A family of embedding spaces

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Abstract

We study $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$ the space of C^∞-smooth embeddings of spheres in spheres, $\mathcal{K}_{n,j}$ the space of 'long' embeddings of $\mathbb{R}^j$ in $\mathbb{R}^n$, and spaces of embeddings of spheres in euclidean space $\mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$, and their framed analogues. We describe some of the basic features of these spaces: their first non-trivial homotopy-groups, actions of operads of cubes on the most elementary of these spaces, some natural maps between these spaces and their properties. In the process, we give a new geometric description to Haefliger's knots, showing, among other things, that a graphing/spinning construction analogous to the Litherland deform-spun knot construction gives an isomorphism of groups $\pi_2\mathcal{K}_{4,1} \to \pi_0\mathrm{Emb}(\mathrm{S}^3, \mathrm{S}^6)$.

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1 Introduction

This paper was motivated by a rather elementary observation: both $\pi_2\mathcal{K}_{4,1}$ and $\pi_0\mathcal{K}_{6,3}$ are infinite-cyclic groups, so perhaps there is a geometrically-inspired isomorphism between the two. We show that a graphing construction $\Omega^2\mathcal{K}_{4,1} \to \mathcal{K}_{6,3}$ induces an isomorphism $\pi_2\mathcal{K}_{4,1} \to \pi_0\mathcal{K}_{6,3}$. More generally we will study graphing constructions of the form $\Omega^i\mathcal{K}_{n,j} \to \mathcal{K}_{n+i,j+i}$. We describe how these maps fit in with some of the basic geometric properties of these spaces: their first non-trivial homotopy groups, the 'concatenation' monoid structure and the actions of operads of little cubes on $\mathcal{K}_{n,j}$ and their framed analogues $\mathrm{EC}(j, \mathrm{D}^{n-j})$.

Section 2 briefly covers the most elementary relationships between the spaces $\mathcal{K}_{n,j}$, $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$, $\mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$, $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n)$, $\mathcal{P}_{n,j}$ and the framed spaces $\mathrm{EC}(j, \mathrm{D}^{n-j})$ and $\mathrm{PEC}(j, \mathrm{D}^{n-j})$.

Section 3 begins with a 'motivational' proof of an old theorem of Haefliger's, that $\pi_0\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$ is a group provided $n - j > 2$, the group operation being connect-sum. The proof is via a permutation of the main concepts of Haefliger's original argument: A homotopy-equivalence $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n) \simeq \mathrm{SO}_{n+1} \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n,j}$ reduces the problem to the monoid structure of $\pi_0\mathcal{K}_{n,j}$. There is a fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ where $\mathcal{P}_{n,j}$ is a pseudoisotopy embedding space of discs. A similar homotopy-equivalence $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{P}_{n,j}$ tells us that $\pi_1\mathcal{K}_{n-1,j-1} \to \pi_0\mathcal{K}_{n,j}$ is onto provided $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n)$ is connected. That $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n)$ is connected for $n-j > 2$ is a classical theorem originally due to Smale.

Section 4 investigates the extent to which the fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ is equivariant with respect to an action of the operad of $(j-1)$-cubes. These actions extend in a natural way to an action of the operad of j-cubes on appropriate spaces of 'framed' embeddings $\mathrm{EC}(j, \mathrm{D}^n) \to \mathrm{PEC}(j, \mathrm{D}^n) \to \mathrm{EC}(j-1, \mathrm{D}^n)$, which ultimately lead us in Section 5 to a $(j+1)$-cubes equivariant fibre-sequence $\Omega\mathrm{PEC}(j, \mathrm{D}^n) \to \Omega\mathrm{EC}(j-1, \mathrm{D}^n) \to \mathrm{EC}(j, \mathrm{D}^n)$. At present the iterated loop-space structures of the spaces $\mathrm{EC}(j, \mathrm{D}^n)$ are largely mysterious. There are two very different theorems that have something to say about these spaces. One is Morlet's Comparison Theorem which says $\mathrm{EC}(j, \mathrm{D}^0) \simeq \Omega^{j+1}(PL(j)/O_j)$ where $PL(j)$ is a suitable space of PL-automorphisms of $\mathbb{R}^j$, and O_j is the corresponding orthogonal group. The other, a theorem of the author's, says that $\mathrm{EC}(1, \mathrm{D}^2) \simeq \mathbb{Z} \times \mathcal{C}_2(\mathcal{P} \sqcup \{*\})$, where $\mathcal{P} \subset \mathcal{K}_{3,1}$ is the subspace of long knots which are prime. $\mathcal{C}_2(\mathcal{P} \sqcup \{*\})$ indicates the free 2-cubes object on the space $\mathcal{P} \sqcup \{*\}$. Both theorem describe the iterated loop space structure of some of the spaces $\mathrm{EC}(j, \mathrm{D}^n)$, but the disparity in the answers is rather perplexing. In a sense, this paper is an attempt to construct a few elementary connections between the various spaces $\{\mathrm{EC}(j, \mathrm{D}^n) : j, n > 0\}$.

In Section 5 we give a geometric interpretation of the first non-trivial homotopy groups of the spaces $\mathcal{K}_{n,j}$ provided $2n-3j-3 \geq 0$. In Proposition 5.1 we show that the map $\Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ induced from the fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ is homotopic to $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$

$$(\mathrm{gr}_1 f)(t_0, t_1, \cdots, t_{j-1}) = (t_0, f(t_0)(t_1, \cdots, t_{j-1})).$$

In the above formula, we think of a loop space ΩX as being a space of functions $f : \mathbb{R} \to X$ such that $f(\mathbb{R} \setminus \mathbf{I}) = *$. We show this map is epic between the 1st non-trivial homotopy groups of $\Omega\mathcal{K}_{n-1,j-1}$ and $\mathcal{K}_{n,j}$ respectively. The main technical ingredients we need are some computations of Sinha, Scannell and Turchin, together with the dissertation of Goodwillie. The first non-trivial homotopy group of $\mathcal{K}_{n,1}$ is $\pi_{2n-6}\mathcal{K}_{n,1} \simeq \mathbb{Z}$, and we describe this isomorphism as a signed count of quadrisecants, in direct analogy with a previous paper with Conant, Scannell and Sinha [11].

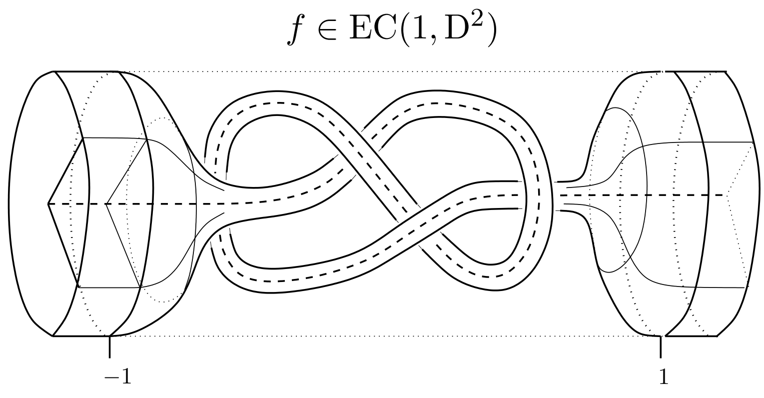
Section 6 mentions, in a very terse fashion, some basic results on the homotopy-type of the spaces $\mathcal{K}_{n,j}$. We describe what is known about several natural maps of the form: $\mathcal{K}_{n,j} \to \mathcal{K}_{n+1,j}$, $\mathcal{K}_{n,j} \to \Omega\mathcal{K}_{n+j,j}$, and $\mathcal{K}_{n,j} \to \Omega\mathcal{K}_{n,j-1}$.

We start with definitions of the spaces studied and notation.

Definition 1.1

- $\mathrm{D}^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ is called either the unit n-disc, respectively. $\partial\mathrm{D}^n = \mathrm{S}^{n-1}$.
- $\mathbf{I} = [-1,1] = \mathrm{D}^1$ is the standard interval.
- The space of proper embeddings of a disc in a disc is denoted $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n)$. We put no requirements on the embeddings other than being proper, ie: $f : \mathrm{D}^j \to \mathrm{D}^n$ satisfies $f(\mathrm{D}^j) \cap \partial\mathrm{D}^n = f(\partial\mathrm{D}^j)$. All our embedding spaces will be endowed with the weak C^∞-topology [35].
- The space of embeddings of a j-sphere in an n-sphere $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$.
- $\mathcal{K}_{n,j}$, the space of long-embeddings of $\mathbb{R}^j$ in $\mathbb{R}^n$. This is the space of all embeddings $f : \mathbb{R}^j \to \mathbb{R}^n$ such that $f(t_1, t_2, \cdots, t_j) = (t_1, t_2, \cdots, t_j, 0, \cdots, 0)$ provided $(t_1, \cdots, t_j) \notin \mathbf{I}^j$.
- Let $\mathcal{P}_{n,j}$ denote the space of embeddings $f : \mathbb{R}^k \to \mathbb{R}^n$ such that:
  - $f(t_1, t_2, \cdots, t_j) = (t_1, t_2, \cdots, t_j, 0, \cdots, 0)$ for $(t_1, \cdots, t_j) \notin [-1, \infty) \times \mathbf{I}^{j-1}$
  - for all $t_1 \geq 1$ $f(t_1, t_2, \cdots, t_j) = (t_1, g(t_2, \cdots, t_j))$ where $g \in \mathcal{K}_{n-1,j-1}$ is fixed and depends only on f.

  In the literature, $\mathcal{P}_{n,j}$ is either given the notation $PE(\mathrm{D}^{j-1}, \mathrm{D}^{n-1})$ [21], $C(\mathrm{D}^{j-1}, \mathrm{D}^{n-1})$ [20] or $cemb(\mathrm{D}^{j-1}, \mathrm{D}^{n-1})$ [23], and is either called the pseudoisotopy embedding space, or concordance embedding space respectively. We will call it the pseudoisotopy embedding space.



- $\mathrm{EC}(j, M)$ is defined to be the space of embeddings $f : \mathbb{R}^j \times M \to \mathbb{R}^j \times M$ such that $supp(f) \subset \mathbf{I}^j \times M$. Here, $supp(f) = \{x \in \mathbb{R}^j \times M : f(x) \neq x\}$. 'EC' stands for 'cubically-supported embeddings'. These embeddings are not required to be proper.
- $\mathrm{PEC}(j, M)$ is the space of embeddings $f : \mathbb{R}^j \times M \to \mathbb{R}^j \times M$ such that $supp(f) \subset [-1, \infty) \times \mathbf{I}^{j-1} \times M$ and there exists some function $g \in \mathrm{EC}(j-1, M)$ such that $f(t_1, t_2, \cdots, t_j, m) = (t_1, g(t_2, \cdots, t_j, m))$ for all $(t_1, t_2, \cdots, t_j, m) \in [-1, \infty) \times \mathbb{R}^{j-1} \times M$. Here the letters 'PEC' stand for cubically-supported embedding pseudo-isotopy space.

- We will say a diagram of two maps $A \to B \to C$ is a homotopy fibre sequence if there exists a commutative diagram
$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$
such that $F \to E \to B$ is a fibration and the vertical maps are homotopy-equivalences.

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2 Basic relations between embedding spaces

This section describes some basic relationships between the spaces: $\mathcal{K}_{n,j}$, $\mathrm{EC}(j,M)$, $\mathrm{Emb}(\mathrm{S}^j,\mathrm{S}^n)$, $\mathrm{Emb}(\mathrm{S}^j,\mathbb{R}^n)$, $\mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n)$, $\mathcal{P}_{n,j}$ and $\mathrm{PEC}(j,M)$. The essential spirit of the results is that all homotopy questions about these spaces reduce to studying the spaces $\mathcal{K}_{n,j}$ and $\mathcal{P}_{n,j}$.

