Linear predictor of discounted aggregated cash flows with dependent inter-occurrence time

by

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Abstract

In this minor dissertation we derive the first two moments and a linear predictor of the compound discounted renewal aggregate cash flows when taking into account dependence within the inter-occurrence times. To illustrate our results, we use specific mixtures of exponential distributions to define the Archimedean copula, the dependence structure between the cash flow inter-occurrence times. The Ho-Lee interest rate model is used to show that the formulas derived can be calculated.

Keywords

Discounted compound renewal aggregate sums, Moments, Archimedean Copula, random interest rate, linear predictor.

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Declaration

I declare that the present minor dissertation: *Linear predictor of discounted aggregated cash flows with dependent inter-occurrence time* is my own work. All work adopted from other authors to support my study have been referenced to acknowledge the authors. This work has not been submitted for any other degree or professional qualification except as specified.

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Signature Date

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CHAPTER 1: INTRODUCTION

1.1 Background and motivation of study

In the insurance space, the distribution of discounted aggregate claims with the use of renewal processes is subject to ongoing study. The instantaneous interest rate used to discount the claims is either stochastic or constant. The distribution of discounted aggregate claims can be used to compute the discounted value of a surplus process, and of moments and the moment generating functions. Most of the literature assumes that the inter-arrival times and claim amounts are independent, although this assumption might be too restrictive and therefore a generalisation would be necessary.

The influence of inflation and the interest rate on the present value of the claims process and some of its functionalities, including the surplus process, has been studied by other authors such as Gerber (1971), Taylor (1979), Delbaen and Haezendonck (1987), Waters (1989), Willmort (1989), Sundt and Teugels (1995), Cai and Dickson (2003), Léveillé and Garrido (2001a, 2001b, 2004), and Leveille, Garrido and Wang (2010). The latter formulated their results in a context of renewal, while, Jang (2004) used martingales and jump diffusion processes to get the moments, and Kim and Kim (2007) studied the problem in a Markovian environment.

Léveillé and Garrido (2001a) determined the first two moments using the discounted aggregate claims in an economic environment. The work they did stimulated other authors to find the distributions of discounted aggregate claims with the use of renewal processes. Léveillé and Garrido (2001b) extended their earlier work and derived the recursive moments of the compound renewal sums with discounted aggregate claims. This was improved on by Léveillé & Adékambi (2011, 2012), who found the simple and higher moments of the discounted compound renewal sums in the presence of a stochastic instantaneous interest rate.

The studies cited above make two important and common assumptions. They assume (i) that the claims inter-arrival times and claim amounts are independent, and (ii) that the claim amounts are independent and identically distributed and the claims inter-arrival times are independent and identically distributed with an exponential distribution. Although these assumptions simplify our derivations, they are too restrictive when it comes to the real-world practical application of our models. For instance, in an ever-changing economic environment the assumption that the inter-arrival times of cash flows (dividends) are independent may not necessarily be the case, since the current state of the economy influences a company's performance. Consequently, it profits to ride out any economy downturn. For example, it thus becomes necessary to generalise these assumptions so that the models obtained are in tandem with real-world peculiarities.

In this study, we expect to predict the discounted aggregate dividends with a linear predictor when there is dependence between dividends inter-occurrence times. Where dependence is assumed to be from an Archimedean copula, we look at four different cases for the copula mixing random variable. The research focuses on finding the conditional moments of the discounted aggregate dividends in the presence of an economic variable.

1.2 Importance of the study

Albrecher, Constantinescu and Loisel (2011) used mixing random variables to compute the bankruptcy probability and allowed for the relaxation of the independence assumption between claims inter-occurrence times. The relaxation of the independence assumption is important to our study since we assume dependence between inter-occurrence times of cash flows such as dividends. Newly formed companies tend to have a high growth rate when they have just started, but as time goes by the growth rate starts decreasing due to the decline in investment opportunities. Dividends inter-occurrence times may be dependent on the economic environment. It is necessary to examine the distribution of dividends inter-occurrence times during a period of economic growth or economic crisis since there could be some element of dependency. Therefore, dependency is incorporated into the model to check if it can help minimise the standard errors when using a linear predictor to predict dividends.

If the introduction of dependence yields a better predictive model, then the model can assist company managers to manage shareholders' expectations when it comes to the expected discounted aggregate value of their future dividends. The linear predictor can assist in understanding how the company might be performing in the future, and will be important in risk management and understanding the different economic cycles of a company.

1.3 Objectives of the study

The objective of this study is to find the explicit formula for the first two moments and the joint moment to be able to predict cash flows using renewal process when dependence is assumed between dividends (the cash flows) and their inter-occurrence times. The real-world motivation underlying the study is the desire for the ability to predict discounted aggregate cash flows for a company.

1.4 Research questions/ Hypothesis

The study is based on the following questions:

- Is the model with dependent inter-occurrence times appropriate for deriving the explicit formulas for the moments?
- What is the impact of dependence on the linear predictor of dividends?

1.5 Overview of the chapters

This minor dissertation is arranged into six chapters, where each chapter has its own subsections which will help the explanation of the chapter. In Chapter 1, we covered the introduction of the thesis which relates to the developments made on the classical risk theory and what we aim to achieve in this minor dissertation.

Chapter 2 covers the literature review relating to the discounted aggregate renewal claims model and the assumptions made on the model.

Chapter 3 gives a background on the discounted compound aggregate model and copulas for dependence assumptions.

Chapter 4 gives a brief discussion of the different papers with the independence assumption on inter-occurrence times.

Chapter 5 discusses the paper and assumes dependence of cash flows inter-occurrence times, which is closely linked to our research

Chapter 6 presents the first two moments and the linear predictor with examples using a Ho-Lee interest rate model and a Pareto distribution with Clayton copula dependence as a mixing random variable. The conclusion with recommendations for future work in the collective risk space is presented.

CHAPTER 2: LITERATURE REVIEW

2.1 Collective risk theory

The theory on collective risk models emerged through a thesis named *On the approximation of the probability function in the insurance of collective risks* by a Swedish actuary, Filip Lundberg (1903). The author developed the collective risk theory to model the total amount paid by an insurance company on all claims occurring in a fixed time horizon. One of the advantages of the model is that it is mathematically tractable, and gives a better approximation of the reality. A collective risk model requires the claim amounts and claim number processes to be independent. The assumption makes it difficult to model car insurance portfolios. For instance, terrible weather conditions can result in significant numbers of small claims, but Kass et al. (2002) show that such events appear to have minimal impact on the portfolio.

Definition 1: Given the interval $(0, t]$, the collective risk model is then defined as the total loss $S(t)$ which is the sum of all the random number claims $N(t)$ within the given interval of the payment amounts $\left(X_1, X_2, \dots, X_{N(t)}\right)$. Then $S(t)$, is as follows:

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where it is assumed that:

- The severity variables $\{X_i\}_s'$ are independent and identically distributed.
- The frequency variable $N(t)$ and X_i is independent
- $S(t) = 0$ for $N(t) = 0$

$S(t)$ is a compound distribution since the frequency variable $N(t)$ follows a random distribution. For example, if $N(t)$ follows a Poisson distribution then $S(t)$ follows a compound Poisson distribution. Then, the distribution of $S(t)$ is given by:

$$F(X, t) = \sum_{i=0}^{\infty} p(n, t) G^{(n)}(X),$$

where $p(n, t)$ is the probability that $N(t)$ claims will occur within the given period $(0, t]$ and $G^{(n)}(X)$ is the n^{th} convolution for all $n > 0$. $G^{(0)}(X) = K(X)$, where $K(X)$ takes a value of 0 if $X < 0$ and a value of 1 if $X \geq 0$. $F(X, t)$ gives the claim distribution of the entire portfolio rather than individual claims since it considers both $G^{(n)}(X)$ and $p(n, t)$. For the case where $N(t)$ follows a Poisson distribution, we have that:

$$F(X, t) = \sum_{i=0}^{\infty} \frac{(\lambda t)^i e^{-\lambda t}}{i!} G^{(n)}(X).$$

One of the advantages of the collective risk model is that one can compute the moments of the total loss distribution using iterated expectations. The expectation of $S(t)$ is given as:

$$E[S(t)] = \mu_1 E[N(t)].$$

For the case where $N(t)$ follows a Poisson distribution, $E[F(X,t)]$ is given as:

$$E[S(t)] = \mu_1 \lambda t.$$

There are different approaches on how one can get the distribution $S(t)$. Refer to Borch (1967, p.432-442) and Kass et al. (2002) for the collective risk model results.

Lundberg's theory on collective risk was developed further by Cramer (1930, 1955) where the distinction between collective and individual risk theory was clarified. Anderson (1957) refined the basic collective risk process into a general process which is the compound ordinary renewal process; the extension was made by Thorin (1975) where a delayed renewal risk process was considered.

There are a significant number of contributions on the classical risk process. As the studies transitioned from infancy to a more mature level, researchers shifted their focus to finding the distribution of discounted aggregate claims or discounted collective risk models for the generalised basic risk processes since the classical risk process has the drawbacks of not accounting for the claims inter-occurrence times, and a force of interest which is used to discount the claims.

Definition 2. Given the interval $(0, t]$, the discounted collective risk model is then defined as the total loss $Z(t)$ which is the sum of all the random number claims $N(t)$ within the given interval of the discounted value of payment amounts $\left(X_1, X_2, \dots, X_{N(t)}\right)$ taking into account the inter-occurrence times $\left(\tau_1, \tau_2, \dots, \tau_{N(t)}\right)$ and a constant instantaneous interest rate δ. Then $Z(t)$ is given by:

$$Z(t) = \sum_{i=1}^{N(t)} X_i e^{-\delta T_i},$$

where it is assumed that:

- The severity variables $\{X_i\}_s'$ are independent, and identically distributed.
- The frequency variable $N(t)$ and X_i are independent.
- $S(t) = 0$ for $N(t) = 0$.
- $\tau_i = T_i - T_{i-1}, k \in \mathbb{N}$ and $T_0 = 0$ are iid.

A typical problem considered in the distribution of discounted aggregate claims is when independence is assumed between the number of claims and they follow an exponential distribution, although other authors have also looked at cases where the independence assumption is relaxed.

The distribution of aggregate discounted claims is well studied in the literature, but for the purpose our study we will constrain the research to the papers which are close to our study. As stated above, the studies on classical risk theory originated from Filip Lundberg (1903). After some decades, Léveillé and Garrido (2001a, 2001b) addressed the problem of the distribution of discounted aggregate claims. While the Lundberg (1903) paper determined the first two moments under inflationary conditions, the Léveillé and Garrido (2001a, 2001b) papers focused on the recursive moments of the compound renewal sums with discounted aggregate claims. The latter results were improved further by Léveillé et al. (2010) by exploring the discounted aggregate claims process distribution with asymptotic and time finite moment generating functions. Examples were provided for the claim inter-occurrence times and the severity of claims assuming a phase-type distribution.

2.2 Extension of the collective risk theory

The papers cited above use common assumptions for the risk model. The assumptions are as follows:

- The severity variables $\{X_i\}_s'$ are independent and identically distributed.
- The frequency variable $N(t)$ and X_i are independent.
- Claims occurrence distribution is a Poisson distribution.
- Neutral economic environment which doesn't account for inflation and interest rates.
- Claims inter-occurrence times are iid.

Although the assumptions above simplify our derivations, they are too restrictive when it comes to real-world practical application of our models.

2.2.1 Economic environment

Taylor (1979) studied the effect of inflation on the income generated from premiums and the claims amount distribution. The probability of ruin (bankruptcy) increases as the rate of inflation increases. The author defines the distribution of aggregate claims under inflationary conditions, and it is used to determine the upper bound of the ruin probability. The numerical examples considered show that the upper bound may not be practically useful due to it not being sharp enough. Although one can use the upper bound to approximate the probability of bankruptcy under inflationary conditions, if inflation is assumed to occur at a constant rate then bankruptcy will occur with certainty, and the magnitude of inflation won't matter in such a case.

Delbaen & Haezendonck (1987) studied the influence of two forces: inflation and interest. It was found that the inclusion of the inflation and interest forces significantly improved the approximation of the probability of bankruptcy for both finite and infinite time horizons. Other notable studies are those of Waters (1989), who considered the effect of claims cost inflation on a model with stochastic variations on the surplus process; Willmot (1989) who considered aggregate claims distribution with dependent claim amounts over a fixed period with constant inflation; Sundit and Teugels (1995) who considered continuous time stochastic variations of the surplus process where the interest and premiums are constant; Cai and Dickson (2003) who considered the Sparre Andersen model where the surplus process incorporates a constant interest rate; Yuen, Wang and Wu (2006) who considered the effect of stochastic interest on the renewal risk process, where the authors derived the expressions penalty function and the probability of ruin; and Léveillé & Adékambi (2011) who considered the effect of a stochastic interest rate when deriving the moments for the distribution of discounted compound renewal sums for an ordinary or delayed renewal risk process.

2.2.2 Dependence

Dependence between the claims inter-arrival times and successive claims amounts in risk theory was introduced in the following literature: Albrecher and Boxman (2004) who considered a general setting of the classical ruin model where there is dependence. The aggregate claims distribution between two claim arrivals depends on the preceding claim amount, which is similar to a claim exceeding a certain margin, then the parameters of the distribution of the next claim inter-occurrence times will be altered. Albrecher and Teugels (2006) used an arbitrary copula to introduce dependence between claim inter-occurrence time and claim amount. The results were derived for two cases: the finite and infinite time bankruptcy probability. Contrary to Albrecher and Boxman (2004), Boudreault, Cossette, Landreault and Marceau (2006) assumed that that if a claim inter-occurrence time is greater than a certain margin, then the parameters of the distribution for the next claim amount is altered. Kim and Kim (2007) and Ren (2008) considered dependence in a two-state Markovian environment where the claim rates and sizes vary according to the risk state of the business. Baird et al. (2009) presumed that the distribution of the claim amount has its parameters modified if preceding claim inter-occurrence times are either greater or lower than a certain margin. Bargès et al. (2011) used the copula approach for dependence to compute the moments for the distribution of the aggregate claims when the instantaneous interest rate is not random. In more recent studies, Adékambi and Dziwa (2016) derived an explicit formula for the discounted compound renewal sums when dependence is assumed by FGM copula, and Adékambi (2017) extended the work to find the second moment.

2.2.3 Our research approach

The proposed model and studies on the present minor dissertation had already been proposed by Albrecher, Constantinescu and Loisel (2011), where the authors used the idea of mixing random variables to derive the explicit formula for ruin in renewal risk models with dependence among claim sizes and among claim inter-occurrence times. In their paper, the authors relaxed the assumption of independence between the claim inter-occurrence times through an arbitrary copula, such as an Archimedean copula.

The goal of our research is to derive explicit expressions for the first two moments of the discounted aggregate claims under the model proposed by Albrecher, Constantinescu and Loisel (2011). We used simpler models for which explicit formulas exist, and then mixed the involved parameters. That is, the mixing parameters could be carried over to the mixing of the moments and the linear predictor under study.

