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Controllability and Observability of Linear Nabla Discrete Fractional Systems

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CONTROLLABILITY AND OBSERVABILITY OF LINEAR NABLA DISCRETE FRACTIONAL SYSTEMS

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The main purpose of this thesis to examine the controllability and observability of the linear discrete fractional systems. First we introduce the problem and continue with the review of some basic definitions and concepts of fractional calculus which are widely used to develop the theory of this subject. In Chapter 3, we give the unique solution of the fractional difference equation involving the Riemann-Liouville operator of real order between zero and one. Additionally we study the sequential fractional difference equations and describe the way to obtain the state-space representation of the sequential fractional difference equations. In Chapter 4, we study the controllability and observability of time-invariant linear nabla fractional systems.We investigate the time-variant case in Chapter 5 and we define the state transition matrix in fractional calculus. In the last chapter, the results are summarized and directions for future work are stated.

Chapter 1
INTRODUCTION

Fractional calculus is a branch of mathematics that generalizes the derivative and integral of a function to any real order. Nowadays, the various tools of fractional calculus are used in several areas including mathematics, engineering, and finance [10, 11, 12, 13, 54, 55]. Due to the existence of the singular kernel in continuous fractional operators, the study of discrete fractional calculus provides a more practical and complete view of certain mathematical models than continuous fractional calculus. Thus, in this work, we focus on discrete fractional calculus.

The study of control systems has become significant and promising in our modern society. From devices as simple as a calculator to complex systems like airplanes and space shuttles, control engineering is a part of our everyday life. There are several methodologies to examine control systems such as classical control theory, modern control theory, robust control, adaptive control, and nonlinear control. In this thesis, we study the modern control theory to obtain the controllability and observability criteria for the linear nabla fractional systems in both time-invariant and time-variant cases. The modern control theory deals with the state-space model of the control system. A state-space model is a set of first-order differential or difference equations that uses state variables (input, output and internal states of the system) to describe a dynamical system.

The study of controllability and observability plays an essential role in modern control theory and engineering. The controllability and observability become particularly important for practical implementations after Kalman [37] introduced the rank condition for these theoretically initiated concepts. They can be roughly defined as follows.

(i) **Controllability:** The ability to transfer any initial state to any arbitrary final state under a control vector of the system.

(ii) **Observability:** The ability to measure or determine the state of the system based on its outputs.

The primary purpose of this study is to discuss the controllability and observability of the linear nabla fractional system,

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1) \\ z(t) = Cy(t) + Du(t) \end{cases}$$

where A is an $n \times n$ constant matrix, B is an $n \times m$ constant matrix, C is an $r \times n$ constant matrix, D is an $r \times m$ constant matrix, $y(t)$ is an $n \times 1$ state vector, $u(t)$ is an $m \times 1$ control vector (control signal) and $z(t)$ is an $r \times 1$ the output vector (response vector).

The motivation for the study of discrete fractional control systems is mainly twofold: First, from an applicability point of view, controllability and observability have close connections to pole assignment, structural decomposition, optimal quadratic control, observer design, controller design and so forth. For this reason, in recent decades, the investigation of control systems has aroused great interest among applied mathematicians and engineers. Second, because of the popularization of the computer, the study of the discrete time case becomes practical and promising.

Furthermore, we extend our works and investigate the necessary and sufficient conditions for the controllability and observability of the linear time-variant nabla fractional difference system,

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = A(t)y(t-1) + B(t)u(t-1) \\ z(t) = C(t)y(t) + D(t)u(t) \end{cases}$$

where $y(t)$ is an $n \times 1$ state vector of the system, $u(t)$ is an $m \times 1$ control input, $z(t)$ is an $r \times 1$ output vector, $A(t)$ is an $n \times n$ matrix valued function, $B(t)$ is an $n \times m$ matrix valued function, $C(t)$ is an $r \times n$ matrix valued function, $D(t)$ is an $r \times m$ matrix valued function and ν is positive real number $0 < \nu < 1$.

In this study, we intend to reduce mathematical derivations and several definitions and produce some precise check-up tests and identify the controllability and observability of discrete fractional systems more rigorously.

Chapter 2
PRELIMINARIES

In this chapter, we recall some fundamental definitions and notations for the discrete fractional nabla calculus.We refer the readers to [1, 6, 8, 32, 33, 42] for further background on this topic.

The backward difference operator, or nabla operator (∇) for a function $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ is defined by

$$(\nabla f)(t) := f(t) - f(\rho(t)), \tag{2.0.1}$$

where $a \in \mathbb{R}$, $\mathbb{N}_a = \{a,\ a+1,\ a+2, \ldots\}$ and $\rho(t) = t - 1$ is known as backward jump operator on time scale calculus [32].

We define a discrete interval as a set of the form

$$\mathbb{N}_a^b := \{a, a+1, ..., b\}$$

where $a, b \in \mathbb{R}$ and $b - a$ a is positive integer.

Let μ be any real number. The rising factorial power $t^{\overline{\mu}}$ (read 't to the μ rising') is defined as

$$t^{\overline{\mu}} := \frac{\Gamma(t+\mu)}{\Gamma(t)},$$

where $t,\ t+\mu \in \mathbb{R} \setminus \{..., -2, -1, 0\}$ and Γ denotes the Gamma function. We accept the convention that if t is a pole of the Gamma function and $t+\mu$ is not a pole of the Gamma function, then $t^{\overline{\mu}} := 0$.

We consider the ν-th order fractional sum of f defined as in [6]

$$\nabla_a^{-\nu} f(t) := \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \tag{2.0.2}$$

where $a \in \mathbb{R}$, $\nu \geq 0$, and $t \in \mathbb{N}_a$. Further, we consider the ν-th order fractional difference (a Riemann-Liouville fractional difference) of f defined by

$$\nabla_a^{\nu} f(t) := \nabla^n (\nabla_a^{-(n-\nu)} f(t)) \tag{2.0.3}$$

where $\nu > 0$, $n-1 < \nu < n$, n denotes a positive integer [6].

Let us recall some basic properties of the rising factorial power function. We refer to the reader [9] for the proof of these basic properties.

Lemma 2.1. *Let a be a real number and ν be a positive real number. Then the following properties hold for those values of t, ν, and μ for which the expressions are well-defined.*

(i). $\nabla t^{\overline{\mu}} = \mu t^{\overline{\mu-1}}$.

(ii). $\nabla_a^{-\nu}(t-a+1)^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a+1)^{\overline{\nu+\mu}}$.

Theorem 2.2. *For $\nu > 0$, the following equality holds:*

$$\nabla_{a+1}^{\nu} \nabla f(t) = \nabla \nabla_a^{\nu} f(t) - \frac{(t-a+1)^{\nu-1}}{\Gamma(\nu)} f(a)$$

where f is defined on $\mathbb{N}_a$.

We refer to the reader [9] for the proof of property stated above.

Theorem 2.3. *(Leibniz Rule [33]) For any ν positive real number, the ν-th order fractional difference of the product fg is given by in the form*

$$\overset{t}{\nabla_a^{\nu}} f(t)g(t) = \sum_{n=0}^{t-a} \binom{\nu}{n} [\overset{t-n}{\nabla_a^{\nu-n}} f(t-n)][\nabla^n g(t)]$$

where f, g are defined on $\mathbb{N}_a$.

Theorem 2.4. *[32] Asumme $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. Then*

$$\nabla \sum_{a}^{t} f(t,\tau) = \sum_{a}^{t} \nabla f(t,\tau) + f(t-1,t)$$
$$\nabla \sum_{a}^{t} f(t,\tau) = \sum_{a}^{t-1} \nabla f(t,\tau) + f(t,t)$$

for $t \in \mathbb{N}_{a+1}$.

Theorem 2.5. *[39] n functions $f_1, f_2, ..., f_n$ are linearly independent in an interval I if there exist a set of n points in I, namely $x_1, x_2, x_3, ..., x_n$ such that the matrix*

$$\begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{bmatrix}$$

is nonsingular.

Chapter 3

NABLA FRACTIONAL DIFFERENCE EQUATIONS

In this chapter, we shall provide a solution for the fractional differential equation of the form

$$\nabla_a^{\nu} y(t) = Ay(t-1) + f(t-1) \tag{3.0.1}$$

where $\nu \in \mathbb{R}$, $0 < \nu < 1$, $t \in \mathbb{N}_{a+1}$, $y(t)$ is an $n \times 1$ vector, A is an $n \times n$ constant matrix and suppose $f(t-1)$ is an $n \times 1$ vector valued function. For the convenience of the reader, we refer related works [8, 1, 18].

In mathematics, the Mittag-Leffler function is a special function with importance in the solution of a general problem of the theory of analytic functions. It can be considered the direct generalization of the exponential function e^x and is essential for the theory of fractional calculus. This function is named after Gösta Mittag-Leffler who defined and studied the particular function in 1903 [43]. The one and two-parametric representations of the Mittag-Leffler function can be defined in terms of a power series

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)},$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)},$$

where α and β are positive real numbers. Agarwal first defined the Mittag-Leffler function with two-arguments in 1953 [3]. Discrete Mittag-Leffler function with one and two-parameters are given by Nagai in 2003 [48],

$$F_\alpha(at) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k}}}{\Gamma(\alpha k + 1)},$$

$$F_{\alpha,\beta}(at) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k}}}{\Gamma(\alpha k + \beta)},$$

where α and β are positive real numbers and $|a| < 1$. For the discrete fractional case given by Atıcı and Eloe in 2011 [8],

$$F_{\alpha,\beta}(at^{\overline{\nu}}) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k\nu}}}{\Gamma(\alpha k + \beta)}$$

where α and β are positive real numbers, ν any real number. As we will present in Section 3.1 the solution of the fractional difference equation (3.0.1) is a Mittag-Leffler type function in fractional difference calculus.

This chapter is organized as follows: In Section 3.1, we give the unique solution of the initial value problem (IVP) of the order up to one. Then, we define the solution as a new function and state the properties of this function. We close Section 3.1 by giving the variation of constants formula for the fractional difference equation. In Section 3.2 we will give the general solution for the sequential fractional difference equations. In Section 3.3, we present the method of obtaining the state-space model of the $n\nu^{th}$ order fractional difference equations. In the final section we will generalize our result for the matrix valued fractional difference equations and give the Putzer Algorithm to evaluate the matrix exponential functions in discrete fractional calculus.

3.1 Up to First Order Fractional Difference Equations

In this section, we first present and then prove the existence of the unique solution of the following initial value problem (IVP)

$$\nabla_a^{\nu} y(t) = \lambda y(t-1) \quad for \quad t = a+1, a+2, a+3, ..., \tag{3.1.1}$$

$$\nabla_0^{-(1-\nu)} y(t)|_{t=a} = y(a) = c. \tag{3.1.2}$$

where $\lambda, c \in \mathbb{R}$ and $\nu \in (0,1)$. Then, we define the solution of this initial value problem and give its properties. Finally, we give the variation of constants formula.

Theorem 3.1. *The solution of the IVP (3.1.1)- (3.1.2) is uniquely determined.*

Proof. We use the definition of the fractional nabla difference operator to obtain the following iteration schema.

$$\nabla_a^{\nu} y(t) = \lambda y(t-1)$$

$$\nabla \nabla_a^{-(1-\nu)} y(t) = \lambda y(t-1) \qquad by \quad 2.0.2$$

$$\nabla \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{-\nu}}}{\Gamma(1-\nu)} y(s) = \lambda y(t-1) \qquad by \quad 2.0.3$$

$$\sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{-\nu}}}{\Gamma(1-\nu)} y(s) - \sum_{s=a}^{t-1} \frac{(t-1-\rho(s))^{\overline{-\nu}}}{\Gamma(1-\nu)} y(s) = \lambda y(t-1) \qquad by \quad 2.0.1$$

$$y(t) = -\sum_{s=a}^{t-1} \frac{(t-\rho(s))^{\overline{-\nu-1}}}{\Gamma(-\nu)} \quad y(s) + \lambda y(t-1),$$

for $t = a+1, a+2, \ldots$. This iteration scheme ensures that the solution of the IVP (3.1.1)-(3.1.2) is uniquely determined. $\square$

Theorem 3.2. *The unique solution of the initial value problem (3.1.1)-(3.1.2) is given by*

$$y(t) = c\sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}. \tag{3.1.3}$$

Proof. We show that

$$c\sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}$$

satisfies the IVP (3.1.1)- (3.1.2). Performing the definition of the nabla fractional difference yields

$$\nabla_a^{\nu} c\sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}$$

$$= c\nabla\nabla_a^{-(1-\nu)} \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}$$

$$= c\nabla\sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{-\nu}}}{\Gamma(1-\nu)} \sum_{n=a}^{s} \frac{\lambda^{n-a}(s-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} = I.$$

Next, we interchange the order of sums and obtain

$$I = c\nabla\sum_{n=a}^{t}\sum_{s=n}^{t} \frac{\lambda^{n-a}\Gamma(t-s+1-\nu)\Gamma(s-n+n\nu-a\nu+\nu)}{\Gamma(1-\nu)\Gamma(t-s+1)\Gamma(s-n+1)\Gamma(n\nu-a\nu+\nu)}$$

$$= c\nabla\sum_{n=a}^{t}\sum_{s=0}^{t-n} \frac{\lambda^{n-a}\Gamma(t-s-n+1-\nu)\Gamma(s+n\nu-a\nu+\nu)}{\Gamma(1-\nu)\Gamma(t-s-n+1)\Gamma(s+1)\Gamma(n\nu-a\nu+\nu)}.$$

By using the formula $\binom{t}{r} = \dfrac{\Gamma(t+1)}{\Gamma(r+1)\Gamma(t-r+1)}$ and the definition of the rising factorial power we get,

$$
\begin{aligned}
I &= c\nabla \sum_{n=a}^{t} \sum_{s=0}^{t-n} \binom{t-n}{s} \frac{\lambda^{n-a}\Gamma(t-s-n+1-\nu)\Gamma(s+n\nu-a\nu+\nu)}{\Gamma(1-\nu)\Gamma(t-n+1)\Gamma(n\nu-a\nu+\nu)} \\
&= c\nabla \sum_{n=a}^{t} \frac{\lambda^{n-a}}{\Gamma(t-n+1)} \sum_{s=0}^{t-n} \binom{t-n}{s} (1-\nu)^{\overline{t-s-n}} (n\nu-a\nu+\nu)^{\overline{s}} \\
&= c\nabla \sum_{n=a}^{t} \frac{\lambda^{n-a}}{\Gamma(t-n+1)} (n\nu-a\nu+1)^{\overline{t-n}} \\
&= c\nabla \sum_{n=a}^{t} \frac{\lambda^{n-a}}{\Gamma(n\nu-a\nu+1)} (t-n+1)^{\overline{(n-a)\nu}},
\end{aligned}
$$

where we used the identity [4]

$$
\sum_{s=0}^{t-n} \binom{t-n}{s} (1-\nu)^{\overline{t-s-n}} (n\nu-a\nu+\nu)^{\overline{s}} = (n\nu-a\nu+1)^{\overline{t-n}}.
$$

Next, we apply the following rule [16] to the above expression

$$
\nabla \sum_{n=a}^{t} f(t,n) = \sum_{n=a}^{t} \nabla f(t,n) + f(\rho(t),t).
$$

Hence, we have

$$
\begin{aligned}
I &= c \sum_{n=a}^{t} \nabla \frac{\lambda^{n-a}}{\Gamma(n\nu-a\nu+1)} (t-n+1)^{\overline{(n-a)\nu}} + \frac{c\lambda^{n-a}(t-n+1)^{\overline{(n-a)\nu}}}{\Gamma(n\nu-a\nu+1)}\Big|_{t=t-1,n=t} \\
&= c \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a)\nu-1}}}{\Gamma((n-a)\nu)}
\end{aligned}
$$

where we used Lemma 2.1 (i). Now, we use the assumption that $\dfrac{1}{\Gamma(0)} = 0$ and obtain

$$
\begin{aligned}
I &= c \sum_{n=a+1}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a)\nu-1}}}{\Gamma((n-a)\nu)} \\
&= c\lambda \sum_{n=a}^{t-1} \frac{\lambda^{n-a}(t-n)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} \\
&= c\lambda y(t-1).
\end{aligned}
$$

Uniqueness of this solution follows from Theorem 3.1. □

Next, we define the following nabla function that will be used throughout the work

$$
\widehat{y}_{\lambda,\nu}(t,a) := \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)},
$$

where λ is any constant number, ν is positive real number and $t \in \mathbb{N}_a$.

We continue with some properties of $\widehat{y}_{\lambda,\nu}(t,a)$

Lemma 3.3. *The following properties hold:*

(i) $\widehat{y}_{\lambda,\nu}(a,a) = 1$, *where* $\lambda \in \mathbb{R}$ *and* ν *is positive real number.*

(ii) $\nabla_a^{\nu}\widehat{y}_{\lambda,\nu}(t,a) = \lambda\widehat{y}_{\lambda,\nu}(t-1,a)$, *where* $0 < \nu < 1$ *and* $\lambda \in \mathbb{R}$.

(iii) $\widehat{y}_{\lambda,\nu}(t,a)$ *is monotone increasing on* $t \in \mathbb{N}_a$, *where* $\nu \geq 1$ *and* λ *is a positive real number.*

(iv) $\widehat{y}_{\lambda,\nu}(t,a)$ *is monotone increasing on* $t \in \mathbb{N}_a$, *where* $\lambda \geq 1$ *and* ν *is any positive real number.*

Proof. *(i)* The proof follows the definition of $\widehat{y}_{\lambda,\nu}(t,a)$.