Given a proper embedding $f : \mathrm{D}^j \to \mathrm{D}^n$ one could restrict that embedding to the boundary and get an embedding of $f_{|\partial \mathrm{D}^j} : \mathrm{S}^{j-1} \to \mathrm{S}^{n-1}$. On a global level, this restriction defines a function:

$$\mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n) \to \mathrm{Emb}(\mathrm{S}^{j-1},\mathrm{S}^{n-1})$$

which is a fibration [62]. If N is an embedded S^{j-1} in S^{n-1}, N is said to be smoothly-slice [41] if there exists a properly embedded manifold $M \subset \mathrm{D}^n$, M diffeomorphic to D^j, so that $\partial M = N$. Thus the above fibration is onto the components of $\mathrm{Emb}(\mathrm{S}^{j-1},\mathrm{S}^{n-1})$ consisting of the embeddings whose images are smoothly-slice knots. In this paper, as in this example, fibrations are not required to have constant fibres, nor are fibrations required to be onto.

We give $\mathrm{Emb}(\mathrm{S}^{j-1},\mathrm{S}^{n-1})$ the base-point of the standard inclusion $\mathrm{S}^{j-1} \equiv \mathrm{S}^{j-1} \times \{0\}^{n-j} \subset \mathrm{S}^{n-1}$. With this base-point, the fibre of the above fibration has the homotopy-type of $\mathcal{K}_{n,j}$.

Similarly, there is a fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$.

Proposition 2.1 *There are homotopy-equivalences:*

$$\mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{P}_{n,j}$$
$$\mathrm{Emb}(\mathrm{S}^{j-1},\mathrm{S}^{n-1}) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1}$$

Moreover, the homotopy fibre sequence $\mathcal{K}_{n,j} \to \mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n) \to \mathrm{Emb}(\mathrm{S}^{j-1},\mathrm{S}^{n-1})$ *fits into a commutative diagram of 6 homotopy fibre sequences:*

$$\begin{array}{ccccc} \mathcal{K}_{n,j} & \longrightarrow & \mathcal{P}_{n,j} & \longrightarrow & \mathcal{K}_{n-1,j-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}_{n,j} & \longrightarrow & \mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n) & \longrightarrow & \mathrm{Emb}(\mathrm{S}^{j-1},\mathrm{S}^{n-1}) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & V_{n,j} & \longrightarrow & V_{n,j} \end{array}$$

Proof The homotopy equivalence $\mathrm{Emb}(\mathrm{S}^{j-1}, \mathrm{S}^{n-1}) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1}$ was given in [12]. We apply the same ideas to study $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n)$. Let $B \subset \partial \mathrm{D}^j$ be an embedded $(j-1)$-disc. Consider the bundle $\mathrm{Emb}(\mathrm{D}^j \text{ rel } B, \mathrm{D}^n) \to \mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n) \to \mathrm{Emb}(B, \mathrm{S}^{n-1})$ given by restriction to B. The base-space has the homotopy-type of $V_{n,j} \simeq \mathrm{SO}_n/\mathrm{SO}_{n-j}$ and there is a natural map [12]

$$\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathrm{Emb}(\mathrm{D}^j \text{ rel } B, \mathrm{D}^n) \to \mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n).$$

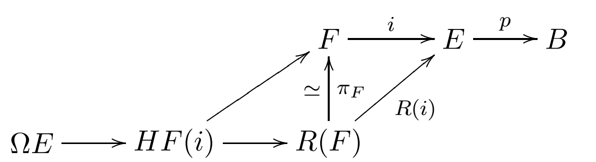
This is a homotopy-equivalence since it is a map of fibrations, which is a homotopy-equivalence on the base and fibres respectively. □

We note a basic fact about homotopy-fibres.

Proposition 2.2 *Let $p : E \to B$ be a fibration. Let $b \in B$ and $e \in E$ be the base-points of E and B respectively, with $p(e) = b$. Take e to be the base-point of $F = p^{-1}(b)$. Let $i : F \to E$ be inclusion. Let $R(F) = \{(a, h) : a \in F, h : [0,1] \to E, h(0) = p(a)\}$ then the map $R(i) : R(F) \to E$ given by evaluation $h(1)$ is a fibration, and $\pi_F : R(F) \to F$ given by projection onto F is a homotopy-equivalence. The fibre of the map $R(i) : R(F) \to E$ is the space $HF(i) = \{h : [0,1] \to E, h(0) \in F, h(1) = e\}$, and the map $p_* : HF(i) \to \Omega B$ given by post-composition with p is a weak homotopy-equivalence, giving a fibration:*

$$\Omega E \to HF(i) \to F$$

and a homotopy-commutative diagram



We mention what is known about the fibrations $\mathrm{EC}(j, \mathrm{D}^n) \to \mathcal{K}_{n+j,j}$. This result is a compilation of observations due to Goodwillie (unpublished), Sinha, Turchin and Salvatore.

Proposition 2.3 *The homotopy fibre sequence*

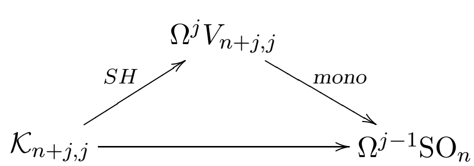
$$\Omega^j \mathrm{SO}_n \to \mathrm{EC}(j, \mathrm{D}^n) \to \mathcal{K}_{n+j,j}$$

is trivial for $j = 1$, and also for $n \leq 2$. There is a pull-back diagram of homotopy fibre sequences:

$$\begin{array}{ccc}
\Omega^j \mathrm{SO}_n & \longrightarrow & \Omega^j \mathrm{SO}_n \\
\downarrow & & \downarrow \\
\mathrm{EC}(j, \mathrm{D}^n) & \longrightarrow & P\Omega^{j-1}\mathrm{SO}_n \\
\downarrow & & \downarrow \\
\mathcal{K}_{n+j,j} & \longrightarrow & \Omega^{j-1}\mathrm{SO}_n
\end{array}$$

Where $\Omega^j \mathrm{SO}_n \to P\Omega^{j-1}\mathrm{SO}_n \to \Omega^{j-1}\mathrm{SO}_n$ is the path-loop fibration of the space $\Omega^{j-1}\mathrm{SO}_n$.

The classifying map $\mathcal{K}_{n+j,j} \to \Omega^j \mathrm{SO}_n$ factors as a composite



where 'SH' is the Smale-Hirsch map, $V_{n+j,j}$ is the Stiefel manifold of linearly independent j-frames in $\mathbb{R}^{n+j}$, and 'mono' is the looping of a certain classifying map.

Framed and unframed pseudoisotopy embedding spaces are more directly related, as the forgetful map $\mathrm{PEC}(j, \mathrm{D}^n) \to \mathcal{P}_{n+j,j}$ is a homotopy-equivalence.

Proof The observation of the existence of the above pull-back diagram first appears in Turchin's work [77]. Turchin also observed that $\mathcal{K}_{n+j,j} \to \Omega^{j-1}\mathrm{SO}_n$ factors through the Smale-Hirsch map, where mono : $\Omega^j V_{n+j,j} \to \Omega^{j-1}\mathrm{SO}_n$ is the $(j-1)$-fold looping of the composite $\alpha \circ \beta$ in the diagram:

$$\begin{array}{ccccc}
\Omega G_{n+j,n} & \xrightarrow{\alpha} & \mathrm{SO}_n & \longrightarrow & V_{n+j,n} \\
\equiv \uparrow \perp & & & & \\
\Omega G_{n+j,j} & & & & \\
\beta \uparrow & & & & \\
\Omega V_{n+j,j} & & & & \\
\uparrow & & & & \\
\Omega \mathrm{SO}_j & & & &
\end{array}$$

Where $\Omega G_{n+j,n} \to \mathrm{SO}_n \to V_{n+j,n}$ is the backing-up of the fibration $\mathrm{SO}_n \to V_{n+j,n} \to G_{n+j,n}$ where $G_{n+j,n}$ is the Grassmanian of n-planes in $\mathbb{R}^{n+j}$.

Goodwillie (unpublished) and Sinha [72] observed that $SH : \mathcal{K}_{n,1} \to \Omega V_{n+1,1}$ is null-homotopic, where their proof is simply an application of the definition of the derivative. We will give a natural extension of their proof in Proposition 2.4. □

The homotopy-class of the Smale-Hirsch map $SH : \mathcal{K}_{n,j} \to \Omega^j V_{n,j}$ is not so well understood. One would expect it to be highly non-trivial as Haefliger [28] has shown that the kernel of the map $\pi_0\mathcal{K}_{n,j} \to \pi_j V_{n,j}$ is the co-kernel of $\pi_{n+1}(\mathrm{SO}, \mathrm{SO}_{n-j}) \to \pi_{n+1}(G, G_{n-j})$, where $\mathrm{SO} = \varinjlim(\mathrm{SO}_1 \to \mathrm{SO}_2 \to \mathrm{SO}_3 \to \cdots)$ is the stable special orthogonal group, G_i is the space of degree one maps of S^{i-1}, and $G = \varinjlim(G_1 \to G_2 \to G_3 \to \cdots)$.

Proposition 2.4 *The Smale-Hirsch map $SH : \mathcal{K}_{n,j} \to \Omega^j V_{n,j}$ fits into a homotopy-commutative diagram*

$$\begin{array}{ccc}
\mathcal{K}_{n,j} & \xrightarrow{SH} & \Omega^j V_{n,j} \\
& \searrow \quad \nearrow_{\Omega^j(i)} & \\
& \Omega^j V_{n-1,j-1} &
\end{array}$$

where $i : V_{n-1,j-1} \to V_{n,j}$ is the fibre-inclusion of the fibration $V_{n-1,j-1} \to V_{n,j} \to S^{n-1}$.

Proof Consider the commutative diagram of spaces and maps:

$$\begin{array}{ccccc}
\mathcal{K}_{n,j} & \longrightarrow & \mathcal{P}_{n,j} & \longrightarrow & \mathcal{K}_{n-1,j-1} \\
\downarrow SH & & \downarrow SH & & \downarrow SH \\
\Omega^j V_{n,j} & \longrightarrow & \Omega^{j-1} HF(i) & \longrightarrow & \Omega^{j-1} V_{n-1,j-1}
\end{array}$$

Here $HF(i)$ is the homotopy-fibre of i. By Proposition 2.2 we can identify $HF(i)$ with ΩS^{n-1}, thus $\Omega^{j-1}HF(i) \simeq \Omega^j S^{n-1}$.

The Smale-Hirsch map $SH : \mathcal{P}_{n,j} \to \Omega^j \mathrm{S}^{n-1}$ is given by differentiation in the vertical direction. The map $h : [0,3] \times \mathbb{R}^j \times \mathcal{P}_{n,j} \to \mathrm{S}^{n-1}$ given by:

$$h(t, x_1, \cdots, x_j, f) = \begin{cases} n(\frac{\partial f}{\partial x_1}(x_1, \cdots, x_j)) & t = 0 \\ n(f(x_1 + t, x_2, \cdots, x_j) - f(x_1, \cdots, x_j)) & 0 < t \leq 2 \\ p_{t-2}(n(f(x_1 + 2, x_2, \cdots, x_j) - f(x_1, \cdots, x_j))) & 2 \leq t \leq 3 \end{cases}$$

is a null-homotopy of the Smale-Hirsch map, giving the result. Here, $p : [0,1] \times \mathrm{S}^{n-1} \setminus \{-1\} \to \mathrm{S}^{n-1} \setminus \{-1\}$ is a deformation-retraction of $\mathrm{S}^{n-1} \setminus \{-1\}$ to $\{1\} \subset \mathrm{S}^{n-1}$. $n : \mathbb{R}^n \setminus \{0\} \to \mathrm{S}^{n-1}$ is the function $n(v) = \frac{v}{|v|}$. □

Lastly, we relate $\mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$ to $\mathcal{K}_{n,j}$. For this proposition we will identify $\dot{\mathbb{R}^n}$ with S^n via stereographic projection. If we consider SO_{n+1} to be a SO_n-bundle over S^n, then we can identify the sub-bundle over $\mathbb{R}^n$ with $\mathbb{R}^n \times \mathrm{SO}_n$.