Chapter 3: Background on the compound renewal process and copulas

This minor dissertation is an extension to the collective risk model and therefore makes use of the theory on the compound renewal process and copulas to attain the desired results. The aim of this chapter is to give a brief discussion of the theory, assuming that the reader has basic knowledge of stochastic processes and probability theory, since these are prerequisites for understanding the discussion. We hope that the discussion on compound renewal processes and copulas will be sufficiently clear for the reader to understand the following chapters. The reader can also consult Denuit. Dhaene, Goovaerts, and Kass (2001, pp. 29-75) and Dhaene, Goovaerts and Kass (2005) for collective risk models, and Embrechts, Klüpperberg & Mikosh (1997, Chapter 5, pp. 184-234) for the dependence models i.e. copulas.

3.1 The compound renewal risk model

One of the roles of insurance companies is to adopt and use mathematical models to price their products i.e. premium calculations of death benefits. Claims incurred by insurance companies can be modelled by stochastic/random variable X which maps all the events/claims into a set of real numbers. The random variable can then be linked to a probability distribution $F_X: \mathbb{R} \to [0,1]$ which is defined as $F_X(x) = P(\omega \in \Omega | \mathrm{X}(\omega) \leq x) = P(X \leq x)$ for $x \in \mathbb{R}$. $F_X(x)$ and gives information on the distribution of X for all the possible values. Using the context of insurance, $F_X(x)$ denotes the probability of the insured damage being less than or equal x.

The model is defined based on the following assumptions:

1. $N(t)$ is a random variable which represents the total number of claims received at time t. Individual claims inter-occurrence times are in $0 \leq T_1 \leq T_2 \leq T_3 \leq T_4 \leq \cdots$,

2. T_i is the claim inter-occurrence time which induces a claim payment X_i. The random variables $\{X_i\}_{i \geq 1}$ are non-negative.

3. X_i claim size and T_i claim inter-occurrence times are independent from each other.

There are two important stochastic processes for renewal processes. They are the claims number process $N(t)$ and the total claim amount process.

3.1.1 Counting processes

$N(t)$ models the claim number process. $\{N(t), t \geq 0\}$ is a stochastic process which is known as a counting process if $N(t)$ denotes all the claims which occurred in the interval $(s, t]$. By definition,

$$N(t) = \sup\{N(t) \geq 1 : T_n \leq t\}, \qquad t \geq 0$$

By accord, $\sup\{\emptyset\} = 0$. Then the process has the following properties:

(A1) $N(0) = 0$ with probability 1,

(A2) $N(t) \geq 0$,

(A3) $N(t)$ has integer values,

(A4) If $0 \leq s < t$ then $N(t) \geq N(s)$ and

(A5) For $s < t, N(t) - N(s)$ denotes the number of claims/events in the interval $(s, t]$.

The random variables from the counting process have independent increments over the disjoint interval. For $0 < t_1 < t_2 < t_3 < \cdots < t_n < \infty$ we have $N(t_1)$, $N(t_2) - N(t_1)$, $N(t_3) - N(t_2), \dots, N(t_n) - N(t_{n-1})$ as independent variables for all $n \geq 1$.

Likewise, $\{N(t), t \geq 0\}$ has stationary increments if the probability distribution of the number of claims/events only depends on the length of the given time interval. If $0 \leq s < t$, $k > 0$ then the probability distribution of the random variables $N(t) - N(s)$ and $N(t + k) - N(s + k)$ is the same.

3.1.2 Poisson process

The Poisson process is one of the basic processes of the counting process $\{N(t), t \geq 0\}$.

Definition 3.1: $\{N(t), t \geq 0\}$ is a Poisson with the intensity/rate λ if

(B1) $N(0) = 0$ with probability 1,

(B2) $N(t)$ has stationary and independent increments,

(B3) $P(N(h) = 1) = \lambda h + o(h)$ and

(B4) $P(N(h) \geq 2) = o(h)$

Remark: $f(h) = o(h)$ which means $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

Combining the properties of the stochastic process $N(t)$ (A1)-(A5) with (B1)-(B2) we get a theorem by Ross (1980, Theorem 2.1.1, p.61) which follows below:

Theorem 3.1: Let $\{N(t), t \geq 0\}$ be a stochastic process which satisfies the properties (A1)-(A5) and (B1)-(B2). For any $t \geq 0, s \geq 0, x \geq 0$

$$P(N(t + s) - N(s) = x) = P(N(t) = x) = \frac{(\lambda t)^x}{x!} e^{-\lambda t},$$

The process $N(t)$ is then said to be a time homogeneous Poisson process with intensity/rate λ. Then we have $E[N(t)] = \lambda t$ and $Var\big(N(t)\big) = \lambda t, \forall\ t \geq 0$.

Then, theorem 3.1 dispenses a substitute for a Poisson process.

Definition 3.2: $\{N(t), t \geq 0\}$ is a Poisson with the intensity/rate λ if

(C1) $N(0) = 0$ with probability 1,

(C2) $N(t)$ has independent increments,

(C3) $N(t) - N(s)$ has Poisson distribution with parameter $\lambda(t - s)$ for $s < t$:

$$P(N(t) - N(s) = x) = \frac{\big(\lambda(t-s)\big)^x}{x!} e^{-\lambda(t-s)}, \ \ \forall x \geq 0.$$

3.1.3 Renewal process

The fundamental property of the Poisson process $N(t)$ states that the claims inter-occurrence times $\{\tau_i = T_i - T_{i-1},\ \ i \in \mathbb{N},\ \text{ and } T_0 = 0\}$ are random variables which are independently and identically distributed (iid) with an exponential distribution of parameter λ. Let us assume that variable K_i is random, non-negative, independent and is distributed like the random variable T. From the assumption of identical distribution, the process is then said to have the memoryless property and so it renews itself at each occurrence. Then, $N(t)$ is called a Poisson process. In renewal theory, the distribution of $N(t)$ does not necessarily need to follow a Poisson distribution but here we used it as one of the special cases of $N(t)$.

Definition 3.3: The counting process $N(t)$ is called a renewal process if the claims inter-occurrence times $\{\tau_i = T_i - T_{i-1},\ \ i \in \mathbb{N},\ \text{ and } T_0 = 0\}$ are independently and identically distributed (iid) with the same probability distribution function.

3.1.4 Distribution of the aggregate claim process

The process $\{Z(t)\}$ of the total amount of claims to paid is defined as:

$$Z(t) = \sum_{i=1}^{N(t)} X_i, \quad Z(t) = 0 \text{ if } \ N(t) = 0 \tag{3.1}$$

If the claims number process is deterministic and denoted by n

$$Z_n = X_1 + X_2 + X_3 + \cdots + X_n$$

the deterministic variable n is then replaced by a stochastic variable $N(t)$ and the $\{Z(t)\}$ is known as a compound process. If we use the special case that $\{N(t)\}$ follows a Poisson distribution, then (3.1) becomes a compound Poisson process. The distribution of $\{Z(t)\}$ depends on $N(t)$ and $\{X_i\}_{i\geq 1}$ which are stochastic in nature. Then, the CDF of $\{Z(t)\}$ is as follows:

$$F_{\boldsymbol{Z(t)}}(x) = P(Z(t) \leq x) = P\left(\sum_{i=1}^{N(t)} X_i \leq x\right).$$

If one doesn't have the knowledge of the interdependence between the random variables $N(t)$ and $\{X_i\}_{i\geq 1}$ then it will be complex to find the explicit form of $F_{\boldsymbol{Z(t)}}(x)$. Due to the problem stated above, we then use numerical/recursive algorithms to approximate $F_{\boldsymbol{Z(t)}}(x)$. Reader can refer to Léveillé & Garrido (2001b).

In summary, section 3.1 outlined the important elements of the compound renewal risk model which is the claims number process $N(t)$, inter-occurrence times $\{\tau_i = T_i - T_{i-1},\ i \in \mathbb{N},\ \text{and}\ T_0 = 0\}$ which are iid and the distribution of the aggregate claims process $\{Z(t)\}$. The process $\{Z(t)\}$ is then a double stochastic process and hence, it is referred to as a compound process.

3.2 Copula for modelling dependence

A copula is a joint distribution of a finite number of random variables, where each variable is uniformly distributed between 0 and 1. The joint distribution can be used to model dependence, for instance financial risk factors or operational risk factors. In the sub-sections that follow, we give a brief description of the marginals for the individual risk factors, dependence structure and an introduction of a mixing random variable.

3.2.1 Bivariate copulas

Given a bivariate/joint density function F_x with univariate marginal densities F_1 and F_2, then we can link together three numbers with each pair of real numbers $x = (x_1, x_2)$: $F_1(x_1)$, $F_2(x_2)$ and $F_x(x)$, where each these numbers lie in the unit interval [0,1]. Said differently, each pair x of $\mathbb{R}$ to a point $\big(F_1(x_1), F_2(x_2)\big)$ in the unit square, and this pair is analogous to a number $F_x(x)$ in [0,1]. This analogy, which assigns the value of the bivariate/joint density function to each of the ordered pair of values of the individual density functions, is a called copula.

Definition 3.4 {Dhaene et al. (2005, p.194)}: A bivariate copula C is a function mapping the unit square $[0,1]^2 := [0,1] \times [0,1]$ to the unit interval $[0,1]$ that is non-decreasing and right-continuous, and satisfies the following conditions:

(i) $\lim_{u_i \to 0} C(u_1, u_2) = 0$ for $i = 1,2$;

(ii) $\lim_{u_1 \to 1} C(u_1, u_2) = u_2$ and $\lim_{u_2 \to 1} C(u_1, u_2) = u_1$;

(iii) C is supermodular, that is, the inequality

$$C(v_1, v_2) - C(u_1, v_2) - C(v_1, u_2) + C(u_1, u_2) \geq 0$$

is valid for any $u_1 \leq v_1$, $u_2 \leq v_2$.

The results above are for a bivariate copula. A multivariate copula is discussed in a later section.

3.2.2.1 Sklar's theorem for continuous marginals

Sklar's theorem provides the theoretical base for the applications. Refer to theorem 3.3 for the definition. It explains the role that copulas play in the relationship between multivariate density functions and their univariate marginal density functions.

Theorem 3.2 {Dhaene et al. (2005, Theorem 4.2.2, p.194)}: Let $F_X \in \boldsymbol{\mathcal{R}_2}(\boldsymbol{F_1}, \boldsymbol{F_2})$ have continuous cumulative distribution functions F_1 and F_2. There then exists a unique copula C such that for all $x \in \mathbb{R}^2$,

$$F_{\boldsymbol{X}}(x_1, x_2) = C\big(F_1(x_1), F_2(x_2)\big) \tag{3.2}$$

Conversely, if C is a copula and F_1 and F_2 are density functions, then the function F_X defined by (3.2) is a bivariate distribution function with marginals F_1 and F_2.

The copula C in (3.2) links the marginal distributions functions of F_1 and F_2 to form the joint distribution F_X of the pair $X = (X_1, X_2)$. Then C describes the dependence structure and dissociates from the marginals F_1 and F_2.

Example 3.1 {The independence copula C_I}

Consider X_1 and X_2 which are random variables, with respective density functions F_1 and F_2 . Their joint distribution is given by $F_X(x) = F_{\boldsymbol{X_1}}(x_1)F_{\boldsymbol{X_2}}(x_2)$, therefore the underlying copula is given by

$$C_I(u_1, u_2) = u_1 u_2, \quad u \in [0,1]^2.$$

The copula above is referred to as the independence copula. If X_1 and X_2 possess the probability distribution function (3.2), then they are independent, if and only if $C \equiv C_I$.

Example 3.2 {The Fréchet upper bound copula C_u}

The Fréchet upper bound copula, is denoted by C_u:

$$C_u(u_1, u_2) = min\{u_1, u_2\}, \;\; u \in [0,1]^2.$$

If X_1 and X_2 possess the probability distribution function (3.2), then X_2 is a non-decreasing function of X_1, if $C \equiv C_u$.

3.2.1.2 Conditional distributions derived from copulas

The conditional distributions can be derived from the copula (3.2) but only if the partial derivatives $\frac{\partial}{\partial u_1}C$ and $\frac{\partial}{\partial u_2}C$ exist for the copula (3.2). The conditional distribution is subject to the following theorem from Dhaene et al. (2005, Proposition 4.2.12, p.199).

Theorem 3.3: Let C be a copula. For any $u_2 \in [0,1]$ the partial derivative $\frac{\partial}{\partial u_1}C(u_1, u_2)$ exists almost everywhere, and for each (u_1, u_2) where it exists, we have

$$0 \le \frac{\partial}{\partial u_1}C(u_1, u_2) \le 1.$$

Similarly, for any $u_1 \in [0,1]$ the partial derivative $\frac{\partial}{\partial u_2}C(u_1, u_2)$ exists almost everywhere, and for each (u_1, u_2) where it exists, we have

$$0 \le \frac{\partial}{\partial u_2}C(u_1, u_2) \le 1.$$

Moreover, the functions $u_1 \mapsto \frac{\partial}{\partial u_2}C(u_1, u_2)$ and $u_2 \mapsto \frac{\partial}{\partial u_1}C(u_1, u_2)$ are defined and non-decreasing almost everywhere on [0,1].

For the proof of theorem (3.3) please refer to Dhaene et al. (2005, p.199).

3.2.1.3 Probability density functions associated with copulas

In conclusion of the bivariate copulas, the copula density function can be written as a product of marginal distributions under appropriate conditions.

If the marginal probability distributions F_1 and F_2 are continuous with respective probability distributions functions f_1 and f_2, then the joint probability density function of $\boldsymbol{X}$ can be written as

$$f_X(x) = f_{X_1}(x_1) f_{X_2}(x_2) c\left(F_{\boldsymbol{X_1}}(x_1), F_{\boldsymbol{X_2}}(x_2)\right), \;\; \boldsymbol{x} \epsilon \mathbb{R}^2,$$

where the copula density function c is given by

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2), \quad \boldsymbol{u} \epsilon [0,1]^2.$$

3.2.2 Multivariate copulas

The bivariate copula results are now extended to a more general case, being the multivariate copula.

Definition 3.5 {Embrechts, McNeil & Frey (2005, p.185)}: A d-dimensional copula is a mapping of the following form $C : [0,1]^d \longmapsto [0,1]$ which satisfies the following properties:

(i) $C(u_1, u_2, \dots, u_d)$ is increasing in each component u_i.

(ii) $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for all $i \epsilon 1,2, \dots, d,\ \ u_i \epsilon [0,1]$.

(iii) For all $(a_1, a_2, \dots, a_d), (b_1, b_2, \dots, b_d)\ \epsilon [0,1]^d$ with $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \sum_{i_2=2}^{2} \dots \sum_{i_d=1}^{2} (-1)^{i_1+i_2+\dots+i_d} C(u_{1i_1}, u_{2i_2}, \dots, u_{di_d}) \geq 0, \tag{3.3}$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in (1,2, \dots, d)$.

Property (i) is a necessary condition for any multivariate distribution, property (ii) is mandatory for the uniform marginal distributions and property (iii), the rectangle inequality (3.3), warrants that the random vector $(U_1, U_2, \dots, U_d)^T$ has a probability density function C, then $P(a_1 \leq U_1 \leq b_1, \dots, a_d \leq U_d \leq b_d)$ can't be negative. The properties above are characteristics of a multivariate copula and if C satisfies all of them, then it is called multivariate copula.

