choice," *Journal of Political Economy*, 121(5), 803–843.

Borgers, T., and R. Sarin (2000): "Naive Reinforcement Learning with Endogenous Aspirations," *International Economic Review*, 41(4), 921–50.

Bushong, B., M. Rabin, and J. Schwartzstein (2015): "A Model of Relative Thinking," *Working paper*.

Camerer, C., J. Chapman, M. Dean, P. Ortoleva, and E. Snowberg (2017): "Willingness-To-Pay and Willingness-To-Accept are Probably Less Correlated than You Think," *Working paper*.

Cherepanov, V., T. Feddersen, and A. Sandroni (2013): "Revealed Preferences and Aspirations in Warm Glow Theory," *Economic Theory*, 54(3), 501–535.

Correll, S. J. (2004): "Constraints into Preferences: Gender, Status, and Emerging Career Aspirations," *American Sociological Review*, 69(1), 93–113.

DelVecchio, D., D. H. Henard, and T. H. Freling (2006): "The Effect of Sales Promotion on Post-Promotion Brand Preference: A Meta-Analysis," *Journal of Retailing*, 82(3), 203–213.

Duesenberry, J. S. (1949): *Income, Saving, and the Theory of Consumer Behavior*. Harvard University Press.

Evren, Ö., and E. A. Ok (2011): "On the Multi-utility Representation of Preference Relations," *Journal of Mathematical Economics*, 47(4), 554–563.

Farquhar, P. H., and A. R. Pratkanis (1992): "A Brief History of Research on Phantom Alternatives: Evidence for Seven Empirical Generalizations About Phantoms," *Basic and Applied Social Psychology*, 13(1), 103–122.

——— (1993): "Decision Structuring with Phantom Alternatives," *Management Science*, 39, 1214–1226.

Gabaix, X., and D. Laibson (2006): "Shrouded Attributes, Consumer Myopia, and Information Suppression in Competitive Markets," *The Quarterly Journal of Economics*, 121(2), 505–540.

Genicot, G., and D. Ray (2017): "Aspirations and Inequality," *Econometrica*, 85(2), 489–519.

Guney, B., and M. Richter (2015): "An Experiment on Aspiration-Based Choice," *Journal of Economic Behavior & Organization*, 119, 512 – 526.

Hendel, I., and A. Nevo (2003): "The Post-Promotion Dip Puzzle: What Do the Data Have to Say," *Quantitative Marketing and Economics*, 1(4), 409–424.

Highhouse, S. (1996): "Context-Dependent Selection: The Effects of Decoy and Phantom Job Candidates," *Organizational Behavior and Human Decision Processes*, 65(1), 68 – 76.

Hopkins, E., and T. Kornienko (2004): "Running to Keep in the Same Place: Consumer Choice as a Game of Status," *The American Economic Review*, 94(4), 1085–1107.

Huber, J., J. W. Payne, and C. Puto (1982): "Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis," *Journal of Consumer Research: An Interdisciplinary Quarterly*, 9(1), 90–98.

Isoni, A., G. Loomes, and R. Sugden (2011): "The Willingness to PayWillingness to Accept Gap, the "Endowment Effect," Subject Misconceptions, and Experimental Procedures for Eliciting Valuations: Comment," *American Economic Review*, 101(2), 991–1011.

Kahneman, D., J. L. Knetsch, and R. H. Thaler (1991): "Anomalies: The Endowment Effect, Loss Aversion, and Status Quo Bias," *Journal of Economic Perspectives*, 5(1), 193–206.

Kalai, E., and M. Smorodinsky (1975): "Other Solutions to Nash's Bargaining Problem," *Econometrica*, 43(3), 513–518.

Kamenica, E. (2008): "Contextual Inference in Markets: On the Informational Content of Product Lines," *The American Economic Review*, 98(5), 2127–2149.

Kao, G., and M. Tienda (1998): "Educational Aspirations of Minority Youth," *American Journal of Education*, 106(3), 349–384.

Karandikar, R., D. Mookherjee, D. Ray, and F. Vega-Redondo (1998): "Evolving Aspirations and Cooperation," *Journal of Economic Theory*, 80(2), 292–331.

Kircher, P., and A. Postlewaite (2008): "Strategic Firms and Endogenous Consumer Emulation," *The Quarterly Journal of Economics*, 123(2), 621–661.

Kőszegi, B., and M. Rabin (2007): "Reference-Dependent Risk Attitudes," *The American Economic Review*, 97(4), 1047–1073.

Kőszegi, B., and A. Szeidl (2013): "A model of focusing in economic choice," *The Quarterly Journal of Economics*, 128(1), 53–104.

Levin, V. (1983): “Measurable Utility Theorems for Closed and Lexicographic Preference Relations,” *Soviet Math. Dokl.*, 27, 639–643.

Manzini, P., and M. Mariotti (2007): “Sequentially Rationalizable Choice,” *The American Economic Review*, 97(5), 1824–1839.

Masatlioglu, Y., D. Nakajima, and E. Y. Ozbay (2012): “Revealed Attention,” *The American Economic Review*, 102(5), 2183–2205.

Masatlioglu, Y., and E. A. Ok (2005): “Rational Choice with Status Quo Bias,” *Journal of Economic Theory*, 121(1), 1–29.

Masatlioglu, Y., and E. A. Ok (2014): “A Canonical Model of Choice with Initial Endowments,” *The Review of Economic Studies*, 81(2), 851–883.

Masatlioglu, Y., and C. Raymond (2016): “A Behavioral Analysis of Stochastic Reference Dependence,” *The American Economic Review*, 106(9), 2760–82.

Natenzon, P. (2018): “Random Choice and Learning,” *Journal of Political Economy, forthcoming.*

Ok, E. A., P. Ortoleva, and G. Riella (2015): “Revealed (P) Reference Theory,” *The American Economic Review*, 105(1), 299–321.

Pettibone, J. C., and D. H. Wedell (2000): “Examining Models of Nondominated Decoy Effects Across Judgment and Choice,” *Organizational Behavior and Human Decision Processes*, 81, 300–328.

Pollak, R. A. (1970): “Habit Formation and Dynamic Demand Functions,” *The Journal of Political Economy*, 78(4, Part 1), 745–763.

——— (1976): “Interdependent Preferences,” *The American Economic Review*, 66(3), 309–320.

Rader, T. (1963): “The Existence of a Utility Functionsto Represent Preferences,” *The Review of Economic Studies*, 30, 229–232.

RAY, D. (2006): "Aspirations, Poverty, and Economic Change," *Understanding Poverty*, pp. 409–421.

RAY, D., AND A. ROBSON (2012): "Status, Intertemporal Choice, and Risk-Taking," *Econometrica*, 80(4), 1505–1531.

RUBINSTEIN, A., AND Y. SALANT (2006): "Two Comments on the Principle of Revealed Preference," *Working Paper*.

——— (2008): "(A,f): Choice with Frames," *The Review of Economic Studies*, 75(4), 1287–1296.

RUBINSTEIN, A., AND L. ZHOU (1999): "Choice Problems with a 'Reference' Point," *Mathematical Social Sciences*, 37(3), 205–209.

SAMUELSON, W., AND R. ZECKHAUSER (1988): "Status Quo Bias in Decision Making," *Journal of Risk and Uncertainty*, 1, 7–59.

SOLTANI, A., B. DE MARTINO, AND C. CAMERER (2012): "A Range-Normalization Model of Context-Dependent Choice: A New Model and Evidence," *PLoS Computational Biology*, 8(7), 1–15.

SPIEGLER, R. (2006): "Competition Over Agents with Boundedly Rational Expectations," *Theoretical Economics*, 1(2), 207–231.

——— (2011): *Bounded Rationality and Industrial Organization*. Oxford University Press.

STUTZER, A. (2004): "The Role of Income Aspirations in Individual Happiness," *Journal of Economic Behavior and Organization*, 54(1), 89–109.

TSERENJIGMID, G. (2017): "Choosing with the Worst in Mind: A Reference-Dependent Model," *Mimeo*.

TVERSKY, A., AND D. KAHNEMAN (1991): "Loss Aversion in Riskless Choice: A Reference-Dependent Model*," *The Quarterly Journal of Economics*, 106(4), 1039–1061.

Van Heerde, H. J., P. S. Leeflang, and D. R. Wittink (2000): "The Estimation of Pre- and Postpromotion Dips with Store-Level Scanner Data," *Journal of Marketing Research*, 37(3), 383–395.

Veblen, T. (1899): *The Theory of the Leisure Class*. Oxford University Press.

5 Appendix A

For all proofs below, a *preference relation* is a reflexive and transitive binary relation.