Proposition 2.5 *[12] Let $C \rtimes \mathcal{K}_{n,j} = \{(p, f) : f \in \mathcal{K}_{n,j}, p \in \mathbb{R}^n \setminus img(f)\}$. There is a homotopy-equivalence*

$$\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} (C \rtimes \mathcal{K}_{n,j}) \to \mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$$

provided $n - j > 1$. Given $f \in \mathcal{K}_{n,j}$ let $\dot{f} \in \mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$ be the one-point compactification of f. A pair $(A, p) \in \mathrm{SO}_n \times \mathbb{R}^n$ can naturally be considered an element $T_{(A,p)} \in \mathrm{SO}_{n+1}$. The homotopy-equivalence is given by the map which sends $(A, p, f) \in \mathrm{SO}_n \times (C \rtimes \mathcal{K}_{n,j})$ to $T_{(A,p)} \circ \dot{f} \in \mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$.

3 A motivational proof

We prove for $n - j > 2$ every proper embedding of D^j in D^n is isotopic, through proper embeddings, to a linear inclusion, ie: $\pi_0 \mathrm{Emb}(\mathrm{D}^k, \mathrm{D}^n) = 0$. This is an old result for which there are several references [36, 28]. Our proof is 'elementary' in the sense that it uses only elementary theorems about handlebody decompositions of manifolds which can be found in textbooks [43, 57]. An elementary corollary is that $\pi_0 \mathcal{K}_{n,j} \simeq \pi_0 \mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$ is a group, since the map $\pi_1 \mathcal{K}_{n-1,j-1} \to \pi_0 \mathcal{K}_{n,j}$ is an epic map of monoids.

Proposition 3.1 *The spaces $\mathrm{Emb}(\mathrm{D}^j, \mathrm{D}^n)$ and $\mathcal{P}_{n,j}$ are path-connected provided $n - j > 2$.*

Proof Let $e : \mathrm{D}^j \to \mathrm{D}^n$ be a proper embedding, and let U be an open tubular neighbourhood of $e(\mathrm{D}^j)$. Let $F \simeq \mathrm{S}^{n-j-1}$ be a fibre for the unit normal bundle of e. Since $n - j > 2$, the inclusion $F \hookrightarrow \mathrm{D}^n \setminus U$ is a homotopy-equivalence.

- Assume $n \geq 5$. By the minimal handle presentation theorem we know $\mathrm{D}^n \setminus U$ has a handle presentation with precisely two handles: one a 0-handle and the other a $(n - j - 1)$-handle. So $\mathrm{D}^n \setminus U$ is a D^{j+1}-disc bundle over F and therefore a tubular neighbourhood of $F \hookrightarrow \mathrm{D}^n \setminus U$. As a bundle, $\mathrm{D}^n \setminus U$ is trivial since F bounds a disc. So we have a diffeomorphism $\mathrm{D}^n \setminus U \simeq \mathrm{S}^{n-j-1} \times \mathrm{D}^{j+1}$. Consider the dual handle presentation of

$D^n \setminus U$. This allows us to think of $D^n \setminus U$ as $S^{n-j-1} \times S^j \times \mathbf{I}$, union a $(j+1)$-handle and an n-handle. Consider a handle presentation for $S^{n-j-1} \times S^j$. By the Whitney trick [43, 57], the core sphere of the $(j+1)$-handle attachment can be isotoped to intersect the belt-sphere of the j-handle transversely in a point. Thus one can isotope the core-disc for the $(j+1)$-handle attachment to be a submanifold $M \subset D^n$, $M \simeq D^{j+1}$ with $\partial M = A \cup e(D^j)$ with $A \subset \partial D^n$ a disc. M provides us with an isotopy from $e(D^j)$ to a linear embedding of D^j in D^n.

- Consider $n \leq 4$. The path-connectivity of $\mathrm{Emb}(D^1, D^4)$ is well-known and appears in many places, for example, it is a special case of Proposition 5.6.

□

As a point of comparison, $\pi_0\mathcal{K}_{j+2,j}$ is never a group [64, 41, 34, 67]. To see this, let $f \in \mathcal{K}_{j+2,j}$ be a knot whose complement C_f has a non-trivial fundamental group, then if $g \in \mathcal{K}_{j+2,j}$ is any knot, $\pi_1 C_{f\#g} \simeq \pi_1 C_f *_{\mathbb{Z}} \pi_1 C_g$, in particular, $\pi_1 C_f$ is a subgroup of $\pi_1 C_{f\#g}$ (see for example Proposition 2.3.4 of [88]).

$\pi_0\mathcal{K}_{j+1,j}$ is known to be trivial for all j except perhaps $j = 3$. This follows from the topological Schoenflies theorem [7, 8, 53] and uniqueness of smooth structures on D^{j+1} [74, 43]. Of course, $\mathcal{K}_{j,j}$ is always a group.

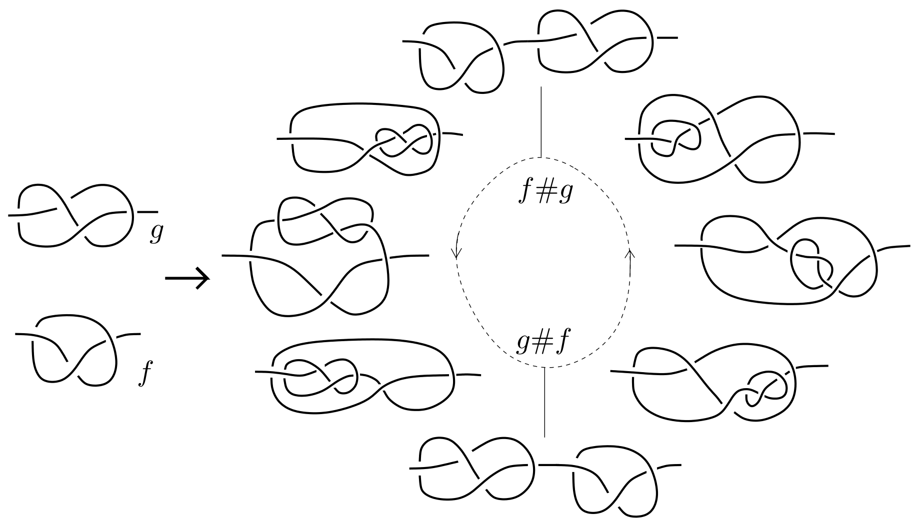
Proposition 3.1 is a very special case of the results in Goodwillie's dissertation [20] where he gets sharp lower-bounds on the connectivity of arbitrary pseudoisotopy embedding spaces.

4 Actions of operads of little cubes on embedding spaces

The work of Boardman, Vogt and May [6, 51, 52] gives one a very simple criterion for recognising if a space X has the homotopy-type of an n-fold loop-space, being that X admits an action of the operad of little n-cubes, and that the induced monoid structure on $\pi_0 X$ is that of a group. A good reference for operads relevant to topology is the book of Markl, Shnider and Stasheff [48].

There is an action of the operad of j-cubes on the spaces $\mathrm{EC}(j, M)$ and $\mathcal{K}_{n,j}$ given by concatenation (see Definition 4.2). The first instance of an action of the operad of $(j+1)$-cubes on any space of the form $\mathrm{EC}(j, M)$ was given by Morlet [59]. Morlet's 'Comparison Theorem' states that $\mathrm{EC}(j, *) \simeq \Omega^{j+1}(PL_j/O_j)$ (see [14] for a proof). Here PL_j is the group of PL-automorphisms of $\mathbb{R}^j$ (given a suitable topology) and O_j is the group of linear isometries of $\mathbb{R}^j$.

The first 'hint' of a higher cubes action on the spaces $\mathrm{EC}(j, M)$ for M non-trivial would perhaps be the work of Schubert [67]. Schubert demonstrated that the connect-sum pairing turns $\pi_0\mathcal{K}_{3,1}$ into a free commutative monoid on the isotopy-classes of prime long knots, where the demonstration of commutativity involved 'pulling one knot through another'.



In 'little cubes and long knots' [9] this idea was extended to construct a $(j+1)$-cubes action on the spaces $\mathrm{EC}(j, M)$ for an arbitrary compact manifold M. By an elementary construction, this also gave an action of the operad of $(j+1)$-cubes on $\mathcal{K}_{n,j}$ for all $n-j \leq 2$. Schubert's theorem that $\pi_0\mathcal{K}_{3,1}$ is a free commutative monoid over the isotopy classes of prime long knots was extended to the theorem that $\mathcal{K}_{3,1}$ is a free 2-cubes object over the based space $\mathcal{P} \sqcup \{*\}$ where $\mathcal{P} \subset \mathcal{K}_{3,1}$ is the subspace of prime long knots.

The freeness result $\mathcal{K}_{3,1} \simeq \mathcal{C}_2(\mathcal{P} \sqcup \{*\})$ implies that the group-completion of $\mathcal{K}_{3,1}$, $\Omega B\mathcal{K}_{3,1}$ has the homotopy-type of $\Omega^2\Sigma^2(\mathcal{P} \sqcup \{*\})$ [51]. Moreover, one can compute (recursively) the homotopy-type of the path-components of $\mathcal{P}$ by the theorems in [10]. For applications, see [12].

There is a major 'conceptual gap' between the Morlet Comparison Theorem and the above result on $\Omega B\mathcal{K}_{3,1}$. This gap is one of the motivations of this paper.

In this section we define actions of operads of little cubes on various pseudo-isotopy embedding spaces, extending previous constructions [9].

Definition 4.1

- A (single) little n-cube is a function $L : \mathbf{I}^n \to \mathbf{I}^n$ such that $L = l_1 \times \cdots \times l_n$ where each $l_i : \mathbf{I} \to \mathbf{I}$ is affine-linear and increasing ie: $l_i(t) = a_i t + b_i$ for some $0 \leq a_i < 1$ and $b_i \in \mathbb{R}$.
- Let CAut_n denote the monoid of affine-linear automorphisms of $\mathbb{R}^n$ of the form $L = l_1 \times \cdots \times l_n$ where $l_i : \mathbb{R} \to \mathbb{R}$ affine linear and increasing, and $L(\mathbf{I}^n) \subset \mathbf{I}^n$.
- Given a little n-cube L, we sometimes abuse notation and consider $L \in \mathrm{CAut}_n$ by taking the unique affine-linear extension of L to $\mathbb{R}^n$.
- The space of j little n-cubes $\mathcal{C}_n(j)$ is the space of maps $L : \sqcup_{i=1}^{j} \mathbf{I}^n \to \mathbf{I}^n$ such that the restriction of L to the interior of its domain is an embedding, and the restriction of L to any connected component of its domain is a little n-cube. Given $L \in \mathcal{C}_n(j)$, denote the restriction of L to the i-th copy of $\mathbf{I}^n$ by L_i. By convention $\mathcal{C}_n(0)$ is taken to be a point. This makes the union $\sqcup_{j=0}^{\infty}\mathcal{C}_n(j)$ into an operad, called the operad of little n-cubes $\mathcal{C}_n$ [51].

- There is an action of CAut_n on $\mathrm{EC}(n, M)$ given by

$$\mu : \mathrm{CAut}_n \times \mathrm{Emb}(\mathbb{R}^n \times M, \mathbb{R}^n \times M) \to \mathrm{Emb}(\mathbb{R}^n \times M, \mathbb{R}^n \times M)$$
$$\mu(L, f) = (L \times Id_M) \circ f \circ (L^{-1} \times Id_M)$$

In the above formula, L^{-1} is the inverse of L in the group of affine-linear isomorphisms of $\mathbb{R}^n$. We write the above action as $\mu(L, f) = L.f$. There is an action of CAut_j on $\mathcal{K}_{n,j}$ defined in the same way.

