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Reducibility of Equivalence Relations Arising from Nonstationary Ideals under Large Cardinal Assumptions

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Abstract Working under large cardinal assumptions such as supercompactness, we study the Borel reducibility between equivalence relations modulo restrictions of the nonstationary ideal on some fixed cardinal κ. We show the consistency of $E^{\lambda^{++},\lambda^{++}}_{\lambda\text{-club}}$, the relation of equivalence modulo the nonstationary ideal restricted to $S^{\lambda^{++}}_{\lambda}$ in the space $(\lambda^{++})^{\lambda^{++}}$, being continuously reducible to $E^{2,\lambda^{++}}_{\lambda^{+}\text{-club}}$, the relation of equivalence modulo the nonstationary ideal restricted to $S^{\lambda^{++}}_{\lambda^{+}}$ in the space $2^{\lambda^{++}}$. Then we show that for κ ineffable $E^{2,\kappa}_{\text{reg}}$, the relation of equivalence modulo the nonstationary ideal restricted to regular cardinals in the space 2^{κ} is Σ^1_1-complete. We finish by showing that, for Π^1_2-indescribable κ, the isomorphism relation between dense linear orders of cardinality κ is Σ^1_1-complete.

1 Introduction

Throughout this article we assume that κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$. The equivalence relations modulo (restrictions of) the nonstationary ideal have provided a very useful tool, and a main focus of study, in generalized descriptive set theory. Friedman, Hyttinen, and Kulikov [2] showed that the relation of equivalence modulo the nonstationary ideal is not a Borel relation, and that if $V = L$, then it is not Δ^1_1. The equivalence relation modulo the nonstationary ideal restricted to a stationary set S, denoted $E^{2,\kappa}_S$ (see Definition 1.3), is useful when it comes to

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studying the complexity of the isomorphism relations of first-order theories ($\cong_T$; see Definition 1.5). In [2] it was proved that, under some cardinality assumptions, $E^{2,\kappa}_{S^\kappa_\omega}$ is Borel reducible to $\cong_T$ for every first-order stable unsuperstable theory T, where S^κ_λ is the set of λ-cofinal ordinals below κ. Similar results were obtained in [2] for the other nonclassifiable theories. This motivates the study of the Borel reducibility properties of $E^{2,\kappa}_S$.

Theorem 1.1 ([2, Theorem 56]) *The following is consistent: for all stationary S and S', $E^{2,\kappa}_S$ is Borel reducible to $E^{2,\kappa}_{S'}$ if and only if $S \subseteq S'$.*

Theorem 1.2 ([2, Theorem 55]) *The following is consistent: $E^{2,\omega_2}_{S^{\omega_2}_\omega}$ is Borel reducible to $E^{2,\omega_2}_{S^{\omega_2}_{\omega_1}}$.*

In [6] the second and third authors used the Borel reducibility properties of the equivalence relation modulo the nonstationary ideal to prove that in L, all Σ^1_1-equivalence relations are reducible to $\cong_{\mathrm{DLO}}$, where DLO is the theory of dense linear orderings without endpoints, which means that this equivalence relation is on top of the Borel reducibility hierarchy among Σ^1_1-equivalence relations, that is, it is Σ^1_1-complete. This result stands in contrast to the classical, countable case, $\kappa = \omega$, for which it is known that all other isomorphism relations are reducible to $\cong_{\mathrm{DLO}}$ (see Friedman and Stanley [1]), but far from all Σ^1_1-equivalence relations are reducible to it; even some Borel equivalence relations such as E_1 are not reducible to any isomorphism relations in the countable case. So the question remains: Is the Σ^1_1-completeness of $\cong_{\mathrm{DLO}}$ just a manifestation of the pathological behavior of L or is it a more robust property in the generalized realm? One of the contributions of this article is that the Σ^1_1-completeness of $\cong_{\mathrm{DLO}}$ is indeed a rather robust phenomenon and that it holds whenever κ has certain large cardinal properties (see Theorem 3.10).