Theorem 3.4 {Sklar's theorem}: Let F denote a joint probability distribution function with marginals $F_1, F_2, \dots, F_d$. Then there exists a copula $C: [0,1]^d \to [0,1]$ such that, $\forall\, x_1, x_2, \dots, x_d$ in $\mathbb{R} \in (-\infty, \infty)$,

$$F(x_1, x_2, \dots, x_d) = C\left(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_d}(x_d)\right). \tag{3.4}$$

If the marginal distributions are continuous, then C is unique, or else C can be determined uniquely on $RanF_1 \times RanF_2 \times \dots \times RanF_d$, where $RanF_i = F_i(\boldsymbol{R})$ represents the range of F_i. Conversely, if C is a copula and $F_1, F_2, \dots, F_d$ uniform univariate distributions functions, then the function F defined in (3.4) is called a joint probability distribution function with marginals $F_1, F_2, \dots, F_d$.

The Sklar theorem gives closed form definition of the multivariate copula and as in 3.2.1, we look at the following results

Definition 3.5. If the random variable $\boldsymbol{X}$ has a joint probability density function F with marginal distributions $F_1, F_2, \dots, F_d$ which are continuous, then the copula C of F or $\boldsymbol{X}$ is the distribution function C of $\left(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_d}(x_d)\right)$.

Example 3.3 {The independence copula C_I}

We look at independent random variables $\boldsymbol{X} = (X_1, X_2, \dots, X_d)$, with respective marginal distributions $F_1, F_2, \dots, F_d$. Then, the joint probability distribution function is given by $F_X(x) = \prod_{i=1}^{d} F_i$ and the copula is as follows

$$C_I(u_1, u_2, \dots, u_d) = \prod_{i=1}^{d} u_i \, .$$

Example 3.4 {The Fréchet upper bound copula C_u}

The Fréchet upper bound copula, is denoted by C_u:

$$C_u(u_1, u_{2,} \dots, u_d) = min\{u_1, u_2, \dots, u_d\}, \;\; \boldsymbol{u} \in [0,1]^2.$$

3.2.3 Archimedean copula

The Archimedean copula is the copula we used in this minor dissertation to model the dependence between cash flows / dividends inter-occurrence times, since it allowed us to introduce a random mixing variable representing the state of the economy. Please note that the dividends inter-occurrence times are independent if we are given the random mixing variable or the state of the economy is known.

Definition 3.6: A d-dimensional copula C is an Archimedean copula if

$$C(u_1, u_2, \dots, u_d) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \dots + \psi^{-1}(u_d)\right), \;\; \text{for } \boldsymbol{u} \in [0,1]^d. \qquad (3.5)$$

The function ψ is called a copula generator function which is continuous and strictly decreasing. Where ψ: $[0, \infty) \to [0,1]$, $\psi(0) = 1$ and then $\lim_{t \to \infty} \psi(t) = 0$. In similar manner, we have $\psi^{-1}: [0,1] \to [0, \infty)$, for which $\psi^{-1}(0) = inf\{t : \psi(t) = 0\}$, where ψ^{-1} represents the inverse function of the generator ψ. The proof that (3.5) is a d-dimensional copula if and only if ψ is a d-monotone function, can be found in McNeil and Nešlehová (2009).

We considered Archimedean copulas, which have completely monotone generator functions ψ. By co-opting the Bernstein's theorem, we found that such generator functions correspond to the Laplace-Stieltjes Transform (LST) of a random variable Θ which is strictly positive, and the cumulative distribution $F_\Theta(.)$ Is given by

$$\mathcal{L}_{\Theta}(t) = \int_{0}^{\infty} e^{-t\theta}\, dF_{\Theta}(\theta) = E\left[e^{-t\theta}\right] \tag{3.6}$$

Then, the Archimedean copula (3.5) becomes

$$C(u_1, u_2, \dots, u_d) = \mathcal{L}_{\Theta}\left({\mathcal{L}_{\Theta}}^{-1}(u_1) + {\mathcal{L}_{\Theta}}^{-1}(u_2) + \dots + {\mathcal{L}_{\Theta}}^{-1}(u_d)\right) \tag{3.7}$$

The random variable Θ which is strictly positive can either have a discrete or continuous distribution. Θ corresponds to a mixing random variable which has a one-to-one relation with the Archimedean copula (3.5) and its distribution. The special case (3.7) of Archimedean copulas is linked to common mixtures. Further explanation can be found in Embrechts et al. (2005) and Denuit et al. (2006). The Archimedean copula with common mixtures allows us to recognise the univariate conditional cumulative distributions of $(U_1|\Theta = \theta), (U_2|\Theta = \theta), \dots, (U_d|\Theta = \theta)$, where the vector of random variables $\boldsymbol{U} = (U_1, U_2, \dots, U_d)$ are independently and identically distributed with a uniform distribution on an interval (0,1). Using the combination of (3.6) and (3.7), we have the representation of an Archimedean copula as common mixture:

$$\begin{aligned} C(u_1, u_2, \dots, u_d) &= \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \dots + \psi^{-1}(u_d)\right) \\ &= \mathcal{L}_{\Theta}\left({\mathcal{L}_{\Theta}}^{-1}(u_1) + {\mathcal{L}_{\Theta}}^{-1}(u_2) + \dots + {\mathcal{L}_{\Theta}}^{-1}(u_d)\right) \\ &= \int_{0}^{\infty} \prod_{i=1}^{d} e^{-\theta {\mathcal{L}_{\theta}}^{-1}(u_i)} dF_{\Theta}(\theta). \end{aligned}$$

which becomes

$$C(u_1, u_2, \dots, u_d) = F_{\boldsymbol{U}}(u_1, u_2, \dots, u_d) = \int_{0}^{\infty} \prod_{i=1}^{d} F_{U_i|\Theta=\theta}(u_i|\theta) dF_{\Theta}(\theta)\,. \tag{3.8}$$

From (3.8), we see that the conditional cumulative distribution of $(U_i|\Theta = \theta)$ is given by $F_{U_i|\Theta=\theta}(u_i|\theta) = e^{-\theta {\mathcal{L}_{\theta}}^{-1}(u_i)}\ \forall\ i = 1,2, \dots, d, u_i \in [0,1]$ and $\theta > 0$ which is the common mixture representation of a copula C. For examples on this special case of Archimedean copulas, refer to Cossette et al. (2017).

In the following section we give a discussion of the common mixture representation.

3.2.3.1 Common mixture representation

Let $\boldsymbol{X} = (X_1, X_2, \dots, X_d)$ be a vector which contains random variables with a multivariate distribution defined by (3.7). Then, the multivariate cumulative distribution of $F_{\boldsymbol{X}}$ of $\boldsymbol{X}$ can be defined with the copula C with univariate marginal cumulative distributions $F_{\boldsymbol{X_1}}(x_1), F_{\boldsymbol{X_2}}(x_2), \dots, F_{\boldsymbol{X_d}}(x_d)$ as

$$F_{\boldsymbol{X}}(x_1, x_2, \dots, x_d) = C\left(F_{\boldsymbol{X_1}}(x_1), F_{\boldsymbol{X_2}}(x_2), \dots, F_{\boldsymbol{X_d}}(x_d)\right) \tag{3.9}$$

therefore, the common mixture representation of $F_{\boldsymbol{X}}$ is given by

$$F_{\boldsymbol{X}}(x_1, x_2, \dots, x_d) = \int_{\boldsymbol{0}}^{\infty} \prod_{\boldsymbol{i}=1}^{d} F_{X_i|\Theta=\theta}(x_i|\theta) dF_\Theta(\theta) = \int_0^\infty \prod_{i=1}^{d} e^{-\theta \mathcal{L}_\theta{}^{-1}\left(F_{X_i}(x_i)\right)} dF_\Theta(\theta) \tag{3.10}$$

where

$$F_{X_i|\Theta=\theta}(x_i|\theta) = e^{-\theta \mathcal{L}_\theta{}^{-1}\left(F_{X_i}(x_i)\right)}, \quad \forall\, i \in [1, d]. \tag{3.11}$$

Hence, we can now define the multivariate distribution of the random variable $\boldsymbol{X}$ using the survival function of the Archimedean copula C and univariate marginal survival functions are denoted as $\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2), \dots, \bar{F}_{X_d}(x_d)$ that is like

$$\bar{F}_{\boldsymbol{X}}(x_1, x_2, \dots, x_d) = C\left(\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2), \dots, \bar{F}_{X_d}(x_d)\right). \tag{3.12}$$

Therefore, the common mixture representation of $\bar{F}_{\boldsymbol{X}}$ is given by

$$\bar{F}_{\boldsymbol{X}}(x_1, x_2, \dots, x_d) = \int_{\boldsymbol{0}}^{\infty} \prod_{\boldsymbol{i}=1}^{d} \bar{F}_{X_i|\Theta=\theta}(x_i|\theta) dF_\Theta(\theta) = \int_0^\infty \prod_{i=1}^{d} e^{-\theta \mathcal{L}_\theta{}^{-1}\left(\bar{F}_{X_i}(x_i)\right)} dF_\Theta(\theta). \tag{3.13}$$

where

$$\bar{F}_{X_i|\Theta=\theta}(x_i|\theta) = e^{-\theta \mathcal{L}_\theta{}^{-1}\left(\bar{F}_{X_i}(x_i)\right)}, \quad \forall\, i \in [1, d] \tag{3.14}$$

and $(S|\Theta = \theta) = \sum_{i=1}^{d}(X_i|\Theta = \theta)$

Example 3.4

Let $\boldsymbol{X} = (X_1, X_2, \dots, X_d)$ be a vector which contains exchangeable Bernoulli random variables such that $X_i \sim Bernoulli(q)\ \forall\, i = 1,2, \dots, d$ and

$$F_{\boldsymbol{X}}(t_1, t_2, \dots, t_d) = C\left(F_{X_1}(t_1), F_{X_2}(t_2), \dots, F_{X_d}(t_d)\right), \quad \text{for } t_1, t_2, \dots, t_d \in [0,1].$$

from 3.11, we get that:

$$(X_i|\Theta = \theta) \sim Bernoulli\left(1 - e^{-\theta \mathcal{L}_\Theta^{-1}(1-q)}\right), \text{for } i = 1,2, \dots, d.$$

Therefore, $(S|\Theta = \theta)$ follows a binomial distribution, which follows below

$$f_{S|\Theta=\theta}(t) = \binom{d}{t}\left(1 - e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)}\right)^t e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)(d-t)}$$
$$= \binom{d}{t}\sum_{i=0}^{t}\binom{t}{i}(-1)^i\, e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)(i+d-t)},$$

Then we can conclude that

$$f_{S|\Theta=\theta}(t) = \binom{d}{t}\sum_{i=0}^{t}\binom{t}{i}(-1)^i\,\mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(1-q)(i+d-t)\right),\ \ for\ t = 1,2,\dots,d.$$

3.2.3.2 Closed-form expressions for the multivariate mixed exponential distributions

Let $\boldsymbol{X} = (X_1, X_2, \dots, X_d)$ be a random vector which consists of independent and identically distributed distributions conditional on $\Theta = \theta$. Then we have the stochastic representation of X_i as

$$X_i|\Theta = \theta \sim Exp(\theta)\ for\ i = 1,2,\dots,d$$
$$\Theta \sim F_\Theta(.),$$

Please note that $Exp(\theta)$ denotes an exponential distribution with mean $\frac{1}{\Theta}$. So, the joint conditional distribution of the survival function is given by,

$$P(X_1 > x_1, X_2 > x_2, \dots, X_d > x_d) = \prod_{i=1}^{d} e^{-\theta x_i},\ \ x_i > 0,\ \ i = ,2,\dots,d.$$

Since the components of the vector $\boldsymbol{X}$ are independently and identically distributed when conditional on Θ they are then exchangeable. The unconditional joint distribution of the survival function is given by

$$P(X_1 > x_1, X_2 > x_2, \dots, X_d > x_d) = \int_0^\infty e^{-\theta(x_1+x_2+\cdots.+x_d)}\, dF_\Theta(\theta)$$
$$= \mathcal{L}_\Theta(x_1 + x_2 + \cdots. + x_d)$$

For $x_i > 0,\ \ i = ,2,\dots,d$. The joint survival function can also be written as,

$$P(X_1 > x_1, X_2 > x_2, \dots, X_d > x_d) = C\left(\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2), \dots, \bar{F}_{X_d}(x_d)\right),$$

Since the copula is the Archimedean copula, we can introduce the generator function ψ and the survival copula will be given by,

$$C\left(\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2), \dots, \bar{F}_{X_d}(x_d)\right) = \psi^{-1}\left(\psi\left(\bar{F}_{X_1}(x_1)\right) + \psi\left(\bar{F}_{X_2}(x_2)\right) + \cdots + \psi\left(\bar{F}_{X_d}(x_d)\right)\right)$$

from the survival function the marginals are defined as

$$\bar{F}_{X_i}(x_i) = \int_0^\infty e^{-\theta x_i}\, dF_\Theta(\theta) = \mathcal{L}_\Theta(x_i), \;\; i = ,2, \dots, d.$$

We can obtain the probability distribution function and survival functions of the aggregate risk S_d.

Theorem 3.5 {Sarabia, Gomez-Deniz, Prieto & Jordá, (2017, p.7)}: Let Θ be a positive random variable with cumulative distribution $F_\Theta(.)$ and Laplace transform $\mathcal{L}_\Theta(.)$. Assume that, given $\Theta = \theta$, the random variables $(X_1, X_2, \dots, X_d)$ are conditionally independent and distributed as exponential $Exp(\theta)$. Then, the probability distribution function of the aggregated random variable $S_d = X_1 + X_2 + \cdots + X_d$ is given by

$$f_{S_d}(x) = \frac{x^{d-1}}{\Gamma(d)} \left\{ (-1)^d \frac{d^d}{dx^d} \mathcal{L}_\Theta(x) \right\}, \;\; x \geq 0 \tag{3.15}$$

Otherwise $f_{S_d}(x) = 0$ if $x < 0$.

Proof: The unconditional distribution of S_d is given by

$$f_{S_d}(x) = \int_0^\infty f_{S_d|\Theta=\theta}(x|\theta) dF_\Theta(\theta).$$

Since the conditional distribution of $S_d|\Theta \sim \Gamma(d, \theta)$, then we have

$$\begin{aligned} f_{S_d}(x) &= \int_0^\infty \frac{\theta^{\,d} x^{d-1} e^{-\theta x}}{\Gamma(d)} dF_\Theta(\theta) \\ &= \frac{x^{d-1}}{\Gamma(d)} \int_0^\infty \theta^d\, e^{-\theta x}\, dF_\Theta(\theta) \\ &= \frac{x^{d-1}}{\Gamma(n)} \left\{ (-1)^d \frac{d^d}{dx^d} \mathcal{L}_\Theta(x) \right\} \end{aligned}$$

The distributional of the unconditional survival aggregated random variable $S_d = X_1 + X_2 + \cdots + X_d$ is given for $x > 0$

$$\begin{aligned} P(S_d > x) &= \int_0^\infty P(S_d > x|\Theta = \theta)\, dF_\Theta(\theta) \\ &= \int_0^\infty \sum_{k=1}^{d-1} \frac{(\theta x)^k e^{-\theta x}}{k!} dF_\Theta(\theta) \\ &= \sum_{k=1}^{d-1} \frac{x^k}{k!} \left\{ (-1)^k \frac{d^k}{dx^k} \mathcal{L}_\Theta(x) \right\} \end{aligned}$$

Where the unconditional survival probability function can only be computed using $(d - 1)$ derivatives of $\mathcal{L}_{\Theta}(.)$.