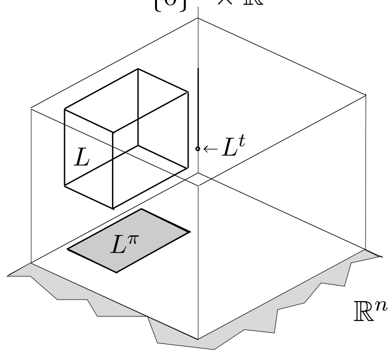
We now define an 'obvious' j-cubes action on $\mathcal{K}_{n,j}$ and $\mathrm{EC}(j, M)$. The associated multiplication in $\pi_0 \mathcal{K}_{n,j}$ is called the connect-sum operation.

Definition 4.2 $k_i : \mathcal{C}_j(i) \times (\mathcal{K}_{n,j})^i \to \mathcal{K}_{n,j}$, $k_i : \mathcal{C}_j(i) \times \mathrm{EC}(j, M)^i \to \mathrm{EC}(j, M)$ is defined by the rule $k_i(L_1, \cdots, L_i, f_1, \cdots, f_i) = L_1.f_1 \circ \cdots \circ L_i.f_i$.

We will give an extension of the above j-cubes action on $\mathrm{EC}(j, M)$ to a $(j+1)$-cubes action in the next definition.

Definition 4.3

- Given j little $(n+1)$-cubes, $L = (L_1, \cdots, L_j) \in \mathcal{C}_{n+1}(j)$ define the j-tuple of (non-disjoint) little n-cubes $L^\pi = (L_1^\pi, \cdots, L_j^\pi)$ by the rule $L_i^\pi = l_{i,1} \times \cdots \times l_{i,n}$ where $L_i = l_{i,1} \times \cdots \times l_{i,n+1}$. Similarly define $L^t \in \mathbf{I}^j$ by $L^t = (L_1^t, \cdots, L_j^t)$ where $L_i^t = l_{i,n+1}(-1)$.



- The action of the operad of little $(n+1)$-cubes on the space $\mathrm{EC}(n, M)$ is given by the maps $\kappa_j : \mathcal{C}_{n+1}(j) \times \mathrm{EC}(n, M)^j \to \mathrm{EC}(n, M)$ for $j \in \{1, 2, \cdots\}$ defined by

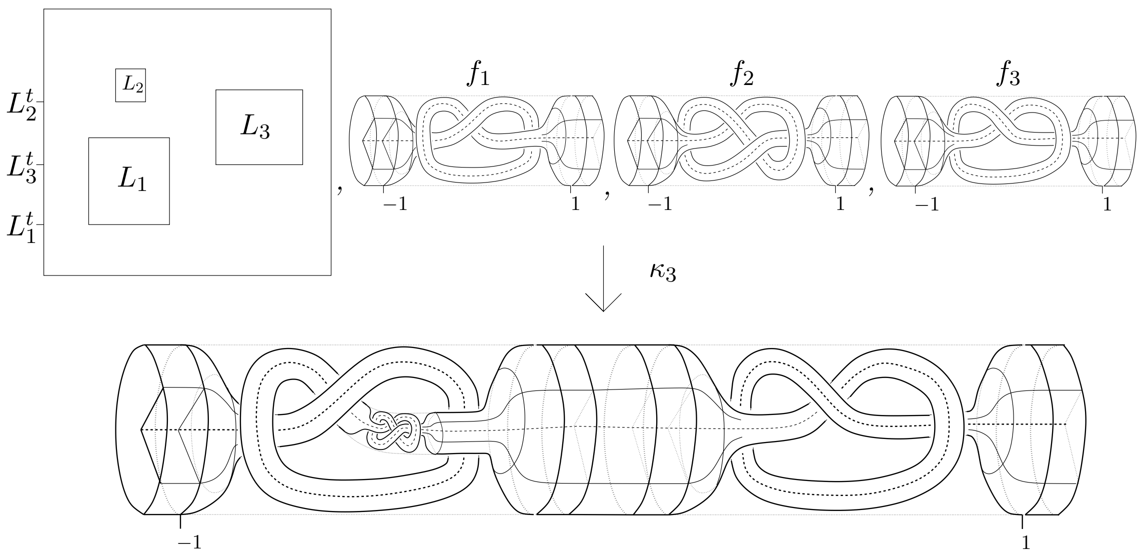
$$\kappa_j(L_1, \cdots, L_j, f_1, \cdots, f_j) = L^\pi_{\sigma(1)}.f_{\sigma(1)} \circ L^\pi_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L^\pi_{\sigma(j)}.f_{\sigma(j)}$$

where $\sigma : \{1, \cdots, j\} \to \{1, \cdots, j\}$ is any permutation such that $L^t_{\sigma(1)} \leq L^t_{\sigma(2)} \leq \cdots \leq L^t_{\sigma(j)}$. The map $\kappa_0 : \mathcal{C}_{n+1}(0) \times \mathrm{EC}(n, M)^0 \to \mathrm{EC}(n, M)$ is the inclusion of a point $*$ in $\mathrm{EC}(n, M)$, defined so that $\kappa_0(*) = Id_{\mathbb{R}^n \times M}$.

Theorem 4.4 *[9] For any compact manifold M and any integer $n \geq 0$ the maps κ_j for $j \in \{0, 1, 2, \cdots\}$ define an action of the operad of little $(n+1)$-cubes on $\mathrm{EC}(n, M)$.*

In the definition of $\mathrm{EC}(n, M)$, if one replaces the condition that the support of f is contained in $\mathbf{I}^n \times M$ with it being contained in $\mathrm{D}^n \times M$ one obtains a homotopy-equivalent space $\mathrm{ED}(n, M)$ and by the same constructions in Definition 4.3, one also obtains an action of the operad of unframed little $(n+1)$-discs on $\mathrm{ED}(n, M)$.

Example 4.5



$L_1^t < L_3^t < L_2^t$ so $\sigma = (23)$ and $\kappa_3(L_1, L_2, L_3, f_1, f_2, f_3) = L_1^\pi.f_1 \circ L_3^\pi.f_3 \circ L_2^\pi.f_2$, which explains why we see the figure-8 knot 'inside' of the trefoil on the left hand side of the picture.

We give an analogous action of $\mathcal{C}_n$ on $\mathrm{PEC}(n, M)$.

Definition 4.6 $\kappa_j : \mathcal{C}_n(j) \times \mathrm{PEC}(n, M)^j \to \mathrm{PEC}(n, M)$ for $j \in \{1, 2, \cdots\}$ is defined by

$$\kappa_j(L_1, \cdots, L_j, f_1, \cdots, f_j) = L_{\sigma(1)}.f_{\sigma(1)} \circ L_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L_{\sigma(j)}.f_{\sigma(j)}$$

where $\sigma : \{1, \cdots, j\} \to \{1, \cdots, j\}$ is any permutation such that $L^t_{\sigma(1)} \leq L^t_{\sigma(2)} \leq \cdots \leq L^t_{\sigma(j)}$.

Proposition 4.7 *The maps κ_* define an action of the operad of little n-cubes on $\mathrm{PEC}(n, M)$.*

Proof We need to verify the three axioms of a cubes action.

(1) Identity. Let $Id_{\mathbf{I}^n}$ be the identity n-cube, then $\kappa_1(Id_{\mathbf{I}^n}, f) = Id_{\mathbf{I}^n}.f = f$ by design.

(2) Symmetry. We need to verify that $\kappa_n(L.\alpha, f.\alpha) = \kappa_n(L, f)$.
Let
$$\kappa_j(L, f) = L_{\sigma(1)}.f_{\sigma(1)} \circ L_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L_{\sigma(j)}.f_{\sigma(j)}$$
and
$$\kappa_j(L.\alpha, f.\alpha) = L_{\alpha\sigma'(1)}.f_{\alpha\sigma'(1)} \circ L_{\alpha\sigma'(2)}.f_{\alpha\sigma'(2)} \circ \cdots \circ L_{\alpha\sigma'(j)}.f_{\alpha\sigma'(j)}$$
Thus we are assuming let $\sigma, \sigma' \in S_n$ satisfy $L^t_{\sigma(1)} \leq \cdots \leq L^t_{\sigma(n)}$ and $L^t_{\alpha\sigma'(1)} \leq \cdots \leq L^t_{\alpha\sigma'(n)}$. Up to the ambiguity in our choice of σ and σ' we can assume $\sigma' = \alpha^{-1}\sigma$, giving the result.

(3) Associativity. We need to verify the diagram below commutes:

$$\begin{array}{ccc}
\mathcal{C}_n(m) \times \left(\mathcal{C}_n(j_1) \times \mathrm{PEC}(n, M)^{j_1} \times \cdots \times \mathcal{C}_n(j_m) \times \mathrm{PEC}(n, M)^{j_m}\right) & \longrightarrow & \mathcal{C}_n(m) \times \mathrm{PEC}(n, M)^m \\
\downarrow & & \downarrow \\
\mathcal{C}_n(j_1 + \cdots + j_m) \times \mathrm{PEC}(n, M)^{j_1 + \cdots + j_m} & \longrightarrow & \mathrm{PEC}(n, M)
\end{array}$$

Given something in the top-left corner, consider what it maps to in the bottom-right corner, going around both ways. One gets a composite of functions of the form $L_i.L_{i,p}.f_{i,p}$ in some order. The point being, either way around the diagram, one gets a giant composite of the same collection of functions, perhaps in different orders. The point being, the order of composition is irrelevant as our definition only allows re-ordering of functions with disjoint supports.

□

Proposition 4.8 *Both the fibre-inclusion and projection maps in the fibration*

$$\mathrm{EC}(n,M) \to \mathrm{PEC}(n,M) \to \mathrm{EC}(n-1,M)$$

are maps of little n-cubes objects.

Proof We need to check equivariance of the map $\mathrm{PEC}(n,M) \to \mathrm{EC}(n-1,M)$, where to be precise, we are identifying $\{1\}\times\mathbb{R}^{n-1}\times M$ with $\mathbb{R}^{n-1}\times M$ via the map $(1,t_2,t_3,\cdots,t_n,m) \longmapsto (t_2,t_3,\cdots,t_n,m)$.

If we restrict

$$\kappa_j(L_1,\cdots,L_j,f_1,\cdots,f_j) = L_{\sigma(1)}.f_{\sigma(1)} \circ L_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L_{\sigma(j)}.f_{\sigma(j)}$$

to $\{1\}\times\mathbb{R}^{n-1}\times M$ we get the composite

$$L^{\pi}_{\sigma(1)}.f_{\sigma(1)|\{1\}\times\mathbb{R}^{n-1}\times M} \circ L^{\pi}_{\sigma(2)}.f_{\sigma(2)|\{1\}\times\mathbb{R}^{n-1}\times M} \circ \cdots \circ L^{\pi}_{\sigma(j)}.f_{\sigma(j)|\{1\}\times\mathbb{R}^{n-1}\times M}$$

which is just

$$\kappa_j(L_1,\cdots,L_j,f_{1|\{1\}\times\mathbb{R}^{n-1}\times M},\cdots,f_{j|\{1\}\times\mathbb{R}^{n-1}\times M})$$

□

5 Knot graphing and spinning

In this section we investigate the 'graphing' maps

$$\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}, \qquad \mathrm{gr}_1 : \Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to \mathrm{EC}(j,\mathrm{D}^n)$$

defined in the introduction. We compute their effect on the first non-trivial homotopy groups of $\Omega\mathcal{K}_{n-1,j-1}$ and $\mathcal{K}_{n,j}$ respectively and show they are equivariant with respect to an action of an operad of cubes.