(ii) $\widehat{y}_{\lambda,\nu}(t,a)$ satisfies the fractional difference equation (3.1.1).

(iii) The function is monotone increasing if the first nabla difference is positive on given discrete interval. Thus we take the nabla difference of our function by using following rule [16]

$$\nabla \sum_{n=a}^{t} f(t,n) = \sum_{n=a}^{t} \nabla f(t,n) + f(\rho(t),t).$$

Hence, we have

$$\begin{aligned}
\nabla \widehat{y}_{\lambda,\nu}(t,a) &= \nabla \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} \\
&= \sum_{n=a}^{t} \frac{\nabla \lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} + \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}\Big|_{t=t-1,n=t} \\
&= \sum_{n=a}^{t} \frac{\Gamma(t-n+(n-a+1)\nu-1)\lambda^{n-a}}{\Gamma(t-n+1)\Gamma((n-a+1)\nu-1)}.
\end{aligned}$$

The last quantity is positive if $\nu \geq 1$.

(iv) Let t be any real number $t \in \mathbb{N}_a$. $\widehat{y}_{\lambda,\nu}(t,a)$ is monotone increasing if $\widehat{y}_{\lambda,\nu}(t+1,a) > \widehat{y}_{\lambda,\nu}(t,a)$.

$$\begin{aligned}
\widehat{y}_{\lambda,\nu}(t+1,a) &= \sum_{n=a}^{t+1} \frac{\lambda^{n-a}(t-n+2)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} \\
&= \frac{\Gamma(t-a+\nu+1)}{\Gamma(t-a+2)\Gamma(\nu)} + \sum_{n=a+1}^{t+1} \frac{\lambda^{n-a}(t-n+2)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} \\
&> \sum_{n=a}^{t} \frac{\lambda^{n-a+1}\Gamma(t-n+(n-a+2)\nu)}{\Gamma(t-n+1)\Gamma((n-a+2)\nu)}.
\end{aligned}$$

Next, we use a property of Gamma function [32] and we get

$$
\begin{aligned}
\frac{\Gamma(t-n+(n-a+2)\nu)}{\Gamma((n-a+2)\nu)} &= \frac{1}{\Gamma((n-a+2)\nu)}\prod_{s=1}^{t-n}(t-n+(n-a+2)\nu-s)\Gamma((n-a+2)\nu) \\
&\geq \frac{1}{\Gamma((n-a+1)\nu)}\prod_{s=1}^{t-n}(t-n+(n-a+1)\nu-s)\Gamma((n-a+1)\nu) \\
&= \frac{\Gamma(t-n+(n-a+1)\nu)}{\Gamma((n-a+1)\nu)}
\end{aligned}
$$

since ν is positive integer. Using this inequality, we obtain

$$
\begin{aligned}
\widehat{y}_{\lambda,\nu}(t+1,a) &> \sum_{n=a}^{t}\frac{\lambda^{n+1-a}\Gamma(t-n+(n-a+1)\nu)}{\Gamma(t-n+1)\Gamma((n-a+1)\nu)} \\
&\geq \sum_{n=a}^{t}\frac{\lambda^{n-a}\Gamma(t-n+(n-a+1)\nu)}{\Gamma(t-n+1)\Gamma((n-a+1)\nu)} \\
&= \sum_{n=a}^{t}\frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} \\
&= \widehat{y}_{\lambda,\nu}(t,a)
\end{aligned}
$$

since $\lambda \geq 1$. □

We conclude this section by giving the following useful theorem and remark.

Theorem 3.4. *Assume $\lambda \in \mathbb{R}$. The fractional difference equation of order ν where $\nu \in$ (0,1)*

$$
\nabla_a^{\nu}y(t) = \lambda y(t-1) + f(t-1) \quad for \quad t = a+1, a+2, a+3, ..., \tag{3.1.4}
$$

has the general solution

$$y(t) = \widehat{y}_{\lambda,\nu}(t,a)c + \sum_{s=a}^{t-1} \widehat{y}_{\lambda,\nu}(t+a-s-1,a)f(s), \quad t = a, a+1, a+2, ..., \tag{3.1.5}$$

where c is constant.

Proof. A direct substitution gives that $\sum_{s=a}^{t-1} \widehat{y}_{\lambda,\nu}(t+a-s-1,a)f(s)$ is a particular solution of equation

$$\nabla_a^\nu y(t) = \lambda y(t-1) + f(t-1).$$

We show that

$$\nabla_a^\nu \sum_{s=a}^{t-1} \widehat{y}_{\lambda,\nu}(t+a-s-1,a)f(s) = \lambda \sum_{s=a}^{t-2} \widehat{y}_{\lambda,\nu}(t+a-s-2,a)f(s) + f(t-1).$$

Using the definition of the nabla fractional difference operator we have

$$\begin{aligned}\nabla_a^\nu \sum_{s=a}^{t-1} \widehat{y}_{\lambda,\nu}(t+a-s-1,a)f(s) &= \nabla \nabla_a^{-(1-\nu)} \sum_{s=a}^{t-1} \widehat{y}_{\lambda,\nu}(t+a-s-1,a)f(s) \\ &= \underbrace{\nabla \sum_{u=a}^{t} \frac{(t-\rho(u))^{\overline{-\nu}}}{\Gamma(1-\nu)} \sum_{s=a}^{u-1} \widehat{y}_{\lambda,\nu}(u+a-s-1,a)f(s).}_{I}\end{aligned}$$

Next we interchange the order of sums and obtain

$$I = \nabla \sum_{s=a}^{t-1} \sum_{u=s+1}^{t} \frac{(t-\rho(u))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(u+a-s-1,a)f(s).$$

Next we apply the following rule [16] to the above expression

$$\nabla \sum_{s=a}^{t-1} f(t,s) = \sum_{s=a}^{t-2} \nabla f(t,s) + f(t,t-1). \tag{3.1.6}$$

Hence, we have

$$I = \sum_{s=a}^{t-2} \nabla \sum_{u=s+1}^{t} \frac{(t-\rho(u))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(u+a-s-1,a) f(s)$$

$$+ \sum_{u=s+1}^{t} \frac{(t-\rho(u))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(u+a-s-1,a) f(s) \,|_{t=t,s=t-1}$$

$$= \sum_{s=a}^{t-2} \nabla \sum_{u=s+1}^{t} \frac{(t-\rho(u))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(u+a-s-1,a) f(s) + f(t-1)$$

since $\widehat{y}_{\lambda,\nu}(a,a) = 1$.

Next we use the substitution $u+a-s-1=\tau$, we obtain

$$\sum_{u=s+1}^{t} \frac{(t-\rho(u))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(u+a-s-1,a) = \sum_{\tau=a}^{t+a-s-1} \frac{(t-(\tau+s-a+1-1))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(\tau,a)$$

$$= \nabla_a^{-(1-\nu)} \widehat{y}_{\lambda,\nu}(t+a-s-1,a).$$

Thus,

$$I = \sum_{s=a}^{t-2} \nabla \nabla_a^{-(1-\nu)} \widehat{y}_{\lambda,\nu}(t+a-s-1,a) f(s) + f(t-1)$$

$$= \sum_{s=a}^{t-2} \nabla_a^{\nu} \widehat{y}_{\lambda,\nu}(t+a-s-1,a) f(s) + f(t-1)$$

$$= \lambda \sum_{s=a}^{t-2} \widehat{y}_{\lambda,\nu}(t+a-s-2,a) f(s) + f(t-1)$$

We use Theorem 3.2 to complete the proof. □

Remark 3.5. *Assume $\lambda \in \mathbb{R}$. The first order nabla difference equation*

$$\nabla y(t) = \lambda y(t-1) + f(t-1) \quad for \quad t = a+1, a+2, a+3, .., \tag{3.1.7}$$

has the general solution

$$y(t) = (1+\lambda)^{t-a}c + \sum_{s=a}^{t-1}(1+\lambda)^{t+a-s-1}f(s), \quad t = a, a+1, a+2, ..., \tag{3.1.8}$$

where c is constant.

If we take the $\nu = 1$, then the solution of (3.1.7) and the our function value coincides. Thus:

$$\widehat{y}_{\lambda,1}(t,a) = (1+\lambda)^{t-a}.$$

3.2 Sequential Fractional Difference Equations

In this section we establish the solution of the sequential fractional equation with constant coefficients

$$p\nabla_a^\nu \nabla_a^\nu y(t) + q\nabla_a^\nu y(t-1) + ry(t-2) = 0 \quad for \quad t = a+2, a+3, a+4, \ldots, \tag{3.2.1}$$

where $\nu \in (0,1)$ and p, q, r are constant. The characteristic equation of (3.2.1) is given as

$$p\lambda^2 + q\lambda + r = 0.$$

Assume that λ_1 and λ_2 are the roots of the characteristic equation. By using the fact that any given equation can be represented by its characteristic roots, we have characteristic polynomial

$$\nabla_a^\nu \nabla_a^\nu y(t) - (\lambda_1 + \lambda_2)\nabla_a^\nu y(t-1) + \lambda_1\lambda_2 y(t-2) = 0. \tag{3.2.2}$$

CASE I. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

By using the fact that, λ_1 and λ_2 are solutions of characteristic equation and part (ii) of Lemma 3.3 we obtain

$$p\nabla_a^\nu\nabla_a^\nu\widehat{y}_{\lambda_i,\nu}(t,a)+q\nabla_a^\nu\widehat{y}_{\lambda_i,\nu}(t-1,a)+r\widehat{y}_{\lambda_i,\nu}(t-2,a)=(p\lambda_i^2+q\lambda_i+r)\widehat{y}_{\lambda_i,\nu}(t-2,a)=0$$

which implies that $\widehat{y}_{\lambda_i,\nu}(t,a)$, $i=1,2$ are solutions of (3.2.1). The functions $\widehat{y}_{\lambda_1,\nu}(t,a)$ and $\widehat{y}_{\lambda_2,\nu}(t,a)$ are linearly independent, otherwise there would be nonzero constant c_1 and c_2 such that

$$c_1\widehat{y}_{\lambda_1,\nu}(t,a)+c_2\widehat{y}_{\lambda_2,\nu}(t,a)=0$$

for all t. First let $t=a$

$$\widehat{y}_{\lambda_1,\nu}(a,a)=\widehat{y}_{\lambda_2,\nu}(a,a)=1.$$

$$c_1+c_2=0.$$

Now let $t=a+1$. Then we have

$$\widehat{y}_{\lambda_1,\nu}(a+1,a)=\nu+\lambda_1,\qquad \widehat{y}_{\lambda_2,\nu}(a+1,a)=\nu+\lambda_2$$

which implies that

$$c_1(\nu+\lambda_1)+c_2(\nu+\lambda_2)=0.$$

This is a system of two equations and two unknowns. The determinant of the corresponding matrix is

$$(\nu+\lambda_1)-(\nu+\lambda_2)=\lambda_1-\lambda_2\neq 0.$$

Since determinant is nonzero the only solution is the trivial solution. That is $c_1=c_2=0$.

Since $\widehat{y}_{\lambda_1,\nu}(t,a)$ and $\widehat{y}_{\lambda_2,\nu}(t,a)$ are linearly independent we have that

$$y(t)=c_1\widehat{y}_{\lambda_1,\nu}(t,a)+c_2\widehat{y}_{\lambda_2,\nu}(t,a)$$

is the general solution of (3.2.1).

CASE II. If $\lambda_1 = \lambda_2 = \lambda$ and $\lambda \in \mathbb{R}$.

We claim that $\widehat{y}_{\lambda,\nu}(t,a)$ and $t\widehat{y}_{\lambda,\nu}(t,a)$ are solutions of linear homogeneous nabla fractional equation (3.2.1). In Case I we determined that $\widehat{y}_{\lambda,\nu}(t,a)$ is a solution of the equation (3.2.1) and we also need to show $t\widehat{y}_{\lambda,\nu}(t,a)$ is a solution of (3.2.1).

To continue the proof, we use Leibniz rule [33]

$$\overset{t}{\nabla}{}_a^{\nu} f(t)g(t) = \sum_{n=0}^{t-a} \binom{\nu}{n} [\overset{t-n}{\nabla}{}_a^{\nu-n} f(t-n)][\nabla^n g(t)].$$

For $g(t) = t$ and $f(t) = \widehat{y}_{\lambda,\nu}(t,a)$ we have

$$\overset{t}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t,a)t = \sum_{n=0}^{1} \binom{\nu}{n} [\overset{t-n}{\nabla}{}_a^{\nu-n} \widehat{y}_{\lambda,\nu}(t-n,a)][\nabla^n t].$$

By using

$$\binom{\nu}{n} = \frac{\Gamma(\nu+1)}{\Gamma(n+1)\Gamma(\nu-n+1)}$$

we obtain $\binom{\nu}{0} = 1$, $\binom{\nu}{1} = \nu$.

Since $\overset{t}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t,a) = \lambda \widehat{y}_{\lambda,\nu}(t-1,a)$, we obtain the following form

$$\overset{t}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t,a)t = [\overset{t}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t,a)]t + \nu \overset{t-1}{\nabla}{}_a^{\nu-1} \widehat{y}_{\lambda,\nu}(t-1,a) \tag{3.2.3}$$

$$= \lambda \widehat{y}_{\lambda,\nu}(t-1,a)t + \nu \overset{t-1}{\nabla}{}_a^{\nu-1} \widehat{y}_{\lambda,\nu}(t-1,a). \tag{3.2.4}$$

Next, we consider the second nabla fractional derivative of $t\widehat{y}_{\lambda,\nu}(t,a)$ to obtain

$$\nabla_a^{\nu}\nabla_a^{\nu}\widehat{y}_{\lambda}(t)t = \overset{t}{\nabla}{}_{a}^{\nu}\lambda\widehat{y}_{\lambda,\nu}(t-1,a)t + \nu\,\overset{t}{\nabla}{}_{a}^{\nu}\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-1,a) \tag{3.2.5}$$

$$= \lambda^2\widehat{y}_{\lambda,\nu}(t-2,a)t + \nu\lambda\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-2,a) + \nu\,\overset{t}{\nabla}{}_{a}^{\nu}\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-1,a). \tag{3.2.6}$$

<u>Claim:</u>

$$\overset{t}{\nabla}{}_{a}^{\nu}\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-1,a) = \lambda\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-2,a). \tag{3.2.7}$$

We start to prove our claim by writing the left side as

$$\nabla\,\overset{t}{\nabla}{}_{a}^{-(1-\nu)}\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-1,a). \tag{3.2.8}$$

Then we use the lemma

$$\nabla_{a+1}^{-\nu}\nabla f(t) = \nabla\nabla_a^{-\nu}f(t) - \frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}f(a).$$

Define $\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-1,a) = f(t)$. So $f(a) = a$ and we conclude that

$$\overset{t}{\nabla}{}_{a+1}^{\nu-1}\nabla f(t) = \nabla\,\overset{t}{\nabla}{}_{a}^{\nu-1}f(t).$$

Thus (3.2.8) can be written as

$$\overset{t}{\nabla}{}_{a+1}^{\nu-1}\nabla\,\overset{t-1}{\nabla}{}_{a}^{\nu-1}\widehat{y}_{\lambda,\nu}(t-1,a) = \overset{t}{\nabla}{}_{a+1}^{\nu-1}\,\overset{t-1}{\nabla}{}_{a}^{\nu}\widehat{y}_{\lambda,\nu}(t-1,a).$$

Since $\overset{t}{\nabla}{}_{a}^{\nu}\widehat{y}_{\lambda,\nu}(t-1,a) = \lambda\widehat{y}_{\lambda,\nu}(t-2,a)$, by shifting one unit left we obtain

$$\overset{t-1}{\nabla}{}_{a}^{\nu}\widehat{y}_{\lambda,\nu}(t-1,a) = \lambda\widehat{y}_{\lambda,\nu}(t-3,a)$$

Thus 3.2.8 can be written as

$$\overset{t}{\nabla}{}_{a+1}^{\nu-1} \overset{t-1}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t-1,a) = \overset{t}{\nabla}{}_{a+1}^{\nu-1} \lambda \widehat{y}_{\lambda,\nu}(t-3,a).$$

Using the definition of nabla sum , we have

$$\overset{t}{\nabla}{}_{a+1}^{\nu-1} \overset{t-1}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t-1,a) = \lambda \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(s-3,a)$$

By using $s-1=\tau$, we have

$$\overset{t}{\nabla}{}_{a+1}^{\nu-1} \overset{t-1}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t-1,a) = \lambda \sum_{\tau=a}^{t-1} \frac{(t-1-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)} \widehat{y}_{\lambda,\nu}(\tau-2,a).$$

which equals to

$$\overset{t}{\nabla}{}_{a+1}^{\nu-1} \overset{t-1}{\nabla}{}_a^{\nu} \widehat{y}_{\lambda,\nu}(t-1,a) = \lambda \overset{t-1}{\nabla}{}_a^{\nu-1} \widehat{y}_{\lambda,\nu}(t-2,a).$$

Next we need to show that $t\widehat{y}_{\lambda,\nu}(t,a)$ satisfies (3.2.2). Hence, by (3.2.4)-(3.2.6)-(3.2.7) we have

$$\begin{aligned}
&\nabla_a^{\nu}\nabla_a^{\nu}\widehat{y}_{\lambda,\nu}(t,a)t - (2\lambda)\nabla_a^{\nu}\widehat{y}_{\lambda,\nu}(t-1,a)(t-1) + \lambda^2\widehat{y}_{\lambda,\nu}(t-2,a)(t-2) \\
&= \lambda^2\widehat{y}_{\lambda,\nu}(t-2,a)t + \nu\lambda \overset{t-1}{\nabla}{}_a^{\nu-1}\widehat{y}_{\lambda,\nu}(t-2,a) + \nu\lambda \overset{t-1}{\nabla}{}_a^{\nu-1}\widehat{y}_{\lambda,\nu}(t-2,a) \\
&- 2\lambda^2\widehat{y}_{\lambda,\nu}(t-2,a)t + 2\lambda^2\widehat{y}_{\lambda,\nu}(t-2,a) - 2\nu\lambda \overset{t-1}{\nabla}{}_a^{\nu-1}\widehat{y}_{\lambda,\nu}(t-2,a) \\
&+ \lambda^2\widehat{y}_{\lambda,\nu}(t-2,a)t - 2\lambda^2\widehat{y}_{\lambda,\nu}(t-2,a) \\
&= 0.
\end{aligned}$$

As a result, $t\widehat{y}_{\lambda}(t)$ is also a solution of (3.2.1).