Chapter 4: Review of moments from discounted compound renewal sums with independence

The aim of this chapter is to highlight or give or review of the results on collective risk theory where the model assumes that the distribution of the cash flow and it's inter-occurrence times are independent. The review is limited to only a few papers closely related to our research. The papers to be reviewed in a sequential form are: Léveillé and Garrido (2001a, 2001b), Léveillé et al. (2010) and Léveillé and Adékambi (2011, 2012).

4.1 Léveillé and Garrido (2001a): Moments of compound renewal sums with discounted claims

Léveillé and Garrido (2001a) proposed an extension to the compound renewal risk process as a generalised classical risk model by Andersen (1957). The extension is on discounting the claims amount to the relevant point in time so that we can have a present value of the claims amount. The model that the authors proposed is called the compound renewal present value risk model (CRPVR). Renewal theory arguments were used to derive the first two moments of the CRPVR model, considering both the ordinary and delayed renewal cases in the presence of regularity conditions such as inflation and force of interest.

4.1.1 Definition of the risk model and it's assumptions

The risk model is assembled as follows. Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote a complete probability space with the following variables and assumptions.

1. The claim counting $N(t,\omega) = \sup\{n \in \mathbb{N};\ T_k(\omega) \leq t\}$, where $\omega \in \Omega$, $t > 0$ and $\sup\{\theta\} = 0$ forms an ordinary renewal process. Then,

- the claim occurrence times are given by $\{T_k\}_{k\geq 1}$;
- the independent and identically distributed times $\tau_k = T_k - T_{k-1},\ k \geq 2,\ T_0 = 0$ and $\tau_1 = T_1$, have a common continuous distribution function F;
- $\mathcal{L}_F$ denotes the Laplace transform of F and $E[\tau_1] = \lambda^{-1} < \infty$.

2. The corresponding inflated claim severities $\{Y_k\}_{k\geq 1}$ are stochastic. We then looked at the deflated random variables $\{X_k\}_{k\geq 1}$ where the units of measure at time point 0 is the currency:

$$X_k = e^{-A(T_k)} Y_k, \quad k \geq 1,$$

where $A(t) = \int_0^t \alpha_s ds$ for any $t \geq 0$. There are also restrictions which are imposed and they imply that force of inflation α_t is bounded and non-negative and the constant net force of interest is positive $\delta_s = \beta_s - \alpha_s = \delta > 0$. For each fixed k, we have $A(T_k)$ which is a random variable on the complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, as A is continuous for every (measurable) inflation rate function α.

3. $\{X_k\}_{k\geq 1}$ are independent and identically distributed;

4. $\{X_k, \tau_k\}_{k\geq 1}$ are mutually independent;

5. $E[X_1] = \mu_1$, and $0 < E[X_1^2] = \mu_2$

6. The discounted aggregate value at time t of the deflated random variables over the period $[0, t]$ is defined as follows

$$Z(t) = \sum_{k=1}^{N(t)} e^{-D(T_k)} X_k$$

where $D(t) = B(t) - A(t) = \int_0^t (\beta_s - \alpha_s)\, ds = \int_0^t \delta_s ds$ and $Z(t) = 0$ if $N(t) = 0$.

4.1.2 Results for the ordinary renewal case

The single premium which is paid at time 0 for a contract of a duration $t > 0$ is given by the following equation

$$\pi_0(t) = E[Z(t)] = E\left[\sum_{k=1}^{N(t)} e^{-\delta T_k} X_k\right]. \tag{4.1}$$

It is worth noting that $\{D(t) = \delta t, \forall\, t > 0\}$ from (4.1).

Application of conditional expectations and assumptions (3) -(5) to (4.1) results in following

$$\begin{aligned} \pi_0(t) = E[Z(t)] &= E\left\{E\left[\sum_{k=1}^{N(t)} e^{-\delta T_k} X_k | N(t)\right]\right\} \\ &= E[X_1] E\left[\sum_{k=1}^{N(t)} e^{-\delta T_k}\right] \\ &= \mu_1 E\left[\sum_{k=1}^{N(t)} e^{-\delta T_k}\right] \end{aligned} \tag{4.2}$$

Since T_k and $N(t)$ are not independent a problem results as one wants to compute the value of (4.2), but there are several approximations which can used.

4.1.2.1 First moment of the CRPVR model

The following theorem shows the explicit results for the first moment

Theorem 4.1 {Léveillé and Garrido (2001a, Theorem 4.1, p.222)}

For any $t > 0$ and $\delta \geq 0$,

$$E[Z(t)] = \mu_1 \sum_{k=1}^{\infty} H_\delta^{*k}(t) = \mu_1 \int_0^t e^{-\delta v}\, dm(v)$$

where from {Léveillé and Garrido (2001a, Lemma 4.1, p.222)}

- $H_\delta^{*k}(t) = \int_0^t e^{-\delta v} dF^{*k}(v)$ for any $k \geq 0$;
- $\sum_{k=1}^{\infty} H_\delta^{*k}(t) = \int_0^t e^{-\delta v} dm(v)$, where $m(t) = E[N(t)] = \sum_{k=1}^{\infty} F^{*k}(t)$ is the renewal function associated with F.

4.1.2.2 Second moment of the CRPVR model

Here we consider the second moment. Then, using conditional expectations and assumptions (3) – (5) we get the following

$$\begin{aligned} E[Z^2(t)] &= E\left\{E\left[\left(\sum_{k=1}^{N(t)} e^{-\delta T_k} X_k\right)^2 |N(t)\right]\right\} \\ &= E\left\{E\left[\sum_{k=1}^{N(t)} e^{-2\delta T_k} X_k^2 + \sum_{i=1}^{N(t)} \sum_{j=1, j\neq i}^{N(t)} e^{-\delta(T_i+T_j)} X_i X_j |N(t)\right]\right\} \\ &= \mu_2 E\left[\sum_{k=1}^{N(t)} e^{-2\delta T_k}\right] + \mu_1^2 E\left[\sum_{i=1}^{N(t)} \sum_{j=1}^{N(t)} e^{-\delta(T_i+T_j)}\right]. \end{aligned}$$

Using the same arguments as in Theorem 4.1, the expectation above can be evaluated. The following theorem shows the explicit results for the second moment

Theorem 4.2 {Léveillé and Garrido (2001a, Theorem 4.2, p.223)}

For any $t > 0$ and $\delta \geq 0$,

$$E[Z^2(t)] = \mu_2 \sum_{k=1}^{\infty} H_{2\delta}^{*k}(t) + 2\mu_1^2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} H_{\delta}^{*k} * H_{2\delta}^{*k}(t)$$
$$= \mu_2 \int_0^{\infty} e^{-2\delta v} \, dm(v) + 2\mu_1^2 \int_0^{t} \int_0^{t-v} e^{-\delta(2v+u)} dm(u) dm(v)$$

4.1.3 Results for the ordinary renewal case

A delayed renewal process is a process where the distribution of the claims inter-occurrence times vary, such that $\{\tau_1\}$ is an independent random variable and $\{\tau_k\}_{k\geq 2}$ are iid with distribution function K and τ_1 has distribution F.

We adapt the results in (4.1.3) to the delayed renewal process and used the following notation to differentiate it from the ordinary case

- $Z_d(t)$ and $m_d(t)$ represents the discounted aggregate process and renewal function associated with the delayed renewal process.
- $Z_O(t)$ and $m_O(t)$ represents the discounted aggregate process and renewal function associated with the embedded ordinary renewal process.

Below are the results of the first moment:

Theorem 4.3 {Léveillé and Garrido (2001a, Theorem 6.1, p.227)}

For any $t > 0$ and $\delta \geq 0$,

$$E[Z_d(t)] = \mu_1 \int_0^{t} e^{-\delta v} \, dm_d(t)$$

Results for the second moment are given by the theorem below:

Theorem 4.4 {Léveillé and Garrido (2001a, Theorem 6.2, p.227)}

$$E[Z_d^2(t)] = \mu_2 \sum_{k=0}^{\infty} H_{2\delta} * I_{2\delta}^{*k}(t) + 2\mu_1^2 \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} H_{2\delta} * I_{2\delta}^{*n} * I_{\delta}^{*k}(t)$$
$$= \mu_2 \int_0^{t} e^{-2\delta v} \, dm_d(t) + 2\mu_1^2 \int_0^{t} \int_0^{t-v} e^{-\delta(2v+u)} \, dm_O(t) dm_d(t)$$

where,

- $H_\delta\,(t) = \int_0^t e^{-\delta v} dF(v)$ and $I_\delta(t) = \int_0^t e^{-\delta v} dG(v)$.
- $H_\delta * I_\delta^{*k}(t) = \int_0^t e^{-\delta v}\, dF * G^{*k}(v)$ for any $k \geq 0$.
- $\sum_{k=0}^{\infty} H_\delta * I_\delta^{*k}(t) = \int_0^t e^{-\delta v} d\, m_d(t)$ and $m_d(t) = \sum_{k=0}^{\infty} F * G^{*k}(t)$

4.2 Léveillé and Garrido (2001b): Recursive Moments of Compound Renewal Sums with Discounted Claims

Léveillé and Garrido (2001b) is an extension of Léveillé and Garrido (2001a) under regularity conditions. They derive the equations of moments using recursive formulas.

4.2.1 Definition of the risk model and its assumptions

1. The claim counting process $N(t,\omega) = \sup\{n \in \mathbb{N}, T_k(\omega)\}$ where $\omega \in \Omega$ and $sup\{\emptyset\} = 0$ forms an ordinary renewal process.

- The claim occurrence times are given $\{T_k\}_{k\geq 1}$;
- Claims inter-occurrence times are positive, and iid given by $\tau_k = T_k - T_{k-1},\ k \geq 2,\ T_0 = 0$ and $\tau_1 = T_1$, have a common continuous distribution function F;
- The Laplace transform of $T_1, \mathcal{L}_T$ exists over a subset of $\mathbb{R}$.

2. The corresponding deflated claim severities $\{X_k\}_{k\geq 1}$ are such that:

- $\{X_k\}_{k\geq 1}$ are independent and identically distributed;
- $\{X_k, \tau_k\}_{k\geq 1}$ are mutually independent;
- The moment generating function of X_1, M_x, exists over a subject Ω of $\mathbb{R}$;
- $\mu_k = E[X_1^k] > 0$

3. The aggregated discounted value at time 0 of the inflated claims recorded over the period $[0,t]$ yields

$$Z(t) = \sum_{k=1}^{N(t)} e^{\delta T_k} X_k$$

where $Z(t) = 0$ if $N(t) = 0$.

4.2.2 Results for the ordinary renewal case

The moments are obtained recursively using the following theorem:

Theorem 4.5 {Léveillé and Garrido (2001b, Theorem 2.1, p100}

For any $\delta > 0$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
M^n_{Z(t)}(t) &= \sum_{k=1}^{n-1} \binom{n}{k} \mu_{n-k} M^k_{Z(.)}(0) * \sum_{i=1}^{\infty} H^i_{n\delta}(t) \\
&= \sum_{k=1}^{n-1} \binom{n}{k} \mu_{n-k} \int_0^t e^{-n\delta v} M^k_{Z(t-v)}(0) dm(v)
\end{aligned}
$$

4.2.3 Results for the delayed renewal case

We adapt the results in (4.2.1) to the delayed renewal process and use the following notation to differentiate it from the ordinary case:

- $Z_d(t)$ and $m_d(t)$ represents the discounted aggregate process and renewal function associated with the delayed renewal process.
- $Z_O(t)$ and $m_O(t)$ represents the discounted aggregate process and renewal function associated with the embedded ordinary renewal process.

Results of the moment are shown below

Theorem 4.6 {Léveillé and Garrido (2001b, Theorem 4.1, p.104)}

For any $\delta > 0$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
M^n_{Z_d(t)}(0) &= \sum_{k=0}^{n-1} \binom{n}{k} \mu_{n-k} M^k_{Z_0(.)}(0) * \sum_{i=1}^{\infty} H^i_{n\delta} * I^i_{n\delta}(t) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \mu_{n-k} \int_0^t e^{-n\delta v} M^k_{Z_0(t-v)}(0) dm_d(v).
\end{aligned}
$$

4.3 Léveillé et al. (2010): Moment generating functions of compound renewal sums with discounted claims

Léveillé et al. (2010) refined the results by Léveillé and Garrido (2001a, 2001b), where the main focus was on the distribution of the discounted aggregate claims. They give the asymptotic results of the moments, which are finite.

4.3.1 Definition of the risk model and it's assumptions

1. The claim counting process $N(t,\omega) = \sup\{n \in \mathbb{N}, T_k(\omega)\}$ where $\omega \in \Omega$ and $sup\{\emptyset\} = 0$ forms an ordinary renewal process.

- The claim occurrence times are given $\{T_k\}_{k\geq 1}$;
- Claims inter-occurrence times are positive, and iid given by $\tau_k = T_k - T_{k-1},\ k \geq 2,\ T_0 = 0$ and $\tau_1 = T_1$, have a common continuous distribution function F;
- The Laplace transform of $T_1, \mathcal{L}_T$ exists over a subset of $\mathbb{R}$.

2. The corresponding deflated claim severities $\{X_k\}_{k\geq 1}$ are such that

- $\{X_k\}_{k\geq 1}$ are independent and identically distributed;
- $\{X_k, \tau_k\}_{k\geq 1}$ are mutually independent;
- The moment generating function of X_1, M_x, exists over a subject Ω of $\mathbb{R}$, including a neighbourhood of the origin;
- $\mu_k = E[X_1^k] > 0$

3. $P_{N(t)}\big(M_X(t)\big)$ is a probability generating function which exists over a subset Ω of $\mathbb{R}$.

4. The aggregated discounted value at time 0 of the inflated claims recorded over the period $[0,t]$ yields

$$Z(t) = \sum_{k=1}^{N(t)} e^{\delta T_k} X_k,$$

where $Z(t) = 0$ if $N(t) = 0$.