Proposition 5.1 *The fibrations*

$$\mathrm{EC}(j,\mathrm{D}^n) \to \mathrm{PEC}(j,\mathrm{D}^n) \to \mathrm{EC}(j-1,\mathrm{D}^n) \quad \textit{and} \quad \mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$$

'back up' to homotopy fibre sequences

$$\Omega\mathrm{PEC}(j,\mathrm{D}^n) \to \Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to \mathrm{EC}(j,\mathrm{D}^n) \quad \textit{and} \quad \Omega\mathcal{P}_{n,j} \to \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$$

where the fibre-inclusions $\mathrm{gr}_1 : \Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to \mathrm{EC}(j,\mathrm{D}^n)$ and $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ are given by the formulae $(\mathrm{gr}_1 f)(t_0,t_1,\cdots,t_{j-1},m) = (t_0,f(t_0)(t_1,\cdots,t_{j-1},m))$ and $(\mathrm{gr}_1 f)(t_0,t_1,\cdots,t_{j-1}) = (t_0,f(t_0)(t_1,\cdots,t_{j-1}))$ respectively. $\mathrm{gr}_1 : \Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to \mathrm{EC}(j,\mathrm{D}^n)$ commutes with the action of the operad of little $(j+1)$-cubes on the domain and range respectively.

Proof In the case of the fibration $\mathrm{EC}(j,\mathrm{D}^n) \to \mathrm{PEC}(j,\mathrm{D}^n) \to \mathrm{EC}(j-1,\mathrm{D}^n)$

$$HF(i) = \{f : [0,1] \to \mathrm{PEC}(j,\mathrm{D}^n), f(0) = Id_{\mathbb{R}^j \times \mathrm{D}^n}, f(1) \in \mathrm{EC}(j,\mathrm{D}^n)\}$$

and the map $HF(i) \to \Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ defined in Proposition 2.2 is a weak homotopy equivalence. All embedding spaces are dominated by CW-complexes [63], and if a space is dominated by a CW-complex, it has the homotopy-type of a CW-complex [85]. The class of spaces having the homotopy-type of CW-complexes is closed under the kinds path and loop-space constructions given above [56], thus $HF(i) \to \Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ is a homotopy-equivalence as both of the spaces involved have the homotopy-type of CW-complexes.

We compute an explicit homotopy-inverse for $\Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to HF(i)$. Let $b_{\epsilon,t} : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function such that $b_{\epsilon,t}(x) = t$ for all $x \geq t$, $b_{\epsilon,t}(x) = x$ for all $x \leq t-\epsilon$ with $b'_{\epsilon,t}(x) > 0$ for $x \in (-\infty, t)$. Consider an element of $f \in \Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ to be a function from $f : \mathbb{R} \to \mathrm{EC}(j-1,\mathrm{D}^n)$ that is constant the base-point outside of $[-1,1]$. Given $f \in \Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ we define $\tilde{f} \in HF(i)$

$$\tilde{f}(t)(t_1,\cdots,t_j,m) = (t_1, f(b_{\epsilon,2t-1}(t_1))(t_2,t_3,\cdots,t_j,m))$$

This is a homotopy-inverse since the composite $\Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to HF(i) \to \Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ is homotopic to the identity (take $\epsilon \to 0$). The composite $\Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to HF(i) \to \mathrm{EC}(j,\mathrm{D}^n)$ is also homotopic to gr_1, by taking $\epsilon \to 0$.

We can now verify that $\mathrm{gr}_1 : \Omega\mathrm{EC}(j-1,\mathrm{D}^n) \to \mathrm{EC}(j,\mathrm{D}^n)$ is a map of $(j+1)$-cubes objects. First, we describe the $(j+1)$-cubes action on $\Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ induced from the j-cubes action on $\mathrm{EC}(j-1,\mathrm{D}^n)$. Given i little $(j+1)$-cubes $L = (L_1,\cdots,L_i)$ let $L^\alpha = (L_1^\alpha,\cdots,L_i^\alpha) \in \mathcal{C}_1(1)^i$ be their projections on the 1st coordinate, and let $L^\beta = (L_1^\beta,\cdots,L_i^\beta) \in \mathcal{C}_j(1)^i$ be their projections on the remaining j-coordinates. The $(j+1)$-cubes action on $\Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ is given by κ' defined below:

$$\kappa'_i(L_1,\cdots,L_i,F_1,\cdots,F_i) := \kappa_i(L_1^\beta,\cdots,L_i^\beta,L_1^\alpha.F_1,\cdots,L_i^\alpha.F_i) \quad (1)$$

$$= L^{\beta\pi}_{\sigma(1)}.L^\alpha_{\sigma(1)}.F_{\sigma(1)} \circ L^{\beta\pi}_{\sigma(2)}.L^\alpha_{\sigma(2)}.F_{\sigma(2)} \circ \cdots \circ L^{\beta\pi}_{\sigma(i)}.L^\alpha_{\sigma(i)}.F_{\sigma(i)} \quad (2)$$

Here $L_i^\alpha.F_i$ is the $\mathcal{C}_1$-action on $\Omega\mathrm{EC}(j-1,\mathrm{D}^n)$ and L_i^β acts on this via the $\mathcal{C}_j$-action on $\mathrm{EC}(j-1,\mathrm{D}^n)$. The composition operation $\circ$ is induced from composition, and $\sigma \in S_i$ is any permutation such that $L^{\beta t}_{\sigma(1)} \leq L^{\beta t}_{\sigma(2)} \leq \cdots \leq L^{\beta t}_{\sigma(i)}$.

Consider applying the map gr_1:

$$\mathrm{gr}_1 : \Omega\mathrm{EC}(j-1,\mathrm{D}^n) \ni F \longmapsto ((t_0,t,v) \longmapsto (t_0, F(t_0)(t,v))) \in \mathrm{EC}(j,\mathrm{D}^n)$$

Observe that $\mathrm{gr}_1(L^{\beta\pi}_{\sigma(p)}.L^\alpha_{\sigma(p)}.F_{\sigma(p)}) = L^\pi_{\sigma(p)}.\mathrm{gr}_1(F_{\sigma(p)})$ thus

$$\mathrm{gr}_1(\kappa'_i(L_1,\cdots,L_i,F_1,\cdots,F_i)) = L^\pi_{\sigma(1)}.\mathrm{gr}_1(F_{\sigma(1)}) \circ L^\pi_{\sigma(2)}.\mathrm{gr}_1(F_{\sigma(2)}) \circ \cdots \circ L^\pi_{\sigma(i)}.\mathrm{gr}_1(F_{\sigma(i)}) \quad (3)$$

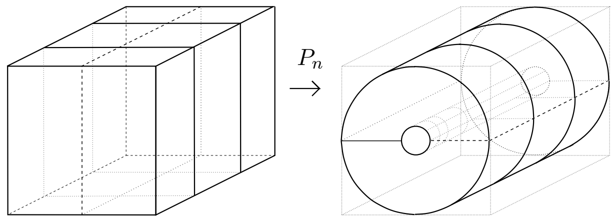
$$= \kappa_i(L_1,\cdots,L_i,\mathrm{gr}_1(F_1),\cdots,\mathrm{gr}_1(F_i)) \quad (4)$$

since gr_1 commutes with $\circ$. □

We explore the connection between the maps gr_1 and the Litherland deform-spun knot construction.

Given a topological space X, we will denote the space of continuous functions $f : S^1 \equiv \mathbb{R}/2\mathbb{Z} \to X$ by LX, and call LX the free loop space on X. Consider the map $P_n : \mathbf{I}^n \to \mathbf{I}^n$ given by

$(t_1, t_2, \cdots, t_n) \longmapsto (\frac{t_2+2}{3}\cos(\pi t_1), \frac{t_2+2}{3}\sin(\pi t_1), t_3, \cdots, t_n)$ P_n is an embedding on the interior of $\mathbf{I}^n$, and is globally one-to-one except for the equality $P_n(-1, t_2, t_3, \cdots, t_n) = P_n(1, t_2, \cdots, t_n)$.



Definition 5.2 Given $f \in L\mathcal{K}_{n-1,j-1}$, if $f' : \mathbb{R}^j \to \mathbb{R}^n$ is the function $f'(t_0, t_1, \cdots, t_{j-1}) = (t_0, f(t_0)(t_1, \cdots, t_{j-1}))$, observe that $P_n \circ f' \circ P_j^{-1}$ is defined on the image of P_j, and on $\partial img(P_j)$ it agrees with the standard inclusion $\mathbb{R}^j \to \mathbb{R}^n$. Let $\mathrm{gr}_1(f) \in \mathcal{K}_{n,j}$ be the unique extension of $P_n \circ f' \circ P_j^{-1}$ such that $\mathrm{gr}_1(f)_{|\mathbb{R}^j \setminus img(P_j)}$ agrees with the standard inclusion $\mathbb{R}^j = \mathbb{R}^j \times \{0\}^{n-j} \subset \mathbb{R}^n$ on $\overline{\mathbb{R}^j \setminus img(P_j)}$.

Proposition 5.3 *The diagram*

$$\begin{array}{ccc} L\mathcal{K}_{n-1,j-1} & \xrightarrow{\mathrm{gr}_1} & \mathcal{K}_{n,j} \\ \uparrow & \nearrow_{\mathrm{gr}_1} & \\ \Omega\mathcal{K}_{n-1,j-1} & & \end{array}$$

is homotopy-commutative.

Proof There exists a 1-parameter family $P_n(t) : \mathbf{I}^n \to \mathbf{I}^n$ for $t \in [0,1]$ satisfying $P_n(0) = P_n$, $P_n(1) = Id_{\mathbf{I}^n}$, such that for all $t \in (0,1]$ the function $P_n(t) : \mathbf{I}^n \to \mathbf{I}^n$ is an embedding. Substituting $P_n(t)$ for P_n in the definition of $\mathrm{gr}_1 : L\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ gives the desired homotopy. □

We assume for all $f \in \mathcal{K}_{n,j}$, $img(f) \cap (\mathbb{R}^n \setminus D^n) = \mathbb{R}^j \setminus D^n$. Technically, this condition describes a subspace of $\mathcal{K}_{n,j}$ but since it is homotopy-equivalent, it causes no harm. Consider the fibration $\mathrm{Diff}(\mathrm{D}^n, f) \to \mathrm{Diff}(\mathrm{D}^n) \to \mathcal{K}_{n,j}(f)$ given by restriction. The map $\mathrm{Diff}(\mathrm{D}^n) \to \mathcal{K}_{n,j}$ is null homotopic. So the induced map $\Omega\mathcal{K}_{n,j}(f) \to \mathrm{Diff}(\mathrm{D}^n, f)$ gives an injection $\pi_{i+1}\mathcal{K}_{n,j}(f) \to \pi_i\mathrm{Diff}(\mathrm{D}^n, f)$ for all i. Given an element $g \in \pi_1\mathcal{K}_{n,j}(f)$, let $\tilde{g} \in \pi_0\mathrm{Diff}(\mathrm{D}^n, f)$ be the corresponding element.

Proposition 5.4 *Given $f \in \mathcal{K}_{n-1,j-1}$ and $g \in \pi_1\mathcal{K}_{n-1,j-1}(f)$ as above, denote the 1-point compactification of $\mathrm{gr}_1 g$ by $\overline{\mathrm{gr}_1 g} \in \mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$. Provided $\tilde{g} \in \pi_0\mathrm{Diff}(\mathrm{D}^{n-1}, f)$ corresponds to g under the map $\pi_1\mathcal{K}_{n-1,j-1}(f) \to \pi_0\mathrm{Diff}(\mathrm{D}^{n-1}, f)$, the manifold pair*

$$\left[(\partial\mathrm{D}^{n-1}, \partial\mathrm{D}^{j-1}) \times \mathrm{D}^2\right] \cup \left[(\mathrm{D}^{n-1}, img(f) \cap \mathrm{D}^{n-1}) \times_{\tilde{g}} \mathrm{S}^1\right]$$

(called the $\tilde{g}$-spun knot, as in [19, 47, 41]) is diffeomorphic to the pair $(\mathrm{S}^n, img(\overline{\mathrm{gr}_1 g}))$.

