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On the rigidity of stable maps to Calabi–Yau threefolds

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If $X \subset Y$ is a nonsingular curve in a Calabi–Yau threefold whose normal bundle $N_{X/Y}$ is a generic semistable bundle, are the local Gromov–Witten invariants of X well defined? For X of genus two or higher, the issues are subtle. We will formulate a precise line of inquiry and present some results, some positive and some negative.

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1 Introduction

In 1998, Gopakumar and Vafa [5] proposed a duality between SU(N) Chern–Simons theory on the 3–sphere and topological string theory on the resolved conifold. As evidence, Gopakumar and Vafa showed the large–N free energy in Chern–Simons theory exactly matches (after a change of variables) the topological string partition function on the resolved conifold.

Mathematically, the *topological string partition function* is just the natural generating function for the Gromov–Witten invariants. The *resolved conifold* is the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, considered as a Calabi–Yau threefold. The Gromov–Witten theory of the noncompact total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is well-defined: all (nonconstant) stable maps have image contained in the zero section and thus their moduli spaces are compact.

The Gromov–Witten invariants of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ are often regarded as the *local Gromov–Witten invariants* of $\mathbb{P}^1$. Indeed, if $X \subset Y$ is any smoothly embedded rational curve in a Calabi–Yau threefold Y with normal bundle isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then the contribution of X to the Gromov–Witten invariants of Y is well-defined and is given by the corresponding invariants of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We consider here the local theory of higher genus curves. If $X \subset Y$ is a nonsingular curve in a Calabi–Yau threefold whose normal bundle $N_{X/Y}$ is a generic semistable bundle, are the local Gromov–Witten invariants of X well defined? For X of genus two or higher, the issues are subtle. We will formulate a precise line of inquiry and present some results, some positive and some negative.

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2 Definitions and results

Let $X \subset Y$ be a nonsingular genus–g curve in a threefold Y with normal bundle $N_{X/Y}$ of degree $2g-2$. If Y is Calabi–Yau, the condition on the normal bundle is always satisfied. We define the following notions of rigidity:

Definition 2.1

(i) A curve $X \subset Y$ is *(d,h)–rigid* if for every degree–d, genus–h stable map $f\colon C \to X$, we have $H^0(C, f^*N_{X/Y}) = 0$.

(ii) A curve $X \subset Y$ is *d–rigid* if $X \subset Y$ is (d,h)–rigid for all genera h.

(iii) A curve $X \subset Y$ is *super-rigid* if $X \subset Y$ is d–rigid for all $d > 0$.

For example, a nonsingular rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is super-rigid. An elliptic curve $E \subset Y$ is d–rigid if and only if $N_{E/Y} \cong L \oplus L^{-1}$ where $L \to E$ is a flat line bundle which is not d–torsion (see Pandharipande [7]).

For a (d,h)–rigid curve $X \subset Y$, the contribution of X to $N^h_{d[X]}(Y)$, the genus–h Gromov–Witten invariant of Y in the class $d[X]$, is well-defined and given by

$$\int_{[\overline{M}_h(X,d[X])]^{\mathrm{vir}}} c_{\mathrm{top}}(R^1\pi_* f^*N), \tag{1}$$

where $\overline{M}_h(X,d)$ is the moduli space of degree–d, genus–h stable maps to X,

$$\pi\colon\ U \to \overline{M}_h(X,d)$$

is the universal curve,

$$f\colon\ U \to X$$

is the universal map, and $[\ ]^{\mathrm{vir}}$ denotes the virtual fundamental class. The (d,h)–rigidity of X guarantees that $R^1\pi_* f^*N$ is a *bundle*. See Bryan–Pandharipande [2] for an expanded discussion.

By definition, (d,h)–rigidity is a condition on the normal bundle $N_{X/Y}$. Assuming $N_{X/Y}$ is generic, we may ask for which pairs (d,h) does (d,h)–rigidity hold. The 1–rigidity of a generic normal bundle is straightforward and was used in [7]. We prove the following positive result.