4.3.2 Risk model results

$\{N(t), t \geq 0\}$ is the claim number process which is assumed to form an ordinary renewal process, and the claims inter-occurrence times τ_k have a distribution F. The results are as follows:

Theorem 4.7 {Léveillé et al. (2010, Theorem 2.1, p.168)}

For any $t > 0,\ \delta \geq 0$ and $s \in \Omega$,

$$M_{Z(t)}(s) = \sum_{k=0}^{\infty} H_k(t,s),$$

where

- $H_k(t,s) = \int_0^t M_x(se^{-\delta v}) H_{k-1}(t-v, se^{-\delta v}) dF(v)$,
- $H_0(t,s) = \bar{F}(t)$,

and

Theorem 4.8 {Léveillé et al. (2010, Theorem 2.2, p.168)}

For any $t > 0$, $\delta \geq 0$ and $s \in \Omega$,

$$M_{Z(t)}(s) = \sum_{k=0}^{\infty} I_k(t,s),$$

where

- $I_k(t,s) = \int_0^t [M_X(se^{-\delta v}) - 1] I_{k-1}(t-v, se^{-\delta v}) dm(v)$,
- $H_0(t,s) = 1$.

4.4 Léveillé and Adékambi (2011): Covariance of discounted compound renewal sums with stochastic interest rate

Léveillé and Adékambi (2011) derive the first two moments and the joint moment of the discounted compound renewal sums in the presence of a stochastic force of interest.

4.4.1 Definition of the risk model and its assumptions

1. The number of claims $\{N(t), t \geq 0\}$ and $\{N_d(t), t \geq 0\}$ respectively form ordinary and delayed renewal processes and, for $k \in \mathbb{N} = \{1,2,3,4, \ldots\}$:

- The positive claim occurrence times are given $\{T_k\}_{k \geq 1}$;
- Claims inter-occurrence times are positive, and iid given by $\tau_k = T_k - T_{k-1}$, $k \geq 2$, $T_0 = 0$ and $\tau_1 = T_1$.

2. The k^{th} random claim is given by X_k, and

- $\{X_k\}_{k \geq 1}$ are independent and identically distributed;
- $\{X_k, \tau_k\}_{k \geq 1}$ are mutually independent;
- The first moment of X_1 exists.

3. The aggregated discounted value at time 0 of the inflated claims recorded over the period $[0, t]$ yields respectively, for the ordinary and delayed renewal processes:

$$Z(t) = \sum_{k=1}^{N(t)} D(T_k)X_k\,, \qquad Z_d(t) = \sum_{k=1}^{N_d(t)} D(T_k)X_k,$$

where

- $Z(t) = Z_d(t) = 0$, if $N(t) = N_d(t) = 0$.
- $D(T_k) = e^{-I(T_k)}$.
- $I(T_k) = \int_0^{T_k} \delta(x)dx$.

The stochastic force of interest is defined by $\delta(t)$ and the integral of each sample is finite over the interval $[0, \infty)$.

4.4.2 Results for the first moment

Theorem 4.9 {Léveillé and Adékambi (2011, Theorem 2.1, p.144)}

Using the assumptions of the risk model the discounted aggregate claims model first moment is given, $t > 0$, by:

1. For the ordinary renewal case:

$$E[Z(t)] = E[X_1]\int_0^t E[D(v)]dm(v).$$

2. For the delayed renewal case:

$$E[Z_d(t)] = E[X_1]\int_0^t E[D(v)]dm_d(v).$$

4.4.3 Results for the second moment

Theorem 4.10 {Léveillé and Adékambi (2011, Theorem 3.1, p.146)}

Using the assumptions of the risk model the discounted aggregate claims model second moment is given, $t > 0$, by:

1. For the ordinary renewal case:

$$E[Z^2(t)] = E[X_1^2]\int_0^t E[D^2(v)]dm(v) + 2E^2[X_1]\int_0^t\int_0^{t-v} E[D(v)D(v+u)]dm(u)dm(v).$$

2. For the delayed renewal case:

$$E\left[{Z_d}^2(t)\right] = E[X_1^2]\int_0^t E[D^2(v)]dm_d(v)$$
$$+ 2E^2[X_1]\int_0^t\int_0^{t-v} E[D(v)D(v+u)]dm_o(u)dm_d(v).$$

It is worth noting that theorem 4.10 gives a proof of theorem (4.4) if the discounted aggregate claims are discounted using a constant force of interest.

4.5 Léveillé and Adékambi (2012): Joint moments of discounted compound renewal sums

Léveillé and Adékambi (2012) is an extension of 4.4, which derives the recursive formula for joint moments of discounted aggregate claims with a constant force of interest and the non-recursive formulas for higher moments when the force of interest is stochastic.

4.5.1 Definition of the risk model and its assumptions

1. The number of claims $\{N(t), t \geq 0\}$ and $\{N_d(t), t \geq 0\}$ form, respectively, an ordinary and delayed renewal processes and, for $k \in \mathbb{N} = \{1,2,3,4, \ldots\}$:

- The positive claim occurrence times are given $\{T_k\}_{k\geq 1}$;
- Claims inter-occurrence times are positive, and iid given by $\tau_k = T_k - T_{k-1},\ k \geq 2,\ T_0 = 0$ and $\tau_1 = T_1$.

2. The k^{th} random claim is given by X_k, and

- $\{X_k\}_{k\geq 1}$ are independent and identically distributed;
- $\{X_k, \tau_k\}_{k\geq 1}$ are mutually independent;
- The first moment of X_1 M_x, exists over a subject Ω of $\mathbb{R}$, including a neighbourhood of the origin.

3. The aggregated discounted value at time 0 of the inflated claims recorded over the period $[0, t]$ yields respectively, for the ordinary and delayed renewal processes:

$$Z(t) = \sum_{k=1}^{N(t)} D(T_k)X_k, \qquad Z_d(t) = \sum_{k=1}^{N_d(t)} D(T_k)X_k,$$

where

- $Z(t) = Z_d(t) = 0$ if $N(t) = N_d(t) = 0$,
- $D(T_k) = e^{-I(T_k)}$ and
- $I(T_k) = \int_0^{T_k} \delta(x)dx$.

4.5.2 Results for the recursive moments (constant interest rate)

Here we present the recursive moments assuming a constant force of interest. For the results on stochastic force of interest, refer to Léveillé and Adékambi (2012, Theorem 3.2, p.10).

Theorem 4.11 {Léveillé and Adékambi (2012, Theorem 2.1, p.6)}

The joint moments between $Z(t)$ and $Z(t+h)$ are given, respectively, for $n, m \in \mathbb{N}$, by:

1. For the ordinary renewal case:

$$E[Z^n(t)Z^m(t+h)] = \sum_{k=1}^{n+m} E[X_1^k] \sum_{i=[k-m]_+}^{\min(k,n)} \binom{n}{i}\binom{m}{k-i} \times \int_0^t e^{-(n+m)\delta v} E\left[Z^{n-i}(t-v)Z^{m-(k-i)}(t+h-v)\right] dm(v)$$

2. For the delayed renewal case:

$$E[{Z_d}^n(t){Z_d}^m(t+h)] = \sum_{k=1}^{n+m} E[X_1^k] \sum_{i=[k-m]_+}^{\min(k,n)} \binom{n}{i}\binom{m}{k-i} \times \int_0^t e^{-(n+m)\delta v} E\left[{Z_o}^{n-i}(t-v){Z_o}^{m-(k-i)}(t+h-v)\right] dm_d(v)$$

where $[k-m]_+ = \max\{0, k-m\}$.

The above reviewed literature does not constitute the only papers covering independence: Jang (2004) used martingales and jump diffusion processes to get the moments, and Kim and Kim (2007) studied the problem in a Markovian environment. Their approaches differ from the one proposed in this minor dissertation, since their risk models assumed Markovian environment and jump diffusion processes.

As mentioned at the beginning of the chapter, our review was limited to specific papers that closely linked to our research. The papers cited above, derived the moments of discounted compound renewal sums where independence of inter-arrival times was assumed. The following chapter is a review on how to dispense with the independence assumption and looks at dependent inter-occurrence times.

Chapter 5: Relaxation of the independence assumption of inter-occurrence times

This chapter discusses the research by Albrecher, Constantinescu and Loisel (2011), since it is the paper which introduces the relaxation of the independence assumption of inter-occurrence times.

5.1 Albrecher et al. (2011): Explicit ruin formulas for models with dependence among risks

Albrecher et al. (2011) use a simple mixing idea to derive explicit formulas for the ruin probabilities and other quantities which relate to the collective risk models when dependence between cash flow sizes and cash flow inter-occurrence times are assumed. Examples given in the paper include a compound Poisson risk process and renewal risk models with dependent cash flow inter-occurrence times.

5.1.1 Definition of the risk model and it's assumptions

The risk model used by Albrecher et al. (2011) is the one proposed in (1997, p.22) with an exception that the cash flow sizes and cash flow inter-occurrence times are dependent, and the assumption that the cash flow inter-occurrence times are dependent.

5.1.2 Results

In this section we discuss the results for completely monotone claim sizes and Archimedean dependence as well as completely monotone inter-occurrence time distributions and Archimedean dependence.

5.1.2.1 Compound Poisson models with completely monotone cash flow sizes and Archimedean dependence

Let Θ denote a positive random variable with a cumulative distribution function F_Θ and consider the model $R(t) = u + ct - \sum_{k=1}^{N(t)} X_k$, where cash flow sizes are exponential, for each n,

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n | \Theta = \theta) = \prod_{k=1}^{n} e^{-\theta x_k} \quad (5.1)$$

We see that the cash flow sizes $\{X_k\}$ are conditionally independent and have an exponential distribution with parameter θ. Now the marginal distributions of $\{X_k\}$ will no longer be an exponential distribution and the cash flow sizes will be dependent. Let $\psi_\theta(u)$ denote the bankruptcy/ruin probability with independent cash flow amounts, which are exponentially distributed with parameter Θ. Then, $\psi_\theta(u)$ is given by:

$$\psi_\theta(u) = min\left\{\frac{\lambda}{\theta c} exp\left\{-\left(\theta - \frac{\lambda}{c}\right)u\right\}, 1\right\}. \tag{5.2}$$

For a dependence model (5.1), the ruin probability is given by:

$$\psi(u) = \int_0^\infty \psi_\theta(u)\, dF_\Theta(\theta). \tag{5.3}$$

If $\theta \leq \theta_0 = \frac{\lambda}{c}$, the net profit condition is breached and therefore $\psi_\theta(u) = 1$ for all $u \geq 0$, which can be written as:

$$\psi(u) = F_\Theta(\theta_0) + \int_0^\infty \psi_\theta(u)\, dF_\Theta(\theta). \tag{5.4}$$

due to the dependence induced in the model, we have:

$$\lim_{u\to\infty} \psi(u) = F_\Theta(\theta_0)\,, \tag{5.5}$$

if the random variable Θ probability mass is at or below $\theta_0 = \frac{\lambda}{c}$, then (5.5) is positive.

Example 3.5 {Pareto cash flows with Clayton copula dependence}

If $\Theta \sim Gamma(\alpha, \beta)$ with the following probability density function,

$$f_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \qquad \text{for all } \theta > 0,$$

then, the mixing distribution for the marginal cash flow size is

$$\bar{F}_X(x) = \int_0^\infty e^{-\theta x} f_\Theta(\theta) d\theta = \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \text{for } x \geq 0.$$

We can observe that X_k follows a Pareto (α, β) and due to Albrecher et al. (2011, proposition 2.1, p.3), the Archimedean survival copula has the following generator function,

$$\phi(t) = t^{-1/\alpha} - 1,$$

which is a Clayton copula with a parameter of α. Therefore, the upper tail dependence index between two cash flow amounts is

$$\lambda_U = 2 - 2^{-1/\alpha},$$

then, from (5.4) it now follows that for this model the ruin probability function is given by:

$$\psi(u) = 1 - \frac{\Gamma(\alpha, \beta\theta_0)}{\Gamma(\alpha)} + \theta_0 e^{\theta_0 u}\beta\left(1 + \frac{u}{\beta}\right)\frac{\Gamma(1-\alpha, (\beta+u)\theta_0)}{\Gamma(\alpha)},$$

where $\Gamma(\alpha, t) = \int_t^\infty k^{\alpha-1}e^{-k}dk$ is an incomplete Gamma function and $\theta_0 = \frac{\lambda}{c}$. From (5.5),

$$\lim_{u\to\infty} \psi(t) = 1 - \frac{\Gamma\left(\alpha, \frac{\beta\lambda}{c}\right)}{\Gamma(\alpha)}. \tag{5.6}$$

We use $\lim_{k\to\infty} \frac{\Gamma(s,k)}{x^{s-1}e^{-k}}$ and then one can simply deduce that the convergence towards this constant is asymptotic and of order u^{-1}. Set $u = 0$ and we obtain,

$$\psi(0) = 1 - \frac{\Gamma(\alpha, \beta\theta_0)}{\Gamma(\alpha)} + \beta\theta_0\frac{\Gamma(\alpha-1, \beta\theta_0)}{\Gamma(\alpha)}.$$

For more examples on $\{X_k\}$, refer to Albrecher et al. (2011, p.5-7)

5.1.2.2 Renewal risk models with completely monotone cash flow inter-occurrence time distributions and Archimedean dependence

Adapting the approach in (5.1.2.1) and mixing over the Poisson distribution with parameter λ. Let F_Λ denote the cumulative distribution function after mixing, then the probability function of ruin with dependent inter-occurrence times will be given by

$$\psi(u) = \int_0^\infty \psi_\Lambda(u)dF_\Lambda(\lambda), \qquad \text{for all } u \geq 0. \tag{5.7}$$

Albrecher et al. (2011, proposition 2.1, p.3) will still hold, but we must replace θ with λ. Then, the dependence structure between cash flows inter-arrival times $\{T_1, T_2, \dots\}$ will be described by an Archimedean copula with a generator function $\phi(t) = \left(\tilde{F}_\Lambda(t)\right)^{-1}$. Cash flows inter-arrival times $\{T_1, T_2, \dots\}$ are no longer exponentially distributed but they follow a completely monotone distribution $P(T_i > t) = \tilde{F}_\Lambda(t)$. Note that the net profit condition is breached whenever $\Lambda > \lambda_0 = {}^{c}/_{E[X_i]}$. Then, we have

$$\psi(u) = \int_0^{\lambda_0} \psi_\Lambda(u)dF_\Lambda(\lambda) + \bar{F}_\Lambda(\lambda_0), \qquad \text{for all } u \geq 0.$$

Likewise,

$$\lim_{u\to\infty}\psi(u)=\bar{F}_\Lambda(\lambda_0),$$

is positive, if and only if Λ has a positive probability which is larger or equal to λ_0.