Proof Let $U \simeq \mathrm{D}^2 \times \mathrm{S}^{n-2}$ be a closed tubular neighbourhood of $\overline{\{0\}^2 \times \mathbb{R}^{n-2}} \subset \dot{\mathbb{R}}^n$, then $\left(\overline{\mathrm{S}^n \setminus U}, img(\overline{\mathrm{gr}_1 g})\right)$ is diffeomorphic to the pair $\left[(\mathrm{D}^{n-1}, img(f) \cap \mathrm{D}^{n-1}) \times_{\tilde{g}} \mathrm{S}^1\right]$. □

Definition 5.5 An element $f \in \pi_0\mathcal{K}_{n,j}$ is said to have (Gromoll) degree i if $f \in img(\mathrm{gr}_i)$ and $f \notin img(\mathrm{gr}_{i+1})$. If f is not in the image of gr_1, we say it has degree 0.

The terminology of 'degree' comes from the pseudoisotopy theory of discs and spheres [4, 5, 84, 25]. In the next proposition we compute the first non-trivial homotopy groups of the spaces $\mathcal{K}_{n,j}$, $\mathrm{Emb}(\mathrm{S}^j,\mathrm{S}^n)$, and $\mathrm{Emb}(\mathrm{S}^j,\mathbb{R}^n)$. The key step is to show that every element of $\pi_0\mathcal{K}_{n,j}$ has degree at least $n-j-2$, which reduces to Goodwillie's dissertation.

Proposition 5.6 (1) *$\mathcal{K}_{n,j}$ is $(2n-3j-4)$-connected for all n and j. If we assume that $2n-3j-3 \geq 0$, then the first non-trivial homotopy group of $\mathcal{K}_{n,j}$ is in dimension $2n-3j-3$ and* $\pi_{2n-3j-3}\mathcal{K}_{n,j} \simeq \begin{cases} \mathbb{Z} & j=1 \text{ or } n-j \text{ is odd} \\ \mathbb{Z}_2 & j>1 \text{ and } n-j \text{ is even} \end{cases}$

(2) $\mathrm{Emb}(\mathrm{S}^j,\mathrm{S}^n)$ *is* $\min\{(2n-3j-4),(n-j-1)\}$*-connected. If we let* $m = \min\{2n-3j-3, n-j-1\}$*, provided* $2n-3j-3 \geq 0$ *the first non-trivial homotopy-group of* $\mathrm{Emb}(\mathrm{S}^j,\mathrm{S}^n)$ *is in dimension* m *and is isomorphic to:*

$$\pi_m\mathrm{Emb}(\mathrm{S}^j,\mathrm{S}^n) \simeq \begin{cases} \mathbb{Z} & 2n-3j-3 < n-j-1, (j=1 \text{ or } n-j \text{ odd}) \\ \mathbb{Z} & 2n-3j-3 > n-j-1, n-j \text{ even} \\ \mathbb{Z}_2 & 2n-3j-3 < n-j-1, j>1 \text{ and } n-j \text{ even} \\ \mathbb{Z}_2 & 2n-3j-3 > n-j-1, n-j \text{ odd} \\ \mathbb{Z}\oplus\mathbb{Z}_2 & 2n-3j-3 = n-j-1 \end{cases}$$

(3) $\mathrm{Emb}(\mathrm{S}^j,\mathbb{R}^n)$ *is* $\min\{2n-3j-4, n-j-2\}$ *connected for all* n *and* j*. Provided* $2n-3j-3 \geq 0$ *the first non-trivial homotopy group of* $\mathrm{Emb}(\mathrm{S}^j,\mathbb{R}^n)$ *is in dimension* $m = \min\{2n-3j-3, n-j-1\}$ *and is given by:*

$$\pi_m\mathrm{Emb}(\mathrm{S}^j,\mathbb{R}^n) \simeq \begin{cases} \mathbb{Z} & 2n-3j-3 < n-j-1, (j=1 \text{ or } n-j \text{ odd}) \\ \mathbb{Z}_2 & 2n-3j-3 < n-j-1, j>1 \text{ and } n-j \text{ even} \\ \mathbb{Z} & 2n-3j-3 > n-j-1 \\ \mathbb{Z}^2 & 2n-3j-3 = n-j-1, (j=1 \text{ or } n-j \text{ odd}) \\ \mathbb{Z}\oplus\mathbb{Z}_2 & 2n-3j-3 = n-j-1, j>1 \text{ and } j \text{ even} \end{cases}$$

(4) *The space* $\mathcal{P}_{n,j}$ *is* $(2n-2j-5)$*-connected, and* $\mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n)$ *is* $\min\{(2n-2j-5),(n-j-1)\}$*-connected for all* n *and* j*.*

Proof (4) The result on $\mathcal{P}_{n,j}$ follows directly from Goodwillie's dissertation [20]. The result on $\mathrm{Emb}(\mathrm{D}^j,\mathrm{D}^n)$ is then a corollary of Proposition 2.1.

(1) There is a computation of the 3rd stage of the Goodwillie tower for $\mathcal{K}_{n,1}$ in [11]. This is a $(2n-6)$-connected map $\mathcal{K}_{n,1} \to AM_3$. AM_3 is shown to have the homotopy-type of the 3-fold loop-space on the homotopy fibre of the inclusion $\mathrm{S}^{n-1}\vee\mathrm{S}^{n-1} \to \mathrm{S}^{n-1}\times\mathrm{S}^{n-1}$. The first non-trivial integral homology group of $\mathcal{K}_{n,1}$ is computed by Victor Turchin [78] (see the computations for the homology of the complexes CT_0D^{even} and CT_0D^{odd} for $j=4$, $i=2$), $H_{2n-6}(\mathcal{K}_{n,1};\mathbb{Z}) \simeq \mathbb{Z}$, thus $\pi_{2n-6}\mathcal{K}_{n,1} \simeq \mathbb{Z}$ by the Hurewicz Theorem. We inductively compute the first non-trivial homotopy groups of $\mathcal{K}_{n,j}$. Consider the fibre-sequence $\mathcal{K}_{n+1,j+1} \to \mathcal{P}_{n+1,j+1} \to \mathcal{K}_{n,j}$ with base-case $j=1$. Thus, for all $j \geq 2$ $\pi_{2n-6-j}\mathcal{K}_{n+j,j+1} \simeq \pi_{2n-6}\mathcal{K}_{n,1}/img(\pi_{2n-6}\mathcal{P}_{n+1,2})$. When $j=2n-6$ this gives $\pi_0\mathcal{K}_{3n+j-6,2n-5} \simeq \pi_{2n-6}\mathcal{K}_{n,1}/img(\pi_{2n-6}\mathcal{P}_{n+1,2})$. Haefliger's [28] has shown that $\pi_0\mathcal{K}_{2n+j-6,2n-5} \simeq \mathbb{Z}$ if n is even and $\mathbb{Z}_2$ if n is odd, giving the result.

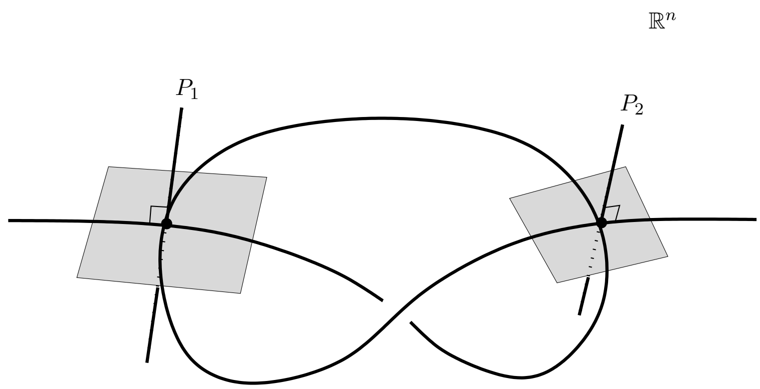
(2) Proposition 2.1 gives us a homotopy equivalence $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n) \simeq \mathrm{SO}_{n+1} \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n,j}$. Since $\mathrm{SO}_{n+1}/\mathrm{SO}_{n-j} \equiv V_{n+1,j+1}$ is $(n-j-1)$-connected, the homotopy LES of the fibration $\mathcal{K}_{n,j} \to \mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n) \to V_{n+1,j+1}$ tells us that $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$ is $min\{n-j-1, 2n-3j-4\}$-connected. Since the bundle $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n) \to V_{n+1,j+1}$ is split, we can compute the first non-trivial homotopy group of $\mathrm{Emb}(\mathrm{S}^j, \mathrm{S}^n)$ directly.

(3) For $\mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$ we use the homotopy equivalence $\mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} (C \rtimes \mathcal{K}_{n,j})$ from Proposition 2.5. The bundles $C \rtimes \mathcal{K}_{n,j} \to \mathcal{K}_{n,j}$ and $\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} (C \rtimes \mathcal{K}_{n,j}) \to V_{n,j}$ are split, so the computation follows as in case (2). □

Proposition 5.6 proves that Munson's lower bound [61] on the connectivity of $\mathrm{Emb}(\mathrm{S}^j, \mathbb{R}^n)$ of $\min\{2n-3j-4, n-j-2\}$ is sharp.

We devote the rest of this section to a geometric description of the generator of $\pi_{2n-6}\mathcal{K}_{n,1}$.

Take a 'long' immersion $f : \mathbb{R} \to \mathbb{R}^3 \subset \mathbb{R}^n$ having two regular double points $f(t_1) = f(t_3)$, $f(t_2) = f(t_4)$ with $t_1 < t_2 < t_3 < t_4 \in \mathbb{R}$. Let Tf_i be the tangent space to $img(f)$ at t_i. Let P_1 be the orthogonal complement to $Tf_1 \oplus Tf_3$, and P_2 the orthogonal complement of $Tf_2 \oplus Tf_4$.



P_1 and P_2 are $(n-2)$-dimensional, so if S_1 and S_2 are the unit sphere of P_1 and P_2 respectively they are both $(n-3)$-dimensional. There is a 'resolution function' $r : S_1 \times S_2 \to \mathcal{K}_{n,1}$ given by perturbing f near the double points via a bump-function prescribed by $(v_1, v_2) \in S_1 \times S_2$. We claim r is a generator of $H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$.

To verify that r generates $H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ one could work back through the computations of Turchin and Vassiliev [78, 81]. We supply an alternative 'geometric' argument which is inspired by the work [11]. The idea is to construct an integral co-homology class $\nu_2 \in H^{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ such that if $x \in H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ is suitably represented by a manifold then $\nu_2(x)$ is a signed count of the number of alternating quadrisecants along the family of long knots x.

Definition 5.7 Given two points $x, y \in \mathbb{R}^n$ let $[x, y]$ denote the oriented line segment in $\mathbb{R}^n$, starting at x and ending at y. An alternating quadrisecant in $C_4(\mathbb{R}^n)$ is a point $(x_1, x_2, x_3, x_4) \in C_4(\mathbb{R}^n)$ such that $[x_1, x_4] \subset [x_3, x_2]$ as an oriented subinterval. We are using the notation $C_k M = \{x \in M^k : x_i \neq x_j \ \forall\, i \neq j\}$.

Let $AQ_n \subset C_4(\mathbb{R}^n)$ denote the set of all alternating quadrisecants. Let $C_4'(\mathbb{R}) = \{t \in C_4(\mathbb{R}) : t_1 < t_2 < t_3 < t_4\}$. Given $f \in \mathcal{K}_{n,1}$ let $AQ_n(f) \subset C_4'(\mathbb{R})$ denote the pull-back of AQ_n. More generally, if $f : X \to \mathcal{K}_{n,1}$ define $AQ_n(f) \subset X \times C_4'(\mathbb{R})$ as the pull-back of AQ_n.