The functions $t\widehat{y}_{\lambda,\nu}(t,a)$ and $\widehat{y}_{\lambda,\nu}(t,a)$ are linearly independent, otherwise there would be nonzero constants c_1 and c_2 such that

$$c_1\widehat{y}_{\lambda,\nu}(t,a) + c_2 t\widehat{y}_{\lambda,\nu}(t,a) = 0,$$

for all $t \in \mathbb{N}_a$. First let $t = a$, we have

$$t\widehat{y}_{\lambda,\nu}(t,a)|_{t=a} = a, \quad \widehat{y}_{\lambda,\nu}(t,a)|_{t=a} = 1,$$

hence

$$c_1 + ac_2 = 1.$$

Now let $t = a + 1$. Then

$$t\widehat{y}_{\lambda,\nu}(t,a)|_{t=a+1} = (a+1)\nu + (a+1)\lambda, \quad \widehat{y}_{\lambda,\nu}(t,a)|_{t=a+1} = \nu + \lambda,$$

implies that

$$c_1(\nu + \lambda) + c_2((a+1)\nu + (a+1)\lambda) = 0.$$

This is a system of two equations and two unknowns. Since the determinant of the corresponding matrix is nonzero the only solution is the trivial solution.Thus $t\widehat{y}_{\lambda,\nu}(t,a)$ and $\widehat{y}_{\lambda,\nu}(t,a)$ are linearly independent. Since $t\widehat{y}_{\lambda,\nu}(t,a)$ and $\widehat{y}_{\lambda,\nu}(t,a)$ are linearly independent we have that

$$y(t) = c_1 t\widehat{y}_{\lambda,\nu}(t,a) + c_2\widehat{y}_{\lambda,\nu}(t,a)$$

is general solution of (3.2.1).

CASE III. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

Consider the following equation

$$\nabla_a^\nu \nabla_a^\nu y(t) + \beta^2 y(t-2) = 0, \qquad t = a+1, a+2, \ldots \tag{3.2.9}$$

where $\nu \in (0,1)$. The characteristic equation of (3.2.9) is

$$\lambda^2 + \beta^2 = 0$$

and the roots of the characteristic equation are $\lambda_1, \lambda_2 = \pm i\beta$. We claim that the general solution of the equation (3.2.9) is

$$y(t) = c_1 \widehat{\cos}_{\beta,\nu}(t,a) + c_2 \widehat{\sin}_{\beta,\nu}(t,a)$$

where c_1, c_2 are arbitrary constants and

$$\widehat{\cos}_{\beta,\nu}(t,a) = \frac{\widehat{y}_{i\beta,\nu}(t,a) + \widehat{y}_{-i\beta,\nu}(t,a)}{2}, \qquad \widehat{\sin}_{\beta,\nu}(t,a) = \frac{\widehat{y}_{i\beta,\nu}(t,a) - \widehat{y}_{-i\beta,\nu}(t,a)}{2i}.$$

Firstly, we need to show that $\widehat{\cos}_{\beta,\nu}(t,a)$ and $\widehat{\sin}_{\beta,\nu}(t,a)$ are solutions of (3.2.9). By using the fact that ∇_a^ν is linear we obtain

$$\begin{aligned}
\nabla_a^\nu \widehat{\cos}_{\beta,\nu}(t,a) &= \frac{\nabla_a^\nu \widehat{y}_{i\beta,\nu}(t,a) + \nabla_a^\nu \widehat{y}_{-i\beta,\nu}(t,a)}{2} \\
&= -\beta \frac{\nabla_a^\nu \widehat{y}_{i\beta,\nu}(t-1,a) - \nabla_a^\nu \widehat{y}_{-i\beta,\nu}(t-1,a)}{2i} \\
&= -\beta \widehat{\sin}_{\beta,\nu}(t-1,a).
\end{aligned}$$

Using a similar technique, we obtain the following identities.

$$\nabla_a^{\nu}\widehat{\sin}_{\beta,\nu}(t,a) = \beta\widehat{\cos}_{\beta,\nu}(t-1,a)$$
$$\nabla_a^{\nu}\nabla_a^{\nu}\widehat{\cos}_{\beta,\nu}(t,a) = -\beta^2\widehat{\cos}_{\beta,\nu}(t-2,a)$$
$$\nabla_a^{\nu}\nabla_a^{\nu}\widehat{\sin}_{\beta,\nu}(t,a) = -\beta^2\widehat{\sin}_{\beta,\nu}(t-2,a).$$

Using the identities we get

$$\nabla_a^{\nu}\nabla_a^{\nu}\widehat{\cos}_{\beta,\nu}(t,a) + \beta^2\widehat{\cos}_{\beta,\nu}(t-2,a) = 0$$
$$\nabla_a^{\nu}\nabla_a^{\nu}\widehat{\sin}_{\beta,\nu}(t,a) + \beta^2\widehat{\sin}_{\beta,\nu}(t-2,a) = 0.$$

Thus shows that $\widehat{\cos}_{\beta,\nu}(t,a)$ and $\widehat{\sin}_{\beta,\nu}(t,a)$ satisfy (3.2.9) thus both of them are solutions of (3.2.9). The proof of linearly independence follows a similar technique as case-I and case-II. Since $\widehat{\cos}_{\beta,\nu}(t,a)$ and $\widehat{\sin}_{\beta,\nu}(t,a)$ are linearly independent we have that

$$y(t) = c_1\widehat{\cos}_{\beta,\nu}(t,a) + c_2\widehat{\sin}_{\beta,\nu}(t,a)$$

is general solution of (3.2.9).

3.3 State-Space Representation of Nabla Fractional Difference Equations

A dynamic system consisting of a finite number of inputs and outputs may be described by ordinary differential equations in which time is the independent variable. The state space representation of any linear system helps us to analyse the stability, controllability, observability etc. For instance, n^{th} order circuit systems, mechanical rotating systems, and mechanical translating systems can be represented as an n piece first order linear differential system. Similarly, by using of the vector-matrix notation, an $n\nu^{th}$ order linear nabla fractional difference equation may be expressed by a ν^{th} order matrix valued linear nabla fractional difference equation. In this section, we shall present methods for obtaining state-space representations of $n\nu^{th}$ order linear nabla fractional difference equations.

Define

$$\nabla_a^{(n\nu)}y(t) := \underbrace{\nabla_a^{\nu}\nabla_a^{\nu}\cdots\nabla_a^{\nu}}_{\text{n}\,times}y(t), \qquad n \in \mathbb{N}. \tag{3.3.1}$$

This is known as a sequential fractional difference operator [1, 30, 31].

Consider that we have been given the following $n\nu^{th}$ order nabla fractional equation,

$$a_n\nabla_a^{(n\nu)}u(t)+a_{n-1}\nabla_a^{((n-1)\nu)}u(t-1)+a_{n-2}\nabla_a^{((n-2)\nu)}u(t-2)+\cdots+a_0u(t-n) = f(t), \tag{3.3.2}$$

for all $t \in \mathbb{N}_{a+n}$ and given values

$$u(a), \nabla_a^{\nu}u(a), \nabla_a^{(2\nu)}u(a), \ldots, \nabla_a^{((n-1)\nu)}u(a).$$

Let us define

$$\begin{aligned}
y_1(t) &= u(t-n)\\
y_2(t) &= \nabla_a^{\nu}u(t-n+1)\\
y_3(t) &= \nabla_a^{(2\nu)}u(t-n+2)\\
\vdots \quad & \quad \vdots\\
y_n(t) &= \nabla_a^{((n-1)\nu)}u(t-1).
\end{aligned}$$

Then we have

$$
\begin{aligned}
\nabla_a^{\nu} y_1(t) &= \nabla_a^{\nu} u(t-n) = y_2(t-1) \\
\nabla_a^{\nu} y_2(t) &= \nabla_a^{(2\nu)} u(t-n+1) = y_3(t-1) \\
\vdots \quad & \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
\nabla_a^{\nu} y_{n-1}(t) &= \nabla_a^{((n-1)\nu)} u(t-2) = y_n(t-1) \\
\nabla_a^{\nu} y_n(t) &= \nabla_a^{(n\nu)} u(t-1) \\
&= -\frac{a_{n-1}}{a_n} \nabla_a^{((n-1)\nu)} u(t-2) - \cdots - \frac{a_0}{a_n} u(t-n-1) + \frac{f(t-1)}{a_n} \\
&= -\frac{a_{n-1}}{a_n} y_n(t-1) - \frac{a_{n-2}}{a_n} y_{n-1}(t-1) - \cdots - \frac{a_0}{a_n} y_1(t-1) + \frac{f(t-1)}{a_n}.
\end{aligned}
$$

Then equation (3.3.1) can be written as

$$
\begin{aligned}
&\nabla_a^{\nu} y(t) = Ay(t-1) + Bf(t-1) \\
&\nabla_a^{-(1-\nu)} y(t)|_{t=a} = y(a)
\end{aligned}
$$

where

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-1}(t) \\ y_n(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{a_n} \end{bmatrix}.$$

3.4 Vector Fractional Difference Equations

In this section, we study the following fractional difference equation

$$\nabla_a^{\nu} y(t) = Ay(t-1) + f(t-1), \qquad t \in \mathbb{N}_{a+1}$$

where where A is an $n \times n$ constant matrix, and y_0 and $y(.)$ are $n \times 1$ vectors and we present a method to compute the matrix exponential function $\widehat{y}_{A,\nu}(t,a)$ in fractional calculus.

Now, we give the following theorem, and the proof follows similar techniques as Theorem 3.2.

Theorem 3.6. *The unique solution of the initial value problem*

$$\nabla_a^\nu y(t) = Ay(t-1), \qquad t = a+1, a+2, \ldots \tag{3.4.1}$$

$$\nabla_a^{-(1-\nu)} y(t)|_{t=a} = y(a) = y_0 \tag{3.4.2}$$

is given by

$$y(t) = \sum_{n=a}^{t} \frac{A^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)} y_0, \quad t \in \mathbb{N}_a$$

where A is an $n \times n$ constant matrix, y_0 and $y(.)$ are $n \times 1$ vectors.

Next, we define following nabla function that we will be used sequel.

$$\widehat{y}_{A,\nu}(t,a) := \sum_{n=a}^{t} \frac{A^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}, \quad t \in \mathbb{N}_a \tag{3.4.3}$$

In the following lemma, we mention some important properties of the $\widehat{y}_{A,\nu}(t,a)$.

Lemma 3.7. *For given any A an $n \times n$ matrix, the following properties hold:*

(i) $\widehat{y}_{A,\nu}(a,a) = I_n$.

(ii) $\nabla_a^\nu \widehat{y}_{A,\nu}(t,a) = A\widehat{y}_{A,\nu}(t-1,a) \qquad t \in \mathbb{N}_{a+1}$.

Proof. *(i)* The proof follows from the definition of the new nabla function.

(ii) $\widehat{y}_{A,a}(t,a)$ satisfies the IVP (3.4.1)-(3.4.2). □

We omit the proof of the next theorem since it follows from the technique used in the proof of Theorem 3.4 and the properties of $\widehat{y}_{A,\nu}(t,a)$ in Lemma 3.7.

Theorem 3.8. *(Variation of Constants) Let $\nu \in \mathbb{R}$, $0 < \nu < 1$, A be an $n \times n$ constant matrix and suppose $f(t-1)$ is an $n \times 1$ vector valued function. Then the initial value* *problem*

$$\nabla_a^\nu y(t) = Ay(t-1) + f(t-1), \qquad t \in \mathbb{N}_{a+1}$$

$$\nabla_a^{-(1-\nu)} y(t)|_{t=a} = y(a) = y_0,$$

has a unique solution. Moreover, this solution is given by

$$y(t) = \widehat{y}_{A,\nu}(t,a)y_0 + \sum_{s=a}^{t-1} \widehat{y}_{A,\nu}(t+a-s-1,a)f(s), \quad t \in \mathbb{N}_a \tag{3.4.4}$$

3.4.1 Putzer Algorithm

The Putzer Algorithm is an analytic method for evaluating matrix exponential functions using eigenvalues and components in the solution of a relatively simple linear system. This algorithm was defined by E. J. Putzer who studied the matrix exponential and presented the method to compute e^{At}, where A is an $n \times n$ constant matrix in 1966 [57]. Elaydi and Harris [22] presented a method for the computation of A^n for non-singular A based on the Cayley–Hamilton theorem [29] and the precise determination of the eigenvalues of A. We shall present the proof of the Putzer Algorithm to evaluate the matrix exponential in discrete fractional calculus.

Next, we want to give an algorithm to calculate $\widehat{y}_{A,\nu}(t,a)$ in terms of $\widehat{y}_{\lambda,\nu}(t,a)$ where λ is an eigenvalue of the matrix A. For this purpose, we first define the matrix exponential function in discrete fractional calculus. Then we will state and prove the Putzer algorithm for any $n \times n$ matrix.

Definition 3.9. *(Matrix Exponential Function) Let A be an $n \times n$ constant matrix.*

The unique matrix valued solution of the initial value problem(IVP)

$$\nabla_a^{\nu} Y(t) = AY(t-1) \quad for \quad t \in \mathbb{N}_{a+1} \tag{3.4.5}$$

$$\nabla_a^{-(1-\nu)} Y(t)|_{t=a} = Y(a) = I_n, \tag{3.4.6}$$

where I_n denotes the $n \times n$ identity matrix, is called the matrix exponential function.

Theorem 3.10. *If $\lambda_1, \lambda_2, ..., \lambda_n$ are (not necessarily distinct) eigenvalues of the $n \times n$ matrix A, with each eigenvalue repeated as many times as its multiplicity, then*

$$\widehat{y}_{A,\nu}(t,a) = \sum_{i=0}^{n-1} p_{i+1}(t) M_i,$$

where

$$M_0 = I_n$$

$$M_i = (A - \lambda_i I_n) M_{i-1}, \quad (1 \le i \le n-1)$$

$$M_n = 0$$

and the vector valued function p defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ \vdots \\ p_n(t) \end{bmatrix}$$

is the solution of the initial value problem

$$\nabla_a^\nu p(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} p(t-1) \qquad for \qquad t \in \mathbb{N}_{a+1} \tag{3.4.7}$$

$$\nabla_a^{-(1-\nu)} p(t)|_{t=a} = p(a) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.4.8}$$

Proof. Let $\Phi(t) = \sum_{i=0}^{n-1} p_{i+1}(t) M_i$. We first show that Φ solves the IVP (3.4.5)-(3.4.6). First note that

$$\begin{aligned} \nabla_a^{-(1-\nu)} \Phi(a) &= \nabla_a^{-(1-\nu)} p_1(a) M_0 + \nabla_a^{-(1-\nu)} p_2(a) M_1 + \cdots + \nabla_a^{-(1-\nu)} p_n(a) M_{n-1} \\ &= I_n \end{aligned}$$

since we are given the initial values $p(a) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T$.

$$\begin{aligned} \nabla_a^\nu \Phi(t) - A\Phi(t-1) &= \nabla_a^\nu \sum_{i=0}^{n-1} p_{i+1}(t) M_i - A \sum_{i=0}^{n-1} p_{i+1}(t-1) M_i \\ &= \nabla_a^\nu p_1(t) M_0 + \nabla_a^\nu p_2(t) M_1 + \cdots + \nabla_a^\nu p_n(t) M_{n-1} - A \sum_{i=0}^{n-1} p_{i+1}(t-1) M_i, \end{aligned}$$

since ∇_a^ν is a linear operator. Next we use (3.4.7), so the last quantity equals

$$\nabla_a^\nu \Phi(t) - A\Phi(t-1) = \lambda_1 p_1(t-1)M_0 + [p_1(t-1) + \lambda_2 p_2(t-1)]M_1$$

$$+ [p_2(t-1) + \lambda_3 p_3(t-1)]M_2 + \cdots + [p_{n-1}(t-1) + \lambda_n p_n(t-1)]M_{n-1}$$

$$- A\sum_{i=0}^{n-1} p_{i+1}(t-1)M_i$$

$$= [\lambda_1 M_0 + M_1 - AM_0]p_1(t-1) + [\lambda_2 M_1 + M_2 - AM_1]p_2(t-1)$$

$$+ \cdots + [\lambda_n M_{n-1} - AM_{n-1}]p_n(t-1)$$

$$= [\lambda_n I_n - A]M_{n-1}p_n(t-1),$$

since $M_i = (A - \lambda_i I_n)M_{i-1}$ for $(1 \le i \le n)$. The last quantity is zero matrix by Cayley-Hamilton Theorem. In fact,

$$(\lambda_n I_n - A)M_{n-1}p_n(t-1) = -(A - \lambda_n I_n)(A - \lambda_{n-1} I_n)\cdots(A - \lambda_1 I_n)p_n(t-1)$$

$$= 0_{n\times n}.$$

Since $\widehat{y}_{A,\nu}(t,a)$ satisfies the IVP (3.4.5)-(3.4.6), we have

$$\Phi(t) = \widehat{y}_{A,\nu}(t,a)$$

by the unique solution of the given initial value problem. $\square$

Next, we will give an example to illustrate the use of the Putzer algorithm for a 2×2 matrix.