Proof of Theorem 2.1

[$\Rightarrow$] Suppose C is a choice correspondence that satisfies Axioms 2.1-2.5. Define $a(Z) := C(Z, Z)$. By Axioms 2.1 and 2.4, there exists an anti-symmetric total order $\succeq$ such that $a(Z) = \max(Z, \succeq)$. Notice that $x \succeq y$ if and only if $\{x\} = C(\{x, y\}, \{x, y\})$. Moreover, by Axiom 2.5, $\succeq$ and a are continuous.

Suppose $x, y, z \in X$. For each z, we define the aspiration based preference as $x \succeq_z y$ if $x \in C(\{x, y\}, \{x, y, z\})$ and $C(\{x, y, z\}, \{x, y, z\}) = \{z\}$. A ranking over pairs $\succeq^*$ is defined on X^2 as: $(x, y) \succeq^* (z, y)$ if $x \succeq_y z$. Note that both of these preferences are generally incomplete. To obtain a representing utility function, we could appeal to Levin (1983)'s Theorem 1, but for readability, we instead rely upon Corollary 1 of Evren and Ok (2011). That corollary requires $\succeq^*$ to be closed-continuous and $T := \{(x, y) : y \succeq x\}$ to be a locally compact separable metric space (which is an implication of T being a compact metric space). Below we show that $\succeq^*$ and T meet these conditions.

First, $\succeq^*$ is closed-continuous. To see it, take any $x, y, z, x_n, y_n, z_n \in X$ and $n = 1, 2, \ldots$ such that $x_n \to x, y_n \to y, z_n \to z$ and $(x_n, y_n) \succeq^* (z_n, y_n)$ for all n. By the definition of $\succeq^*$, we have $x_n \succeq_{y_n} z_n$, i.e. $x_n \in C(\{x_n, z_n\}, \{x_n, y_n, z_n\})$ and $\{y_n\} = C(\{x_n, y_n, z_n\}, \{x_n, y_n, z_n\})$. By Axioms 2.4 and 2.5, we have that $x \in C(\{x, z\}, \{x, y, z\})$ and $\{y\} = C(\{x, y, z\}, \{x, y, z\})$. Therefore $x \succeq_y z \Rightarrow (x, y) \succeq^* (z, y)$.

Second, T is compact. To see this, notice that the Axioms 2.4 and 2.5 imply that T is a closed subset of the compact metric space X^2. Therefore, there exists a continuous $u : T \to \mathbb{R}$ so that if $(x, y) \succeq^* (z, y)$, then $u(x, y) \geq u(z, y)$ and likewise for the strict relation. Define the similarity measure $\hat{d}$ as

$$\hat{d}(x, y) := \begin{cases} u(y, y) - u(x, y) & \text{if } y \succeq x, \\ u(x, x) - u(y, x) & \text{otherwise} \end{cases}$$

This similarity measure represents an agent's choices, that is, $C(S,Y) = \text{argmin}_{s\in S}\hat{d}(s, a(Y))$.

"$C(S,Y) \subseteq \text{argmin}_{s\in S}\ \hat{d}(s, a(Y))$"

For ease of notation, denote $a(Y)$ by a. Take $y \in C(S,Y)$ and suppose $y \notin \text{argmin}_{s\in S}\ \hat{d}(s,a)$. Then, there exists $z \in S$ s.t. $\hat{d}(z,a) < \hat{d}(y,a)$. As a is the aspiration alternative in Y, $a \succeq z, y$. By the definition of $\hat{d}$, it is the case that $u(z,a) > u(y,a)$. So, $z \succ_a y$ which implies that $\{z\} = C(\{z,y\}, \{z,y,a\})$ and $\{a\} = C(\{z,y,a\}, \{z,y,a\})$. Axiom 2.2 guarantees that $y \in C(\{z,y\}, Y)$. By Axiom 2.3, it is the case that $C(\{z,y\}, Y) = C(\{z,y\}, \{z,y,a\})$. This is a contradiction as $y \in C(\{z,y\}, Y) = C(\{z,y\}, \{z,y,a\}) = \{z\}$.

"$\text{argmin}_{s\in S}\ \hat{d}(s, a(Y)) \subseteq C(S,Y)$"

Consider $z \in \text{argmin}_{s\in S}\ \hat{d}(s,a)$ and assume $z \notin C(S,Y)$. As C is non-empty valued, there must exist $y \in C(S,Y)$. By Axioms 2.2 and 2.3, it is the case that $\{y\} = C(\{z,y\}, \{a,z,y\})$. Then $y \succ_a z$ by the definition of $\succeq_a$. Thus, $\hat{d}(z,a) > \hat{d}(y,a)$ by the definition of $\hat{d}$, a contradiction.

By construction, $\hat{d}$ is continuous, reflexive and symmetric, but need not satisfy the triangle inequality. Lemma 5.1 shows that there exists a continuous metric $d : X^2 \to \mathbb{R}_+$ with the same distance ordering, that is,

$$\hat{d}(x,y) \leq \hat{d}(z,w) \text{ if and only if } d(x,y) \leq d(z,w).$$

Thus, $C(S,Y) = \underset{s\in S}{\text{argmin}}\ d(s, a(Y))$ for all $(S,Y) \in \mathcal{C}(X)$.

[$\Leftarrow$] Axioms 2.1, 2.2, 2.3, 2.4 all follow trivially.

To show Axiom 2.5, suppose $x_n \in C(S_n, Y_n)$ for all n, where $S_n \overset{H}{\to} S$, $Y_n \overset{H}{\to} Y$ and $x_n \to x$. Since $S_n \overset{H}{\to} S$, we know that $\forall s \in S$, there exist $s_n \in S_n$ for $n = 1, 2, \ldots$ such that $s_n \to s$. By $x_n \in C(S_n, Y_n)$, we have that $d(x_n, a(Y_n)) \leq d(s_n, a(Y_n))$ for all n. Continuity of $a(\cdot)$ guarantees that $a(Y_n) \to a(Y)$ and taking limits of both sides gives $d(x,y) \leq d(s,y)$. Since s was chosen arbitrarily, it is the case that $x \in \text{argmin}_{s\in S}\ d(s,y)$ and thus $x \in C(S,Y)$. $\square$

The previous theorem constructs a semimetric (a distance function without the triangle inequality) and the following lemma shows that any semimetric can be transformed into a metric while preserving its distance ordering over pairs.

Definition: A function $\phi : X^2 \to \mathbb{R}$ is a *semimetric* if

1. $\forall x, y \in X,\ \phi(x, y) = 0 \Leftrightarrow x = y$ (Reflexivity)
2. $\forall x, y \in X,\ \phi(x, y) = \phi(y, x)$ (Symmetry)
3. $\forall x, y \in X,\ \phi(x, y) \geq 0$ (Non-Negativity)

Lemma 5.1. *Take (X, D) a compact metric space and suppose $\phi : X^2 \to \mathbb{R}_+$ is a semimetric. Then, there exists a continuous metric $D_\phi : X^2 \to \mathbb{R}_+$ such that*

$$\phi(x, y) \leq \phi(z, w) \text{ if and only if } D_\phi(x, y) \leq D_\phi(z, w) \text{ for any } x, y, z, w \in X.$$

Around the same time we proved this lemma, Ben Yaacov, Berenstein, and Ferri (2011) independently proved an equivalent result (Theorem 2.8) in a mathematics journal. Our proof is different and relies upon more elementary techniques. To keep the analysis to a minimum, we omit it from the current appendix and provide it in the online appendix.

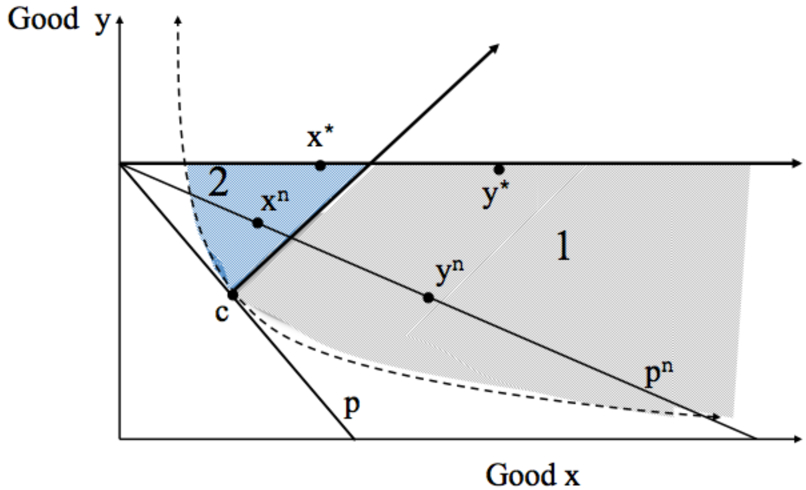


Figure 7: Continuity argument for Proposition 3.3.