It was asked in Friedman, Hyttinen, and Kulikov [3] and in Khomskii, Laguzzi, Löwe, and Sharankou [8, Question 3.46] whether or not the equivalence relation modulo the nonstationary ideal on the Baire space can be reduced to the Cantor space for some fixed cofinality: in our notation, whether or not $E^{\kappa,\kappa}_{S^\kappa_\mu} \le E^{2,\kappa}_{S^\kappa_\mu}$. We approach the problem by proving several results in this direction. Our results have the forms

$$E^{\kappa,\kappa}_{S^\kappa_\mu} \le E^{2,\kappa}_{S^\kappa_{\mu*}},$$

$$E^{\kappa,\kappa}_{S^\kappa_\mu} \le E^{2,\kappa}_{\mathrm{reg}(\kappa)},$$

and

$$E^{\kappa,\kappa}_{\mathrm{reg}(\kappa)} \le E^{2,\kappa}_{\mathrm{reg}(\kappa)},$$

where μ^* is larger than μ and $\mathrm{reg}(\kappa)$ is the set of regular cardinals below κ, for κ Mahlo. These results are obtained under various assumptions and sometimes in forcing extensions.

Many of the results in the area of reducibility of equivalence relations modulo nonstationary ideals use combinatorial principles, like $\diamondsuit$, and other reflection principles. In this article we also bring some large cardinal principles into the picture.

The generalized Baire space is the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set. The open sets are of the form $\bigcup X$ where X is a collection of basic open sets. The collection of κ-Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections of length κ. Since in this article we do not consider any other kind of Borel sets besides κ-Borel, we will omit the prefix "κ-".

The generalized Cantor space is the subspace $2^\kappa \subset \kappa^\kappa$ with the relative subspace topology. For $X, Y \in \{\kappa^\kappa, 2^\kappa\}$, we say that a function $f: X \to Y$ is *Borel* if for every open set $A \subseteq Y$ the inverse image $f^{-1}[A]$ is a Borel subset of X. Let E_1 and E_2 be equivalence relations on X and Y, respectively. We say that E_1 is *Borel reducible* to E_2 if there is a Borel function $f: X \to Y$ that satisfies $(\eta, \xi) \in E_1 \Leftrightarrow (f(\eta), f(\xi)) \in E_2$. We call f a *reduction* of E_1 to E_2. This is denoted by $E_1 \leq_B E_2$, and if f is continuous, then we say that E_1 is *continuously reducible* to E_2, which is denoted by $E_1 \leq_c E_2$.

For every stationary $S \subset \kappa$, we define the equivalence relation modulo the non-stationary ideal restricted to a stationary set S, on the space λ^κ for $\lambda \in \{2, \kappa\}$, as follows.

Definition 1.3 For every stationary $S \subset \kappa$ and $\lambda \in \{2, \kappa\}$, we define $E_S^{\lambda,\kappa}$ as the relation

$$E_S^{\lambda,\kappa} = \big\{(\eta, \xi) \in \lambda^\kappa \times \lambda^\kappa \bigm| \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is not stationary}\big\}.$$

Note that $E_S^{2,\kappa}$ can be identified with the equivalence relation on the power set of κ in which two sets A and B are equivalent if their symmetric difference restricted to S is nonstationary. This can be done by identifying a set $A \subset \kappa$ with its characteristic function.

For every regular cardinal $\mu < \kappa$, we denote $\{\alpha < \kappa \mid cf(\alpha) = \mu\}$ by S^κ_μ. A set C is μ-club if it is unbounded and closed under μ-limits. For brevity, when $S = S^\kappa_\mu$, we will denote $E^{\lambda,\kappa}_{S^\kappa_\mu}$ by $E^{\lambda,\kappa}_{\mu\text{-club}}$. Note that $(f, g) \in E^{\lambda,\kappa}_{\mu\text{-club}}$ if and only if the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ contains a μ-club.

For a Mahlo cardinal κ, the set $\operatorname{reg}(\kappa) = \{\alpha < \kappa \mid \alpha \text{ a regular cardinal}\}$ is stationary. We will denote the equivalence relation $E^{\lambda,\kappa}_{\operatorname{reg}(\kappa)}$ by $E^{\lambda,\kappa}_{\operatorname{reg}}$.