Example 3.11. *Let the matrix* $A = \begin{bmatrix} -0.2 & 0.5 \\ 0.6 & -0.1 \end{bmatrix}$ *be given with the eigenvalues* $\lambda_1 = 0.4,\ \lambda_2 = -0.7.$

By using Theorem 3.4 we find that the solution of the IVP (3.4.7)-(3.4.8) is given by

$$p_1(t) = \widehat{y}_{.4,\nu}(t,a) \qquad \textit{and}$$

$$p_2(t) = \sum_{s=a}^{t-1} \widehat{y}_{-.7,\nu}(t+a-s-1,a)\widehat{y}_{.4,\nu}(s,a).$$

Now we compute $\widehat{y}_{A,\nu}(t,a)$ *by using Theorem 3.10*

$$\widehat{y}_{A,\nu}(t,a) = \widehat{y}_{.4,\nu}(t,a) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + p_2(t) \begin{bmatrix} -0.6 & 0.5 \\ 0.6 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{y}_{.4,\nu}(t,a) - .6p_2(t) & .5p_2(t) \\ .6p_2(t) & \widehat{y}_{.4,\nu}(t,a) - .5p_2(t) \end{bmatrix}.$$

Chapter 4

CONTROLLABILITY AND OBSERVABILITY OF LINEAR TIME-INVARIANT NABLA FRACTIONAL SYSTEMS

Nowadays the concepts of fractional order derivative and integrals have attracted increasing attention from various fields of science and engineering communities. The main reason for this is that many physical materials and processes can be properly described by using fractional order calculus. It has been proven by scientific findings that many fractional-order mathematical models are the best description for natural phenomena. Most of the research on the applications of the fractional difference/differential calculus are focused on the temporal state of physical change, image processing, viscoelastic theory, controller design, and random fractional dynamics [51, 54, 55, 56].

The study of controllability and observability plays an important role in control theory and engineering. They have close connections to pole assignment, structural decomposition, quadratic optimal control, observer design, controller design, and so forth. For this reason, many active scholars contributed to controllability of continuous time systems [5, 36, 37, 50] and controllability of dynamic systems on time scales [12, 13, 24, 27, 63]. Bartosiewicz and Pawluszewicz [12] proposed the controllability criteria for linear time-invariant dynamic systems on time scales, whereas Fausett and Murty [27] not only studied the controllability of dynamic systems but also obtained the observability and realizability criteria for linear time-invariant dynamic systems on time scales. Davis et al. [24] proved some basic results on controllability, observability, and realizability of linear time-invariant dynamic systems, and then extended their results to time-variant systems. Pawluszewicz [13] proposed a necessary and sufficient condition for positive reachability of a positive system on an arbitrary time scale considering the Gramian matrix. However, when studying the controllability of dynamic systems [12, 24, 27], one must assume that the graininess function is differentiable, an assumption that is not satisfied in general for all time scales. For this reason, Wintz and Bohner [63] altered the system and obtained controllability of time-invariant linear dynamic systems without assuming differentiability of the graininess function. Due to these solid works, controllability theory on continuous time systems, dynamic systems, and continuous fractional order systems [14, 15, 41, 47] all have been well developed.

In contrast to that for the continuous-time case, the amount of literature which focus on controllability of time-invariant linear discrete systems is much less. The controllability of the linear discrete-time systems have been investigated in [21, 49, 51], and the necessary and sufficient conditions for discrete fractional order systems with the Grünwald-Letnikov operator are given in [34, 36, 59]. Kaczorek [36] introduced the notion of the positive fractional discrete-time linear system and proposed the necessary and sufficient conditions for the positivity, reachability, and controllability to zero. Guermah et al. [34] studied controllability and observability of linear discrete-time fractional-order systems that are modeled by a discrete-time linear system with delays in states. Mozyrska et al. [44] proposed the properties of the h-difference linear control systems with fractional order and developed the rank conditions for controllability and observability of fractional order systems with the Caputo-Type operator. Then they extended their results to h-difference linear control systems with n different fractional orders in [45]. Mozyrska et al. [46] investigated the local controllability and observability of nonlinear discrete-time systems considering the Caputo, the Riemann-Liouville, and the Grünwald-Letnikov-type h-difference fractional operators. In [7] Atici and Nguyen studied the controllability and observability of the discrete Δ-fractional time-invariant linear systems.

Fractionalizing of mathematical models in the field of applied mathematics is a method which improves the descriptive meaning of the mathematical models of real world problems, as illustrated in many papers in the area of applied mathematics, physics, computer science, and bioengineering [11, 40, 51, 55]. So the natural question follows: Do we keep or lose the controllability of the discrete system if we fractionalize it?

Motivated by this question and the recent work in discrete time, we shall continue to develop the control theory in discrete time and search for an answer to this question in this chapter. We discuss the controllability and observability of a linear time-invariant nabla discrete fractional state-space model, which is represented by

$$\begin{cases} \nabla^{\nu}_{t_0} y(t) = Ay(t-1) + Bu(t-1) \\ z(t) = Cy(t) + Du(t) \end{cases}, \tag{4.0.1}$$

where A is an $n \times n$ constant matrix, B is an $n \times m$ constant matrix, C is an $r \times n$ constant matrix, D is an $r \times m$ constant matrix, $y(t)$ is an $n \times 1$ state vector, $u(t)$ is an $m \times 1$ control vector (control signal) and $z(t)$ is an $r \times 1$ the output vector (response vector).

Throughout this chapter we assume that $t_0, t_1 \in \mathbb{R}^+$ and $t_1 - t_0 \in \mathbb{Z}^+$.

This chapter is organized as follows: In Section 4.1, we give the definition of controllability and controllability to the origin. We give a necessary and sufficient condition for the linear time-invariant fractional difference system (4.0.1) to be controllable via the controllability Gramian matrix. In particular, it requires more computations of $\widehat{y}_{A,\nu}(\cdot, t_0)$(see Chapter-3). For this reason, we give the Kalman rank condition to be controllable via the controllability matrix. We close Section 4.1 by giving the necessary condition for the two notions completely controllable and controllability to the origin to coincide. While observability of (4.0.1) is studied in Section 4.2 and similarly necessary and sufficient conditions to be observable are given in terms of the observability Gramian matrix and observability matrix. Section 4.3 discusses the relationship between the controllability and observability of the system (4.0.1) with the duality principle.

4.1 Controllability

In this section, we establish the criterion for controllability of the linear discrete-fractional time-invariant system

$$\nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1), \quad t \in \mathbb{N}_{t_0+1}^{t_1}, \tag{4.1.1}$$

where $y(t_0) = y_0$ is the initial state, A is an $n \times n$ constant matrix, $y(t)$ is an $n \times 1$ state vector, B is an $n \times m$ constant matrix and $u(t)$ is an $m \times 1$ control vector, $m \leq n$ and $0 < \nu < 1$. Because the output does not play any role in controllability, the output equation is disregarded in this study. By Theorem 3.8 the corresponding solution of the system (4.1.1) is

$$y(t) = \widehat{y}_{A,\nu}(t, t_0) y_0 + \sum_{s=t_0}^{t-1} \widehat{y}_{A,\nu}(t + t_0 - s - 1, t_0) Bu(s). \tag{4.1.2}$$

We say that a system is controllable if we can transfer any initial state to any arbitrary final state under the control vector of the system. Now we present a formal definition of this and controllability to the origin.

Definition 4.1. *A system modeled by (4.1.1) or pair $\{A, B\}$ is said to be completely controllable, if it is possible to construct a control signal $u(t)$ that will transfer any initial state $y(t_0)$ to any final state $y(t_1)$ in finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1-1}$. Otherwise the system (4.1.1) or $\{A, B\}$ is said to be uncontrollable.*

Definition 4.2. *If every non-zero initial state $y(t_0)$ can be transferred to final state $y(t_1) = 0_{n\times 1}$, by control signal $u(t)$ in finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1-1}$, then the system (4.1.1) is said to be controllable to the origin.*

To give necessary and sufficient conditions for controllability of the linear system (4.1.1), we will define the controllability matrix and the controllability Gramian matrix of the given fractional control system (4.1.1).

The *controllability matrix* $\widehat{W}$ of the system (4.1.1) is defined as an $n \times (nm)$ matrix

$$\widehat{W} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

and we define *controllability Gramian matrix* $\mathcal{P}$ of the system (4.1.1) as an $n \times n$ matrix

$$\mathcal{P}(t, t_0) := \sum_{s=t_0}^{t-1} \widehat{y}_{A,\nu}(s, t_0) B B^T [\widehat{y}_{A,\nu}(s, t_0)]^T.$$

Theorem 4.3. *The following statements are equivalent:*

(i) The system $\nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1)$ is completely controllable on the discrete time interval $\mathbb{N}_{t_0+1}^{t_1}$.

(ii) The controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ has rank n.

(iii) The controllability matrix $\widehat{W}$ has rank n.

Proof. $(i) \Leftrightarrow (ii)$

First we show that if a given system is completely controllable then controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ of the given system has rank n. Let us prove this by contradiction. Suppose that $rank(\mathcal{P}(t_1, t_0)) < n$. And then there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T \mathcal{P}(t_1, t_0) = 0_{1\times n}$. Then it follows that

$$\begin{aligned}
0 &= \eta^T \mathcal{P}(t_1, t_0)\eta \\
&= \sum_{s=t_0}^{t_1-1} \eta^T \widehat{y}_{A,\nu}(s, t_0) B B^T [\widehat{y}_{A,\nu}(s, t_0)]^T \eta \\
&= \sum_{s=t_0}^{t_1-1} \|\eta^T \widehat{y}_{A,\nu}(s, t_0) B\|_2^2,
\end{aligned}$$

where $\|\cdot\|_2$ defines the Euclidean norm. Hence

$$\eta^T \widehat{y}_{A,\nu}(t, t_0) B = 0_{1\times m}, \qquad t \in \mathbb{N}_{t_0}^{t_1-1}. \tag{4.1.3}$$

From the controllable assumption there exists a control signal $u(t)$ that will transfer initial state $y(t_0) = y_0$ to final state $y(t_1) = y_f = \widehat{y}_{A,\nu}(t_1, t_0)y_0 + \eta$. By substitution of initial and final state in to (4.1.2) the solution of the given system becomes

$$\begin{aligned}
\widehat{y}_{A,\nu}(t_1, t_0)y_0 + \eta = \widehat{y}_{A,\nu}(t_1, t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0) B u(s) \\
\eta = \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0) B u(s).
\end{aligned}$$

Multiplying though by η^T and using (4.1.3) yields

$$\eta^T \eta = \sum_{s=t_0}^{t_1-1} \eta^T \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0) B u(s) = 0,$$

which contradicts the assumption that η is a nonzero vector in $\mathbb{R}^n$ Thus, the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ has rank n.

Conversely, suppose $\mathcal{P}(t_1, t_0)$ has rank n. Then it follows that $\mathcal{P}(t_1, t_0)$ is invertible. Therefore, for the given any initial state $y(t_0) = y_0$ and final state $y(t_1) = y_f$

we can choose the control signal $u(t)$ as

$$u(t) = B^T[\widehat{y}_{A,\nu}(t_1 + t_0 - t - 1, t_0)]^T[\mathcal{P}(t_1, t_0)]^{-1}[y_f - \widehat{y}_{A,\nu}(t_1, t_0)y_0].$$

The corresponding solution of the system at $t = t_1$ can be written as

$$\begin{aligned}
y(t_1) &= \widehat{y}_{A,\nu}(t_1, t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)Bu(s) \\
&= \widehat{y}_{A,\nu}(t_1, t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)BB^T[\widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)]^T \\
&\times [\mathcal{P}(t_1, t_0)]^{-1}[y_f - \widehat{y}_{A,\nu}(t_1, t_0)y_0].
\end{aligned}$$

By performing the above last summation we obtain

$$\begin{aligned}
&\sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)BB^T[\widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)]^T \\
&= \widehat{y}_{A,\nu}(t_1 - 1, t_0)BB^T[\widehat{y}_{A,\nu}(t_1 - 1, t_0)]^T + \widehat{y}_{A,\nu}(t_1 - 2, t_0)BB^T[\widehat{y}_{A,\nu}(t_1 - 2, t_0)]^T \\
&+ \cdots + \widehat{y}_{A,\nu}(t_0, t_0)BB^T[\widehat{y}_{A,\nu}(t_0, t_0)]^T \\
&= \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(s, t_0)BB^T[\widehat{y}_{A,\nu}(s, t_0)]^T = \mathcal{P}(t_1, t_0).
\end{aligned}$$

Hence we have

$$y(t_1) = \widehat{y}_{A,\nu}(t_1, t_0)y_0 + \mathcal{P}(t_1, t_0)[\mathcal{P}(t_1, t_0)]^{-1}[y_f - \widehat{y}_{A,\nu}(t_1, t_0)y_0] = y_f.$$

This shows that if the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ has rank n, then a given system is completely controllable on given discrete time interval.

$(i) \Leftrightarrow (iii)$ First we note that for all $N \geq n$ the rank of matrix

$$\widehat{W}(N) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{N-1}B \end{bmatrix}$$

is equal to the rank of the controllability matrix $\widehat{W}$. By the Cayley-Hamilton's theorem

$$A^n = \sum_{s=0}^{n-1} p_s A^s,$$

where $-p_s$ are coefficients of the characteristic polynomial of A. Multiplying the above last expression by the matrix B we obtain

$$A^n B = \sum_{s=0}^{n-1} p_s A^s B.$$

Thus columns of $A^n B$ are linearly dependent on the columns of $\widehat{W}$ and

$$rank(\widehat{W}(n+1)) = rank(\widehat{W}).$$

Multiplying the last equation by the matrix A we obtain

$$A^{n+1} B = \sum_{s=0}^{n-1} p_s A^{s+1} B.$$

Consequently, $rank(\widehat{W})(n+2) = rank(\widehat{W})(n+1) = rank(\widehat{W})$. Proceeding forward, we can conclude that $rank(\widehat{W}(N)) = rank(\widehat{W})$ for all $N \geq n$. Thus, here we assume that $t_1 - t_0 = n$.

First we show that if the given system is completely controllable, then the controllability matrix has full rank n. Since given system is completely controllable, there exists a control signal $u(t)$ that will transfer any given initial state $y(t_0) = y_0 \in \mathbb{R}^n$ to any final state $y(n + t_0) = y_f \in \mathbb{R}^n$. Plugging $t_1 = n + t_0$ into the solution (4.1.2) yields

$$y_f = \widehat{y}_{A,\nu}(n+t_0,t_0)y_0 + \sum_{s=t_0}^{n+t_0-1} \widehat{y}_{A,\nu}(n+2t_0-s-1,t_0)Bu(s).$$

By performing the sum we obtain

$$\begin{aligned}
y(n+t_0) - \widehat{y}_{A,\nu}(n+t_0,t_0)y_0 &= \sum_{s=t_0}^{n+t_0-1} \widehat{y}_{A,\nu}(s,t_0)Bu(n+2t_0-s-1) \\
&= \sum_{s=t_0}^{n+t_0-1} \sum_{\tau=t_0}^{s} \frac{A^{\tau-t_0}(s-\tau+1)^{\overline{(\tau-t_0+1)\nu-1}}}{\Gamma((\tau-t_0+1)\nu)} Bu(n+2t_0-s-1) \\
&= \sum_{\tau=t_0}^{n+t_0-1} A^{\tau-t_0}B \sum_{s=\tau}^{n+t_0-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)\nu-1}}}{\Gamma((\tau-t_0+1)\nu)} u(n+2t_0-s-1),
\end{aligned}$$

where we interchanged the order of the summations. Next, we define $F(\tau)$ for $t_0 \le \tau \le n+t_0-1$ by

$$F(\tau) = \sum_{s=\tau}^{n+t_0-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)\nu-1}}}{\Gamma((\tau-t_0+1)\nu)} u(n+2t_0-s-1).$$

Substituting back $F(\tau)$ into equation, we have

$$y_f - \widehat{y}_{A,\nu}(n+t_0,t_0)y_0 = \sum_{\tau=t_0}^{n+t_0-1} A^{\tau-t_0}BF(\tau)$$

$$y_f - \widehat{y}_{A,\nu}(n+t_0,t_0)y_0 = \begin{bmatrix} B & AB & A^2B \cdots A^{n-1}B \end{bmatrix} \begin{bmatrix} F(t_0) \\ F(t_0+1) \\ F(t_0+2) \\ \vdots \\ F(t_0+n-1) \end{bmatrix} = \widehat{W} F_1(n). \quad (4.1.4)$$

Suppose the controllability matrix $\widehat{W}$ has rank less than n, then this implies that there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T \widehat{W} = 0_{1\times(mn)}$. Hence, multiplying both sides of (4.1.4) by η^T yields $\eta^T(y_f - \widehat{y}_{A,\nu}(n+t_0,t_0)y_0) = 0_{1\times n}$ regardless of control signal $u(t)$. Since the given system is completely controllable, we choose $y_f = \widehat{y}_{A,\nu}(n+t_0,t_0)y_0+\eta$. Then $\eta^T\eta = 0$ which contradicts the assumption that η is a nonzero vector. Therefore, $rank(\widehat{W}) = n$.