Proof of Proposition 3.3:

Let c denote the consumed bundle with price p and income I. Denote the Regions $R_1 = \{x : x_2 \leq I, U(x) > U(c), (x - c) \cdot (p, -1) > 0\}$ and $R_2 = \{x : x_2 \leq I,\ U(x) > U(c), (x - c) \cdot (p, -1) < 0\}$. Take a sequence of prices less than p so that $p_1, p_2 \ldots \to 0$. Let $x_n = C(B(I, p_N))$. For each n, choose y_n in R_1 so that $p_n \cdot y_n = I$ and so that $\lim_{n \to \infty} y_n = y^*$ is in R_1 with $y_2 = I$. Suppose that $x_n \in R_2$ infinitely often. Then, passing to that subsequence, by compactness $\lim_{n \to \infty} x_n = x^*$ exists and $x_n \succ y_n$ for all n. Therefore, continuity implies that $x^* \succsim y^*$, but monotonicity stipulates that y^* is strictly preferred to all members of R_2, including x^*, a contradiction. Therefore, x_n must eventually lie in R_1 and then any such p_n represents a sales price which will have consumption in R_1 and consequently a positive impact on x consumption after the sale ends. $\square$

Proof of Proposition 3.1

Denote the chosen prices of the blue and red firms as $p_H^B, p_L^B, p_H^R, p_L^R$ and the price of the generic firms as $p_L^{colorless}, p_H^{colorless}$. Recall that $0 < c_L < c_H$ and $0 < q_L - c_L$. The following notation will be used extensively. Let

$$\Pi(p) = (1-p)(p-c_H) + \frac{(p-c_L)^2}{4} \tag{1}$$

denote the profit that a firm would obtain if it sold its high-quality good for p and acts as a monopolist for buyers with $w < p$ and charges $(p+c_L)/2$ for its low-quality good.

First, in any equilibrium, the low-quality good is sold by the colorless firms at price $p_L^{colorless} = c_L$. If it were not, then some firm is selling a good at a price beneath c_L, which cannot happen.

Second, all firms selling only the low-quality good (at marginal cost) is an equilibrium if and only if $q_H - q_L < \Delta$ where Δ is to be defined.

To see it, suppose that all firms follow the above strategy. Clearly, the colorless firms have no profitable deviation. If the red firm introduces a high-quality good at price p, then $p \leq q_H - q_L + c_L$. Otherwise, $q_H - p < q_L - c_L$ and no buyer would purchase it. Any such price attracts buyers in the high-quality market, and through aspirations, the red firm becomes a monopolist for all buyers with wealth below p. This deviation is profitable when $\Pi(p) \geq 0$.

Notice that the function Π is a hump-shaped parabola with left root

$$\rho = \left(2 + 2c_H - c_L - 2\sqrt{1 - c_H + c_H^2 - c_L - c_H c_L + c_L^2}\right)/3.$$

Thus, a deviation p is profitable exactly when $\rho \leq p \leq q_H - q_L + c_L$ and there are no profitable deviations exactly when

$$\Delta := \rho - c_L > q_H - q_L.$$

Finally, $\Pi(c_L) < 0 < \Pi(c_H)$, so $c_L < \rho < c_H$ and $0 < \Delta < c_H - c_L$.

We now show that when $q_H - q_L < \Delta$ this equilibrium is unique. Notice that the high-quality good is never sold in equilibrium, because there is no price for that good which is both attractive to agents and which can be profitably cross-subsidized by monopoly pricing of the low-quality good. Thus, only the low-quality good is sold and Bertrand competition drives the price of this good down to marginal cost.

Third, when the valuation gap is large, that is $q_H - q_L > \Delta$, the following strategies constitute an equilibrium: The generic firms only sell the low-quality good at marginal cost, c_L. The red and blue firms sell both the high and the low quality goods and charge the same prices given by $p_H = \rho$ and $p_L = (\rho + c_L)/2$. These prices are the unique admissible solution to i) Monopoly price of low-quality, $p_L = (p_H + c_L)/2$ and ii) Zero profits, $\Pi(p_H) = 0$ and are given by:

$$p_H = \left(\frac{1}{3}\right)\left(2 + 2c_H - c_L - 2\sqrt{c_H^2 - c_H c_L - c_H + c_L^2 - c_L + 1}\right) = \rho$$
$$p_L = \left(\frac{1}{3}\right)\left(1 + c_H + c_L - \sqrt{c_H^2 - c_H c_L - c_H + c_L^2 - c_L + 1}\right)$$

At the above prices, agents prefer the high-quality good to the generic low-quality good because the gap is large. That is, $q_H - p_H > q_L - c_L$ because $q_H - q_L > \Delta = \rho - c_L = p_H - c_L$. Furthermore, the above strategy is not an equilibrium when the gap is small because agents will not purchase the high-quality good.

The colorless firms clearly have no profitable deviations. The red firm cannot profit by increasing its price for the high-quality good because no agent would purchase it. If the red firm loses the high-quality market, then it is essentially no different from the colorless firms and cannot profit in the low-quality market. Following a deviation to a lower price $p'_H < p_H$, the red firm captures the entire high-quality market and has profit $\Pi(p'_H) < 0$. Therefore, the above prices constitute an equilibrium.

Uniqueness is established by the following points.

1. When the gap is large, the high-quality good is sold in equilibrium.

Only the low-quality good being sold is not an equilibrium because $q_H - q_L > \Delta$ and as shown in the low-gap case, the red firm would then have a profitable deviation.

2. Both the red and blue firms must charge the prices given above.

WLOG assume that $p_H^R \leq p_H^B$. If $p_H^R < p_H$, then its profit is either $\Pi(p_H^R)$ or $\Pi(p_H^R)/2$, both of which are negative. If $p_H < p_H^R$, then the blue firm can profitably undercut either the red firm or the generic firm (depending upon which minimally prices the high-quality good). Finally, if $p_H = p_H^R < p_H^B$, then the red firm can profitably increase p_H^R. □

Proof of Proposition 3.2

As firms make zero profit in both the rational and aspirational equilibria, the following focuses on social welfare but covers consumers' welfare as well. In the rational and aspirational equilibria, welfare respectively is

$$\mathcal{W}_R = (1 - c_H)(q_H - c_H) + (c_H - c_L)(q_L - c_L)$$

$$\mathcal{W}_A = (1 - p_H^*)(q_H - c_H) + (p_H^* - c_L)(q_L - c_L).$$

Then $\mathcal{W}_A - \mathcal{W}_R = (c_H - p_H^*)(q_H - c_H - (q_L - c_L))$. As $c_H > p_H^*$, the first term is positive and thus $\mathcal{W}_A > \mathcal{W}_R$ if and only if the high-quality good is more efficient, that is, $q_H - c_H > q_L - c_L$. □

Appendix B -FOR ONLINE PUBLICATION

Lemma 6.1. *Take (X, D) a compact metric space and suppose $\phi : X^2 \to \mathbb{R}_+$ is a semimetric. Then, there exists a continuous metric $D_\phi : X^2 \to \mathbb{R}_+$ such that*

$$\phi(x, y) \leq \phi(z, w) \text{ if and only if } D_\phi(x, y) \leq D_\phi(z, w) \text{ for any } x, y, z, w \in X.$$

Proof of Lemma 5.1

If X is a singleton, then ϕ is the constant 0 function and $D = \phi$ satisfies all of the above properties.

If X is not a singleton, then WLOG, assume $\max_{x,y \in X} \phi(x, y) = 1$. This is because $\max_{x,y \in X} \phi(x, y)$ exists since X^2 is a compact space and ϕ is a continuous function on that space. Therefore $\psi := \frac{\phi}{\max_{x,y \in X} \phi(x, y)}$ is an order-preserving transformation where $\max_{x,y \in X} \psi(x, y) = 1$.

The nature of the problem is that ϕ may fail the triangle inequality and the goal is to find a continuous increasing transformation f on the distances so that the triangle inequality will be satisfied. As in the figure, it could be the case that $\phi(x, y)$, $\phi(x, z)$ are very small (but not equal to 0) and $\phi(y, z)$ is close to 1 and it must be that $f(\phi(x, y)) + f(\phi(x, z)) \geq f(\phi(y, z))$.