Given an closed, oriented $(2n-6)$-dimensional manifold X and a map $f : X \to \mathcal{K}_{n,1}$ such that $f_* : X \times C_4'(\mathbb{R}) \to C_4(\mathbb{R}^n)$ is transverse to AQ_n, $AQ_n(f) \subset X \times C_4(\mathbb{R})$ is a 0-dimensional submanifold with oriented normal bundle. A well-defined integer invariant of f, $\nu_2(f) \in \mathbb{Z}$ would then be given by the difference of the number of positively oriented points vs. the number of negatively oriented points. The sign of each point of $AQ_n(f)$ could be computed by a formula analogous to the one in Proposition 6.2 of [11].

In the next two lemmas, we prove that every $f : X \to \mathcal{K}_{n,1}$ is approximated by $\tilde{f}$ such that $\tilde{f}_*$ is transverse to AQ_n, thus $\nu_2(f)$ is well-defined.

Lemma 5.8 *Given $f \in \mathcal{K}_{n,1}$, let $Nf = \{(t,v) : t \in \mathbb{R}, v \in \mathbb{R}^n, v \perp img(f'(t))\}$ be the normal bundle of f, and let $N_\epsilon f \subset Nf$ the disc-bundle of radius ϵ. Let $exp_{(\epsilon,f)} : N_\epsilon f \to \mathbb{R}^n$ be the exponential map $(t,v) \longmapsto f(t)+v$. Let $\Gamma : \mathcal{K}_{n,1} \to (0,\infty]$ be the exponential radius function, $\Gamma(f) = \sup\{\epsilon : exp_{(\epsilon,f)}$ is an embedding$\}$. We claim Γ is continuous.*

Proof Given $f \in \mathcal{K}_{n,1}$ with $\Gamma(f)$ finite, at least one of the two following statements are true.

(1) The derivative of $exp_{(\Gamma(f),f)}$ has a critical point on its boundary. These are called focal-points of f (see [58] §6) and are known to occur at distances $1/\kappa_{f(t)}$ from $f(t)$ where $\kappa_{f(t)}$ is the curvature of f at t.

(2) There are points $(t_1,v_1),(t_2,v_2) \in N_{\Gamma(f)}f$ with $t_1 < t_2$ such that $f(t_1)+v_1 = f(t_2)+v_2$ with $\Gamma(f) = |v_1| = |v_2|$.

In case (1), $\min_{t\in\mathbb{R}}\{1/\kappa_{f(t)}\}$ is a continuous function of $f \in \mathcal{K}_{n,1}$ since κ only depends on the 1st and 2nd derivatives of f.

All functions f that satisfy case (2) have a neighbourhood that also satisfy case (2). To see this, make the definition $\xi_{n,i} = \{(L,v) : L \in G_{n,i}$ and $v \in L^\perp\}$ here $G_{n,i}$ is the Grassman manifold of i-dimensional subspaces of $\mathbb{R}^n$. We think of $\xi_{n,i}$ as the space of affine i-dimensional subspaces of $\mathbb{R}^n$, since one obtains every i-dimensional subspace uniquely as a sum $L+v$. Intersection defines a continuous function $C_2(\xi_{n,n-1}) \to \dot{\xi}_{n,n-2}$ where $\dot{\xi}_{n,n-2}$ is the one-point compactification of $\xi_{n,n-2}$. □

Given $f : X \to \mathcal{K}_{n,1}$ with X compact, we call $\min\{\Gamma(f(x)) : x \in X\}$ the exponential radius of f.

Lemma 5.9 *Every $x \in H_{2n-6}(\mathcal{K}_{n,1};\mathbb{Z})$ represented by a manifold $f : M \to \mathcal{K}_{n,1}$ can be perturbed so that f_* is transverse to AQ_n. Thus ν_2 is a well-defined element of $H^{2n-6}(\mathcal{K}_{n,1};\mathbb{Z})$.*

Proof Let R be the exponential radius of f. Let $b_\epsilon : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function such that $b_\epsilon(x) = 0$ for all $|x| \geq \epsilon$, $b_\epsilon(x) > 0$ for all $x \in (-\epsilon,\epsilon)$, and $b(0) = 1$. Let $b_{\epsilon,t}(x) = b_\epsilon(x-t)$.

Partition the interval $\mathbf{I}$ into m equal-length sub-intervals $I_1, \cdots, I_m$ such that for every $x \in X$, the convex hull of $f(x)(I_j)$ is contained in $\nu_{R/2}f(x)$. We can do this because there is a uniform upper bound on $|f(x)'(t)|$ for all $x \in X$, $t \in \mathbb{R}$, and let $\epsilon = 2/m$. Consider the function $\tilde{f} : X \times (\mathbb{R}^n)^k \times \mathbb{R} \to \mathbb{R}^n$ given by $\tilde{f}(x,v_1,\cdots,v_k,t) = f(t) + \sum_{i=1}^k b_{\epsilon,t_i}(t)v_i$. In some neighbourhood U of 0 in $(\mathbb{R}^n)^k$, the restriction of $\tilde{f}$ to $X \times U \times \mathbb{R}$ is adjoint to a map $\overline{f} : X \times U \to \mathcal{K}_{n,1}$, this is because embeddings form an open subset of the space of all 'long' smooth maps from $\mathbb{R}$

to $\mathbb{R}^n$ [35]. Consider a point (x,T) of $AQ_n(\overline{f})$. Let $T=(t_1,t_2,t_3,t_4)$. For each i, t_i and t_{i+1} cannot both be elements of some common I_j.

Thus $\overline{f}_* : X\times U\times C_4'(\mathbb{R}) \to C_4(\mathbb{R}^n)$ is transverse to AQ_n. By the Transversality Theorem [26], f can be approximated by a map $X\to \mathcal{K}_{n,1}$ such that the induced map $X\times C_4(\mathbb{R})\to C_4(\mathbb{R}^n)$ is transverse to AQ_n. □

Proposition 5.10 *$\nu_2(r)=\pm 1$, thus $H_{2n-6}(\mathcal{K}_{n,1};\mathbb{Z})$ is generated by r.*

Proof In our picture of the 'immersed trefoil' $f:\mathbb{R}\to\mathbb{R}^3\subset\mathbb{R}^n$ there are no quadrisecants, excepting the 'degenerate' one consisting of the double-points. Thus, only one point of the image of r contains a quadrisecant – the one which is an embedded trefoil in $\mathbb{R}^3$. □

Since $\mathcal{K}_{n,1}$ is $(2n-7)$-connected, by the Hurewicz Theorem $\pi_{2n-6}\mathcal{K}_{n,1}\simeq\mathbb{Z}$ is generated by any map $\tilde{r}:\mathrm{S}^{2n-6}\to\mathcal{K}_{n,1}$ homologous to r. Attachment of an $(n-3)$-handle to $S_1\times S_2\times[0,1]$ along $S_1\times\{*\}\times\{1\}$ gives a cobordism between $S_1\times S_2$ and S^{2n-6}. Since $r_{|S_1\times\{*\}}$ is null, r extends over the cobordism. Let $\tilde{r}$ be the restriction of this cobordism to S^{2n-6}.

6 Survey

So far, much of this paper has been devoted to studying the map $\mathrm{gr}_1:\Omega\mathcal{K}_{n,j}\to\mathcal{K}_{n+1,j+1}$. As we have seen, these maps fit into the pseudoisotopy formalism in a rather natural way. We mention other natural maps between the spaces $\mathcal{K}_{n,j}$ and some of their basic properties.

Proposition 6.1 *The natural inclusion $\mathbb{R}^n\to\mathbb{R}^{n+1}$ induces an inclusion $i:\mathcal{K}_{n,1}\to\mathcal{K}_{n+1,1}$. There exists a map $\tilde{i}:\mathcal{K}_{n,1}\to\Omega\mathcal{K}_{n+1,1}$ and a commutative diagram*

$$\begin{array}{ccc} \mathcal{K}_{n,1} & \xrightarrow{\quad i \quad} & \mathcal{K}_{n+1,1} \\ & {\scriptstyle\tilde{i}}\searrow \quad \nearrow{\scriptstyle p} & \\ & \Omega\mathcal{K}_{n+1,1} & \end{array}$$

where $p:\Omega\mathcal{K}_{n+1,1}\to\mathcal{K}_{n+1,1}$ is an evaluation map, meaning that if $f\in\Omega\mathcal{K}_{n+1,1}$, then $p(f)=f(0)$. p is null-homotopic, thus i is null-homotopic.

Proof Let $b_t:\mathbb{R}\to\mathbb{R}$ be a C^∞-smooth function satisfying $supp(b_t)\subset[t-1,t+1]$, with $b_t(x)\geq 0$ for all $x\in\mathbb{R}$, and $b_t'(x)=0$ precisely when $x\in(\mathbb{R}\setminus(t-1,t+1))\cup\{t\}$. Let $B_t(x)=(0,0,\cdots,0,b_t(x))\in\mathbb{R}^{n+1}$.

Given $f\in\mathcal{K}_{n,1}$, consider the function $F:\mathbf{I}\times[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}^{n+1}$ defined as

$$F(t,a,c,x)=a\,(i(f)(x))+c\,(B_t(x))$$

we define

$$\tilde{i}(f)(x)=\begin{cases} \frac{1}{2}F(1,1,3t,2x) & 0\leq t\leq\frac{1}{3} \\ \frac{1}{2}F(1,2-3t,1,2x) & \frac{1}{3}\leq t\leq\frac{2}{3} \\ \frac{1}{2}F(1,0,3-3t,2x) & \frac{2}{3}\leq t\leq 1 \\ \frac{1}{2}F(-1,1,-3t,2x) & \frac{-1}{3}\leq t\leq 0 \\ \frac{1}{2}F(-1,2+3t,1,2x) & \frac{-2}{3}\leq t\leq\frac{-1}{3} \\ \frac{1}{2}F(-1,0,3+3t,2x) & -1\leq t\leq\frac{-2}{3} \end{cases}$$

□

At present it is not known if $\tilde{i} : \mathcal{K}_{n,1} \to \Omega\mathcal{K}_{n+1,1}$ is null-homotopic. The adjoint of $\tilde{i}$, $\Sigma\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$ is the direct-analogue of the 'Freudenthal suspension map for configuration spaces' [17] $\Sigma C_k\mathbb{R}^n \to C_k\mathbb{R}^{n+1}$ which is known to induce an isomorphism on the 1st non-trivial homology groups provided $n > 1$, thus one might expect $\tilde{i}$ to be non-trivial. But $\Sigma\mathcal{K}_{n,1}$ and $\mathcal{K}_{n+1,1}$ do not have isomorphic first homology groups, $\Sigma^2\mathcal{K}_{n,1}$ and $\mathcal{K}_{n+1,1}$ do. Thus one might suspect that there are two distinct null-homotopies of $\tilde{i}$ that give an essential map $\mathcal{K}_{n,1} \to \Omega^2\mathcal{K}_{n+1,1}$. It is not immediately clear how to construct such a map, if one exists.

There are null-homotopies of the inclusions $\mathcal{K}_{n,j} \to \mathcal{K}_{n+j,j}$ for all $j > 0$ defined analogously.

Question 6.2

- *For each n and j, what is the smallest i such that $\mathcal{K}_{n,j} \to \mathcal{K}_{n+i,j}$ is null-homotopic?*
- *Is there a 'Freudenthal suspension map' $\Sigma^2\mathcal{K}_{n,j} \to \mathcal{K}_{n+1,j}$ provided $2n - 3j - 3 \geq 0$? or $j = 1$?*

Another natural map relating the spaces $\mathcal{K}_{n,j}$. has the form $R : \mathcal{K}_{n,j} \to \Omega\mathcal{K}_{n,j-1}$. Given $f \in \mathcal{K}_{n,j}$, let $\tilde{f} \in \Omega\mathcal{K}_{n,j-1}$ is given by $\tilde{f}(t)(t_1, \cdots, t_{j-1}) = f(t_1, \cdots, t_{j-1}, t) - (0, \cdots, 0, t)$. Clearly if $2n - 3j - 3 \geq 0$ this map is exactly $(2n - 3j - 3)$-connected. Analogous maps in pseudoisotopy theory have been studied using the Morlet Disjunction Lemma (see for example [20]), but this map does not appear to be studied in any depth. Notice the relation $R \circ \mathrm{gr}_1 = \Omega i$.