For the converse, suppose $rank(\widehat{W}) = n$, but for the sake of a contradiction, we assume that the given system is uncontrollable. Since the system is uncontrollable, then the controllability Gramian matrix $\mathcal{P}(t_0+n,t_0)$ has rank less than n. Hence there exists $\eta \in \mathbb{R}^n$ such that $\eta^T\mathcal{P}(t_0+n,t_0) = 0_{1\times n}$. Then we have

$$\begin{aligned}0 &= \eta^T\mathcal{P}(t_0+n,t_0)\eta \\ &= \sum_{s=t_0}^{n+t_0-1} \eta^T\widehat{y}_{A,\nu}(s,t_0)BB^T[\widehat{y}_{A,\nu}(s,t_0)]^T\eta \\ &= \sum_{s=t_0}^{n+t_0-1} \|\eta^T\widehat{y}_{A,\nu}(s,t_0)B\|_2^2,\end{aligned}$$

which implies that

$$\eta^T\widehat{y}_{A,\nu}(t,t_0)B = 0_{1\times m} \qquad for \quad all \quad t \in \mathbb{N}_{t_0}^{t_0+n-1}. \tag{4.1.5}$$

Setting $t = t_0$ and using Lemma 3.7 (i) we have

$$\eta^T B = 0_{1\times m}.$$

Applying ν-th order fractional difference operator to the each side of the last equality and using Lemma 3.7 we have $\eta^T A\widehat{y}_{A,\nu}(t-1,t_0)B = 0_{1\times m}$ for all $t \in \mathbb{N}_{t_0+1}^{t_0+n-1}$. Hence we have

$$\eta^T A\widehat{y}_{A,\nu}(t,t_0)B = 0_{1\times m} \quad for \quad all \quad t \in \mathbb{N}_{t_0}^{t_0+n-2}.$$

Setting $t = t_0$ and using Lemma 3.7 we have

$$\eta^T AB = 0_{1\times m}.$$

Repeating the same step up to $n-1$ times, we have

$$\eta^T A^k B = 0_{1\times m} \qquad for \quad k = 0, 1, ..., n-1.$$

Then we have

$$\eta^T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = \eta^T \widehat{W} = 0_{1\times(mn)}.$$

This contradicts the assumption that $rank(\widehat{W}) = n$. Thus the controllability Gramian matrix has rank n implies that the given system is completely controllable. □

Remark 4.4. *Note that, for every $\eta \in \mathbb{R}^n$*

$$\eta^T \mathcal{P}(t_1, t_0)\eta = \sum_{s=t_0}^{t_1-1} \|\eta^T \widehat{y}_{A,\nu}(s, t_0)B\|^2.$$

Hence the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ is a non-negative symmetric matrix. In particular $\mathcal{P}(t_1, t_0)$ has rank n if there exits $p > 0$ such that

$$\eta^T \mathcal{P}(t_1, t_0)\eta \geq p$$

for all $\eta \in \mathbb{R}^n$.

Since the controllability Gramian matrix is a non-negative symmetric matrix, in the statement of Theorem 4.3, (i) being equivalent to (ii) (i.e $(i) \Leftrightarrow (ii)$) can be interpreted as saying the given time-invariant linear nabla fractional difference system is completely controllable on the given discrete time interval if and only if *the controllability Gramian matrix is positive definite (or the controllability Gramian matrix is invertible) [24, 63].*

In the statement of Theorem 4.3, (i) *being equivalent to* (iii) *(i.e* $(i) \Leftrightarrow (iii)$*) is called the Kalman Rank Condition after Kalman [37] established the proofs for the continuous (*$\mathbb{R}$*) and discrete cases (*$\mathbb{Z}$*). However, it is essential to point out that the proof here is not the one that Kalman gave. In continuous (*$\mathbb{R}$*) and discrete cases (*$\mathbb{Z}$*), the results are more straightforward while we use the matrix exponential on* $\mathbb{Z}$ *and the discrete fractional case requires additional argument as illustrated above.*

Next, we provide an example to illustrate the applicability of the Theorem 4.3.

Example 4.5. *Consider the following system*

$$\nabla_{t_0}^{\nu} y(t) = \begin{bmatrix} -1 & 1 & 1 \\ 5 & -9 & 1 \\ 6 & -3 & -1 \end{bmatrix} y(t-1) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t-1),$$

where $0 < \nu < 1$.

From the given condition we obtain the controllability matrix of system

$$\widehat{W} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & -4 & 1 & 39 & -5 \\ 0 & 1 & 3 & -1 & 9 & 4 \end{bmatrix}.$$

It can be easily verified that the rank of $\widehat{W}$ *is 3. Thus by Theorem 4.3 the given linear fractional order system is completely controllable.*

Next we give an extra assumption on $\widehat{y}$ to prove that completely controllability and controllability to the origin are equivalent concepts for the given system (4.1.1).

Theorem 4.6. *If $\widehat{y}_{A,\nu}(\cdot,t_0)$ in (4.1.1) is non-singular on discrete time interval $t \in \mathbb{N}_{t_0+1}^{t_1}$, then the given system is completely controllable if and only if the system is controllable to the origin.*

Proof. Suppose that the system (4.1.1) is completely controllable. Choose final state as $y(t_1) = 0_{n\times 1}$. Then by the Definition 4.2 the given system is controllable to the origin.

Assume that $\widehat{y}_{A,\nu}(\cdot,t_0)$ in (4.1.2) is non-singular on discrete time interval $\mathbb{N}_{t_0+1}^{t_1}$ and the system (4.1.1) is controllable to the origin. For given any initial state $y(t_0)$ and any final state $y(t_1)$, define

$$x(t_0) := y(t_0) - [\widehat{y}_{A,\nu}(t_1,t_0)]^{-1}y(t_1) \qquad x(t_1) := 0_{n\times 1}.$$

Then we obtain a system with initial state $x(t_0)$ and final state $x(t_1)$, by assumption there exists $u(t)$ in finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1-1}$, such that $x(t_0)$ can be transferred to $x(t_1)$. By Theorem 3.8 we have,

$$x(t_1) = \widehat{y}_{A,\nu}(t_1,t_0)x(t_0) + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1+t_0-s-1,t_0)Bu(s)$$

$$0_{n\times 1} = \widehat{y}_{A,\nu}(t_1,t_0)[y(t_0) - [\widehat{y}_{A,\nu}(t_1,t_0)]^{-1}y(t_1)] + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1+t_0-s-1,t_0)Bu(s)$$

$$0_{n\times 1} = \widehat{y}_{A,\nu}(t_1,t_0)y(t_0) - y(t_1) + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1+t_0-s-1,t_0)Bu(s)$$

$$y(t_1) = \widehat{y}_{A,\nu}(t_1,t_0)y(t_0) + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1+t_0-s-1,t_0)Bu(s)$$

so for any given initial state $y(t_0)$ and any final state $y(t_1)$ there exists a control vector $u(t)$. This means that the given system is completely controllable. □

4.2 Observability

In this section we discuss the observability of the following linear discrete fractional system

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1), \quad t \in \mathbb{N}_{t_0+1}^{t_1} \\ z(t) = Cy(t) + Du(t) \end{cases} \tag{4.2.1}$$

where $z(t)$ is an $r \times 1$ the output vector, C is an $r \times n$ constant matrix, D is an $r \times m$ constant matrix and $A, B, y(\cdot), u(\cdot)$ are defined as in (4.1.1).

Suppose we are given $z(t)$ and $u(t)$ for $t \in \mathbb{N}_{t_0}^{t_1}$. We substitute the solution of state system (4.1.2) into the output measurement and we obtain

$$\begin{aligned} z(t) &= Cy(t) + Du(t) \\ &= C[\widehat{y}_{A,\nu}(t_1, t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)Bu(s)] + Du(t). \end{aligned}$$

Hence we have

$$C\widehat{y}_{A,\nu}(t_1, t_0)y_0 = z(t) - C\sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(t_1 + t_0 - s - 1, t_0)Bu(s) - Du(t).$$

Since A, B, C, D matrices and control vector $u(t)$ are given, the last two terms on the right-hand side of this equation are known quantities. Thus, we can subtract known terms from the observed value of output vector $z(t)$ and we define right-hand side by $z_1(t)$. Then response of the system (4.2.1) can be written as

$$C\widehat{y}_{A,\nu}(t_1, t_0)y_0 = z_1(t). \tag{4.2.2}$$

We say that a system is observable if we can measure or determine the state of the system based on its outputs. Now we present a formal definition of observability.

Definition 4.7. *The system (4.2.1) is said to be completely observable, if every state $y(t_0)$ can be uniquely determined from the observation of $z(t)$ over a finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1}$. Otherwise the system (4.2.1) or $\{A, C\}$ is said to be unobservable.*

To give necessary and sufficient conditions for observability of the system 4.2.1, we define the observability matrix and the observability Gramian matrix of the control system.

We define the *observability matrix* $\widehat{O}$ of this system as an $(nr) \times n$ matrix

$$\widehat{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Next, let us define *observability Gramian matrix* $\mathcal{R}(t, t_0)$ of the system (4.2.1) as an $n \times n$ matrix

$$\mathcal{R}(t, t_0) := \sum_{s=t_0}^{t-1} \widehat{y}_{A,\nu}(s, t_0)^T C^T C \widehat{y}_{A,\nu}(s, t_0).$$

Theorem 4.8. *The following statements are equivalent.*

(i) The system (4.2.1) is completely observable on $\mathbb{N}_{t_0}^{t_1}$.

(ii) The observability Gramian matrix $\mathcal{R}(t_1, t_0)$ has rank n.

(iii) The observability matrix $\widehat{O}$ has rank n.

Proof. $(i) \Longleftrightarrow (ii)$.

First we show that if the given system is completely observable, then $rank(\mathcal{R}(t_1,t_0)) = n$. We will prove the contrapositive, suppose $rank(\mathcal{R}(t_1,t_0)) < n$, then there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\mathcal{R}(t_1,t_0)\eta = 0_{n\times 1}$. Then we have

$$\begin{aligned}
0 &= \eta^T \mathcal{R}(t_1,t_0)\eta \\
&= \sum_{s=t_0}^{t_1-1} \eta^T \widehat{y}_{A,\nu}(s,t_0)^T C^T C \widehat{y}_{A,\nu}(s,t_0)\eta \\
&= \sum_{s=t_0}^{t_1-1} \|C\widehat{y}_{A,\nu}(s,t_0)\eta\|_2^2.
\end{aligned}$$

which implies $C\widehat{y}_{A,\nu}(t,t_0)\eta = 0_{r\times 1}$ for all $t \in \mathbb{N}_{t_0}^{t_1-1}$. Thus $y(t_0) = y_0 + \eta$ yields the same response for the system as $y(t_0) = y_0$ and contradicts the assumption that the given system is completely observable. Therefore $rank(\mathcal{R}(t_1,t_0)) = n$.

On the other hand, suppose the matrix $\mathcal{R}(t_1,t_0)$ has rank n. Multiplying both sides of (4.2.2) by $\widehat{y}_{A,\nu}(t,t_0)^T C^T$ and taking summation over the discrete interval $t \in \mathbb{N}_{t_0}^{t_1-1}$, we obtain

$$\begin{aligned}
\sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(s,t_0)^T C^T C \widehat{y}_{A,\nu}(s,t_0) y_0 &= \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(s,t_0)^T C^T z_1(s) \\
\mathcal{R}(t_1,t_0) y_0 &= \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(s,t_0)^T C^T z_1(s).
\end{aligned}$$

Since $rank(\mathcal{R}(t_1,t_0)) = n$, the matrix is invertible and

$$y_0 = \mathcal{R}(t_1,t_0)^{-1} \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,\nu}(s,t_0)^T C^T z_1(s).$$

Hence, the given system is completely observable.

$(i) \Leftrightarrow (iii)$.

Firstly, for all $N \geq n$ the rank of matrix

$$\begin{bmatrix} C & CA & CA^2 & \cdots & CA^{N-1} \end{bmatrix}^T$$

is equal to the rank of observability matrix $\widehat{O}$. The proof follows from the Cayley-Hamilton theorem and is similar to the controllability case. Here we assume $t_1 - t_0 = n$.

Assume that the system (4.2.1) is completely observable. Multiplying both sides of the state response (4.2.2) by $\widehat{y}_{A,\nu}(t,t_0)^T C^T$ and taking summation over the discrete interval $t \in \mathbb{N}_{t_0}^{t_0+n-1}$, we obtain

$$\begin{aligned}
\mathcal{R}(t_0+n,t_0)y_0 &= \sum_{s=t_0}^{t_0+n-1} \widehat{y}_{A,\nu}(s,t_0)^T C^T z_1(s) \\
&= \sum_{s=t_0}^{t_0+n-1} \sum_{\tau=t_0}^{s} \frac{[A^{\tau-t_0}]^T (s-\tau+1)^{\overline{(\tau-t_0+1)\nu-1}}}{\Gamma((\tau-t_0+1)\nu)} C^T z_1(s) \\
&= \sum_{\tau=t_0}^{t_0+n-1} [A^{\tau-t_0}]^T C^T \sum_{s=\tau}^{t_0+n-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)\nu-1}}}{\Gamma((\tau-t_0+1)\nu)} z_1(s),
\end{aligned}$$

where we interchanged order of the summation. Next, we define $G(\tau)$ for all $\tau \in \mathbb{N}_{t_0}^{t_0+n-1}$ by

$$G(\tau) = \sum_{s=\tau}^{t_0+n-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)\nu-1}}}{\Gamma((\tau-t_0+1)\nu)} z_1(s).$$

Substituting back $G(\tau)$ into the equality, we obtain

$$\mathcal{R}(t_0+n,t_0)y_0 = \sum_{s=t_0}^{t_0+n-1} [A^{\tau-t_0}]^T C^T G(\tau)$$

$$\mathcal{R}(t_0+n,t_0)y_0 = \begin{bmatrix} C^T & A^TC^T & (A^2)^TC^T & \cdots & (A^{n-1})^TC^T \end{bmatrix} \begin{bmatrix} G(0) \\ G(1) \\ G(2) \\ \vdots \\ G(n-1) \end{bmatrix} = \widehat{O}^T G_1(n). \tag{4.2.3}$$

Since the system is completely observable and $(i) \Leftrightarrow (ii)$, then $\mathcal{R}(t_0+n,t_0)$ has full rank n, thus $\mathcal{R}(t_0+n,t_0)y_0 \in \mathbb{R}^n$. Since $rank(\widehat{O}^T G_1(n)) \le rank(\widehat{O}^T)$ we have $\mathbb{R}^n \subseteq Im(\widehat{O}^T) \subseteq \mathbb{R}^n$. Therefore, $rank(\widehat{O}^T) = n = rank(\widehat{O})$.

Conversely, we show that if $rank(\widehat{O}) = n$, then the given system is completely observable. We assume to the contrary that the given system is unobservable. Since the given system is unobservable, by $(i) \Leftrightarrow (ii)$ the observability Gramian matrix has rank less than n, and there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T\mathcal{R}(t_0+n,t_0) = 0_{1\times n}$. Then we have

$$\begin{aligned} 0 &= \eta^T\mathcal{R}(t_0+n,t_0)\eta \\ &= \sum_{s=t_0}^{t_0+n-1} \eta^T\widehat{y}_{A,\nu}(s,t_0)^T C^T C\widehat{y}_{A,\nu}(s,t_0)\eta \\ &= \sum_{s=t_0}^{t_0+n-1} \|C\widehat{y}_{A,\nu}(s,t_0)\eta\|_2^2 \end{aligned}$$

which implies that

$$C\widehat{y}_{A,\nu}(t,t_0)\eta = 0_{r\times 1} \qquad for \quad all \quad t \in \mathbb{N}_{t_0}^{t_0+n-1}.$$

Now setting $t = t_0$ and using Lemma (3.7) (i) yields

$$C\eta = 0_{r\times 1}.$$

Applying the ν-th order fractional difference operator to both sides of the last equality and using Lemma 3.7 yield $CA\widehat{y}_{A,\nu}(t-1,t_0)\eta = 0_{r\times 1}$ for all $t \in \mathbb{N}_{t_0+1}^{t_0+n-1}$ and shifting each side one unit left we obtain

$$CA\widehat{y}_{A,\nu}(t,t_0)\eta = 0_{r\times 1} \quad for \quad all \quad t \in \mathbb{N}_{t_0}^{t_0+n-2}.$$

Setting $t = t_0$ and using Lemma 3.7 one has

$$CA\eta = 0_{r\times 1}.$$

Repeating the same step up to $n-1$ times, we have

$$CA^k\eta = 0 \qquad for \qquad k = 0,1,2,...,n-1.$$

Then

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\eta = \widehat{O}\eta = 0_{(rn)\times 1}.$$

This contradicts the assumption $rank(\widehat{O}) = n$. Therefore the observability Gramian matrix having full rank implies that the given system is completely observable. $\square$

We now provide some remarks regarding the observability Gramian matrix and the observability matrix.

Remark 4.9. *Note that, for every $\eta \in \mathbb{R}^n$*

$$\eta^T \mathcal{R}(t_1, t_0)\eta = \sum_{s=t_0}^{t_1-1} \|C\widehat{y}_{A,\nu}(s, t_0)\eta\|_2^2.$$

Therefore like the controllability Gramian matrix, the observability Gramian matrix $\mathcal{R}(t_1, t_0)$ is a non-negative symmetric matrix. In particular $\mathcal{R}(t_1, t_0)$ has rank n if there exits $c > 0$ such that

$$\eta^T \mathcal{R}(t_1, t_0)\eta \geq c$$

for all $\eta \in \mathbb{R}^n$.