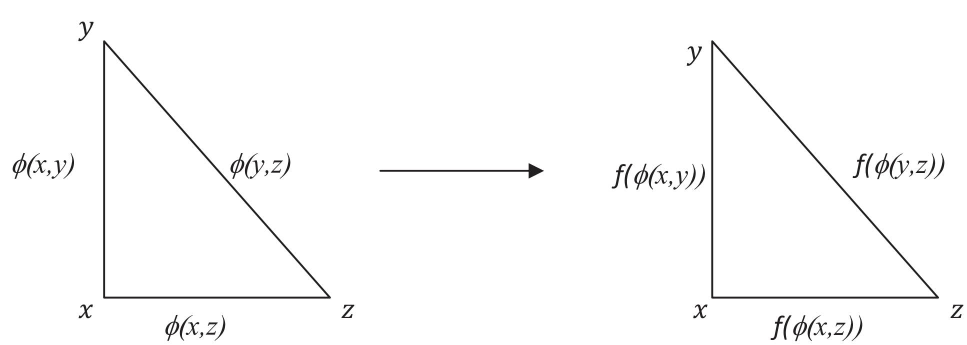


Figure 8: Converting a triangle to satisfy the triangle inequality.

To further analyze the problem, define the operator Δ by $\underset{(x,y,z) \to (\psi(x,y), \psi(x,z), \psi(y,z))}{\Delta(\psi) : X^3 \to \mathbb{R}_+^3}$.

This map, for any triple outputs the length of the edges according to ψ of the triangle defined by the triple. Additionally, notice that the operator Δ's output is ordered, so in general Δ is not invariant to permutations of its inputs.

Let q denote a constant such that $q \in \left(\frac{1}{2}, 1\right)$.

Definition of D:

Now, we will define an increasing sequence of D_n such that their limit defines D. The approach that we take fixes triangles with a "long" side first, and fixes more and more triangles as $n \to \infty$. Formally, at stage n; D_n will be such that all triangles with longest side at least q^n will satisfy the triangle inequality according to D_n.

For the base case, let $D_0 = \phi$.

To proceed with the inductive construction of D_n, we define a few auxiliary functions h_i, g_i, k_i, and f_i such that $D_i = f_i \circ D_{i-1}$.

- $h_i(a) := \max \left\{a, c - b : \exists x, y, z \in X^3 \text{ s.t. } (a, b, c) = \Delta(D_{i-1})(x, y, z) \text{ and } a \leq b \leq c\right\}$. Define $h_i(a) := a$ if $\nexists (a, b, c) \in \Delta(D_{i-1})$ where $a \leq b \leq c$.
- $g_i(a) := \min \left(h_i(a), \frac{q^{i-1}}{2}\right)$ on $[0, q^i)$. Define $g_i(q^i) = q^i$.
- $k_i : [0, q^i] \to [0, q^i]$ to be the Upper Concave Envelope of g_i. Recall that the upper concave envelope of another function is the least concave function that dominates the specified function.
- $f_i(a) := \begin{cases} a & a \in (q^i, 1] \\ k_i(a) & a \in [0, q^i] \end{cases}$

The motivation for the preceding functions is as follows. h_i guarantees that $h_i(D_{i-1}(x, y)) + h_i(D_{i-1}(y, z)) \geq D_{i-1}(x, z)$. So, h_i is enough to fix an unsatisfied triangle inequality if it is to be applied to only the two shorter sides of the triangle. But h_i presents two difficulties: i) it may be that one of the two augmented sides may now be longer than the sum of the other two sides and ii) h_i cannot be applied to only two of the triangle's sides. g_i addresses the first difficulty as $g_i(a) \leq q^i, \forall a \leq q^i$. Then, k_i addresses the second difficulty as it transforms g_i into a continuous concave positive increasing function so that $k_i(q^i) = q^i$ and $k_i(D_{i-1}(x,y)) + k_i(D_{i-1}(y,z)) \geq D_{i-1}(x,z)$ whenever $D_{i-1}(x,z) \leq q^{i-1}$. Notice that this could not be done directly to h_i because $h_i(x)$ may be greater than q^i. Finally, f_i applies k to "small distances" and leaves "large distances" unchanged.

Formally, we next prove that the following inductive properties hold true:

1. D_i is continuous on X^2.
2. $\forall i \geq 1\ \forall x, y \in X,\ D_i(x,y) \geq D_{i-1}(x,y)$.
3. $\forall i, j \geq 0, \forall x, y \in X,\ \phi(x,y) \geq q^{\min(i,j)} \Rightarrow D_i(x,y) = D_j(x,y)$.
4. If $x, y, z \in X$ and $D_{i-1}(y,z) \geq q^i$, then $D_i(y,z) \leq D_i(x,y) + D_i(x,z)$.
5. $\forall i \geq 1$, $D_i \sim_X D_{i-1}$
6. $\forall i \geq 0$, $D_i : X^2 \to \mathbb{R}_+$ is symmetric, reflexive, and non-negative.

Proof of Properties 1,2,5:

Suppose $(0, b, c) = \Delta(D_{i-1})(x, y, z)$. Then $0 = D_{i-1}(x,y) \Rightarrow x = y$ by the reflexivity of D_{i-1}.

Therefore $h_i(0) = \max D_{i-1}(z,x) - D_{i-1}(z,x) = 0$.

So, $g_i(0) = \min(0, \frac{q^{i-1}}{2}) = 0$ and $g_i(q^i) = q^i$.

Moreover, $\forall z < q^i,\ g_i(z) \leq \frac{q^{i-1}}{2} = q^i \left(\frac{1}{2q}\right) < q^i$.

Claim: h_i is upper semi-continuous. Thus g_i is upper semi-continuous as well.

Consider $a_n \to a$ such that $h_i(a_n) \to z$. If it is the case that $h_i(a_n) = a_n$ infinitely often, then $z = a$ and by definition $h_i(a) \geq a$. On the other hand, if it is the case that $h_i(a_n) > a_n$ infinitely often, then there exist triangles x_n, y_n, z_n such that $D_{i-1}(x_n, y_n) = a_n$ and $D_{i-1}(y_n, z_n) - D_{i-1}(x_n, z_n) = h_i(a_n)$. By the compactness of X and the continuity of D_{i-1}, one can pass to convergent subsequences of x_n, y_n, z_n and thus $h_i(a_n) = D_{i-1}(y_n, z_n) - D_{i-1}(x_n, z_n) \to D_{i-1}(y, z) - D_{i-1}(x, z) \leq h_i(a)$ where the last inequality is because $D_{i-1}(x, z) = a$. The claim is proven.

Importantly, non-negativity of g_i, h_i, $g_i(0) = h_i(0) = 0$, and upper semi-continuity of g_i, h_i guarantee full continuity of g_i, h_i at 0. So, it is the case that k_i is strictly increasing, concave, and continuous on $[0, q^i]$ where $k_i(0) = 0$ and $k_i(q^i) = q^i$. Thus f_i is strictly increasing, continuous on $[0, 1]$ which implies that $D_i \sim_X D_{i-1}$ (Properties 1 & 5).

By the concavity of k_i on $[0, q^i]$ and $k_i(x) = x$ for $x \in \{0, q^i\}$, it is the case that $k_i(x) \geq x \ \forall x \in (0, q^i)$. Therefore $f_i(a) \geq a$ on $[0, 1]$ which implies $\forall x, y \in X,\ D_i(x, y) \geq D_{i-1}(x, y)$ (Property 2).

Proof of Property 6:

$D_i(x, y) \geq D_{i-1}(x, y) \geq 0$ implies that D_i is non-negative.

$D_i(x, y) = f_i \circ D_{i-1}(x, y) = f_i \circ D_{i-1}(y, x) = D_i(y, x)$ where the middle equality follows from the symmetry of D_{i-1}.

Finally $0 = D_i(x, y) \Leftrightarrow 0 = f_i(D_{i-1}(x, y)) \Leftrightarrow 0 = D_{i-1}(x, y) \Leftrightarrow x = y$ where the second implication comes from f_i being a strictly increasing (hence invertible) function on $[0, 1]$ with $f_i(0) = 0$ and the third implication is due to the reflexivity of D_{i-1}.

Proof of Property 3:

Let $i < j$. Then $\phi(x, y) = D_0(x, y) \leq \ldots \leq D_i(x, y) \Rightarrow q^i \leq D_i(x, y) = D_{i+1}(x, y) = \ldots = D_j(x, y)$.

Proof of Property 4:

Take x, y, z such that $D_{i-1}(y, z) \geq q^i$.

Case 1: Suppose $D_{i-1}(y, z) \geq q^{i-1}$.

Then, by the inductive hypothesis, definition of D_i and property 2, it is the case that

$$D_i(y, z) = D_{i-1}(y, z) \leq D_{i-1}(x, y) + D_{i-1}(x, z) \leq D_i(x, y) + D_i(x, z)$$

Case 2: Suppose $D_{i-1}(x,y), D_{i-1}(x,z), D_{i-1}(y,z) \geq q^i$ and $D_{i-1}(y,z) < q^{i-1}$.