Example 3.6 {Pareto inter-occurrence times with Clayton copula dependence}

As in example 3.5, $\Lambda\sim\Gamma(\alpha,\beta)$ and therefore the marginal distribution of the mixing distribution for inter-occurrence times is a Pareto distribution,

$$\bar{F}_T(t)=\int_0^\infty e^{-\lambda t}f_\Lambda(\lambda)d\lambda=\left(1+\frac{t}{\beta}\right)^{-\alpha}, \text{for all } t\geq 0.$$

From Albrecher et al. (2011, proposition 2.1, p.3) the Archimedean survival copula has the following generator function,

$$\phi(t)=t^{-1/\alpha}-1.$$

Consider a special case where the cash flow amounts with parameter θ, the ruin probability function will be:

$$\psi_\lambda(u)=\min\left\{\frac{\lambda}{\theta c}\exp\left\{-\left(\theta-\frac{\lambda}{c}\right)u\right\},1\right\},\qquad u\geq 0.$$

Incorporating (5.7) and $\lambda_0=c\theta$, we get the following formula,

$$\psi(u)=\frac{\beta^\alpha e^{-\theta u}}{\theta c}\left(\beta-\frac{u}{c}\right)^{-1-\alpha}\left(\alpha-\frac{\Gamma(\alpha+1,c\theta\beta-\theta u)}{\Gamma(\alpha)}\right)+\frac{\Gamma(\alpha,c\theta\beta)}{\Gamma(\alpha)},\qquad u\geq 0.$$

If $u=0$, we have

$$\psi(0)=\frac{1}{c\theta\beta}\left(\alpha-\frac{\Gamma(\alpha+1,c\theta\beta)}{\Gamma(\alpha)}\right)+\frac{\Gamma(\alpha,c\theta\beta)}{\Gamma(\alpha)},$$

and

$$\lim_{u\to\infty}\psi(u)=\frac{\Gamma(\alpha,c\theta\beta)}{\Gamma(\alpha)}.$$

We used $\lim_{k\to\infty}\frac{\Gamma(s,k)}{x^{s-1}e^{-k}}$ and then one can simply deduce that the convergence towards this constant is asymptotic again, and of order u^{-1}. For more examples on dependent inter-occurrence times, refer to Albrecher et al. (2011, p.9-10).

Since the goal of Chapter 5 was to review the paper with dependent inter-occurrence times, this is summarised above. In Chapter 6 we will answer the research question posed in Chapter 1.

Chapter 6: Moments of the discounted compound renewal sums with dependent inter-occurrence times.

The aim of this chapter is to solve the research problem mentioned in Chapter 1.

6.1 Definition of the risk model and it's assumptions

1. The number of cash flows $\{N(t), t \geq 0\}$ form ordinary and delayed renewal processes and, for $k \in \mathbb{N} = \{1,2,3,4, \dots\}$:

- The positive cash flow occurrence times are given $\{T_k\}_{k\geq 1}$;
- Cash flows inter-occurrence times are positive, and iid given by $w_k = T_k - T_{k-1}$, $k \geq 2$, and $w_1 = T_1$ (since $T_0 = 0$).

2. The k^{th} random cash flow is given by X_k, and

- $\{X_k\}_{k\geq 1}$ are independent and identically distributed;
- $\{X_k, w_k\}_{k\geq 1}$ are mutually independent;
- The first moment of X_1 M_x, exists over a subset Ω of $\mathbb{R}$, including a neighbourhood of the origin.

3. The aggregated discounted value at time 0 of the inflated cash flow recorded over the period $[0, t]$ yields:

$$Z(t) = \sum_{k=1}^{N(t)} D(T_k) X_k,$$

where

- $Z(t) = Z_d(t) = 0$ if $N(t) = N_d(t) = 0$,
- $D(T_k) = e^{-I(T_k)}$ and
- $I(T_k) = \int_0^{T_k} \delta(x) dx$.

6.2 Dependent inter-occurrence times

In the usual ordinary renewal risk process, the sequence $\{W_j\}_{j=1}^{\infty}$ is assumed to be mutually independent. Therefore, in this paper we assume that $W_1, W_2, W_{3,\dots}$ are dependent and joined by Archimedean Copulas. Let Θ be a random variable with pdf $f_\Theta(\theta)$ and we suppose that the Laplace transformation of Θ given by

$f_\Theta^*(s) = \int_0^\infty e^{-s\theta} f_\Theta(\theta) d\theta$ exists over a subset $K \subset \mathbb{N}$ including a neighbourhood of the origin.

For a general setup, the results obtained above by using an exponential distribution for the conditional distribution of the time between successive cash flows, can be extended to other conditionally independent distributions. For example, the conditional distribution of the inter-claims time can be written in the power from $P(W_i \geq w_i|\Theta = \theta) = (\bar{H}(w_i))^{\theta}$ for some distribution function $H(x_i)$ and

$$P(W_1 \geq w_1, W_2 \geq w_2, \dots, W_n \geq w_n|\Theta = \theta) = \prod_{i=1}^{n}\left(\bar{H}(w_i)\right)^{\theta}. \quad (6.2.1)$$

For all n, i.e. is the common mixture parameter, then

$$\bar{F}_{W_1,\dots W_n}(w_1, \dots, w_n) = \int_0^{\infty} P(W_1 \geq w_1, \dots, W_n \geq w_n|\Theta = \theta) f_{\Theta}(\theta)d\theta$$

$$\begin{aligned}\bar{F}_{W_1,\dots W_n}(w_1, \dots, w_n) &= \int_0^{\infty} \left(\bar{H}(w_1)\right)^{\theta} \dots \left(\bar{H}(w_n)\right)^{\theta} f_{\Theta}(\theta)d\theta \\ &= f_{\Theta}^{*}\left(-\log(\bar{H}(w_1) - \dots - \log(\bar{H}(w_n)\right) \\ &= f_{\Theta}^{*}\left(f_{\Theta}^{*\,-1}\left(\bar{F}_{W_1}(w_1)\right) + \dots + f_{\Theta}^{*\,-1}\left(\bar{F}_{W_n}(w_n)\right)\right) \quad (6.2.2)\end{aligned}$$

This an Archimedean dependence structure with generator $\phi(t) = f_{\Theta}^{*-1}(t)$, and where $\bar{F}_{W_i}(x) = f_{\Theta}^{*}\left(-\log\left(\bar{H}(x_i)\right)\right)$.

6.2.1 Remark: Specific mixture of exponential distributions.

If the random variables $W_1, W_2, \dots, W_n$ are n dependent, positive and continuous random variables, and that given $\Theta = \theta$, the random variables $W_1, W_2, \dots, W_n$ are conditionally independent and distributed as $Exp(\theta)$ and

$$\begin{aligned}P(W_1 \geq w_1, W_2 \geq w_2, \dots, W_n \geq w_n|\Theta = \theta) &= P(W_1 \geq w_1|\Theta = \theta) \dots P(w_n \geq x_n|\Theta = \theta) \\ &= e^{-\theta w_1} \dots e^{-\theta w_n} \quad (6.2.3)\end{aligned}$$

It follows that

$$\bar{F}_{w_1}(x_1) = P(W_1 \geq x_1) = \int_0^{\infty} e^{-\theta x_1} f_{\Theta}(\theta)d\theta = f_{\Theta}^{*}(x) = \int_0^{\infty} P(W_1 \geq x_1|\Theta = \theta) f_{\Theta}(\theta)d\theta \ (6.2.4)$$

The joint distribution of the tail of $W_1, W_2, \dots, W_n$ is given by:

$$\bar{F}_{W_1,\dots,W_n}(w_1,\dots,w_n) = \int_0^\infty P(W_1 \ge w_1, W_2 \ge w_2, \dots, W_n \ge w_n | \Theta = \theta) f_\Theta(\theta) d\theta$$
$$= \int_0^\infty e^{-\theta \sum_{i=1}^n w_i} f_\Theta(\theta) d\theta$$
$$= f_\Theta^*\left(\sum_{i=1}^n w_i\right)$$
$$= f_\Theta^*\left(f_\Theta^{*-1}\left(\bar{F}_{W_1}(w_1)\right) + \dots + f_\Theta^{*-1}\left(\bar{F}_{W_n}(w_n)\right)\right). \tag{6.2.5}$$

Otherwise, from the Sklar Theorem,

$$P(W_1 \ge w_1, \dots, W_n \ge w_n) = \bar{C}\left(\bar{F}_{W_1}(w_1), \dots, \bar{F}_{W_n}(w_n)\right). \tag{6.2.6}$$

It follows that $\bar{C}(u_1, \dots, u_n)$ is an Archimedean copula with,

$$\bar{C}(u_1, \dots, u_n) = f_\Theta^*\left(f_\Theta^{*-1}(u_1) + \cdots f_\Theta^{*-1}(u_n)\right) = \phi^{-1}\left(\phi(u_1) + \dots + \phi(u_n)\right), \tag{6.2.7}$$

where $\phi(t) = f_\Theta^{*-1}(t)$ is the generator of the Archimedean copula $\bar{C}$.

As in H. Albrecher et.al (2011), if $\Theta \sim Gamma(\alpha, \beta)$ with pdf $f_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$, for $\theta > 0$. It follows that $W_i \sim Pareto(\alpha, \beta)$ with survival function,

$\bar{F}_{W_i}(w) = P(W_i \ge w) = \mathcal{L}_\Theta(w) = \left(1 + \frac{w}{\beta}\right)^{-\alpha}$, $w > 0$ and $f_\Theta^{*-1}(t) = \beta\left(t^{-\frac{1}{\alpha}} - 1\right)$.

The multivariate Pareto survival function of $W_1, W_2, \dots, W_n$ can be written as,

$\bar{F}_{W_1, W_2, \dots, W_n}\left(w_1, w_2, \dots, w_n\right) = \left(1 + \sum_{i=1}^n \frac{w_i}{\beta}\right)^{-\alpha}$ $w_i > 0, \forall i = 1, \dots, n$ and $\alpha, \beta > 0$. The associated copula is the Clayton copula given by:

$$C_\alpha(u_1, \dots, u_n) = \left(u_1^{-\frac{1}{\alpha}} + \dots + u_n^{-\frac{1}{\alpha}} - n + 1\right)^{-\alpha}.$$

6.2.1.2 Dependent Gamma inter-arrival Cash flows

There is total monotonicity for the gamma distributions, and we can fit it to the model with the dependence introduced in section 6.2 Where the shape parameter takes values $\alpha \in (0,1]$. For more information on the distribution see Gleser (1989) and Albrecher & Kortschak (2009). The results are as follows;

Let $W \sim \Gamma(\alpha, \lambda)$ denote a gamma distribution where the shape parameter is $\alpha \in (0,1]$ and scale parameter is λ. The probability density function is given by:

$$f_W(w) = \frac{\lambda^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w}, w > 0$$

then,

$$f_W(w) = \int_0^\infty e^{-\theta w} f_\Theta(\theta) d\theta \tag{6.2.8}$$

where

$$f_\Theta(\theta) = \begin{cases} \frac{(\theta - \lambda)^{-\alpha} \lambda^\alpha}{\theta \Gamma(1-\alpha)\Gamma(\alpha)}, \lambda \leq \theta < \infty \\ 0 \qquad\qquad\qquad otherwise \end{cases}$$

Lemma 6.2.1 The Laplace transformation of random variables with probability density function (6.2.8) is given by:

$$\mathcal{L}_\Theta(s) = \frac{\Gamma(\alpha, \lambda s)}{\Gamma(\alpha)}, s \geq 0.$$

Where $\Gamma(s, w) = \int_w^\infty t^{s-1} e^{-t} dt$ denotes a gamma function which is incomplete from the upper limit.

Proof

From the Laplace transform of Θ, $f_\Theta^*(s) = \int_0^\infty e^{-s\theta} f_\theta(\theta) d\theta$. We take the derivative of $f_\Theta^*(s)$ with respect to s and then:

$$\begin{aligned} \frac{d f_\Theta^*(s)}{ds} &= \int_0^\infty -\theta e^{-s\theta} f_\Theta(\theta) d\theta \\ &= -f_W(s) \\ &= -\frac{\lambda^\alpha}{\Gamma(\alpha)} s^{\alpha-1} e^{-\lambda s}. \end{aligned}$$

Then,

$$f_\Theta^*(s) = \int_s^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} s^{\alpha-1} e^{-\lambda s} ds.$$

With the change of variable $\lambda s = u$, we have: $f_\Theta^*(s) = \int_{\lambda s}^\infty \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du = \frac{\Gamma(\alpha, \lambda s)}{\Gamma(\alpha)}$.

Using Lemma 3.2.1, we get the following generator function

$$\phi(t) = f_{\Theta}^{*-1}(t) = Q_{G_\alpha}(1-t),$$

Where Q_{G_α} is the quantile function of a gamma distribution where the scale parameter is 1 and the mean is α.

From Lemma 3.2.1, the joint survival function is, $P(W_1 \geq w_1, \dots, W_n \geq w_n) = \frac{\Gamma(\alpha, \lambda \sum_{i=1}^{n} w_i)}{\Gamma(\alpha)}$, if $w_1, \dots, w_n \geq 0$, with marginal survival functions, $P(W_i \geq w) = \frac{\Gamma(\alpha, \lambda w)}{\Gamma(\alpha)}$, for $w \geq 0$,

$i = 1,2, \dots, n$.

6.2.1.3 General Weibull inter-arrival cash flows with Gumbel Copula Dependence

Let us consider a mixing random variable which follows a positive stable distribution (for more information on positive stable distribution see Feller 1971). We have the following probability distribution function:

$$f_{\Theta}(\theta) = -\frac{1}{\pi\theta}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{\mathrm{k}!}(-\theta^{\alpha})^{k}\sin(\alpha k\pi),$$

and the Laplace transform transformation, $f_{\Theta}^{*}(s) = e^{-s^{\alpha}}, s \geq 0$, and $\alpha \in (0,1]$. Using the expression of $f_{\Theta}^{*}(s)$, we have $P(W_1 \geq w_1, W_2 \geq w_2, \dots, W_n \geq w_n) = e^{-(w_1+w_2+\cdots+w_n)^{\alpha}}$, with marginal distribution $P(W_i > w) = \frac{\Gamma(\alpha, \lambda w)}{\Gamma(\alpha)}, w \geq 0$, for $i = 1,2, \dots, n,$. These are Weibull distributions and have the shape parameter $\alpha \in (0,1]$. The generator of the Archimedean copula is $\phi(t) = f_{\Theta}^{*-1}(t) = (-\log(t))^{\frac{t}{\alpha}}$ and the survival copula are given by:

$$\bar{C}_{\alpha}(u_1, \dots, u_n) = \exp\left(-\sum_{k=1}^{\infty}(-log(u_{k)})^{\frac{1}{\alpha}}\right).$$

6.2.1.4 Inverse Gaussian Mixture of exponential inter-arrival cash flows

If $\Theta \sim IG(\lambda, \mu)$ the mixing random variable is an inverse Gaussian distribution with the parameters $\mu > 0$ and $\lambda > 0$. The probability distribution function is given as

$$f_\Theta(\theta) = \sqrt{\frac{\lambda}{2\pi}}\theta^{-\frac{3}{2}}\exp\left(-\frac{\lambda(\theta-\mu)^2}{2\mu^2\theta}\right), \theta > 0.$$

The mixing distribution which corresponds to the marginal inter-arrival times W_j is

$$P(W_i > x) = \int_0^\infty e^{-\theta x} f_\Theta(\theta)d\theta = \exp\left(-\frac{\lambda}{\mu}\left(\sqrt{1+\frac{2\mu^2 x}{\lambda}} - 1\right)\right), x \geq 0, \forall i = 1,2,\ldots,n$$

And we also have the following joint survival function

$P(W_1 \geq w_1, W_2 \geq w_2, \ldots, W_n \geq w_n) = \exp\left(-\frac{\lambda}{\mu}\left(\sqrt{1+\frac{2\mu^2\sum_{i=1}^n w_i}{\lambda}} - 1\right)\right)$,

if $w_1, w_2, \ldots, w_n \geq 0$. This model was introduced by Whitemore (1988) and then extended by Whitemore and Lee (1991) to a multivariate case. The generator is given as,

$\phi(t) = f_\Theta^{*-1}(t) = \frac{\lambda}{2\mu^2}\left\{\left(1 - \frac{\mu}{\lambda}\log(t)\right)^2 - 1\right\}$, the survival copula is given by:

$\bar{C}(u_1, \ldots, u_n) = exp\left(-\frac{\lambda}{\mu}\left[\left(\sum_{i=1}^n \left(1 - \frac{\mu}{\lambda}\log(u_i)\right)^2 - n + 1\right)^{\frac{1}{2}} - 1\right]\right)$,

if $0 \leq u_i \leq 1, i = 1,2,\ldots,n$.