Given an equivalence relation E on $X \in \{\kappa^\kappa, 2^\kappa\}$, we can define the λ-product relation of E for any $0 < \lambda < \kappa$. The λ-product relation $\Pi_\lambda E$ is the relation defined on $X^\lambda \times X^\lambda$ by $\eta\ \Pi_\lambda E\ \xi$ if $\eta_\gamma\ E\ \xi_\gamma$ holds for every $\gamma < \lambda$, where $\eta = (\eta_\gamma)_{\gamma<\lambda}$ and $\xi = (\xi_\gamma)_{\gamma<\lambda}$. We endow the space X^λ, $X \in \{\kappa^\kappa, 2^\kappa\}$, with the box topology generated by the basic open sets

$$\big\{\Pi_{\alpha<\lambda}\mathcal{O}_\alpha \bigm| \forall\alpha < \lambda\ (\mathcal{O}_\alpha \text{ is an open set in } X)\big\}.$$

One of the motivations for studying Borel reducibility in generalized Baire spaces is the connection with model theory. This connection allows for the possibility of studying the Borel reducibility of the isomorphism relation of theories by coding structures with universe κ via elements of κ^κ. We may fix this coding, relative to a given countable relational vocabulary $\mathcal{L} = \{P_n \mid n < \omega\}$, as in the following definition.

Definition 1.4 Fix a bijection $\pi: \kappa^{<\omega} \to \kappa$. For every $\eta \in \kappa^\kappa$, define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with universe κ as follows. For every relation P_m with arity n, every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n satisfies

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Longleftrightarrow \eta\big(\pi(m, a_1, a_2, \ldots, a_n)\big) \geq 1.$$

When we describe a complete theory T in a vocabulary $\mathcal{L}' \subseteq \mathcal{L}$, we think of it as a complete $\mathcal{L}$-theory extending $T \cup \{\forall \bar{x} \neg P_n(\bar{x}) \mid P_n \in \mathcal{L} \backslash \mathcal{L}'\}$.

Definition 1.5 (The isomorphism relation) Assume that T is a complete first-order theory in a countable vocabulary. We define $\cong_T$ as the relation

$$\big\{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \;\big|\; (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\big\}.$$

In the second section we will study the reducibility between different cofinalities, and in the last section we will study the reducibility of $E^{\kappa,\kappa}_{\text{reg}}$ and $E^{2,\kappa}_{\text{reg}}$. Here is the list of the main results in this article:

(a) (Theorem 2.11) *Suppose that κ is a $\Pi^{\lambda^+}_1$-indescribable cardinal for some $\lambda < \kappa$ and $V = L$. Then there is a forcing extension where κ is collapsed to become λ^{++} and $E^{\lambda^{++},\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{2,\lambda^{++}}_{\lambda^+\text{-club}}$.*

(b) (Corollary 2.14) *Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$ and such that $E^{\omega_2,\omega_2}_{\omega\text{-club}} \leq_c E^{\omega_2,\omega_2}_{\omega_1\text{-club}}$, and for every $n > 2$ and every $0 \leq k \leq n-3$, $E^{\omega_n,\omega_n}_{\omega_k\text{-club}} \leq_c E^{\omega_n,\omega_n}_{\omega_{n-1}\text{-club}}$.*

This corollary follows from [7, Theorem 1.3] and gives a model (different from L or the one in Theorem 1.2) in which reducibility between different cofinalities holds.

(c) (Theorem 3.5) *Suppose that $S = S^\kappa_\lambda$ for some regular cardinal $\lambda < \kappa$, or $S = \text{reg}(\kappa)$ with κ weakly compact. If κ has the weakly compact diamond (Definition 3.2), then $E^{\kappa,\kappa}_S \leq_c E^{2,\kappa}_{\text{reg}}$.*

(d) (Corollary 3.6) *Suppose that $V = L$ and that κ is weakly compact. Then $E^{2,\kappa}_{\text{reg}}$ is Σ^1_1-complete.*

(e) (Corollary 3.7) *Suppose that κ is a weakly ineffable cardinal. Then $E^{\kappa,\kappa}_{\text{reg}} \leq_c E^{2,\kappa}_{\text{reg}}$.*