Then $D_{i-1}(x,y) + D_{i-1}(x,z) \geq 2q^i = 2q(q^{i-1}) > q^{i-1} > D_{i-1}(y,z)$ where the second to last inequality holds because $q > 1/2$.

Case 3: Suppose $q^i \leq D_{i-1}(y,z) < q^{i-1}$ and WLOG both of the following hold: $D_{i-1}(x,y) \leq D_{i-1}(x,z)$ and $D_{i-1}(x,y) < q^i$.

If $h_i(D_{i-1}(x,y)) \geq \frac{q^{i-1}}{2}$, then $D_i(x,y) = k_i(D_{i-1}(x,y)) \geq g_i(D_{i-1}(x,y)) = \frac{q^{i-1}}{2}$. By Property 5), it is the case that $D_i(x,z) \geq D_i(x,y) \geq \frac{q^{i-1}}{2}$, hence $D_i(x,y) + D_i(x,z) \geq 2\left(\frac{q^{i-1}}{2}\right) = q^{i-1} > D_{i-1}(y,z)$.

If $h_i(D_{i-1}(x,y)) < \frac{q^{i-1}}{2}$, then $h_i(D_{i-1}(x,y)) = g_i(D_{i-1}(x,y)) \leq k_i(D_{i-1}(x,y)) = D_i(x,y)$. So $D_i(x,y) + D_i(x,z) \geq h_i(D_{i-1}(x,y)) + D_{i-1}(x,z) \geq D_{i-1}(y,z) = D_i(y,z)$ where the last inequality is due to the definition of h_i.

D is well-defined:

Define $D(x,y) := \lim_{n\to\infty} D_n(x,y)$.

Notice that $D_n \leq D_{n+1} \leq 1$ where the first inequality is due to property 2 and the last inequality is by definition.

Additionally, notice that D could instead be defined as $D = f \circ \phi$ where $f = \ldots \circ f_2 \circ f_1$. The function f is well-defined because $f_i(x) \geq x$, and $f_i(x) = x$ if $x \geq q^i$. Thus, each $x > 0$ is touched by at most $\lceil \log_q(x) \rceil$ iterations of f and $f_i(0) = 0 \Rightarrow f(0) = 0$.

D is continuous:

By properties 3 and 5, it is the case that $||D_i - D_j||_\infty < q^{\min(i,j)}$ which implies that D_i converges uniformly. Property 1 stipulates that each D_i is continuous and thus D is continuous because the uniform limit of a sequence of continuous functions is continuous.

$\mathbf{D(x,y) < D(z,w)}$ iff $\boldsymbol{\phi(x,y) < \phi(z,w)}$:

If $\phi(x,y) < \phi(z,w)$, then $\forall i$, it is the case that $D_i(x,y) < D_i(z,w)$ by Property 5. Moreover, if $x \neq y$, then $\exists n$ such that $q^n < \phi(x,y)$. By the definition of D and property 3). one finds that $D(x, y) = D_n(x, y) < D_n(z, w) = D(z, w)$. On the other hand, if $x = y$, then $\forall i,\ D_i(x, y) = 0$ by the definition of f_i and therefore $D(x, y) = 0$. Moreover, $D(x, y) = 0 < \phi(z, w) = D_0(z, w) < D_1(z, w) < \ldots < D_i(z, w) < D_{i+1}(z, w) < \ldots < D(z, w)$. The above argument holds exactly the same for the case where $\phi(x, y) \leq \phi(z, w)$.

D satisfies symmetry, reflexivity and non-negativity:

This is immediate from properties 2, 6 and the definition of D.

D satisfies the triangle inequality:

Suppose $y \neq z$, then $\exists n$ s.t. $q^n < \phi(y, z)$. Then, by Property 3, $q^n < D_n(y, z)$ and by the definition of D and Properties 3-5, it is the case that $D(y, z) = D_n(y, z) \leq D_n(x, y) + D_n(x, z) \leq D(x, y) + D(x, z)$. If $y = z$, then $D(y, z) = 0 \leq D(x, y) + D(x, z)$ by the reflexivity and non-negativity of D. □

The main representation relies upon five axioms, three of which captured our notion of aspirations: *WARP Without Unavailability*, *WARP Given Potential Sets* and *Independence of Irrelevant Unavailable Alternatives* (2.1 - 2.3). The other two, *Single Aspiration Point* (2.4) and *Continuity* (2.5), were technical and necessary to derive the specific representation. In this section, we consider relaxations of the technical axioms. A *preference relation* is a reflexive and transitive binary relation.

5.1 Relaxing the Continuity Axiom

Axiom 2.5 required upper hemi-continuity of the choice correspondence when both the feasible and potential sets vary independently. We relax this assumption by requiring upper hemi-continuity when (i) only the feasible sets vary and (ii) both the feasible and potential sets are the same and vary together.

Axiom 5.1. *(Upper Hemi-Continuity Given Aspiration) For any* $(S, Y) \in \mathcal{C}(X)$ *and* $x, x_1, x_2, \ldots \in Y$ *with* $x_n \to x$*, if* $x_n \in c(S \cup \{x_n\}, Y)$ *for each* n*, then* $x \in C(S \cup \{x\}, Y)$*.*

Axiom 5.2. *(Upper Hemi-Continuity of the Aspiration Preference) For any* $Y, Y_n \in \mathcal{X}$*,* $x_n \in X$*,* $n = 1, 2, \ldots$ *such that* $Y_n \overset{H}{\to} Y$ *and* $x_n \to x$*,*

$$\textit{if } x_n \in C(Y_n, Y_n) \textit{ for each } n, \textit{ then } x \in C(Y, Y).$$

The main representation given in Theorem 2.1 is based upon a stronger continuity axiom than those above. The following theorem characterizes a similar representation under the weaker continuity conditions and from a technical point of view, it is much simpler.

Theorem 5.1. *C is an aspirational choice correspondence that satisfies Axioms 2.4, 5.1 and 5.2 if and only if there exist*

1. *a continuous linear order $\succeq$,*
2. *a metric $d : X^2 \to \mathbb{R}_+$, where $d(\cdot, a)$ is lower semi-continuous on $L_{\succeq}(a)$*

such that

$$C(S,Y) = \operatorname*{argmin}_{s \in S} \; d(s, a(Y)) \text{ for all } (S,Y) \in \mathcal{C}(X), \tag{2}$$

where $a(Y)$ is the $\succeq$-maximum element of Y.

The difference between the two representations is that in Theorem 2.1 the metric is continuous whereas here it is lower semi-continuous on the appropriate domain.

Proof of Theorem 5.1

[$\Rightarrow$] Define $a(Z) := C(Z,Z)$. By Axiom 2.1, we have that there exists a total order $\succeq$ such that $a(Z) = \max(Z, \succeq)$. By Axiom 2.4 we have $\succeq$ is a total anti-symmetric order. Moreover, by Axiom 5.2, this order and a are continuous.

Consider $S \subseteq L_{\succeq}(z)$. For any $z \in X$ define $C_z(S) := C(S, L_{\succeq}(z))$. By Axiom 2.2, C_z is rationalizable on $\mathcal{C}(L_{\succeq}(z))$ by a complete preference relation $\succeq_z$. Alternatively, notice that $x \succeq_z y$ iff $x \in C(\{x,y\},\{x,y,z\})$ and $C(\{x,y,z\},\{x,y,z\}) = z$. This preference relation $\succeq_z$ represents the agent's aspiration-dependent preference when he has aspiration z. We need to show that $\succeq_z$ satisfies the conditions for Rader's Representation Theorem (1963) on $L_{\succeq}(z)$. X is compact, hence separable, so its topology has a countable base. Also, $\succeq_z$ is upper hemi-continuous on $L_{\succeq}(z)$ by Axiom 5.1.

Hence, Rader's Representation Theorem guarantees that there exists $u_z : L_{\succeq}(z) \to \mathbb{R}$ an upper semi-continuous function representing $\succeq_z$. WLOG, take $y \succeq x$. Define $\hat{d}(x,y) := -u_y(x) + u_y(y)$. Define other points symmetrically, i.e. if $x \succeq y$, then $\hat{d}(x,y) := \hat{d}(y,x)$. We point out here, that since $a(\cdot)$ is a choice function, we have that $x \sim y \Rightarrow x = y$ and therefore $\hat{d}(x,y) = \hat{d}(x,x) = -u_x(x) + u_x(x) = 0$. Finally let us notice that since u_z is upper semi-continuous on $L_{\succeq}(z)$, we have that $\hat{d}(\cdot, z)$ is lower semi-continuous on $L_{\succeq}(z)$.

While $\hat{d}$ satisfies reflexivity and symmetry, the triangle inequality may not hold for $\hat{d}$. However, if we define $d = f \circ \hat{d}$ where f is the lower semi-continuous transformation defined below, then d will satisfy the triangle inequality.