Theorem 2.2 *If $X \subset Y$ is a genus–g curve in a threefold Y and $N_{X/Y}$ is a generic stable bundle of degree $2g-2$, then X is 2–rigid.*

However, 3–rigidity is *not* satisfied for genus–3 curves.

Theorem 2.3 *If $X \subset Y$ is a genus–3 curve in a threefold Y with*

$$\deg(N_{X/Y}) = 4,$$

then X is not 3–rigid.

Let $N \to X$ be a generic stable bundle of degree $2g-2$. By Theorem 2.2, the degree–2 Gromov–Witten theory of the total space of N considered as a noncompact threefold is well-defined by the integral (1).

In the case when X embeds in a threefold Y with normal bundle N, we may regard the above theory as the degree–2 local Gromov–Witten theory of $X \subset Y$. Such embeddings of X can always be found. For example, let Y be the threefold $\mathbb{P}(\mathcal{O}_X \oplus N)$ with the embedding $X \subset Y$ determined by the trivial factor. It would be interesting to construct a curve in a Calabi–Yau threefold with a 2–rigid normal bundle.

The degree–2 *local* theory of X is a *global* theory for N. Strong global integrality constraints, obtained from the Gopakumar–Vafa conjecture [4] and more recently from the conjectural Gromov–Witten/Donaldson–Thomas correspondence of Maulik–Nekrasov–Okounkov–Pandharipande [6], should therefore hold for the degree–2 local theory of X.

In [3, 2], a formal local theory is defined in all degrees for $X \subset Y$. By Theorem 2.2, the above local theory coincides with the degree–2 formal local theory defined in [3, 2]. However by Theorem 2.3, the formal local theory of [3, 2] does *not* correspond exactly to a well-defined global theory of N in degree 3.

Using degeneration techniques, we have recently obtained the complete computation of the formal local invariants of curves in all degrees [1]. The expected integrality holds for the degree–2 invariants. Integrality for the degree–3 formal local invariants fails, in fact the breakdown occurs for precisely the *same* domain genus–h for which rigidity fails.

3 Proof of Theorem 2.2

Let X be a nonsingular curve of genus g. Let N be a bundle of rank 2 and degree $2g-2$ on X. The bundle N is 2–rigid if

$$H^0(C, f^*(N)) = 0 \tag{2}$$

for all stable maps $f\colon C \to X$ of degree 2. If $g = 0$, then

$$N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

is 2–rigid. For $g > 0$, we will prove 2–rigidity holds on an open set of the irreducible moduli space of semistable bundles on X. As the 2–rigidity statement was already proven for $g = 1$ in [7], we will assume $g > 1$.

Let Λ be a fixed line bundle of degree $2g-2$. Let $\overline{M}_X(2,\Lambda)$ be the moduli space of rank–2, semistable bundles with determinant Λ. Since 2–rigidity is a generic condition, we need only prove the existence of a 2–rigid bundle $[N] \in \overline{M}_X(2,\Lambda)$.

We first prove $H^0(X,N) = 0$ for an open set $V \subset \overline{M}_X(2,\Lambda)$. Since the vanishing of sections is an open condition and the moduli space is irreducible, we need only find an example. If $L \in \mathrm{Pic}^{g-1}(X)$ then the bundle

$$N = L \oplus L^{-1}\Lambda$$

is semistable. Since the locus of Pic^{g-1} determined by bundles with nontrivial sections is a divisor, neither L nor $L^{-1}\Lambda$ have sections for generic L. Thus

$$H^0(X, L \oplus L^{-1}\Lambda) = 0.$$

If there exists a nonzero section $s \in H^0(C, f^*(N))$ for a stable map f, then s must be nonzero on some dominant, irreducible component of $C' \subset C$. Also, s must be nonzero when pulled-back to the normalization of C'. Therefore, to check 2–rigidity for N, we need only prove

$$H^0(C, f^*(N)) = 0$$

for maps $f\colon C \to X$ where the domain is nonsingular and irreducible.