Since the observability Gramian matrix is a non-negative symmetric matrix, in the statement of Theorem 4.3, (i) being equivalent to (ii) (i.e $(i) \Leftrightarrow (ii)$) can be interpreted as saying the given time-invariant linear nabla fractional difference system is completely observable on the given discrete time interval if and only if the observability Gramian matrix is positive definite (or the observability Gramian matrix is invertible) [24, 63].

Similarly, in the statement of Theorem 4.3, (i) being equivalent to (iii) (i.e $(i) \Leftrightarrow (iii)$) is called the Kalman Rank Condition for the observability of the system.

The following example illustrates the applicability of Theorem 4.8.

Example 4.10. *Consider the following system*

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = \begin{bmatrix} -1 & 1 & 1 \\ 5 & -9 & 1 \\ 6 & -3 & -1 \end{bmatrix} y(t-1) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t-1) \\ z(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} y(t) \end{cases},$$

where $0 < \nu < 1$.

Using Theorem 4.8, we get the observability matrix of the system

$$\widehat{O} = \begin{bmatrix} 1 & 0 & 1 \\ 5 & -2 & 0 \\ -15 & 23 & 3 \end{bmatrix}$$

whose has rank 3. Thus, by Theorem 4.8 given linear fractional order system is completely observable.

We continue with the remark by stating that the rank conditions for the dynamic systems on time scales, continuous fractional systems, and discrete fractional systems coincide.

Remark 4.11. *Let* $\mathbb{T}$ *be a time scale and* ν *be a real number such that* $0 < \nu < 1$. *The following control systems have same controllability and observability criteria. Consider the following systems:*

(i) The linear dynamic time-invariant system on $\mathbb{T}$

$$\begin{cases} y^{\Delta}(t) = Ay(t) + Bu(t), \quad t \in [t_0, t_1] \cap \mathbb{T}, \\ z(t) = Cy(t) + Du(t), \end{cases} \tag{4.2.4}$$

where $t_0, t_1 \in \mathbb{T}$.

(ii) The linear ∇*-discrete fractional time-invariant system*

$$\begin{cases} \nabla^{\nu}_{t_0} y(t) = Ay(t-1) + Bu(t-1), \quad t \in \mathbb{N}^{t_1}_{t_0+1}, \\ z(t) = Cy(t) + Du(t). \end{cases} \tag{4.2.5}$$

(iii) The linear Δ*-discrete fractional time invariant system*

$$\begin{cases} \Delta^{\nu}_{\nu-1} y(t) = Ay(t+\nu-1) + Bu(t+\nu-1), \quad t = 0, 1, 2, \dots. \\ z(t) = Cy(t) + Du(t), \quad t = \nu-1, \nu, \dots \end{cases} \tag{4.2.6}$$

(iv) The continuous fractional time-invariant system

$$\begin{cases} D^{\nu} y(t) = Ay(t) + Bu(t) \quad t \in [t_0, t_1], \\ z(t) = Cy(t) + Du(t). \end{cases} \tag{4.2.7}$$

The system (4.2.4) in [24], the system (4.2.5) in this paper, the system (4.2.6) in [7], and the last system (4.2.7) in [41] have the same controllability matrix

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

Additionally, observability also studied in the mentioned papers and all have the same *observability matrix*

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

4.3 Duality Principle

The notions of controllability and observability can be thought of as duals of one another, and so any theorems and concepts for controllability should have analogs in terms of observability and vice-versa. The duality principle is useful to prove any theorem for the observability of the systems by using the doctrines that we obtained for controllability and vice-versa. Thus in this section, we establish the connection between controllability and observability of the linear time-invariant fractional difference systems via the duality concept.

Theorem 4.12. *We consider that the following systems are defined on the discrete interval $t \in \mathbb{N}_{t_0+1}^{t_1}$ and $0 < \nu < 1$. The linear discrete fractional time-invariant system*

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1) \\ z(t) = Cy(t) + Du(t) \end{cases} \tag{4.3.1}$$

is completely controllable (observable) if and only if the linear discrete fractional time-invariant system

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = A^T y(t-1) + C^T u(t-1) \\ z(t) = B^T y(t) + D^T u(t) \end{cases} \tag{4.3.2}$$

is completely observable(controllable).

Proof. By Theorem 4.3 linear system (4.3.1) is completely controllable if and only if controllability matrix

$$\widehat{W} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

has rank n. Obviously this is true if and only if transpose matrix

$$\widehat{W}^T = \begin{bmatrix} B^T \\ B^T A^T \\ B^T (A^2)^T \\ \cdots \\ B^T (A^{n-1})^T \end{bmatrix}.$$

has rank n. Since this transpose matrix is the observability matrix of linear system(4.3.2), $\widehat{W}^T = \widehat{O}$ and Theorem 4.8 yields that indeed if and only if linear system (4.3.2) is completely observable. Similarly, one can easily prove that linear system (4.3.1) is completely observable if and only if linear system (4.3.2) is completely controllable. $\square$

Chapter 5

CONTROLLABILITY AND OBSERVABILITY OF LINEAR TIME-VARIANT NABLA FRACTIONAL SYSTEMS

Control theory, a branch of the systems theory, deals with behaviors of inputs and outputs of a given system. Controllability and observability are two fundamental concepts in the control theory. A system is said to be controllable if it can be transferred from any initial state to any arbitrary final state under the control vector of the system. A system is observable if we can measure or determine the state of the system based on its outputs. The controllability and observability became particularly important for practical implementations after Kalman [37] introduced the rank conditions. There exist many papers in the literature in which these two concepts are investigated for the linear/non-linear time-invariant continuous or discrete-time systems [9, 14, 15, 19, 20, 24, 35, 37, 38, 51]. The research in this area has been developed in two directions: (i) a study on the time-invariant systems and (ii) a study on the time-variant systems.

The time-variant system refers to a system such that output and input parameters are depending on the time. Almost all systems, which model real world problems are time-varying systems. However, the time-varying system may be tough to be satisfied by the controllability and observability conditions or to be shown whether it is stable or not, due to difficulties in computing its solution. Thus, many scholars are trying to approximate the actual plant by a simple model using available techniques for the linear time-invariant continuous or discrete-time systems. Even though such corresponding models may present a good approximation for some actual plants, there are some time-variant systems, which cannot be modeled by assuming that they are time-invariant such as the aerodynamic coefficients of aircraft, the earth's thermodynamic response to incoming solar irradiance, the circuit parameters in electronic circuits and the human vocal track. Therefore, and by the advance of technology all these are motivating us to have more detailed study on the linear time-variant control systems.

In contrast to the time-invariant systems, the amount of literature which focuses on the controllability and observability of linear time-variant systems [20, 24, 26, 58, 60, 61, 62] is much less. Tsakalis and Ioannou [60] have investigated the continuous time systems and introduced the analysis and design techniques for linear time-invariant systems, while Silverman and Meadow [61] introduced the test for the controllability and observability of time-varying systems. This test is a generalization of the familiar Wronskian determinant test for scalar functions. Rumchev and Adeane [58] established the necessary and sufficient conditions for the null-controllability, reachability, and controllability of the time-variant discrete-time positive linear systems, while Engwerda [26] commonly investigated the reachability, output-controllability and target path controllability of the linear discrete time-varying systems. Much more solid work were done by Weiss [62], who not only studied the controllability and observability of the linear time-varying discrete systems but also investigated the sufficient conditions for the local controllability of the non-linear discrete-time systems and obtained the necessary and sufficient conditions for the Lyapunov stability and stability of the non-linear difference systems.

The discrete fractional calculus is a new field for researchers. By interpretation of computers to signal processing, which deals with a tremendous amount of information expressed by discrete numbers, the study of discrete-time systems is becoming practical and promising. Thus in recent years, the discrete fractional calculus [6, 32] has been developed as a counterpart to the classical integer-order calculus.

Motivated by the importance of the study of the linear time-variant control systems and the descriptive power of the fractional order systems in real world problems, we propose the following discrete fractional time-variant system in order to investigate controllability and observability:

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = A(t)y(t-1) + B(t)u(t-1) \\ z(t) = C(t)y(t) + D(t)u(t) \end{cases}$$

where ν is positive real number such that $0 < \nu < 1$, $y(t)$ is an $n \times 1$ state vector of the system, $u(t)$ is an $m \times 1$ control input, $z(t)$ is an $r \times 1$ output vector, $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are an $n \times n$, an $n \times m$, an $r \times n$, and an $r \times m$ matrix valued functions, respectively.

To the best of our knowledge there are no results relating to the study of the above system in the literature.

This chapter is organized as follows. In Section 5.1, we define the state transition matrix of the fractional difference system (5.1.1)-(5.1.2) with the Riemann-Liouville fractional difference operator. Then we state the basic properties of the state transition matrix. Section 5.2 deals with controllability of the linear time-varying fractional difference system, while observability of the control system is given in the final Section 5.3.

5.1 The State Transition Matrix

In this section, we consider the following initial value problem (IVP)

$$\nabla_a^{\nu} y(t) = A(t)y(t-1) \quad for \quad t = a+1, a+2, \ldots \tag{5.1.1}$$

$$\nabla_a^{-(1-\nu)} y(t)|_{t=a} = y(a) = y_0. \tag{5.1.2}$$

First we define the state transition matrix $\Phi_{A(\cdot),\nu}(t,a)$ of the initial value problem (5.1.1)-(5.1.2). Later, we give fundamental properties of the state transition matrix, and we present a unique solution of the initial value problem (5.1.1)-(5.1.2). Finally, to prove necessary and sufficient conditions for the controllability and observability of the system we state and prove the variation of constants formula.

Now we present a precise definition of a state transition matrix.

Definition 5.1. *Let a be any real number and ν be positive number such that $\nu \in (0,1)$. The state transition matrix $\Phi_{A(\cdot),\nu}(t,a)$ of the system (5.1.1) is defined by the unique solution of the following fractional order initial value problem (IVP)*

$$\nabla_a^{\nu}\Phi_{A(\cdot),\nu}(t,a) = A(t)\Phi_{A(\cdot),\nu}(t-1,a) \tag{5.1.3}$$

$$\Phi_{A(\cdot),\nu}(a,a) = I_n, \tag{5.1.4}$$

where $A(t)$ is an $n \times n$ matrix and I_n is an $n \times n$ identity matrix.

The state transition matrix refers to a matrix whose product with the state vector at an initial time gives the state vector at a later time. It plays a vital role in the time-varying control systems. Adamec [2] gave an explicit formula for the generalization matrix exponential for the dynamic systems derived by restricting a principal fundamental matrix of the system. Then, DaCunna [23] investigated the state transition matrix via the Peano-Baker series on time scales has virtually no restriction on the system matrix $A(t)$, while Zhang et al. [64] studied the state transition matrix of linear time-varying fractional differential systems via the Caputo derivative operator. We refer the reader to [17, 25, 28, 52, 53] for further readings on the state transition matrix and the Peano-baker series. To the best of our knowledge, the state transition matrix of the linear time-varying fractional difference systems has not been reported in the literature.

The state transition matrix of the given fractional order initial value problem is expressed as

$$\Phi_{A(\cdot),\nu}(t,a) := \sum_{k=0}^{\infty} \mathcal{I}_{t,a,\nu}^{(k)}, \tag{5.1.5}$$

where

$$\mathcal{I}_{t,a,\nu}^{(0)} := \frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)} I_n,$$

$$\mathcal{I}_{t,a,\nu}^{(1)} := \nabla_{a+1}^{-\nu} A(t) \mathcal{I}_{t-1,a,\nu}^{(0)} = \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} A(s) \frac{(s-a)^{\overline{\nu-1}}}{\Gamma(\nu)} I_n$$

and

$$\mathcal{I}_{t,a,\nu}^{(k)} := \nabla_{a+1}^{-\nu} A(t) \mathcal{I}_{t-1,a,\nu}^{(k-1)}.$$

In this definition we define for $k < 0$ as

$$\mathcal{I}_{t,a,\nu}^{(k)} := O_n$$

We continue with a lemma, which will be useful to prove the properties of the state transition matrix.

Lemma 5.2. *Let I_n and O_n be an identity and a zero square matrice with dimension n respectively and a be any real number and $\nu \in (0,1)$. The following identities hold:*

(i) $\nabla_a^{\nu} \mathcal{I}_{t,a,\nu}^{(0)} = O_n$.

(ii) $\nabla_a^{\nu} \mathcal{I}_{t,a,\nu}^{(k)} = A(t) \mathcal{I}_{t-1,a,\nu}^{(k-1)}$ *for* $k = 1, 2, 3, \ldots$.

(iii) $\mathcal{I}_{a,a,\nu}^{(0)} = I_n$.

(iv) $\mathcal{I}_{a,a,\nu}^{(k)} = O_n$ *for* $k = 1, 2, 3, \ldots$.

Proof. (i) Using the definition of the nabla fractional difference we obtain,

$$\nabla_a^{\nu}\mathcal{I}_{t,a,\nu}^{(0)} = \nabla\nabla_a^{-(1-\nu)}\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}I_n.$$

Applying (ii) of Lemma 2.1 and using the fact that nabla difference of the constant is zero, we obtain

$$\nabla_a^{\nu}\mathcal{I}_{t,a,\nu}^{(0)} = \nabla I_n = O_n.$$

(ii) First, we rewrite the $\nabla_a^{\nu}\mathcal{I}_{t,a,\nu}^{(k)}$ in term of the sum and then we use the definition of the nabla fractional difference operator, and we have

$$\begin{aligned}
\nabla_a^{\nu}\mathcal{I}_{t,a,\nu}^{(k)} &= \nabla_a^{\nu}\nabla_{a+1}^{-\nu}A(t)\mathcal{I}_{t-1,a,\nu}^{(k-1)} \\
&= \nabla\nabla_a^{-(1-\nu)}\sum_{s=a+1}^{t}\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}A(s)\mathcal{I}_{s-1,a,\nu}^{(k-1)} \\
&= \nabla\sum_{\tau=a}^{t}\frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)}\sum_{s=a+1}^{\tau}\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}A(s)\mathcal{I}_{s-1,a,\nu}^{(k-1)}.
\end{aligned}$$

Next, we interchange the order of sums and obtain

$$= \nabla\sum_{s=a+1}^{t}\sum_{\tau=s}^{t}\frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)}\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}A(s)\mathcal{I}_{s-1,a,\nu}^{(k-1)}.$$

Hence, we obtain

$$\begin{aligned}
\sum_{\tau=s}^{t}\frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)}\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} &= \nabla_s^{-(1-\nu)}\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= 1
\end{aligned}$$

by using (ii) of Lemma 2.1.

Hence, we perform the nabla difference of sum and we obtain

$$\begin{aligned}\nabla_a^{\nu}\mathcal{I}^{(k)}_{t,a,\nu} &= \nabla \sum_{s=a+1}^{t} A(s)\mathcal{I}^{(k-1)}_{s-1,a,\nu} \\ &= A(t)\mathcal{I}^{(k-1)}_{t-1,a,\nu}.\end{aligned}$$

(iii) The proof follows from the definition of the rising factorial power.

(iv) Using the definition of $\mathcal{I}^{(k)}_{t,a,\nu}$ we get,

$$\begin{aligned}\mathcal{I}^{(k)}_{a,a,\nu} &= \nabla_{a+1}^{-\nu} A(t)\mathcal{I}^{(k-1)}_{t-1,a,\nu}\big|_{t=a} \\ &= \sum_{s=a+1}^{a} A(s)\mathcal{I}^{(k-1)}_{s-1,a,\nu} = O_n,\end{aligned}$$

using the assumption that if the upper bound of the sum is less than the lower bound, then the quantity is identically equal to zero. □

Now we discuss the convergence of the state transition matrix. Let $T \in \mathbb{N}_a$ and $A(t)$ be a matrix valued function defined on the discrete finite interval $\mathbb{N}_a^{a+T}$. Thus for each $t \in \mathbb{N}_a^{a+T}$, we have that $\|A(t)\|$ is bounded, where $\|\cdot\|$ denotes the spectral norm. Define

$$\alpha := \max_{t \in \mathbb{N}_a^{a+T}} \|A(t)\|.$$

Theorem 5.3. *Let a be any real number and $A(t)$ be an $n \times n$ matrix valued function defined on $\mathbb{N}_a$. Then the following are valid:*

(i) The following series

$$\Phi_{A(\cdot),\nu}(t,a) = \sum_{k=0}^{\infty} \mathcal{I}^{(k)}_{t,a,\nu} \tag{5.1.6}$$

is uniformly convergent on $\mathbb{N}_a$.

(ii) $\|\Phi_{A(\cdot),\nu}(t,a)\| \le (1+\alpha)^{t-a}$.

Proof. (i) First, we prove the following inequality.