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + \dfrac{x}{1+x} & \text{otherwise} \end{cases}$$

Notice that d trivially satisfies the triangle inequality. Consider x, y, z all distinct. Then $d(x, y) + d(y, z) \geq d(x, z)$ for the trivial reason that 2 is a lower bound for the left hand side and an upper bound for the right hand side. If $x = z$, then the right hand side is equal to 0 and we are trivially done. Finally, if $x = y$ or $y = z$, then both sides equal $d(x, z)$ and we have shown the triangle inequality in all cases. Lastly, note that this is an increasing transformation, so as long as $\hat{d}$ represents the preferences, so does d.

Finally, for representability, we need that $C(S, Y) = \text{argmin}_{s \in S} d(s, a(Y)) = \text{argmin}_{s \in S} \hat{d}(s, a(Y))$. For notational ease, let $a = a(Y)$. Note that $\hat{d}(a, \cdot)$ is lower semi-continuous on $L_{\succeq}(a)$ and S is a compact subset of $L_{\succeq}(a(Y))$ and hence the above argmin will exist.[11]

"$C(S, Y) \subseteq \text{argmin}_{s \in S}\ \hat{d}(s, a(Y))$"

Take $y \in C(S, Y)$, suppose $\exists z \in S$ s.t. $\hat{d}(z, a) < \hat{d}(y, a)$. Substituting, we get $u_a(z) > u_a(y) \Rightarrow z \succ_a y \Rightarrow \{z\} = C(\{z, y\}, \{z, y, a\})$ and $\{a\} = C(\{z, y, a\}, \{z, y, a\})$. Axiom 2.2 tells us that $y \in C(S, Y) \Rightarrow y \in C(\{z, y\}, Y)$ and Axiom 2.3 yields $y \in C(\{z, y\}, Y) = C(\{z, y\}, \{z, y, a\}) = \{z\}$, a contradiction.

"$\text{argmin}_{s \in S}\ \hat{d}(s, a(Y)) \subseteq C(S, Y)$"

Consider $z \in \text{argmin}_{s \in S} \hat{d}(s, a(Y))$ and assume $z \notin C(S, Y)$, $y \in C(S, Y)$. By Axioms 2.2 and 2.3 we must then have $\{y\} = C(\{z, y\}, \{a, z, y\}) \Rightarrow z \prec_a y$ $\Rightarrow \hat{d}(z, a) > \hat{d}(y, a)$.

[11]By definition, $a(Y) = \text{argmax}(Y, \succ)$ and therefore $L_{\succeq}(a(Y)) \supseteq Y \supseteq S$.

[$\Leftarrow$] Axioms 2.2, 2.1, 2.3, 2.4 all follow trivially. Axiom 5.2 follows from the continuity of the aspiration map, $a(\cdot)$.

To show Axiom 5.1, we first have that $S, \{x_n\} \subseteq L_{\succeq}(y)$. By continuity of $\succeq$, we therefore have that $x \in L_{\succeq}(y)$. By our representation, we have that $\forall s \in S,\ d(x_n, y) \leq d(s, y)$. Finally, by the lower semi-continuity of $d(\cdot, y)$ on $L_{\succeq}(y)$, we have that $\forall s \in S,\ d(x, y) \leq \text{liminf}_{n\to\infty} d(x_n, y) \leq d(s, y) \Rightarrow x \in \text{argmin}_{s \in S \cup x} \phi(s, y)$. □

5.2 Multiple Aspiration Points

We now also relax the Single Aspiration Point Axiom. As $C(Y, Y)$ need no longer be a singleton, we refer to it as the agent's *aspiration set*. We will characterize a choice procedure based upon a subjective notion of distance d and an aggregator ϕ. When there is a single aspiration point, the procedure coincides with our main theorem: the agent chooses the closest feasible alternatives to her aspiration according to d. However, when an agent aspires to a set of alternatives, each feasible alternative is associated with a vector of distances to each aspiration point. For each feasible alternative, the aggregator translates the vector of distances into a score and the agent chooses the feasible alternative(s) with the minimal score.

This characterization relies on the following axiom:

Axiom 5.3. *(Aggregation of Indifferences) For any $Y \in \mathcal{X}$ and $x, z \in Y$,*

$$\textit{if } C(\{x,z\}, \{x,y,z\}) = \{x,z\}\ \forall y \in C(Y,Y), \textit{ then } C(\{x,z\}, Y) = \{x,z\}.$$

The above axiom stipulates that if an agent is indifferent between x and z with respect to every alternative in her aspiration set, then she is indifferent between x and z when considering that aspiration set as a whole. The axiom provides a link between problems with a single aspiration point and choice problems with multiple aspirations. Note that if the Single Aspiration Point Axiom holds, then the above axiom is trivially satisfied.

A vector of distances $\vec{d} \in \mathcal{R}^{|X|}$ between an alternative x and a set $Z \subseteq X$ is defined as follows:

$$\vec{d}_z(x, Z) := \begin{cases} d(x,z) & \text{if } z \in Z \\ \infty & \text{otherwise} \end{cases}$$

The z^{th} entry of the vector takes the value $d(x, z)$ if z is in Z and ∞ otherwise.

Definition 5.1. *A function $f : \mathbb{R}_+^X \to \mathbb{R}$ is* single-agreeing *if $f(\vec{v}) = v_a$ for all $\vec{v} = (v_x)_{x\in X}$ such that $v_b = \infty$ whenever $b \neq a$.*

If there are several aspirations, then there are several distances to consider and the agent uses an aggregator to assign a score. However, when there is only one aspiration point, there is a single distance to consider and we require that the aggregator coincide with the distance function in that case. The single-agreeing property guarantees this.

Theorem 5.2. *C is an aspirational choice correspondence that satisfies Axioms 5.1, 5.2 and 5.3 if and only if there exist*

1. *a continuous complete preference relation $\succeq$,*
2. *a metric $d : X^2 \to \mathbb{R}_+$, where $d(\cdot, a)$ is lower semi continuous on $L_{\succeq}(a)$ and*
3. *a single-agreeing aggregator $\phi : \mathbb{R}_+^X \to \mathbb{R}_+$ where $\phi \circ \vec{d}(\cdot, \mathcal{A}(Y))$ is lower semi-continuous on $L_{\succeq}(\mathcal{A}(Y))$ for any $Y \in \mathcal{X}$,*

such that

$$C(S, Y) = \operatorname*{argmin}_{s\in S} \phi(\vec{d}(s, \mathcal{A}(Y))) \text{ for all } (S, Y) \in \mathcal{C}(X), \tag{3}$$

where $\mathcal{A}(Y)$ is the $\succeq$-maximal elements of Y and $\phi \circ \vec{d}$ is such that $\mathcal{A}(Y) = \operatorname*{argmin}_{s\in Y} \phi(\vec{d}(s, \mathcal{A}(Y)))$ for any $x \in X$ and $Y \in \mathcal{X}$.

Proof of Theorem 5.2

[⇒] **Step 1:** We define the aspiration preference $\succeq$ and aspiration map $\mathcal{A}$.

Define $x \succeq y$ if $x \in C(\{x, y\}, \{x, y\})$. Since every pair $\{x, y\}$ is compact, we get that $\succeq$ is a total order. Moreover, by Axiom 5.2, the defined aspiration choice correspondence $\mathcal{A}(Y) := \max(Y, \succeq)$ is upper hemi-continuous and the aspiration preference $\succeq$ is continuous.

As before, we will define preferences relative to a single aspiration point. This definition will differ than the one given before because there may in fact be multiple aspiration points for a given choice problem. However, in the case of a single aspiration point, these definitions will be the same.

Step 2: We define the aspiration-based preferences $\succeq_A$ and show that they represent choice.

Definition: For any $A \in \mathcal{X}$, we define

$$x \succeq_A y \text{ if } x \in C(\{x, y\}, \{x, y\} \cup A) \text{ and } A \subseteq C(\{x, y\} \cup A, \{x, y\} \cup A).$$

Claim: For any $(S, Y) \in \mathcal{C}(X)$, $C(S, Y) = \max(S, \succeq_A)$ where $A = \mathcal{A}(Y)$.

Proof: Let $x \in C(S, Y)$ and $y \in Y$. By Axioms 2.1, 2.2 and 2.3, we get $x \in C(\{x, y\}, \{x, y\} \cup A)$. By Axiom 2.3, we have $A = C(\{x, y\} \cup A, \{x, y\} \cup A)$. Hence, by definition of $\succeq_A$, we obtain $x \succeq_A y$. As y is an arbitrary element in Y, we get $x \in \max(S, \succeq_A)$. For the other inclusion, let $x \in \max(S, \succeq_A)$ and suppose further that $x \notin C(S, Y)$. Then there must exist $y \in C(S, Y)$. By definition of $\succeq_A$, we have $x \in C(\{x, y\}, \{x, y\} \cup A)$ and $A \subseteq C(\{x, y\} \cup A, \{x, y\} \cup A)$. Applying Axioms 2.1 and 2.3 gives $x \in C(\{x, y\}, Y)$. By Axiom 2.1, we get $x \in C(S, Y)$, which is a contradiction. $\square$

Step 3: We show that $\succ_z$ are upper semi-continuous and define the distance function.