Let C be nonsingular and irreducible, and let $f\colon C \to X$ be a (possibly ramified) double cover. Let $B \subset X$ be the branch divisor of f. The map f is well-known to determine a square root of $\mathcal{O}(-B)$ in the Picard group of X by the following construction. Let $E = f_*(\mathcal{O}_C)$. E is a rank–2 bundle on X with a $\mathbb{Z}/2\mathbb{Z}$–action induced by the Galois group of f, and decomposes into a direct sum

$$E \cong \mathcal{O} \oplus Q$$

of $+1$ and -1 eigenbundles for the action. The $+1$ eigenbundle is the trivial line bundle $\mathcal{O}$ while the -1 eigenbundle is a line bundle Q with $Q^2 \cong \mathcal{O}(-B)$.

If $[N] \in V$, a sequence of isomorphisms

$$H^0(C, f^*(N)) \cong H^0(X, f_*f^*(N)) \cong H^0(X, (\mathcal{O} \oplus Q) \otimes N) \cong H^0(X, Q \otimes N)$$

is obtained from the geometry of the double cover f. To prove the 2–rigidity of N, it therefore suffices to prove the vanishing of $H^0(X, Q \otimes N)$ for all double covers f.

Suppose $H^0(X, Q \otimes N) \neq 0$ for a double cover f. Then there is a nonzero sheaf map

$$\iota\colon\ Q^{-1} \to N.$$

Let S be the saturation of the image of ι. Then S has the following properties:

(i) S is a subbundle of N,

(ii) S^2 has a section.

Property (ii) is proven as follows. The saturation short exact sequence of sheaves on X is

$$0 \to Q^{-1} \to S \to \mathcal{O}_D \to 0$$

for some effective divisor D on X. Hence $S \cong Q^{-1}(D)$, and

$$S^2 \cong Q^{-2}(2D) \cong \mathcal{O}(B + 2D).$$

The line bundle clearly has a section since both B and D are effective.

For $d \geq 0$, define $\Delta(d) \subset \overline{M}_X(2, \Lambda)$ to be the following locus in the moduli space:

$$\{[N] \mid \text{there exists a subbundle } S \subset N \text{ with } \deg(S) = d \text{ and } H^0(X, S^2) \neq 0\}$$

If $[N] \in V$ and N is not 2–rigid, then we have proven $[N]$ must lie in $\Delta(d)$ for some d. By Lemma 3.1 below, the proof of Theorem 2.2 is complete. □

Lemma 3.1 *For all d, $\dim \Delta(d) < \dim \overline{M}_X(2, \Lambda)$.*

Proof By stability, $\Delta(d)$ is empty if $d \geq g - 1$. We may assume

$$0 \leq d < g - 1.$$

If $[N] \in \Delta(d)$, then N is given by an extension

$$0 \to S \to N \to S^{-1} \otimes \Lambda \to 0.$$

The number of parameters for S is g, the dimension of $\mathrm{Pic}^d(X)$. The number of parameters of the space of extensions is

$$h^1(X, S^2 \otimes \Lambda^{-1}) - 1 = 3g - 3 - 2d - 1,$$

by Riemann–Roch. Therefore

$$\dim \Delta(d) \leq g + 3g - 3 - 2d - 1 = 4g - 4 - 2d.$$

If $d > (g-1)/2$, then $\dim \Delta(d) < \dim M_X(2, \Lambda) = 3g - 3$.