For any given $s \in \mathbb{N}_a$,

$$0 < \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \leq 1 \quad for \quad t-s \in \mathbb{N}_0, \tag{5.1.7}$$

where $\nu \in (0,1)$. If $t = s$, then $\dfrac{(s-(s-1))^{\overline{\nu-1}}}{\Gamma(\nu)} = 1$. Next, we apply the nabla difference to $\dfrac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}$ and we obtain

$$\nabla_t \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} = \frac{(t-s+1)^{\overline{\nu-2}}}{\Gamma(\nu-1)} = \frac{\Gamma(t-s+\nu-1)}{\Gamma(t-s+1)\Gamma(\nu-1)} = \frac{(\nu-1)^{\overline{t-s}}}{(t-s)!} < 0,$$

since $t-s \geq 0$ and $\nu \in (0,1)$. Therefore the given quantity is monotone decreasing on the given interval and using the definition of the rising factorial power we have

$$\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} = \frac{\Gamma(t-s+\nu)}{\Gamma(t-s+1)\Gamma(\nu)} = \frac{(\nu)^{\overline{t-s}}}{(t-s)!} > 0,$$

since $t-s \geq 0$ and $\nu \in (0,1)$. Thus, for any given $s \in \mathbb{N}_a$ we have

$$0 < \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \leq 1 \quad for \quad t-s \in \mathbb{N}_0.$$

Now, using the definition of $\mathcal{I}^{(0)}_{t,a,\nu}$ and inequality (5.1.7) and we obtain

$$||\mathcal{I}^{(0)}_{t,a,\nu}|| = ||\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)} I_n|| \leq 1.$$

Following this we obtain

$$||\mathcal{I}^{(1)}_{t,a,\nu}|| = ||\sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} A(s) \frac{(s-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)} I_n|| \leq \sum_{s=a+1}^{t} ||A(s)|| \leq \alpha(t-a)$$

for $t \in \mathbb{N}_a$. Hence we have

$$\begin{aligned}
\|\mathcal{I}^{(2)}_{t,a,\nu}\| &= \| \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} A(s)\mathcal{I}^{(1)}_{s-1,a,\nu}\| \\
&\le \sum_{s=a+1}^{t} \|A(s)\|\alpha(s-1-a) \\
&\le \sum_{s=a+1}^{t} \alpha^2(s-1-a) \\
&= \frac{\alpha^2(t-1-a)^{\overline{2}}}{2} \\
&\le \frac{\alpha^2(t-a)^2}{2}.
\end{aligned}$$

We can proceed with mathematical induction principle to obtain,

$$\begin{aligned}
\|\mathcal{I}^{(k+1)}_{t,a,\nu}\| &= \| \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} A(s)\mathcal{I}^{(k)}_{s-1,a,\nu}\| \\
&\le \sum_{s=a+1}^{t} \|A(s)\|\alpha^k \frac{(s-k-a)^{\overline{k}}}{k!} \\
&\le \alpha^{k+1} \frac{(t-k-a)^{\overline{k+1}}}{(k+1)!} \\
&\le \alpha^{k+1} \frac{(t-a)^{k+1}}{(k+1)!}, \quad k = 0, 1, 2, 3, 4, \ldots,
\end{aligned}$$

where $t \in \mathbb{N}_a$. For any $T \in \mathbb{N}_a$ and $t \in \mathbb{N}_a^T$ we have

$$\|\mathcal{I}^{(k)}_{t,a,\nu}\| \le \alpha^k \frac{(t-a)^k}{k!} \le \alpha^k \frac{T^k}{k!}, \quad for \quad k = 0, 1, 2, 3, 4, \ldots.$$

Since the series $\sum_{k=0}^{\infty} \alpha^k \frac{T^k}{k!}$ is convergent, the Weierstrass M-Test implies that the series $\Phi_{A(\cdot),\nu}(t,a)$ converges uniformly on the given discrete interval $\mathbb{N}_a^{a+T}$. Since it holds for any $T \in \mathbb{N}_a$, then the given series is convergent on $\mathbb{N}_a$.

(ii) Since $\dfrac{1}{\Gamma(t)} = 0$ for all $t \in \{\cdots, -2, -1, 0\}$

$$\|\mathcal{I}^{(k)}_{t,a,\nu}\| \leq \alpha^k \frac{(t-k-a+1)^{\overline{k}}}{(k)!} = 0, \quad for \quad all \quad k > t-a.$$

Therefore, we have

$$\|\Phi_{A(\cdot),\nu}(t,a)\| \leq \sum_{k=0}^{t-a} \alpha^k \frac{(t-k-a+1)^{\overline{k}}}{\Gamma(k+1)} = (1+\alpha)^{t-a}.$$

□

Subsequently, we give the properties of the state transition matrix.

Lemma 5.4. *Let a be any real number and $A(t)$ be an $n \times n$ matrix valued function defined on $\mathbb{N}_a$. The following properties hold:*

(i) $\nabla_a^\nu \Phi_{A(\cdot),\nu}(t,a) = A(t)\Phi_{A(\cdot),\nu}(t-1,a)$.

(ii) $\Phi_{A(\cdot),\nu}(a,a) = I_n$.

(iii) (Composition) For any given t_0, t_1, and for all t

$$\Phi_{A(\cdot),\nu}(t,t_0) = \Phi_{A(\cdot),\nu}(t,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0).$$

(iv) (Inverse) $\Phi_{A(\cdot),\nu}(t,t_0)$ is a non-singular matrix and

$$[\Phi_{A(\cdot),\nu}(t,t_0)]^{-1} = \Phi_{A(\cdot),\nu}(t_0,t).$$

Proof. (i) Using the linearity property of the nabla fractional difference operator and

$(i),(ii)$ of Lemma 2.1, we get

$$\begin{aligned}\nabla_a^\nu \Phi_{A(\cdot),\nu}(t,a) &= \nabla_a^\nu \sum_{k=0}^{\infty} \mathcal{I}_{t,a,\nu}^{(k)} \\ &= \sum_{k=0}^{\infty} \nabla_a^\nu \mathcal{I}_{t,a,\nu}^{(k)} \\ &= \sum_{k=0}^{\infty} A(t) \mathcal{I}_{t,a,\nu}^{(k-1)} \\ &= A(t)\Phi_{A(\cdot),\nu}(t-1,a).\end{aligned}$$

(ii) The proof is consequences of the (iii) and (iv) of Lemma 5.2.

(iii) Let $y_1(t) := \Phi_{A(\cdot),\nu}(t,t_0)$ and $y_2(t) := \Phi_{A(\cdot),\nu}(t,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0)$, then

$$\begin{aligned}\nabla_a^\nu y_1(t) &= \nabla_a^\nu \Phi_{A(\cdot),\nu}(t,t_0) = A(t)\Phi_{A(\cdot),\nu}(t-1,t_0) = A(t)y_1(t-1) \\ \nabla_a^\nu y_2(t) &= \nabla_a^\nu \Phi_{A(\cdot),\nu}(t,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0) = A(t)\Phi_{A(\cdot),\nu}(t-1,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0) \\ &= A(t)y_2(t-1)\end{aligned}$$

and

$$y_2(t_1) = \Phi_{A(\cdot),\nu}(t_1,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0) = I_n\Phi_{A(\cdot),\nu}(t_1,t_0) = \Phi_{A(\cdot),\nu}(t_1,t_0) = y_1(t_1).$$

Thus, $y_1(t)$ and $y_2(t)$ satisfy the same fractional difference equation and the same initial condition. Hence, we have

$$\Phi_{A(\cdot),\nu}(t,t_0) = \Phi_{A(\cdot),\nu}(t,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0).$$

(iv) By contradiction, suppose there exists a $t_1 \neq t_0$ such that $\Phi_{A(\cdot),\nu}(t_1,t_0)$ is a singular or $det[\Phi_{A(\cdot),\nu}(t_1,t_0)] = 0$. Then using the composition property of the state transition matrix

$$det[\Phi_{A(\cdot),\nu}(t,t_0)] = det[\Phi_{A(\cdot),\nu}(t,t_1)]det[\Phi_{A(\cdot),\nu}(t_1,t_0)] = 0$$

for all t. In particular

$$det[\Phi_{A(\cdot),\nu}(t_0,t_0)] = 0,$$

which contradicts with the fact that $\Phi_{A(\cdot),\nu}(t_0,t_0) = I_n$. Thus, the state transition matrix is a non-singular and

$$\Phi_{A(\cdot),\nu}(t_0,t_0) = I_n = \Phi_{A(\cdot),\nu}(t_0,t_1)\Phi_{A(\cdot),\nu}(t_1,t_0).$$

Hence, we have

$$[\Phi_{A(\cdot),\nu}(t,t_0)]^{-1} = \Phi_{A(\cdot),\nu}(t_0,t).$$

□

Next, we give the following theorem and the proof follows from Lemma 5.4.

Theorem 5.5. *The unique solution of the initial value problem for the nabla fractional difference equation (5.1.1)-(5.1.2) is*

$$y(t) = \Phi_{A(\cdot),\nu}(t,a)y_0.$$

Let us demonstrate the explicit form of the state transition matrix with the following example.

Example 5.6. *Consider the matrix valued function*

$$A(t) = \begin{bmatrix} \frac{1}{2} & t \\ 0 & \frac{1}{2} \end{bmatrix},$$

where $t \in \mathbb{N}_0$ *and* $a = 0$.

First, we obtain the $\mathcal{I}^{(k)}_{t,0,\nu}$ *for* $k = 0, 1, 2, 3, \ldots$.

$$\mathcal{I}^{(0)}_{t,0,\nu} = \begin{bmatrix} \frac{(t+1)^{\overline{\nu-1}}}{\Gamma(\nu)} & 0 \\ 0 & \frac{(t+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \end{bmatrix},$$

$$\mathcal{I}^{(1)}_{t,0,\nu} = \nabla_1^{-\nu} \begin{bmatrix} \frac{1}{2} & t \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{t^{\overline{\nu-1}}}{\Gamma(\nu)} & 0 \\ 0 & \frac{t^{\overline{\nu-1}}}{\Gamma(\nu)} \end{bmatrix} = \begin{bmatrix} \frac{t^{\overline{2\nu-1}}}{2\Gamma(2\nu)} & \nabla_1^{-\nu} t \frac{t^{\overline{\nu-1}}}{\Gamma(\nu)} \\ 0 & \frac{t^{\overline{2\nu-1}}}{2\Gamma(2\nu)} \end{bmatrix},$$

$$\mathcal{I}^{(2)}_{t,0,\nu} = \nabla_1^{-\nu} \begin{bmatrix} \frac{1}{2} & t \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{(t-1)^{\overline{2\nu-1}}}{2\Gamma(2\nu)} & \nabla_1^{-\nu} \frac{(t-1)(t-1)^{\overline{\nu-1}}}{\Gamma(\nu)} \\ 0 & \frac{(t-1)^{\overline{2\nu-1}}}{2\Gamma(2\nu)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(t-1)^{\overline{3\nu-1}}}{4\Gamma(3\nu)} & \nabla_1^{-2\nu} \frac{(t-1)(t-1)^{\overline{\nu-1}}}{2\Gamma(\nu)} + \nabla_1^{-\nu} t \frac{t^{\overline{2\nu-1}}}{2\Gamma(2\nu)} \\ 0 & \frac{(t-1)^{\overline{3\nu-1}}}{4\Gamma(3\nu)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(t-1)^{\overline{3\nu-1}}}{4\Gamma(3\nu)} & \frac{1}{2} \sum_{s=1}^{2} \nabla_1^{-(3-s)\nu} (t-2+s) \frac{(t-1)^{\overline{s\nu-1}}}{\Gamma(s\nu)} \\ 0 & \frac{(t-1)^{\overline{3\nu-1}}}{4\Gamma(3\nu)} \end{bmatrix}.$$

We can proceed with the mathematical induction principle to have,

$$\mathcal{I}^{(k)}_{t,0,\nu} = \begin{bmatrix} \frac{(t-k+1)^{\overline{(k+1)\nu-1}}}{2^k\Gamma((k+1)\nu)} & \frac{1}{2^{k-1}} \sum_{s=1}^{k} \nabla_1^{-(k-s+1)\nu} (t-k+s) \frac{(t-k+1)^{\overline{s\nu-1}}}{\Gamma(s\nu)} \\ 0 & \frac{(t-k+1)^{\overline{(k+1)\nu-1}}}{2^k\Gamma((k+1)\nu)} \end{bmatrix},$$

where $k = 0, 1, 2, 3, \ldots$. *Hence, we obtain the state transition matrix,*

$$\Phi_{A(\cdot),\nu}(t,0) = \begin{bmatrix} \sum_{k=0}^{t} \frac{(t-k+1)^{\overline{(k+1)\nu-1}}}{2^k\Gamma((k+1)\nu)} & \sum_{k=0}^{\infty} \frac{1}{2^{k-1}} \sum_{s=1}^{k} \nabla_1^{-(k-s+1)\nu} (t-k+s) \frac{(t-k+1)^{\overline{s\nu-1}}}{\Gamma(s\nu)} \\ 0 & \sum_{k=0}^{t} \frac{(t-k+1)^{\overline{(k+1)\nu-1}}}{2^k\Gamma((k+1)\nu)} \end{bmatrix}.$$

Here we have

$$\widehat{y}_{\frac{1}{2},\nu}(t,0) = \sum_{k=0}^{t} \frac{(t-k+1)^{\overline{(k+1)\nu-1}}}{2^k\Gamma((k+1)\nu)}$$

as given in the paper [10].

We now provide some remarks regarding the state transition matrix.

Remark 5.7. *The representation of the state transition matrix (5.1.5) is known as the Peano-Baker series in control theory. Although the given form of the state transition matrix is important in theory, in practice it might be quite difficult (or even impossible) to calculate in many cases, even for simple linear fractional difference control systems.*

Corollary 5.8. *Assume that $A(t) = A$ is a constant matrix. Then the state transition matrix for the fractional difference equation (5.1.1)-(5.1.2) is*

$$\Phi_{A,\nu}(t,a) = \widehat{y}_{A,\nu}(t,a)$$

where,

$$\widehat{y}_{A,\nu}(t,a) = \sum_{n=a}^{t} \frac{A^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)}.$$

Proof. By definition of $\Phi_{A,\nu}(t,a)$ we have

$$\mathcal{I}^{(m)}_{t,a,\nu} = \nabla^{-\nu}_{a+1} A \mathcal{I}^{(m-1)}_{t-1,a,\nu}.$$

For $m = 0$,

$$\mathcal{I}^{(0)}_{t,a,\nu} = \frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)} I_n$$

For $m = 1$, using the definition of transition matrix and rising factorial power we obtain,

$$
\begin{aligned}
\mathcal{I}_{t,a,\nu}^{(1)} =& \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} A \frac{(s-a)^{\overline{\nu-1}}}{\Gamma(\nu)} I_n \\
=& A \sum_{s=a+1}^{t} \frac{\Gamma(t-s+\nu)\Gamma(s-a-1+\nu)}{\Gamma(\nu)\Gamma(t-s+1)\Gamma(s-a)\Gamma(\nu)} \\
=& A \sum_{s=0}^{t-a-1} \frac{\Gamma(t-a-1-s+\nu)\Gamma(s+\nu)}{\Gamma(\nu)\Gamma(t-s-a)\Gamma(s+1)\Gamma(\nu)} \\
=& A \sum_{s=0}^{t-a-1} \binom{t-a-1}{s} \frac{\Gamma(t-a-1-s+\nu)\Gamma(s+\nu)}{\Gamma(\nu)\Gamma(\nu)\Gamma(t-a)} \\
=& \frac{A}{\Gamma(t-a)} \sum_{s=0}^{t-a-1} \binom{t-a-1}{s} (\nu)^{\overline{s}} (\nu)^{\overline{t-a-1-s}} \\
=& \frac{A}{\Gamma(t-a)} (2\nu)^{\overline{t-a-1}} \\
=& \frac{A(t-a)^{\overline{2\nu-1}}}{\Gamma(2\nu)}
\end{aligned}
$$

where we used the formula $\binom{t}{r} = \frac{\Gamma(t+1)}{\Gamma(r+1)\Gamma(t-r+1)}$. Suppose for $m = k$ we have

$$
\mathcal{I}_{t,a,\nu}^{(k)} = A^k \frac{(t-a-k+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}.
$$

When $m = k+1$, the proof follows similar techniques as above and we have

$$\begin{aligned}
\mathcal{I}^{(k+1)}_{t,a,\nu} &= \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} AA^k \frac{(t-a-k+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} \\
&= A^{k+1} \sum_{s=k+a+1}^{t} \frac{\Gamma(t-s+\nu)\Gamma(s-a-k-1+(k+1)\nu)}{\Gamma(\nu)\Gamma(t-s+1)\Gamma(s-a-k)\Gamma((k+1)\nu)} \\
&= A^{k+1} \sum_{s=0}^{t-a-k-1} \frac{\Gamma(t-a-k-1-s+\nu)\Gamma(s+(k+1)\nu)}{\Gamma(\nu)\Gamma(t-s-a-k)\Gamma(s+1)\Gamma((k+1)\nu)} \\
&= A^{k+1} \sum_{s=0}^{t-a-k-1} \binom{t-a-k-1}{s} \frac{\Gamma(t-a-k-1-s+\nu)\Gamma(s+(k+1)\nu)}{\Gamma(\nu)\Gamma((k+1)\nu)\Gamma(t-a-k)} \\
&= A^{k+1} \frac{1}{\Gamma(t-a-k)} \sum_{s=0}^{t-a-k-1} \binom{t-a-k-1}{s} ((k+1)\nu)^{\overline{s}} (\nu)^{\overline{t-a-k-1-s}} \\
&= A^{k+1} \frac{((k+2)\nu)^{\overline{t-a-k-1}}}{\Gamma(t-a-k)} \\
&= A^{k+1} \frac{(t-a-k)^{\overline{(k+2)\nu-1}}}{\Gamma((k+2)\nu)}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\Phi_{A(\cdot),\nu}(t,a) &= \sum_{k=0}^{\infty} \mathcal{I}^{(k)}_{t,a,\nu} \\
&= \sum_{k=0}^{\infty} A^k \frac{(t-a-k+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} \\
&= \sum_{k=0}^{t-a} A^k \frac{(t-a-k+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} \\
&= \sum_{k=a}^{t} A^{k-a} \frac{(t-k+1)^{\overline{(k-a+1)\nu-1}}}{\Gamma((k-a+1)\nu)} = \widehat{y}_{A,\nu}(t,a)
\end{aligned}$$

$\square$

We conclude this section with the following additional preliminary result, which is essential to discuss the necessary and sufficient conditions for the controllability and observability of the fractional difference systems in Section 5.2 and Section 5.3.