Consider the case where $x_i \succeq_z y$ for all i and suppose $x_i \to x$. Then, $x_i \in C(\{x_i, y\}, \{x_i, y, z\})$ and $z \in C(\{x_i, y, z\}, \{x_i, y, z\})$. First, we notice that $z \in C(\{x, y, z\}, \{x, y, z\})$ by Axiom 5.2. Now, if $x_i \in C(\{x_i, y, z\}, \{x_i, y, z\})$ happens infinitely often, then by Axiom 5.2, we have $x \in C(\{x, y, z\}, \{x, y, z\})$, which implies that $x \in C(\{x, y\}, \{x, y, z\})$ by Axiom 2.2. If $x_i \notin C(\{x_i, y, z\}, \{x_i, y, z\})$ infinitely often, then $\{z\} = C(\{x_i, y, z\}, \{x_i, y, z\})$ infinitely often because if $y \in C(\{x, y, z\}, \{x, y, z\})$, then $y \sim z \Rightarrow y \in C(\{x_i, y, z\}, \{x_i, y, z\})$ and then by Axiom 2.2 $x_i \in C(\{x_i, y, z\}, \{x_i, y, z\})$ for all i which is a contradiction to $x_i \notin C(\{x_i, y, z\}, \{x_i, y, z\})$ infinitely often. So, we can pass to a subsequence where it is only the case that $\{z\} = C(\{x_i, y, z\}, \{x_i, y, z\})$. Finally, either $x \in C(\bigcup_{i=1}^{\infty}\{x_i\} \cup \{y, z, x\}, \bigcup_{i=1}^{\infty}\{x_i\} \cup \{y, z, x\})$ or $\{z\} = C(\bigcup_{i=1}^{\infty}\{x_i\} \cup \{y, z, x\}, \bigcup_{i=1}^{\infty}\{x_i\} \cup \{y, z, x\})$. If the previous case holds, then we are done since $x \in C(\{x, y, z\}, \{x, y, z\})$ by Axiom 2.2 and 2.3. If the latter case holds, then by Axiom 2.3, we get $x_i \in C(\{x_i, y\}, \bigcup_{i=1}^{\infty}\{x_i\} \cup \{y, z, x\})$ and Axiom 5.1 implies that $x \in C(\{x, y\}, \bigcup_{i=1}^{\infty}\{x_i\} \cup \{y, z, x\})$. Then, by Axiom 2.3, we obtain $x \in C(\{x, y\}, \{x, y, z\})$.

X is compact, hence separable and therefore $L_{\succeq}(x)$ is separable (because subspaces of separable spaces are separable). Upper semi-continuity of $\succeq_x$ was just shown above and $\succeq_x$ is a complete preference relation. Therefore, by Rader's Theorem, there is a $U : X^2 \to \mathbb{R}_+$ such that $U(\cdot, z)$ is upper-semi continuous on $L_{\succeq}(z)$ and represents $\succeq_z$ for any $z \in X$. Hence, $U(x, z) \geq U(y, z)$ if and only if $x \succeq_z y$.[12] Finally, define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & x \neq y \text{ and } x \sim y \\ 2 - \dfrac{U(x,y)}{1+U(x,y)} & x \prec y \\ 2 - \dfrac{U(y,x)}{1+U(y,x)} & x \succ y \end{cases}$$

The above d is symmetric, reflexive, satisfies the Δ-inequality and $d(\cdot, y)$ is lower semi-continuous on $L_{\succeq}(y)$.

Step 4: We show that $\succ_A$ are upper semi-continuous and define a set-based aggregator function.

Notice that we are only concerned with A such that $\forall a, b \in A$, $a \sim b$. For any Y, consider the situation where $A = \mathcal{A}(Y)$ and we have a sequence $x_i \to x$

[12] Rader's theorem does not guarantee non-negativity of U. If U takes negative values, we can always consider e^U instead of U, which is non-negative and order-preserving. Hence, WLOG, we assume non-negativity of U function.

and a z such that $\forall i,\ x_i \succeq_A z$. Suppose there exists $a \in A$ such that $z \succeq a$. Then, it must be the case that $x_i \succeq z \succeq a$ and $x_i \in \mathcal{A}(Y \cup x_i \cup z)$. Then by Axiom 5.2, we have $x \in \mathcal{A}(Y \cup x \cup z)$ and $x \succeq_A z$. Otherwise, consider the case where $x_i \succeq a \succ z$ infinitely often. Then, again by Axiom 5.2, we have that $x \succeq a$ and $x \succeq_A z$. Finally, suppose that $\mathcal{A}(Y \cup \{x_i, z\}) = A$ infinitely often. Then, by Axiom 5.2, we have that $\mathcal{A}(Y \cup x) = A$ or $A \cup x$. In the first case, we can apply Axiom 5.1 and in the second, by virtue of x being an aspiration alternative, we have $x \succeq_A z$.

Now, for any set A of this type, $\succeq_A$ is upper semi-continuous and satisfies the other conditions for Rader's Representation Theorem on $L_{\succeq}(A)$ by analogous arguments to those in the previous paragraph where $A = \{z\}$. Also, by our last claim, we have u_A representing $C(\cdot, Y)$ whenever $\mathcal{A}(Y) = A$.

Hence, Rader's Representation Theorem guarantees that there exists $u_A : L_{\succeq}(A) \to \mathbb{R}_+$ an upper semi-continuous function representing $\succeq_A$.[13]

Suppose that $A = C(Y, Y)$ for some $Y \in \mathcal{X}$ and $x, y \in Y$. Define $\hat{\phi}(\cdot)$ as follows:

$$\hat{\phi}((x, A)) = \begin{cases} d(x,y) & A = \{y\} \\ d(x,y) & |A| = 2,\ x, y \in A, x \neq y \\ 1 & x \in A,\ |A| > 2 \\ 5 - \dfrac{1}{1 + u_A(a) - u_A(x)} & x \notin A,\, a \in A \text{ and } |A| \neq 1 \end{cases}$$

Notice that $u_A(a)$ is constant across all $a \in A$, so the fourth case above is well-defined. It can be checked that the above function is lower semi-continuous due to the upper semi-continuity of u_A and the fact that $0 \leq d \leq 2 < 4 \leq 5 - \dfrac{1}{1 + u_A(a) - u_A(x)}$.

Now, we have only defined $\hat{\phi}$ for certain tuples (x, A). This is because only certain choice problems arise. More formally we make the following definition.

[13] For the non-negativity of u_A, please refer to footnote 12.

Step 5: We show that if two alternatives generate the same distance vectors, then they have the same set-based aggregate score.

Definition: Let $CP = \{(x, A) : x \in Y \text{ for some } Y \in \mathcal{X} \text{ such that } \mathcal{A}(Y) = A\}$

Claim: For any $(x, A), (y, B) \in CP$, where $A = \mathcal{A}(Y)$, $B = \mathcal{A}(Z)$, if $\vec{d}(x, A) = \vec{d}(y, B)$, then $\hat{\phi}(x, A) = \hat{\phi}(y, B)$.

Proof: First, if $\vec{d}(x, A) = \vec{0} = \vec{d}(y, B)$ then $\hat{\phi}(x, A) = \hat{\phi}(y, B) = 0$. Next, we show that $A = B$. Suppose not. Then, WLOG, $\exists z \in B \backslash A$. Since, $\mathbb{1}_{z \in A} = 0$, $\mathbb{1}_{z \in B} = 1$, it must be the case that $y = z$. So, there can be at most one alternative in $B \backslash A$. If B contains only one alternative, then we are in the previous case, so, let's take another alternative $b \in B \cap A$. Now, $y = z \sim b \Rightarrow d(y, b) = 1$. Therefore, it must be the case that $d(x, b) = 1$. Thus $x \sim b$. But, then $0 = d(x, x) = d(y, x)$ when we consider the (now known to be aspirational) alternative x which implies that $y = x$. But, now we have a contradiction because $z \in B \backslash A$ and $z = x \in A$.