(f) (Theorem 3.8) *If κ is a Π^1_2-indescribable cardinal, then $E^{\kappa,\kappa}_{\text{reg}}$ is Σ^1_1-complete.*

(g) (Corollary 3.9) *Suppose that κ is an ineffable cardinal (or weakly ineffable and Π^1_2-indescribable). Then $E^{2,\kappa}_{\text{reg}}$ is Σ^1_1-complete.*

(h) (Theorem 3.10) *Let DLO be the theory of dense linear orderings without endpoints. If κ is a Π^1_2-indescribable cardinal, then $\cong_{\text{DLO}}$ is Σ^1_1-complete.*

2 Reducibility Between Different Cofinalities

In [2], the authors studied the reducibility between the relations $E^{2,\kappa}_{\mu\text{-club}}$ and showed in particular the consistency of $E^{2,\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{2,\lambda^{++}}_{\lambda^+\text{-club}}$. In this section we continue along these lines.

Definition 2.1 We say that a set $X \subset \kappa$ strongly reflects to a set $Y \subset \kappa$ if for all stationary $Z \subset X$ there exist stationary many $\alpha \in Y$ with $Z \cap \alpha$ stationary in α.

In [2, Theorem 55] it is proved that if κ is a weakly compact cardinal, then S^κ_λ strongly reflects to $\text{reg}(\kappa)$, for any regular cardinal $\lambda < \kappa$. This result can be generalized to Π^λ_1-indescribable cardinals.

Definition 2.2 A cardinal κ is Π^λ_1-*indescribable* (for $\lambda < \kappa$) if whenever $A \subset V_\kappa$ and σ is a Π_1-sentence such that

$$\big(V_{\kappa+\lambda}, \in, A, (V_{\kappa+\xi} \mid \xi < \lambda)\big) \models \sigma,$$

then for some $\alpha < \kappa$,

$$\big(V_{\alpha+\lambda}, \in, A \cap V_\alpha, (V_{\alpha+\xi} \, : \, \xi < \lambda)\big) \models \sigma.$$

Note that, in Definition 2.2, the existence of some $\alpha < \kappa$ at which the required reflection is effected is equivalent to the existence of stationary many such $\alpha < \kappa$.

Lemma 2.3 *Suppose that κ is a Π^λ_1-indescribable cardinal. There are λ many disjoint stationary subsets of κ, $\langle S_\gamma \rangle_{\gamma<\lambda}$, such that for every $\gamma < \lambda$, $S_\gamma \subseteq \mathrm{reg}(\kappa)$ and κ strongly reflects to S_γ.*

Proof Let S^*_β denote the set of all Π^β_1-indescribable cardinals below κ. Since "κ is Π^β_1-indescribable" is a Π_1 property of the structure $(V_{\kappa+\lambda}, \in, (V_{\kappa+\xi} \mid \xi < \lambda))$, the set S^*_β is stationary for every $\beta < \lambda$.

Let us show that for every stationary set $X \subseteq \kappa$,

$$B = \{\alpha \in S^*_\beta \mid X \cap \alpha \text{ is stationary in } \alpha\}$$

is stationary. Let C be a club in κ. The sentence

$$(C \text{ is unbounded in } \kappa) \wedge (X \text{ is stationary in } \kappa) \wedge (\kappa \text{ is } \Pi^\beta_1\text{-indescribable})$$

is a Π_1 property of the structure $(V_{\kappa+\lambda}, \in, X, C, (V_{\kappa+\xi} \mid \xi < \lambda))$. By reflection, there is $\gamma < \kappa$ such that $C \cap \gamma$ is unbounded in γ, and hence $\gamma \in C$, $X \cap \gamma$ is stationary in γ, and γ is Π^β_1-indescribable. We conclude that $C \cap B \neq \emptyset$.

Let us denote $S^*_\beta \setminus S^*_{\beta+1}$ by S_β. Let us show that for every stationary set $X \subseteq \kappa$,

$$\{\alpha \in S_\beta \mid X \cap \alpha \text{ is stationary in } \alpha\}$$

is stationary. Let C be a club in κ. Since $\{\alpha \in S^*_\beta \mid X \cap \alpha \text{ is stationary in } \alpha\}$ is stationary, we can pick $\gamma \in C \cap \{\alpha \in S^*_\beta \mid X \cap \alpha \text{ is stationary in } \alpha\}$ such that γ is minimal.