6.3 First moment

From the first moment of Léveillé and Adékambi (2011), we incorporate the mixing random variable Θ.

Lemma 6.3.1

The conditional probability density function of $T_k|N(t) = n, \Theta = \theta$ is given, respectively, for $0 < x \leq t$ and $k \leq n$, by:

$$f_{T_k|N(t)=n,\Theta=\theta}(x|n,\theta) = \frac{P(N(t-x) = n-k|\Theta=\theta)f_{T_k|\Theta=\theta}(x|\theta)}{P(N(t) = n|\Theta=\theta)} \qquad (6.3.1)$$

Proof

For $n \in \mathbb{N} \cup \{0\}$

$$
\begin{aligned}
P(T_k \le x | N(t) = n, \Theta = \theta) &= \frac{P(N(t) = n, \Theta = \theta, T_k \le x)}{P(N(t) = n, \Theta = \theta)} \\
&= \frac{P(N(t) = n | T_k \le x, \Theta = \theta) P(T_k \le x, \Theta = \theta)}{P(N(t) = n, \Theta = \theta)} \\
&= \frac{P(N(t) = n | T_k \le x, \Theta = \theta) P(T_k \le x | \Theta = \theta) f_\Theta(\theta)}{P(N(t) = n | \Theta = \theta) f_\Theta(\theta)} \\
&= \frac{\left\{\int_0^x P(N(t) = n | T_k \le x, T_k = v, \Theta = \theta) f_{T_k | T_k \le x, \Theta = \theta}(\theta) dv\right\} \times P(T_k \le x | \Theta = \theta)}{P(N(t) = n | \Theta = \theta)} \\
&= \frac{\int_0^x P(N(t) - N(v) = n - k | T_k = v, \Theta = \theta) f_{T_k | \Theta = \theta}(v|\theta) dv}{P(N(t) = n | \Theta = \theta)} \\
&= \frac{\int_0^x P(N(t) - N(v) = n - k | \Theta = \theta) f_{T_k | \Theta = \theta}(v|\theta) dv}{P(N(t) = n | \Theta = \theta)} \\
&= \frac{\int_0^x P(N(t - v) = n - k | \Theta = \theta) f_{T_k | \Theta = \theta}(v|\theta) dv}{P(N(t) = n | \Theta = \theta)}
\end{aligned}
$$

which establishes (6.3.1), with $P(N(t) - N(v) = n - k) = P(N(t - v) = n - k)$.

Theorem 6.3.1

Given the assumptions of section 6.2, the first moment of the discounted aggregate cash flows is given, for $t > 0$, by:

$$E[Z(t)] = E[X_1] \int_0^\infty \int_0^t E[D(v)] dm\,(v|\theta) f_\Theta(\theta) d\theta$$

Proof

Conditioning on $N(t)$ and Θ, then using independence between the number and the severity of claims yields:

$$E[Z(t) | N(t) = n, \Theta = \theta] = E[X_1] \sum_{k=1}^{n} E[D(T_k) | N(t) = n, \Theta = \theta]$$

From equation 6.3.1 of lemma 6.3.1, we have:

$$E\big[Z(t) | \delta(x), x \in [0, t]\big] = E\left[E\big[Z(t) | N(t) = n, \Theta = \theta, \delta(x), x \in [0, t]\big]\right]$$

$$= E[X_1]\sum_{k=1}^{n}\int_{0}^{t} D(v) f_{T_k|N(t)=n,\Theta=\theta}(v|n,\theta)dv$$

$$= E[X_1]\sum_{k=1}^{n}\int_{0}^{t} D(v)\frac{P(N(t-v)=n-k|\Theta=\theta) f_{T_k|\Theta=\theta}(v|\theta) f_\Theta(\theta)}{P(N(t)=n|\Theta=\theta) f_\Theta(\theta)}$$

$$= E[X_1]\sum_{k=1}^{n}\int_{0}^{t} D(v)\frac{P(N(t-v)=n-k|\Theta=\theta) f_{T_k|\Theta=\theta}(v|\theta) f_\Theta(\theta)}{P(N(t)=n,\Theta=\theta)}$$

Then,

$$E\big[Z(t)|\delta(x),x\in[0,t]\big] = E\left[E\big[Z(t)|N(t)=n,\Theta=\theta,\delta(x),x\in[0,t]\big]\right]$$

$$= E[X_1]\sum_{n=0}^{\infty}\sum_{k=1}^{n}\int_{0}^{\infty}\int_{0}^{t} D(v)P(N(t-v)=n-k|\Theta=\theta) f_{T_k|\Theta=\theta}(v|\theta) f_\Theta(\theta)dv$$

$$= E[X_1]\sum_{k=1}^{\infty}\int_{0}^{\infty}\int_{0}^{t} D(v) f_{T_k|\Theta=\theta}(v|\theta)\left\{\sum_{n=k}^{\infty} P(N(t-v)=n-k|\Theta=\theta)\right\} f_\Theta(\theta)dv$$

$$= E[X_1]\sum_{k=1}^{\infty}\int_{0}^{\infty}\int_{0}^{t} D(v) f_{T_k|\Theta=\theta}(v|\theta) f_\Theta(\theta)dvd\theta$$

$$= E[X_1]\sum_{k=1}^{\infty}\int_{0}^{\infty}\int_{0}^{t} D(v)d\left\{\sum_{k=1}^{\infty} F^{*k}_{T_k|\Theta=\theta}(v|\theta)\right\} f_\Theta(\theta)d\theta$$

$$= E[X_1]\sum_{k=1}^{\infty}\int_{0}^{\infty}\int_{0}^{t} D(v)dm(v|\theta) f_\Theta(\theta)d\theta$$

Where $m(v|\theta) = \sum_{k=1}^{\infty} F^{*k}_{T_k|\Theta=\theta}(v|\theta) = E[N(v)=k|\Theta=\theta]$.

Since the last integral is a random variable, we use a well-known theorem of stochastic processes theory (see Karatzas & Shreve 1991, p.3) to finally obtain:

$$E[Z(t)] = E\left[E\big[Z(t)|\delta(x),x\in[0,t]\big]\right]$$

$$= E[X_1]E\left[\int_{0}^{\infty}\int_{0}^{t} D(v)dm(v|\theta) f_\Theta(\theta)d\theta\right]$$

$$= E[X_1] \int_0^{\infty} \int_0^{t} E[D(v)]\, dm(v|\theta) f_{\Theta}(\theta) d\theta$$

Example 6.3.1

Let $\{\delta(t), t \geq 0\}$ be an Itô process satisfying the stochastic differential equation of Ho-Lee Merton

$$d\delta(t) = rdt + \sigma dB(t),$$

with constant drift r, constant diffusion coefficient σ, and where $B(t)$ is a standard Brownian motion (see Cairns, 2004, p.87).

From the Itô theory (see Karatzas & Shreve, 1991; Oksendal, 1992), it can be shown that:

$$\int_0^t \delta(x)dx \sim N\left(\delta(0)t + r\frac{t^2}{2}, \sigma^2\frac{t^3}{3}\right).$$

We assume that:

$W_1|\Theta = \theta \sim Exp(\theta)$, $E[X_1 = 1]$, $\delta(0) = 0.06$, $r = 0.004$ and $\sigma = 0.01$.

We set: $\Theta \sim \Gamma(1,1)$

Hence, Theorem 4.1.1 yields:

$$E[Z(t)] = E[X_1]E[\Theta] \int_0^t exp\left\{-\delta(0)v - r\frac{v^2}{2} + \sigma^2\frac{v^3}{6}\right\} dv.$$

Then, with the help of the software MATLAB, we have Table 1 below.

Table 1: First moment of Z(t) Ho-Lee-Merton case

$E[Z(1)]$	$E[Z(5)]$	$E[Z(10)]$	$E[Z(15)]$	$E[Z(20)]$

0.984823097	4.606115332	8.380626312	11.32412846	13.50862841
$E[Z(30)]$	$E[Z(40)]$	$E[Z(50)]$	$E[Z(60)]$	$E[Z(70)]$
16.08951873	17.15895279	17.52411659	17.626955761	17.650864229

Using the same parameter's values as in Léveillé and Adékambi (2011), we get the same results if $E[\Theta] = 1$. The results will vary if $E[\Theta] \neq 1$, for $E[\Theta] > 1$ the value of $E[Z(t)]$ will be higher and lower otherwise. The introduction of an economic variable only affects $E[Z(t)]$ if the value of $E[\Theta] \neq 1$.

6.4 Second moment

From the results of Léveillé and Adékambi (2011), we mix over the involved parameter Θ.

Lemma 6.4.1

Consider a renewal counting process, such as defined in section 6.3. The conditional joint density probability functions of $T_i, T_j | N(t) = n$ are given, for $0 < x < y < t$ and $1 \leq i < j \leq n$, by:

$$f_{T_i,T_j|N(t)=n,\Theta=\theta}(x, y|n, \theta) = \frac{P(N(t-y) = n-j|\Theta = \theta) f_{T_{j-i}|\Theta=\theta}(y-x|\theta) f_{T_i|\Theta=\theta}(x|\theta)}{P(N(t) = n|\Theta = \theta)} \quad (6.4.1)$$

Proof

As in Lemma 4.1.1, we get for $n \in \mathbb{N} - \{0\}$:

$$\begin{aligned} P\left(T_i \leq x, T_j \leq y | N(t) = n, \Theta = \theta\right) &= \frac{P\left(N(t) = n, T_i \leq x, T_j \leq y, \Theta = \theta\right)}{P(N(t) = n, \Theta = \theta)} \\ &= \frac{P\left(N(t) = n | T_i \leq x, T_j \leq y, \Theta = \theta\right) P\left(T_i \leq x, T_j \leq y, \Theta = \theta\right)}{P(N(t) = n, \Theta = \theta)} \end{aligned}$$

The last equation can be written as:

$$P(T_i \le x, T_j \le y | N(t) = n, \Theta = \theta)$$
$$= \frac{\left\{\int_0^x \int_u^y P\big(N(t) = n | T_i = u, T_j = v, \Theta = \theta\big) f_{T_i,T_j|T_i \le x, T_j \le y, \Theta=\theta}(u, v) dv du\right\} \times P\big(T_i \le x, T_j \le y, \Theta = \theta\big)}{P(N(t) = n, \Theta = \theta)}$$
$$= \frac{\int_0^x \int_u^y P(N(t-y) = n-j | \Theta = \theta) f_{T_i,T_j|\Theta=\theta}(u, v|\theta) dv du}{P(N(t) = n | \Theta = \theta)},$$

with:

$$\begin{aligned} P\big(T_i \le u, T_j \le v | \Theta = \theta\big) &= P\big(T_j \le v | T_i \le u, \Theta = \theta\big) P(T_i \le u | \Theta = \theta) \\ &= \int_0^u P\big(T_j \le v | T_i = z, \Theta = \theta\big) f_{T_i|T_i \le u, \Theta=\theta}(z) P(T_i \le u | \Theta = \theta) \\ &= \int_0^u P\big(W_{i+1} + W_{i+2} + \cdots + W_j \le v - z | \Theta = \theta\big) f_{T_i|\Theta=\theta}(z|\theta) dz \\ &= \int_0^u P\big(T_{j-i} \le v - z | \Theta = \theta\big) f_{T_i|\Theta=\theta}(z|\theta) dz \end{aligned}$$

we have

$$f_{T_i,T_j|\Theta=\theta}(u, v|\theta) = f_{T_{j-i}}(v-u|\theta) f_{T_i|\Theta=\theta}(u|\theta).$$

and finally,

$$P(T_i \le x, T_j \le y | N(t) = n, \Theta = \theta)$$
$$= \frac{\int_0^x \int_u^y P(N(t-v) = n-j | \Theta = \theta) f_{T_{j-i}|\Theta=\theta}(v-u|\theta) f_{T_i|\Theta=\theta}(u|\theta) dv du}{P(N(t) = n | \Theta = \theta)}$$

That gives equation (6.4.1).

Theorem 6.5.1

Given the assumptions of section 6.2, the second moment of the discounted aggregate cash flows is given, for $t > 0$, by:

$$E[Z^2(t)] = E[X_1^2]\int_0^{\infty}\int_0^{t} E[D^2(v)]dm(v|\theta)f_{\Theta}(\theta)d\theta$$
$$+2E^2[X_1]\int_0^{\infty}\int_0^{t}\int_0^{t-v} E[D(u+v)D(v)]dm(u|\theta)dm(v|\theta)f_{\Theta}(\theta)d\theta$$

Proof

Conditioning on $N(t)$ and Θ and using independence between the number and the severity of claims yields:

$$E[Z^2(t)|N(t)=n,\Theta=\theta] = E\left[\sum_{k=1}^{n} D^2(T_k)X_k^2\,|N(t)=n,\Theta=\theta\right]$$
$$+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E\big[D(T_i)D(T_j)X_iX_j|N(t)=n,\Theta=\theta\big]$$
$$= E[X_1^2]\sum_{k=1}^{n} E[D^2(T_k)|N(t)=n,\Theta=\theta]$$
$$+2E^2[X_1]\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E\big[D(T_i)D(T_j)|N(t)=n,\Theta=\theta\big]$$

From equation (3.1) of Lemma 3.1 and of equation (4.1) of Lemma 4.1, we have:

$$E\big[Z^2(t)|\delta(x),x\in[0,t]\big] = E\left[E\big[Z^2(t)|N(t)=n,\Theta=\theta,\delta(x),x\in[0,t]\big]\right]$$
$$= E[X_1^2]\sum_{n=0}^{\infty}\sum_{k=1}^{n}\int_0^{\infty}\int_0^{t} D^2(v)f_{T_k|\Theta=\theta}(v|\theta)P(N(t-v)=n-k|\Theta=\theta)dvd\theta$$
$$+2E^2[X_1]\sum_{n=0}^{\infty}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\int_0^{\infty}\int_0^{t}\int_v^{t} D(u)D(v)P(N(t-u)=n-j|\Theta=\theta)$$
$$\times f_{T_{j-i|\Theta=\theta}}(u-v|\theta)f_{T_i|\Theta=\theta}(v|\theta)dudvd\theta$$