Proposition 6.3 *There is a homotopy-equivalence $\mathcal{K}_{n,n} \to \Omega\mathcal{K}_{n,n-1}$.*

Proof There are homotopy-equivalences $\mathcal{K}_{n,n} \simeq \mathrm{EC}(n, *)$ and $\mathcal{K}_{n,n-1} \simeq \mathrm{EC}(n-1, \mathbf{I})$ given by the fibrations in Proposition 2.3. Restriction to $\mathbb{R}^{n-1} \times \mathbf{I}$ gives a map $\mathrm{EC}(n, *) \to \mathrm{EC}(n-1, \mathbf{I})$ which is homotopic to a fibration, whose fibre is $\mathrm{EC}(n, *)^2$. The fibre-inclusion map $\mathrm{EC}(n, *)^2 \to \mathrm{EC}(n, *)$ is homotopic to the composition operation (the homotopy being given by the $(n+1)$-cubes action on $\mathrm{EC}(n, *)$), so the homotopy fibre of $\mathrm{EC}(n, *)^2 \to \mathrm{EC}(n, *)$ is $\mathrm{EC}(n, *)$, but by Proposition 2.2, this is also $\Omega\mathrm{EC}(n-1, \mathbf{I})$. □

Whether or not $\mathcal{K}_{n,n-1}$ is path-connected is called the smooth Schoenflies problem in dimension n. $\mathcal{K}_{n,n-1}$ is known to be connected for all n except perhaps $n = 4$. For $n = 2$ this is the classical smooth Schoenflies theorem (see for example [2], [70] is a good reference for all things closely related to the classical Schoenflies theorem). For $n = 3$ this is Alexander's theorem [2]. For $n \geq 5$ it follows from the affirmative solution to the topological Schoenflies theorem [53, 7, 8] plus the uniqueness of smooth structures on a disc [74, 43]. Scharlemann [69] has some partial results in dimension 4 but progress has been very slow in this realm.

A metric g on S^n is said to be round if it has constant sectional curvature, or equivalently, if the isometry group of (S^n, g) acts transitively on the bundle of oriented orthonormal frames of S^n. Let $\mathbb{M}^n$ denote the space of round Riemann metrics on S^n.

Proposition 6.4 *$\mathbb{M}^n$ has the same homotopy-type as $\mathcal{K}_{n,n}$.*

Proof $\mathbb{M}^n$ is a $\mathrm{Diff}^+(\mathrm{S}^n)$-homogeneous space, where the isotropy subgroup is SO_{n+1}. Proposition 2.1 says $\mathcal{K}_{n,n} \simeq \mathrm{Diff}^+(\mathrm{S}^n)/\mathrm{SO}_{n+1}$. □

Smale [73] and Hatcher [30] have proved that $\mathcal{K}_{n,n}$ is contractible for $n = 2$ and $n = 3$ respectively. That $\mathcal{K}_{1,1}$ is contractible follows from an averaging argument, or equivalently from the 'length' classification of connected closed 1-dimensional Riemann manifolds via Proposition 6.4.

In general, $\mathcal{K}_{n,n}$ is an $(n+1)$-fold loop space [9, 59, 14] whose $(n+1)$-fold delooping is $PL(n)/O_n$ [14, 59]. As of yet, there have been no direct descriptions of the homotopy-type of $PL(n)/O_n$, and essentially nothing seems to be known about $\mathcal{K}_{4,4}$.

Farrell and Hsiang computed the rational homotopy of $\mathcal{K}_{n,n}$ in a range.

Theorem 6.5 *[18] Provided* $0 \leq i < \min\{\frac{n-4}{3}, \frac{n-7}{2}\}$

$$\pi_i \mathcal{K}_{n,n} \otimes \mathbb{Q} \simeq \begin{cases} \mathbb{Q} & 4|i+1 \\ 0 & \text{otherwise} \end{cases}$$

The reason for the bound $i < \min\{\frac{n-4}{3}, \frac{n-7}{2}\}$ is that this is Igusa's stable range [37]. Roughly this is where $\pi_i \mathcal{P}_{n,n}$ can be related to K-theory.

Antonelli, Burghelea and Kahn [4, 5] have shown that $H_* \mathcal{K}_{n,n}$ is not finitely-generated for $n \geq 7$.

The spaces $\mathcal{K}_{j+2,j}$ are in the realm of 'traditional' co-dimension 2 knot theory, on which there is a plethora of literature. The majority of the literature focuses on $\pi_0 \mathcal{K}_{j+2,j}$ in that isotopy classes of knots are the fundamental objects. Some good general references are Kawauchi [41], Hillman [34] and Ranicki [64]. Not much is known about the homotopy-type of the components of $\mathcal{K}_{j+2,j}$ for $j > 1$.

Question 6.6 *Let* $f \in \mathcal{K}_{j+2,j}$ *be a connect-sum of two non-trivial knots. The action of the operad of* $(j+1)$*-cubes on* $\mathcal{K}_{j+2,j}$ *gives a map* $\mathrm{S}^j \to \mathcal{K}_{j+2,j}(f)$*. Is this map essential? Is* $\mathcal{K}_{j+2,j}$ *free as an object over the operad of* $(j+1)$*-cubes?*

The homotopy type of $\mathcal{K}_{3,1}$ has been worked out in the papers of Budney [10, 9] building on the work of Hatcher [29, 31, 32], with [10] being a good reference. In [9] it was shown that Question 6.6 has an affirmative answer. Moreover, $\mathcal{K}_{3,1} \simeq \mathcal{C}_2(\mathcal{P} \sqcup \{*\})$, where $\mathcal{P} \subset \mathcal{K}_{3,1}$ is the subspace of long knots which are prime. This is a space-level analogue of Schubert's connect-sum decomposition of knots [67]. The homotopy-type of $\mathcal{P}$ was worked out in [10]. The description turns out to be recursive, in terms of an indexing of the components of $\mathcal{K}_{3,1}$ by a collection of vertex-labelled trees in [13]. One peculiarity of the homotopy-type of $\mathcal{K}_{3,1}$ is there is a 'fractal-like' $\mathcal{C}_2$-structure, in the sense that there is a map $\mathcal{K}_{3,1} \times S^1 \to \mathcal{P}$ which is a homotopy-equivalence onto a subspace of components of $\mathcal{P}$.

There have been several computations of $\pi_0 \mathcal{K}_{n,j}$. From Proposition 5.6, the first non-trivial homotopy-group of $\mathcal{K}_{n,j}$ is in dimension $2n-3j-3$, thus $\pi_0 \mathcal{K}_{n,j} = 0$ for $2n-3j-3 > 0$.

Along the $2n-3j-3=0$ line there is $\pi_0 \mathcal{K}_{3,1}$ which is the free commutative monoid on $\pi_0 \mathcal{P}$ [67]. Provided $j > 1$, there are Haefliger's computations [28]:

$$\pi_0 \mathcal{K}_{n,j} \simeq \begin{cases} \mathbb{Z} & j \equiv 3 (\text{mod } 4) \\ \mathbb{Z}_2 & j \equiv 1 (\text{mod } 4) \end{cases}$$

The generator being given by Haefliger's Borromean rings construction [27]. This generator is also the image of r via the graphing construction $\mathrm{gr}_{2d-2} : \pi_{2d-2} \mathcal{K}_{d+2,1} \to \pi_0 \mathcal{K}_{n,j}$ (see Propositions 5.6, 5.10) where $n = 3d, j = 2d-1$.

The work of Haefliger [28], Milgram [55], Kreck and Skopenkov [44] gives $\pi_0\mathcal{K}_{n,j}$ along the $2n-3j-3=-1$ line, provided $n-j>2$. Their computations are:

$$\pi_0\mathcal{K}_{n,j} \simeq \begin{cases} 0 & j \equiv 2 (\text{mod } 4) \\ \mathbb{Z}_4 & j \equiv 4 (\text{mod } 8) \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & j \equiv 0 (\text{mod } 8) \end{cases}$$

The technique of Haefliger [28] involves two main steps. The first step is the construction of an isomorphism $\pi_0\mathcal{K}_{n,j} \simeq C_j^{n-j}$ where C_j^{n-j} is the group of concordance classes of embeddings of S^j in S^n. This step is formally analogous to our Proposition 3.1. Using a Thom-type construction, Haefliger constructs an isomorphism between C_j^{n-j} and a multi-relative homotopy group $C_j^n \simeq \pi_{j+1}(G; SO, G_{n-j})$ where $SO = \varinjlim(\mathrm{SO}_1 \to \mathrm{SO}_2 \to \mathrm{SO}_3 \to \cdots)$ is the stable special-orthogonal group, G_n is the space of degree 1 maps $\mathrm{S}^{n-1} \to \mathrm{S}^{n-1}$, and G is the stable analogous $G = \varinjlim(G_1 \to G_2 \to G_3 \to \cdots)$. This reduces the computation of $\pi_0\mathcal{K}_{n,j}$ to rather traditional problems common to surgery theory [64]: homotopy groups of spheres and orthogonal groups.

The recent work of Takase [76] proves that any embedding of $\mathrm{S}^{4k-1} \to \mathrm{S}^{6k}$ can be extended to an embedding of $(\mathrm{S}^{2k} \times \mathrm{S}^{2k}) \setminus \mathrm{D}^{4k} \to \mathrm{S}^{6k}$. Takase gives a rather explicit formula for determining the isotopy class of an element of $\mathrm{Emb}(\mathrm{S}^{4k-1}, \mathrm{S}^{6k})$ that simplifies Haefliger's characteristic class computations [27].

The work of Volic, Lambrechts and Turchin [45] gives the homology of $H_*(\mathcal{K}_{n,1}; \mathbb{Q})$ for $n \geq 4$ as the homology of a DGA by showing the collapse of the Vassiliev spectral sequence. Turchin has found a Poisson Algebra structure for this DGA [77, 78], which motivated the author's construction of the 2-cubes action on $\mathcal{K}_{3,1}$. At present the exact relationship with the induced Poisson algebra structure coming from the 2-cubes action on $\mathcal{K}_{n,1}$ given by Salvatore [65] is not known. Nor has the relationship between Salvatore's 2-cubes action on $\mathrm{EC}(1, \mathrm{D}^n)$ and the author's [9] been worked-out.

One would assume that constructions having the flavour of Mostovoy's [60] or something like Anderson and Hsiang's 'bounded embedding spaces' [3] could give suitable good geometric models for the iterated classifying-spaces $B^j\mathcal{K}_{n,j}$ that could relate to the two theorems:

(1) $B\mathcal{K}_{3,1} \simeq \Omega\Sigma^2(\mathcal{P} \sqcup \{*\})$ [9].

(2) $B^n\mathcal{K}_{n,n} \simeq \Omega(PL(n)/O_n)$ [14, 59].

To be a little less vague, a 'good geometric model' would mean the construction of a space $X_{n,j}$ homotopy-equivalent to $B^j\mathcal{K}_{n,j}$ which is either naturally a subspace of the space embeddings of D^j in D^n or $\mathbb{R}^j$ in $\mathbb{R}^n$ respectively. Ideally, $X_{n,j}$ would be closely related to Salvatore's construction of the iterated classifying space [66].

It would be useful to give a new proof of the Morlet Comparison Theorem $\mathcal{K}_{n,n} \simeq \Omega^{n+1}(PL(n)/O_n)$ that uses the 'innate' $\mathcal{C}_{n+1}$-action on $\mathcal{K}_{n,n} = \mathrm{EC}(n, *)$ given in Theorem 4.4. A sufficiently clear proof would perhaps inform on how to construct geometric models for all the spaces $B^{j+1}\mathrm{EC}(j, \mathrm{D}^n)$.

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