Claim 2.3.1 *We have that γ is not $\Pi^{\beta+1}_1$-indescribable.*

Proof Suppose toward a contradiction that γ is $\Pi^{\beta+1}_1$-indescribable. The sentence

$$(C \cap \gamma \text{ is unbounded in } \gamma) \wedge (X \cap \gamma \text{ is stationary in } \gamma) \wedge (\gamma \text{ is } \Pi^\beta_1\text{-indescribable})$$

is a Π_1 property of the structure $(V_{\gamma+\beta+1}, \in, X \cap \gamma, C \cap \gamma, (V_{\gamma+\xi} \mid \xi < \beta+1))$. By reflection, there is $\gamma' < \gamma$ such that $C \cap \gamma'$ is unbounded in γ', $X \cap \gamma'$ is stationary in γ', and γ' is Π^β_1-indescribable. This contradicts the minimality of γ. □

We conclude that S_β is stationary and $\{\alpha \in S_\beta \mid X \cap \alpha \text{ is stationary in } \alpha\}$ is stationary, for every $\beta < \lambda$. □

The notion of $\diamond$-reflection was introduced in [2] in order to find reductions between equivalence relations modulo nonstationary ideals (see below).

Definition 2.4 ($\diamond$-reflection) Let X, Y be subsets of κ, and suppose that Y consists of ordinals of uncountable cofinality. We say that X *$\diamond$-reflects to* Y if there exists a sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ such that

(i) $D_\alpha \subset \alpha$ is stationary in α for all $\alpha \in Y$;
(ii) if $Z \subset X$ is stationary, then $\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}$ is stationary.

Theorem 2.5 ([2, Theorem 59]) *Suppose that $V = L$ and that $X \subseteq \kappa$ and $Y \subseteq \mathrm{reg}(\kappa)$. If X strongly reflects to Y, then X $\diamond$-reflects to Y.*

Theorem 2.6 ([2, Theorem 58]) *If X $\diamond$-reflects to Y, then $E^{2,\kappa}_X \leq_c E^{2,\kappa}_Y$.*

The notion of $\diamond$-reflection also implies some reductions for the relations $E^{\kappa,\kappa}_{\mu\text{-club}}$ on the space κ^κ. To show this, we first need to introduce some definitions.

Definition 2.7 For every $\alpha < \kappa$ with $\gamma < cf(\alpha)$, define $E^{\kappa,\kappa}_{\gamma\text{-club}} \restriction \alpha$ by

$$E^{\kappa,\kappa}_{\gamma\text{-club}} \restriction \alpha = \big\{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \;\big|\; \exists C \subseteq \alpha \text{ a } \gamma\text{-club}, \forall \beta \in C, \eta(\beta) = \xi(\beta)\big\}.$$

Proposition 2.8 *Suppose that $\gamma < \lambda < \kappa$ are regular cardinals. If S^κ_γ strongly reflects to S^κ_λ, then $E^{\kappa,\kappa}_{\gamma\text{-club}} \leq_c E^{\kappa,\kappa}_{\lambda\text{-club}}$.*

Proof Suppose that for every stationary set $S \subset S^\kappa_\gamma$ it holds that

$$\{\alpha \in S^\kappa_\lambda \mid S \cap \alpha \text{ is stationary in } \alpha\}$$

is a stationary set, and define $F: \kappa^\kappa \to \kappa^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} f_\alpha(\eta) & \text{if } cf(\alpha) = \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_\alpha(\eta)$ is a code in $\kappa\backslash\{0\}$ for the $(E^{\kappa,\kappa}_{\gamma\text{-club}} \restriction \alpha)$-equivalence class of η.