Theorem 5.9. *(Variation of Constants)*
The fractional difference equation of order ν, where $\nu \in (0,1)$

$$\nabla_a^\nu y(t) = A(t)y(t-1) + f(t-1)$$

has general solution

$$y(t) = \Phi_{A(\cdot),\nu}(t,a)c + \sum_{s=a}^{t-1} \Phi_{A(\cdot),\nu}(t,s+1)f(s)$$

where c is constant and $\Phi_{A(\cdot),\nu}(t,a)$ is the state transition matrix for the fractional difference equation $\nabla_a^\nu y(t) = A(t)y(t-1)$.

Proof. A direct substitution gives that $\sum_{s=a}^{t-1} \Phi_{A(\cdot),\nu}(t,s+1)f(s)$ is the particular solution of the given fractional difference equation

$$\underbrace{\nabla_a^\nu \sum_{s=a}^{t-1} \Phi_{A(\cdot),\nu}(t,s+1)f(s)}_{I} = A(t)(\sum_{s=a}^{t-2} \Phi_{A(\cdot),\nu}(t-1,s+1)f(s) + f(t-1).$$

Using the definition of the nabla fractional difference operator we have

$$\begin{aligned} I &= \nabla\nabla_a^{-(1-\nu)} \sum_{s=a}^{t-1} \Phi_{A(\cdot),\nu}(t,s+1)f(s) \\ &= \nabla \sum_{\tau=a}^{t} \frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)} \sum_{s=a}^{\tau-1} \Phi_{A(\cdot),\nu}(\tau,s+1)f(s). \end{aligned}$$

Next, we interchange the order of sums and obtain

$$I = \nabla \sum_{s=a}^{t-1} \sum_{\tau=s+1}^{t} \frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)} \Phi_{A(\cdot),\nu}(\tau, s+1) f(s).$$

We continue by applying the following rule to the above expression

$$\nabla \sum_{s=a}^{t-1} f(t,s) = \sum_{s=a}^{t-2} \nabla f(t,s) + f(t,t-1).$$

Hence, we have

$$\begin{aligned} I &= \sum_{s=a}^{t-2} \nabla \sum_{\tau=s+1}^{t} \frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)} \Phi_{A(\cdot),\nu}(\tau, s+1) f(s) \\ &\quad + \sum_{\tau=s+1}^{t} \frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)} \Phi_{A(\cdot),\nu}(\tau, s+1) f(s)\big|_{t=t,s=t-1} \\ &= \sum_{s=a}^{t-2} \nabla \sum_{\tau=s+1}^{t} \frac{(t-\rho(\tau))^{\overline{-\nu}}}{\Gamma(1-\nu)} \Phi_{A(\cdot),\nu}(\tau, s+1) f(s) + f(t-1) \end{aligned}$$

since $\Phi_{A(\cdot),\nu}(a,a) = I_n$.

Using the definition of the nabla fractional difference operator and (i) of Lemma 5.4 we obtain

$$\begin{aligned} I &= \sum_{s=a}^{t-2} \nabla \nabla_a^{-(1-\nu)} \Phi_{A(\cdot),\nu}(t, s+1) f(s) + f(t-1) \\ &= \sum_{s=a}^{t-2} \nabla_a^{\nu} \Phi_{A(\cdot),\nu}(t, s+1) f(s) + f(t-1) \\ &= A(t) \sum_{s=a}^{t-2} \Phi_{A(\cdot),\nu}(t-1, s+1) f(s) + f(t-1). \end{aligned}$$

We use Theorem 5.5 to complete the proof. □

5.2 Controllability

Throughout this section we assume $t_0, t_1 \in \mathbb{R}^+$ and $t_1 - t_0 \in \mathbb{Z}^+$. This section focuses on the controllability of the linear time-variant discrete fractional system,

$$\nabla_{t_0}^{\nu} y(t) = A(t)y(t-1) + B(t)u(t-1), \quad t \in \mathbb{N}_{t_0+1}^{t_1}, \tag{5.2.1}$$

where ν is a real number such that $0 < \nu < 1$, $y(t_0) = y_0$ is an initial state, $y(t)$ is an $n \times 1$ state vector of the system, $u(t)$ is an $m \times 1$ control input, $A(t)$ and $B(t)$ are an $n \times n$ and an $n \times m$, valued functions, respectively.

We give necessary and sufficient conditions for a linear time-varying discrete fractional control system to be controllable. By Theorem 5.9 the corresponding solution of system (5.2.1) is

$$y(t) = \Phi_{A(\cdot),\nu}(t, t_0)y_0 + \sum_{s=t_0}^{t-1} \Phi_{A(\cdot),\nu}(t_1, s+1)B(s+1)u(s). \tag{5.2.2}$$

Definition 5.10. *A system modeled by (5.2.1) or pair $\{A, B\}$ is said to be completely controllable, if it is possible to construct control signal $u(t)$ that will transfer any initial state $y(t_0)$ to any final state $y(t_1)$ in a finite discrete time interval. Otherwise the system (5.2.1) or $\{A, B\}$ is said to be uncontrollable.*

We define a *controllability Gramian matrix* $\mathcal{P}(t, t_0)$ of the system (5.2.1) as $n \times n$ matrix

$$\mathcal{P}(t, t_0) := \sum_{s=t_0}^{t-1} \Phi_{A(\cdot),\nu}(t_0, s+1)B(s+1)[B(s+1)]^T[\Phi_{A(\cdot),\nu}(t_0, s+1)]^T.$$

Theorem 5.11. *The system (5.2.1) is completely controllable on the discrete time interval $\mathbb{N}_{t_0}^{t_1}$ if and only if the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ is invertible.*

Proof. First, we show that if the given system is completely controllable then the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ is invertible. Let us show by contradiction, suppose that the matrix $\mathcal{P}(t_1, t_0)$ is not invertible and then there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T \mathcal{P}(t_1, t_0) = 0_{1\times n}$. Then it follows that

$$\begin{aligned}
0 &= \eta^T \mathcal{P}(t_1, t_0)\eta \\
&= \sum_{s=t_0}^{t_1-1} \eta^T \Phi_{A(\cdot),\nu}(t_0, s+1)B(s+1)[B(s+1)]^T[\Phi_{A(\cdot),\nu}(t_0, s+1)]^T\eta \\
&= \sum_{s=t_0}^{t_1-1} \|\eta^T \Phi_{A(\cdot),\nu}(t_0, s+1)B(s+1)\|_2^2,
\end{aligned}$$

where $\|\cdot\|_2$ defines Euclidean norm and hence

$$\eta^T \Phi_{A(\cdot),\nu}(t_0, t+1)B(t+1) = 0_{1\times m}, \qquad t \in \mathbb{N}_{t_0-1}^{t_1}. \tag{5.2.3}$$

From the assumption of controllability there exists a control signal $u(t)$ that will transfer the initial state $y(t_0) = y_0 = \Phi_{A(\cdot),\nu}(t_0, t_1)y_f - \eta$ to the final state $y(t_1) = y_f$. By substitution the initial and the final state to (5.2.2) the solution of the given system becomes

$$y_f = \Phi_{A(\cdot),\nu}(t_1, t_0)(\Phi_{A(\cdot),\nu}(t_0, t_1)y_f - \eta) + \sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_1, s+1)B(s+1)u(s).$$

Using the composition property of the state transition matrix and multiplying both sides by $\Phi_{A(\cdot),\nu}(t_0, t_1)$ we obtain

$$\eta = \sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_0, s+1)B(s+1)u(s).$$

Multiplying though by η^T and using (5.2.3) yields

$$\eta^T \eta = \sum_{s=t_0}^{t_1-1} \eta^T \Phi_{A(\cdot),\nu}(t_0, s+1) B(s+1) u(s) = 0,$$

which contradicts the assumption that η is a nonzero vector in $\mathbb{R}^n$. Thus, the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ is invertible.

Conversely, suppose $\mathcal{P}(t_1, t_0)$ is invertible. Therefore, for the given any initial state $y(t_0) = y_0$ and final state $y(t_1) = y_f$ we can choose the control signal $u(t)$ as

$$u(t) = [B(t+1)]^T [\Phi_{A(\cdot),\nu}(t_0, t+1)]^T [\mathcal{P}(t_1, t_0)]^{-1} [\Phi_{A(\cdot),\nu}(t_0, t_1) y_f - y_0].$$

The corresponding solution of the system at $t = t_1$ can be written as

$$\begin{aligned}
y(t_1) =& \Phi_{A(\cdot),\nu}(t_1, t_0) y_0 + \sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_1, s+1) B(s+1) u(s) \\
=& \Phi_{A(\cdot),\nu}(t_1, t_0) y_0 + \sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_1, s+1) B(s+1) [B(s+1)]^T [\Phi_{A(\cdot),\nu}(t_0, s+1)]^T \\
& \times [\mathcal{P}(t_1, t_0)]^{-1} [\Phi_{A(\cdot),\nu}(t_0, t_1) y_f - y_0] \\
=& \Phi_{A(\cdot),\nu}(t_1, t_0) y_0 + \Phi_{A(\cdot),\nu}(t_1, t_0) \sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_0, s+1) B(s+1) [B(s+1)]^T \\
& \times [\Phi_{A(\cdot),\nu}(t_0, s+1)]^T [\mathcal{P}(t_1, t_0)]^{-1} [\Phi_{A(\cdot),\nu}(t_0, t_1) y_f - y_0] \\
=& \Phi_{A(\cdot),\nu}(t_1, t_0) y_0 + \Phi_{A(\cdot),\nu}(t_1, t_0) \mathcal{P}(t_1, t_0) [\mathcal{P}(t_1, t_0)]^{-1} [\Phi_{A(\cdot),\nu}(t_0, t_1) y_f - y_0] \\
=& y_f
\end{aligned}$$

which shows that if the controllability Gramian matrix $\mathcal{P}(t_1, t_0)$ is invertible, then the given system is completely controllable on the given discrete time interval. □

Remark 5.12. *Even though the previous theorem is essential in theory, in practice, it is quite limited. Since computing the controllability Gramian requires explicit knowl-*

edge of the state transition matrix of the given system, the state transition matrix for time-varying problems can be difficult to compute in some cases.

5.3 Observability

In this section, we discuss the observability of the following linear time-variant discrete fractional system

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = A(t)y(t-1) + B(t)u(t-1), \quad t \in \mathbb{N}_{t_0+1}^{t_1} \\ z(t) = C(t)y(t) + D(t)u(t) \end{cases} \tag{5.3.1}$$

where ν is a positive real number such that $0 < \nu < 1$, $y(t)$ is an $n \times 1$ state vector of the system, $u(t)$ is an $m \times 1$ control input, $z(t)$ is an $r \times 1$ output vector, $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are an $n \times n$, an $n \times m$, an $r \times n$, and an $r \times m$ matrix valued functions, respectively.

Since $z(t)$ and $u(t)$ for $t \in \mathbb{N}_{t_0}^{t_1}$ are given, we substitute the solution (5.2.2) of the state system into the output measurement and we obtain

$$\begin{aligned} z(t) &= C(t)y(t) + D(t)u(t) \\ &= C(t)[\Phi_{A(\cdot),\nu}(t,t_0)y_0 + \sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_1, s+1)B(s+1)u(s)] + D(t)u(t). \end{aligned}$$

Then we have

$$C(t)\Phi_{A(\cdot),\nu}(t,t_0)y_0 = z(t) - C(t)\sum_{s=t_0}^{t_1-1} \Phi_{A(\cdot),\nu}(t_1, s+1)B(s+1)u(s) - D(t)u(t)$$

since $A(t), B(t), C(t), D(t)$ matrices and control vector $u(t)$ are given, the last two terms on the right-hand side of this equation are known quantities. Thus, we can subtract known terms from the observed value of output vector $z(t)$. We define the right-hand side of the above equality by $z_1(t)$. Then response of the system (5.3.1) can be written as

$$C(t)\Phi_{A(\cdot),\nu}(t,t_0)y_0 = z_1(t). \tag{5.3.2}$$

Throughout this section we assume $t_0, t_1 \in \mathbb{R}^+$ and $t_1 - t_0 \in \mathbb{Z}^+$.

Definition 5.13. *The system is said to be completely observable, if every state $y(t_0)$ can be uniquely determined from the observation of $z(t)$ over a finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1}$. Otherwise the system (5.3.1) or $\{A, C\}$ is said to be unobservable.*

Let us define a *observability Gramian matrix* $\mathcal{R}(t,t_0)$ of the system (5.3.1) as an $n \times n$ matrix

$$\mathcal{R}(t,t_0) := \sum_{s=t_0}^{t-1} [\Phi_{A(\cdot),\nu}(t,t_0)]^T [C(t)]^T C(t)\Phi_{A(\cdot),\nu}(t,t_0).$$

We conclude this section by giving the necessary and sufficient conditions for control systems to be observable.

Theorem 5.14. *The fractional system (5.3.1) is completely observable on discrete time interval $\mathbb{N}_{t_0}^{t_1}$ if and only if the observability Gramian matrix $\mathcal{R}(t_1,t_0)$ is invertible.*

Proof. First we show that if the given system is completely observable, then the observability Gramian matrix $\mathcal{R}(t_1,t_0)$ is invertible. By contradiction, suppose $\mathcal{R}(t_1,t_0)$ is not invertible then there exists a non zero vector $\eta \in \mathbb{R}^n$ such that $\mathcal{R}(t_1,t_0)\eta = 0_{n\times 1}$.

Then we have

$$
\begin{aligned}
0 &= \eta^T \mathcal{R}(t_1, t_0)\eta \\
&= \sum_{s=t_0}^{t-1} \eta^T [\Phi_{A(\cdot),\nu}(t, t_0)]^T [C(t)]^T C(t) \Phi_{A(\cdot),\nu}(t, t_0)\eta \\
&= \sum_{s=t_0}^{t_1-1} \|C(t)\Phi_{A(\cdot),\nu}(t, t_0)\eta\|_2^2,
\end{aligned}
$$

where $\|\cdot\|_2$ defines Euclidean norm and we have

$$
C(t)\Phi_{A(\cdot),\nu}(t, t_0)\eta = 0_{r\times 1},
$$

for all $t \in \mathbb{N}_{t_0}^{t_1-1}$. Thus $y(t_0) = y_0 + \eta$ yields same response for the system as $y(t_0) = y_0$ and contradicts the assumption that the given system is completely observable. Therefore the observability Gramian $\mathcal{R}(t_1, t_0)$ is invertible.

On the other hand, assume that the matrix $\mathcal{R}(t_1, t_0)$ is invertible. Multiplying both sides of (5.3.2) by $[\Phi_{A(\cdot),\nu}(t, t_0)]^T [C(t)]^T$ and performing the summation over the discrete interval $t \in \mathbb{N}_{t_0}^{t_1-1}$, we obtain

$$
\sum_{s=t_0}^{t_1-1} [\Phi_{A(\cdot),\nu}(t, t_0)]^T [C(t)]^T C(t)\Phi_{A(\cdot),\nu}(t, t_0) y_0 = \sum_{s=t_0}^{t_1-1} [\Phi_{A(\cdot),\nu}(t, t_0)]^T [C(t)]^T z_1(s)
$$

$$
\mathcal{R}(t_1, t_0) y_0 = \sum_{s=t_0}^{t_1-1} [\Phi_{A(\cdot),\nu}(t, t_0)]^T [C(t)]^T z_1(s).
$$

Since the observability Gramian matrix $\mathcal{R}(t_1, t_0)$ is invertible, we have

$$
y_0 = \mathcal{R}(t_1, t_0)^{-1} \sum_{s=t_0}^{t_1-1} [\Phi_{A(\cdot),\nu}(t, t_0)]^T [C(t)]^T z_1(s).
$$

Hence, y_0 is uniquely determined the given system is completely observable. $\square$

Chapter 6

CONCLUSION AND FUTURE WORK

In this thesis, linear fractional difference calculus was considered. In the second chapter, we presented fundamental definitions and formulas in discrete fractional calculus for the convenience of the reader. In the third chapter, we proved existence of the unique solution of one fractional difference equation. Then we stated the properties of the unique solution and generalized for the vector fractional difference equations. We closed the chapter by giving the variation of constants formula to examine the control system. In the fourth chapter, we investigated the controllability and observability of the linear time-invariant nabla fractional systems. We started this chapter by giving the criteria for the controllability of the system via the controllability matrix and controllability Gramian matrix, and then similarly the criteria for the observability of the system and we then gave the connection between the controllability and observability of the system by stating the duality principle. In the fifth chapter, we introduced the state transition matrix in fractional difference calculus and proved some important properties of the state transition matrix. We closed the chapter by investigating the controllability and observability of the linear time-variant nabla discrete fractional systems.

For future work, we would like to see other methods to evaluate the state transition matrix in discrete fractional calculus. There exist other methods to find the transition matrix in continuous time. We will examine the applicability of the other methods in continuous time to the discrete fractional calculus. There are several methodologies to examine the control systems such as classical control theory, modern control theory, robust control, adaptive control, and nonlinear control. Because the majority of the mathematical models are non-linear, we will examine nonlinear control to study the time-variant control systems in discrete fractional calculus.

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