By the above analysis, we may assume $0 \leq d \leq (g-1)/2$. We will now redo the dimension count for $\Delta(d)$ using the condition $H^0(S^2) \neq 0$. By Brill–Noether theory, the dimension of the space of line bundles of degree $2d < g$ having nonzero sections is $2d$. Recomputing, we find

$$\dim \Delta(d) \leq 2d + 3g - 3 - 2d - 1 = 3g - 4.$$

Therefore $\dim \Delta(d) < \dim \overline{M}_X(2, \Lambda)$ for all d. □

4 Proof of Theorem 2.3

Let X be a nonsingular genus–3 curve. For any rank–2 bundle N on X of degree 4, we will show there exists a degree–3 stable map

$$f\colon\ C \to X$$

for which $H^0(C, f^*(N)) \neq 0$.

If N contains a rank–1 subbundle $L \subset N$ of positive degree, then $H^0(X, L^3) \neq 0$ by Riemann–Roch. Let s be a nonzero section of L^3 and let $f\colon\ C \to X$ be the cube root of s in L. Since $H^0(C, f^*(L))$ has a canonical section, we have shown $H^0(C, f^*(N)) \neq 0$. If N is unstable, then it has a destabilizing rank–1 subbundle of positive degree, and is not 3–rigid.

Let Λ be a line bundle on X of degree 4. Let $\overline{M}_X(2, \Lambda)$ be the moduli space of rank–2, semistable bundles with determinant Λ. We will prove, for general

$$[N] \in \overline{M}_X(2, \Lambda),$$

that N contains a rank–1 subbundle of degree 1. Hence the general semistable bundle N with determinant Λ is not 3–rigid. By semicontinuity and the irreducibility of the moduli of semistable bundles, we conclude no semistable bundle on X is 3–rigid.

The dimension of the moduli space $\overline{M}_X(2, \Lambda)$ is 6. The extensions

$$0 \to L \to N \to L^{-1}\Lambda \to 0, \tag{3}$$

where L has rank 1 and degree 1, are parametrized (up to scale) by a projective bundle B with fiber $\mathbb{P}(H^1(X, L^2\Lambda^{-1}))$ over $\mathrm{Pic}^1(X)$. Since the dimensions of $\mathrm{Pic}^1(X)$ and $\mathbb{P}(H^1(X, L^2\Lambda^{-1}))$ are both 3, the dimension of B is 6. An elementary argument shows the map to moduli,

$$\epsilon\colon\ B \to \overline{M}_X(2, \Lambda),$$

is well-defined on a (nonempty) open set of B. If ϵ is dominant, the theorem is proven.

It suffices to prove that the tangent space to B generically surjects onto the tangent space of $\overline{M}_X(2,\Lambda)$. The following argument was provided by Michael Thaddeus.

Let $\mathrm{End}_0 N$ be the bundle of traceless endomorphisms of N. The tangent space to $\overline{M}_X(2,\Lambda)$ at N is given by $H^1(X,\mathrm{End}_0 N)$. The exact sequence (3) induces a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = \mathrm{End}_0 N$$

where

$$E_1/E_0 = L^2\Lambda^{-1}, \quad E_2/E_1 = \mathcal{O}, \quad \text{and } E_3/E_2 = L^{-2}\Lambda.$$

For generic L,

$$H^0(X, L^{-2}\Lambda) = H^1(X, L^{-2}\Lambda) = 0.$$

Hence, the inclusion $E_2 \subset E_3$ induces isomorphisms

$$H^1(X,E_2) \cong H^1(X,\mathrm{End}_0 N) \quad \text{and} \quad H^0(X,E_2) \cong H^0(X,\mathrm{End}_0 N) = 0.$$

Consequently, the exact sequence for the inclusion $E_1 \subset E_2$ reduces to

$$0 \to H^1(X, L^2\Lambda^{-1})/H^0(X,\mathcal{O}) \to H^1(X,\mathrm{End}_0 N) \to H^1(X,\mathcal{O}) \to 0.$$

The tangent space of $\overline{M}_X(2,\Lambda)$ at N is therefore identified with the tangent space of B at N. Indeed, $H^1(X,L^2\Lambda^{-1})/H^0(X,\mathcal{O})$ is the space of deformations of the extension class and $H^1(X,\mathcal{O})$ is the space of deformations of L. □

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