The permutations of the sums give us:

$$E\left[Z^2(t)|\delta(x), x\in[0,t]\right] = E[X_1^2]\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\int_0^t D^2(v) f_{T_{k|\Theta=\theta}(v|\theta)} P(N(t-v)=n-k|\Theta=\theta)dvd\theta$$

$$+2E^2[X_1]\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{n=j}^{\infty}\int_0^t\int_v^t D(u)D(v)P(N(t-u)=n-j|\Theta=\theta)f_{T_{j-i}|\Theta=\theta}(u-v|\theta)\, f_{T_i|\Theta=\theta}(v|\theta)dvd\theta$$

$$= E[X_1^2]\sum_{k=1}^{\infty}\int_0^{\infty}\int_0^t D^2(v)\, dm(v|\theta) f_\theta(\theta)d\theta$$

$$+2E^2[X_1]\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\int_0^{\infty}\int_0^t\int_0^{t-v} D(u+v)D(v)P(N(t-u)=n-j|\theta)f_{T_{j-i}|\Theta=\theta}(u-v|\theta)\, f_{T_i|\Theta=\theta}\,(v|\theta)dudvd\theta$$

Hence, following the same reasoning used in Theorem 6.3.1, we have:

$$E[Z^2(t)] = E\left[E\left[Z^2(t)|\delta(x), x\in[0,t]\right]\right]$$

$$= E[X_1^2]\int_0^{\infty}\int_0^t E[D^2(v)]dm(v|\theta)f_\Theta(\theta)d\theta$$

$$+2E^2[X_1]E\left[\sum_{i=1}^{\infty}\int_0^{\infty}\int_0^t\int_0^{t-v} D(u+v)D(v)f_{T_i|\Theta=\theta}(v|\theta)d\left\{\sum_{j=i+1}^{\infty}F^{*(j-i)}_{\tau_1|\Theta=\theta}\,(u|\theta)\right\}f_\Theta(\theta)dvd\theta\right]$$

$$= E[X_1^2]\int_0^{\infty}\int_0^t E[D^2(v)]dm(v|\theta)f_\Theta(\theta)d\theta$$

$$+2E^2[X_1]E\left[\sum_{i=1}^{\infty}\int_0^{\infty}\int_0^t\int_0^{t-v} D(u+v)D(v)dm(u|\theta)dF^{*i}_{\tau_1}(v|\theta)f_\Theta(\theta)d\theta\right]$$

$$= E[X_1^2]\int_0^{\infty}\int_0^t E[D^2(v)]dm(v|\theta)f_\Theta(\theta)d\theta$$

$$+2E^2[X_1]\int_0^{\infty}\int_0^t\int_0^{t-v} E[D(u+v)D(v)]dm(u|\theta)dm(v|\theta)f_\Theta(\theta)d\theta$$

Example 6.4.1

Let $\{\delta(t), t\geq 0\}$ be an Itô process satisfying the stochastic differential equation of Ho-Lee Merton.

We assume that:

$W_1|\Theta = \theta \sim Exp(\theta), E[X_1 = 1], \delta(0) = 0.06, r = 0.004$ and $\sigma = 0.01$.

We set: $\Theta \sim \Gamma(1,1)$

Hence, we have:

$$E[D^2(v)] = e^{-2\delta(0)v + \frac{2}{3}\sigma^2 v^3},$$

$$E[D(u)D(u+v)] = \exp\left\{-[\delta(0)(u+2v)] - \frac{r}{2}[u^2 + 2uv + 2v^2] + \frac{\sigma^2}{2}\left[\frac{(u+2v)^3 + u^3}{6}\right]\right\}.$$

Then, with the help of the software MATLAB, we have Table 2 below for the second moment.

Table 2: Second moment of Z(t) Ho-Lee-Merton case

$E[Z(1)]$	$E[Z(5)]$	$E[Z(10)]$	$E[Z(15)]$	$E[Z(20)]$
3.879661704	50.94100489	154.7094908	274.2246112	384.7070601
$E[Z(30)]$	$E[Z(40)]$	$E[Z(50)]$	$E[Z(60)]$	$E[Z(70)]$
539.0970931	610.6647840	636.1433241	643.4223167	645.1230498

6.5 Joint moment

As in sections 6.3 and 6.4, we incorporate the mixing random variable Θ.

Theorem 6.5

Given the assumptions of section 3, the joint moments of the compound discounted aggregate cash flows for $t > 0$ and $h > 0$, are given by:

$$E[Z(t)Z(t+h)] = E[Z^2(t)] + E^2[X_1]\int_0^\infty \int_0^t \int_{t-v}^{t+h-v} E[D(u+v)D(v)]dm(u|\Theta=\theta)dm(v|\Theta=\theta)f_\Theta(\theta)\,d\theta$$

Proof

We have:

$$E[Z(t)Z(t+h)] = E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=1}^{N(t+h)} D(T_j)X_j\right]$$
$$= E[Z^2(t)] + E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=N(t)+1}^{N(t+1)} D(T_j)X_j\right]$$

Conditioning on N(t), N(t + h) and Θ, we obtain the following for the second term:

$$E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=N(t)+1}^{N(t+h)} D(T_j)X_j | \delta(x), x \in [0, t+h]\right]$$
$$= E\left[E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=N(t)+1}^{N(t+h)} D(T_j)X_j | \delta(x), x \in [0, t+h] \,| N(t), N(t+h), \Theta, \delta(x), x \in [0, t+h]\right]\right]$$
$$= E[X_1^2]E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D(T_j) | \, \delta(x), x \in [0, t+h]\right]$$

Now,

$$E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D\left(T_j\right) \mid \delta(x), x \in [0, t+h]\right]$$

$$= E\left[E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D\left(T_j\right) \mid N(t), N(t+h), \Theta\, \delta(x), x \in [0, t+h]\right]\right]$$

$$= \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^{\infty} E\left[D(T_i) D\left(T_j\right) | N(t), N(t+h), \Theta\right] P(N(t) = n, N(t+h) = m, \Theta = \theta) d\theta$$

$$= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{m=j}^{\infty} \sum_{j=1}^{m} \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u) D(v) f_{T_i, T_j, N(t), N(t+h), \Theta}(v, u, n, m, \theta) du dv d\theta$$

$$= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{m=j}^{\infty} \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u) D(v)\, f_{T_i, T_j, N(t+h), \Theta}(v, u, n, \theta) du dv d\theta$$

$$= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{m=j}^{\infty} \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u) D(v) f_{T_i, T_j | N(t+h)=m, \Theta=\theta}(v, u | m, \theta) P(N(t+h) = m | \Theta = \theta) \times f_{\Theta}(\theta) du dv d\theta$$

Then from equation (6.4) of Lemma 6.4, we get:

$$E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D(T_j) \mid \delta(x), x \in [0, t+h]\right]$$

$$= \sum_{i=1}^{\infty} \sum_{j=i=1}^{\infty} \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u)D(v) \left[\sum_{m=j}^{\infty} P(N(t+h-u) = m-j|\Theta = \theta)\right]$$

$$\times f_{T_{j-i}|\Theta=\theta}(u-v|\theta) f_{T_i|\Theta=\theta}(v|\theta) f_{\Theta}(\theta) dudvd\theta$$

$$= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u)D(v) f_{T_{j-i}|\Theta=\theta}(u-v|\theta) f_{T_i|\Theta=\theta}(v|\theta) f_{\Theta}(\theta) dudvd\theta$$

$$= \sum_{i=1}^{\infty} \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u)D(v) f_{T_i|\Theta=\theta}(v|\theta) d\left\{\sum_{j=i+1}^{\infty} F_{T_1|\Theta=\theta}^{*(j-i)}(u-v|\theta)\right\} f_{\Theta}(\theta) d\theta$$

$$= \int_0^{\infty} \int_0^{t} \int_t^{t+h} D(u)D(v) dm(u-v|\Theta=\theta) d\left\{\sum_{i=1}^{\infty} F_{T_1|\Theta=\theta}^{*i}(v|\theta)\right\} f_{\Theta}(\theta) d\theta$$

$$= \int_0^{\infty} \int_0^{t} \int_{t-v}^{t+h-v} D(u+v)D(v) dm(u|\Theta=\theta) dm(v|\Theta=\theta) f_{\Theta}(\theta) d\theta$$

From theorems 4.1 and 4.2, we get:

$$E[Z(t)Z(t+h)] = E\big[E[Z(t)Z(t+h)|\delta(x), x \in [0, t+h]]\big]$$

$$= E[Z^2(t)] + E^2[X_1] E\left[\int_0^{\infty} \int_0^{t} \int_{t-v}^{t+h-v} E[D(u+v)D(v)] dm(u|\Theta=\theta) dm(v|\Theta=\theta) f_{\Theta}(\theta) d\theta\right]$$

$$= E[Z^2(t)] + E^2[X_1] \int_0^{\infty} \int_0^{t} \int_{t-v}^{t+h-v} E[D(u+v)D(v)] dm(u|\Theta=\theta) dm(v|\Theta=\theta) f_{\Theta}(\theta) d\theta$$

6.6 Linear predictor

Consider an instantaneous interest rate $\delta > 0$ which is constant and a conditional on the Poisson process with a mixing random variable Θ. Then equation (4.3) yields:

$$E[Z(t)Z(t+h)|\Theta=\theta] = E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)\Theta + E^2[X_1]\left\{\left(\frac{1-e^{-\delta t}}{\delta}\right)^2 + e^{-\delta t}\frac{(1-e^{-\delta t})(1-e^{-\delta h})}{\delta^2}\right\}\Theta^2$$

$$= E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)\Theta + \frac{E^2[X_1]}{\delta^2}(1-e^{-\delta t})(1-e^{-\delta(t+h)})\Theta^2 \qquad (6.6.1)$$

and,

$$E[Z(t)Z(t+h)] = E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)E[\Theta] + \frac{E^2[X_1]}{\delta^2}\left(1-e^{-\delta t}\right)\left(1-e^{-\delta(t+h)}\right)E[\Theta^2]$$

it follows that,

$$\begin{aligned}\mathrm{Cov}[Z(t), Z(t+h)|\Theta = \theta] \\ &= E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)\Theta + \frac{1}{\delta^2}E^2[X_1]\left(1-e^{-\delta t}\right)\left(1-e^{-\delta(t+h}\right)\Theta^2 \\ &- \frac{1}{\delta^2}E^2[X_1]\left(1-e^{-\delta t}\right)\left(1-e^{-\delta(t+h}\right)\Theta^2 \\ &= E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)\Theta\end{aligned}$$

and,

$$\mathrm{Cov}[Z(t), Z(t+h)] = E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)E[\Theta],$$

The covariance is independent of the variable h and this implies that $V[Z(t)]$ is equal to the covariance. $V[Z(t)]$ is almost constant for larger values of t. Let $\rho(t,h)$ denote the correlation between $Z(t)$ and $Z(t+h)$, then we get:

$$\begin{aligned}\rho(t,h) &= \frac{E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)E[\Theta]}{\left[E[X_1^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right)E[\Theta]\right]^{1/2}\left[E[X_1^2]\left(\frac{1-e^{-2\delta(t+h)}}{2\delta}\right)E[\Theta]\right]^{1/2}} \\ &= \left[\frac{1-e^{-2\delta t}}{1-e^{-2\delta(t+h)}}\right]^{1/2}\end{aligned}$$

$\rho(t,h)$ tends to $\left[1-e^{-2\delta t}\right]^{1/2}$ when h approaches ∞. We also note that for a smaller t and a larger h , $\rho(t,h)$ is almost 0 as expected.

This strong result just shows us that the correlation coefficient $\rho(t,h)$ is independent of the specific mixture of exponential distributions for inter-arrival times.

We consider a linear predictor $L(t,h) = a + bZ(t)$, where a and b eventually depends on t and h, by minimising the function $A_{t,h}$ defined by:

$$A_{t,h}(a,b) = E[(Z(t) - Z(t+h))^2].$$

Then the partial derivative of $A_{t,h}$ with respect to a and b, that we equal to 0, gives

$$\frac{\partial A_{t,h}(a,b)}{\partial a} = -2E[Z(t+h) - a - bZ(t)] = 0,$$
$$\frac{\partial A_{t,h}(a,b)}{\partial b} = -E\big[Z(t)\big(Z(t+h) - a - bZ(t)\big)\big] = 0\,.$$

So, we get the following system of linear equations

$$\begin{pmatrix} 1 & E[Z(t)] \\ E[Z(t)] & E[Z^2(t)] \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E[Z(t+h)] \\ E[Z(t)Z(t+h)] \end{pmatrix},$$

with solutions

$$b = \frac{\mathrm{Cov}\big(Z(t), Z(t+h)\big)}{\mathrm{Var}[Z(t)]},$$
$$a = \frac{E[Z^2(t)]E[Z(t+h)] - E[Z(t)]E[Z(t+h)Z(t)]}{\mathrm{Var}[Z(t)]}.$$

Let us assume that the correlation is sufficiently strong on the period $[t, t+h]$, then the formula for the linear predictor $Z(t+h)$, given the value of $Z(t)$, is given by:

$$L(t,h) = E[Z(t+h)] + \rho(t,h)\left[\frac{V[Z(t+h)]}{V[Z(t)]}\right]^{1/2}\big[Z(t) - E[Z(t)]\big].$$

We now consider the case where the cash flow is 1, and the number of cash flows follows a conditional Poisson distribution of the parameter $\Theta \sim \Gamma(1,1)$ and $\delta = 0.005$. Then, from the results of Sections 6.3, 6.4 and 6.5, we have

$$E[Z(t)] = E[\Theta]\left(\frac{1-e^{-\delta t}}{\delta}\right), \qquad Var[Z(t)] = \left(\frac{1-e^{-2\delta t}}{2\delta}\right)E[\Theta],$$

$$\rho(t,h) = \left[\frac{1-e^{-2\delta t}}{1-e^{-2\delta(t+h)}}\right]^{1/2}.$$

We compare the simulated value of $Z(t+h)$ to the values of $L(t,h)$ in Table 3 below for different values of t and h.

Table 3: Comparison between z_simul (t+h)|z(t) and L(t,h)

t	h	$Z(t)$	$Z_{simul}(t+h)\|Z(t)$	$L(t,h)$
1	0.01	0.99177	1.0047	1.0017
1	1	0.99177	1.9974	1.9843
1	10	0.99177	15.7276	10.6972
10	0.01	15.6814	16.6039	15.6912
10	1	15.6814	17.4954	16.6303
10	10	15.6814	29.9252	24.9601
100	0.01	62.6035	63.1082	62.6095
100	1	62.6035	63.7794	63.2085
100	10	62.6035	71.7902	68.5196

The above table shows a comparison between $z_{simul}(t+h)|z(t)$ and $L(t,h)$. We observe that the errors between $z_{simul}(t+h)|z(t)$ and $L(t,h)$ are high in absolute terms. The simulated values are always higher than the actual values. Hence, more work still needs to be done to ensure that the simulated values are in line with the expectations.

6. 7 Conclusion

We have derived explicitly the formulas for the first and second moments of the discounted aggregate compound renewal sums for a stochastic instantaneous interest rate with dependent inter-occurrence times. The techniques used are an extension of those used by Léveillé and Adékambi (2011, 2012). Possible extensions to this research include the computation of higher moments of order $n \in \mathbb{N} - \{0\}$ for the same problem.

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