If $x = a$ for some $a \in A$, then $0 = d(x, a) = d(y, a) \Rightarrow x = y$. Otherwise, $x, y \prec a$ and $\vec{d}(x, A) = \vec{d}(y, A)$, means $d(x, a) = d(y, a)$ for any $a \in A$. This means that $\mathcal{A}(\{x, y, a\}) = a$ and $\{x, y\} = C(\{x, y\}, \{x, y, a\})$. If $|A| = 1$, then $\hat{\phi}(x, A) = d(x, a) = d(y, a) = \hat{\phi}(x, A)$ for $A = \{a\}$. Otherwise, by Axiom 5.3, $\{x, y\} = C(\{x, y\}, A \cup \{x, y\}) \Rightarrow x \sim_A y \Rightarrow u_A(x) = u_A(y)$ and the last case of $\hat{\phi}$ applies, so $\hat{\phi}(x, A) = \hat{\phi}(y, A)$. $\square$

Step 6: We construct a distance based aggregator and show that it satisfies the single-agreeing property.

So, we now know that $\hat{\phi} : CP \to \mathbb{R}$ and $\vec{d} : CP \to \mathbb{R}^X$ and the equivalence relations defined by the inverse image of $\vec{d}$ is a refinement of $\hat{\phi}$. So, by a standard argument, there exists a $\phi : \mathbb{R}^X \to \mathbb{R}$ that makes the diagram above commute.

Therefore $\phi(\vec{d}(x, A)) = \hat{\phi}(x, A)$. The case when a vector has one finite entry is if $|A| \leq 1$. Then $\hat{\phi}$ and hence ϕ is defined by the first case to agree with the distance function. Thus, ϕ has the "single-agreement" property.

Step 7: We show that the constructed objects represent the choice correspondence.

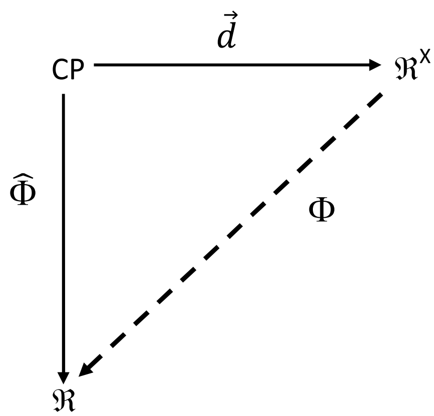


Figure 9: Commuting Graph

For representability, we must show that $C(S,Y) = \underset{s \in S}{\operatorname{argmin}}\ \phi(\vec{d}(s, \mathcal{A}(Y)))$. First, let us note that $\phi \circ \vec{d}$ is lower semi-continuous, S is compact and $S \subseteq L_{\succeq}(\mathcal{A}(Y))$. Hence the above argmin will exist.[14] For notational ease, let $A = \mathcal{A}(Y)$.

"$C(S,Y) \subseteq \underset{s \in S}{\operatorname{argmin}}\ \phi(\vec{d}(s, \mathcal{A}(Y)))$"

Take $y \in C(S,Y)$, suppose $\exists z \in S$ s.t. $\phi(\vec{d}(z,A)) < \phi(\vec{d}(y,A))$. We consider the following cases:

1. $A = \{x\}$. Then the above reduces to $d(z,x) < d(y,x)$. But, then $y \notin C(S,Y)$ ↯

2. $A = \{z,y\}$. Then the above becomes $d(z,y) < d(y,z)$ ↯

3. $A = \{w,y\}$, $w \neq z$. Then the above becomes $4 < 5 - \dfrac{1}{1 + u_A(y) - u_A(z)} < d(w,y) < 2$ ↯

4. $z \in A$, $y \notin A$, then $y \notin C(S,Y)$ ↯

5. $z, y \in A$, $|A| > 2$, then $1 < 1$ ↯

6. $z \notin A$, $y \in A$, $|A| > 2$. Then the above becomes $\phi(\vec{d}(z,A)) < 1$ ↯

[14] By definition, recall $\mathcal{A}(Y) = \operatorname{argmax}(Y, \succeq)$ and therefore $L_{\succeq}(Y) \supseteq Y \supseteq S$.

7. $y, z \notin A$, $|A| > 2$. Then the above becomes
$$5 - \frac{1}{1 + u_A(a) - u_A(z)} < 5 - \frac{1}{1 + u_A(a) - u_A(y)} \Rightarrow u_A(y) < u_A(z) \lightning$$

"$\text{argmin}_{s \in S} \phi(\vec{d}(s, \mathcal{A}(Y))) \subseteq C(S, Y)$"

Consider $z \in \text{argmin}_{s \in S} \phi(\vec{d}(s, \mathcal{A}(Y)))$ and assume $z \notin C(S, Y)$, $y \in C(S, Y)$. We consider the following cases:

1. Suppose $z \prec a \in A$, $|A| > 1$. Then, it must be $y \prec a \in A$ and $5 - \frac{1}{1 + u_A(a) - u_A(z)} \leq 5 - \frac{1}{1 + u_A(a) - u_A(y)} \Rightarrow u_A(y) \leq u_A(z)$ and therefore $y \in C(S, Y) = \max(S, \succeq_A) \Rightarrow z \in \max(S, \succeq_A) = C(S, Y) \lightning$

2. Suppose $z \in A$, then by Axioms 2.1 and 2.3, $z \in C(S, Y)$

3. Suppose $z \notin A$, $|A| = \{a\}$. Then we have $d(z, a) \leq d(y, a) \Rightarrow z \succeq_a y$ and since y was chosen generically, we have $z \in \max(S, \succeq_a) \Rightarrow z \in C(S, Y)$

[$\Leftarrow$] Axioms 2.1, 2.2, 2.3, 5.3 all follow trivially. Axiom 5.2 follows from the continuity of the aspiration preference $\succeq$.

To show Axiom 5.1, let us consider the situation $x_n \in C(S \cup \{x_n\}, Y)$ and $x_n, x \in Y$ and $x_n \to x$, we have $\forall s \in S,\ \phi(\vec{d}(x_n, \mathcal{A}(Y))) \leq \phi(\vec{d}(s, \mathcal{A}(Y)))$. By lower semi-continuity of $\phi(\vec{d}(\cdot, \mathcal{A}(Y)))$ we have that $\phi(\vec{d}(x, \mathcal{A}(Y))) \leq \liminf_{n \to \infty} \phi(\vec{d}(x_n, \mathcal{A}(Y))) \leq \phi(\vec{d}(s, \mathcal{A}(Y)))$. Therefore $x \in C(S \cup \{x\}, Y)$.

□

We interpret the above representation in the following manner. When a decision maker is confronted with a choice problem (S, Y), she first forms her set of aspiration points $\mathcal{A}(Y)$ by maximizing her aspiration preference $\succeq$ over Y. For each feasible alternative s, she considers its distance to each aspiration, giving rise to the vector $\vec{d}(s, \mathcal{A}(Y))$. She aggregates this vector and chooses the alternative with the lowest score. The aggregator ϕ can be interpreted as a measure of dissimilarity between alternatives and aspiration sets.

When the agent always has a single aspiration point, the single-agreeing property implies that choices are made only according to the distance function as in the main representation. This is best illustrated with an example. Consider an agent with a single aspiration a and three alternatives to choose from: x, y, z. The generated distance vectors are then:

$$\begin{bmatrix} d(x,a) \\ \infty \\ \infty \\ \infty \end{bmatrix}, \begin{bmatrix} d(y,a) \\ \infty \\ \infty \\ \infty \end{bmatrix}, \begin{bmatrix} d(z,a) \\ \infty \\ \infty \\ \infty \end{bmatrix}$$

and single-agreement requires that ϕ takes these vectors to $d(x,a)$, $d(y,a)$, and $d(z,a)$, respectively. Formally, $\phi(\vec{d}(s,a)) = d(s,a)$ and therefore minimizing the former is equivalent to minimizing the latter. Put differently, the agent uses her distance function d whenever she can and aggregates otherwise.

A wide range of aggregators are permitted, but desirable properties, such as monotonicity, could be imposed through additional axioms:

Axiom 5.4. *(Monotonicity) For any $Y \in \mathcal{X}$ and $x, z \in Y$,*
(i) $x \in C(\{x,z\},\{x,y,z\})$ for all $y \in C(Y,Y)$ implies $x \in C(\{x,z\},Y)$.
(ii) $C(\{x,z\},\{x,y,z\}) = \{x\}$ for all $y \in C(Y,Y)$ implies $C(\{x,z\},Y) = \{x\}$.

The representation makes clear that Axiom 5.3 is the minimal assumption necessary. If an agent is indifferent between two alternatives with respect to each aspiration in the aspiration set, then both of these alternatives generate the same vector of distances with respect to that aspiration set and thus they must be assigned the same aggregate score.

Further References

Rader, T. (1963): 11 The existence of a Utility Function to Represent Preferences," *The Review of Economic Studies*, 30, 229-232.