Let us prove that if $(\eta, \xi) \in E^{\kappa,\kappa}_{\gamma\text{-club}}$, then $(F(\eta), F(\xi)) \in E^{\kappa,\kappa}_{\lambda\text{-club}}$. Suppose that $(\eta, \xi) \in E^{\kappa,\kappa}_{\gamma\text{-club}}$. There is a γ-club where η and ξ coincide and so there is a club C such that for all $\alpha \in C \cap S^\kappa_\lambda$, the functions η and ξ are $(E^{\kappa,\kappa}_{\gamma\text{-club}} \restriction \alpha)$-equivalent. Thus, by the definition of F, for all $\alpha \in C \cap S^\kappa_\lambda$, $F(\eta)(\alpha) = F(\xi)(\alpha)$. We conclude that $(F(\eta), F(\xi)) \in E^{\kappa,\kappa}_{\lambda\text{-club}}$.

Let us prove that if $(\eta, \xi) \notin E^{\kappa,\kappa}_{\gamma\text{-club}}$, then $(F(\eta), F(\xi)) \notin E^{\kappa,\kappa}_{\lambda\text{-club}}$. Suppose that $(\eta, \xi) \notin E^{\kappa,\kappa}_{\gamma\text{-club}}$. Then there is a stationary $S \subset S^\kappa_\gamma$ on which $\eta(\alpha) \neq \xi(\alpha)$. Since $A = \{\alpha \in S^\kappa_\lambda \mid S \cap \alpha \text{ is stationary in } \alpha\}$ is stationary and for all $\alpha \in A$, $f_\alpha(\eta) \neq f_\alpha(\xi)$, we conclude that $(F(\eta), F(\xi)) \notin E^{\kappa,\kappa}_{\lambda\text{-club}}$. □

Corollary 2.9 *Suppose that $\gamma < \lambda < \kappa$ are regular cardinals. If S^κ_γ $\diamond$-reflects to S^κ_λ, then*

1. $E^{2,\kappa}_{\gamma\text{-club}} \leq_c E^{2,\kappa}_{\lambda\text{-club}}$,
2. $E^{\kappa,\kappa}_{\gamma\text{-club}} \leq_c E^{\kappa,\kappa}_{\lambda\text{-club}}$.

Proof

1. This follows from Theorem 2.6.
2. By the definition of $\diamond$-reflection, S^κ_γ $\diamond$-reflecting to S^κ_λ implies that for all $S \subseteq S^\kappa_\gamma$, the set $\{\alpha \in S^\kappa_\lambda \mid S \cap \alpha \text{ is stationary in } \alpha\}$ is a stationary set. The result follows from Proposition 2.8. □

In [2], the consistency of $S^{\lambda^{++}}_\lambda$ $\diamond$-reflecting to $S^{\lambda^{++}}_{\lambda^+}$ was shown. This gives a model in which $E^{2,\kappa}_{\lambda\text{-club}} \leq_c E^{2,\kappa}_{\lambda^+\text{-club}}$ and $E^{\lambda^{++},\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{\lambda^{++},\lambda^{++}}_{\lambda^+\text{-club}}$.

Theorem 2.10 ([2, Theorem 55]) *Suppose that κ is a weakly compact cardinal and that $V = L$. Then:*

1. *$E^{2,\kappa}_{\lambda\text{-club}} \leq_c E^{2,\kappa}_{\text{reg}}$ holds for all regular $\lambda < \kappa$.*
2. *For every regular $\lambda < \kappa$, there is a forcing extension where κ is collapsed to become λ^{++} and $E^{2,\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{2,\lambda^{++}}_{\lambda^+\text{-club}}$.*

The proof of this theorem can be generalized using Lemma 2.3 to show the consistency of $E^{\lambda^{++},\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{2,\lambda^{++}}_{\lambda^{+}\text{-club}}$.

Theorem 2.11 *Suppose that κ is a $\Pi^{\lambda^+}_1$-indescribable cardinal and that $V = L$. Then there is a forcing extension where κ is collapsed to become λ^{++} and $E^{\lambda^{++},\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{2,\lambda^{++}}_{\lambda^{+}\text{-club}}$.*

Proof Let us collapse κ to λ^{++} with the Levy collapse

$$\mathbb{P} = \big\{f\colon \operatorname{reg}(\kappa) \to \kappa^{<\lambda^+} \mid \operatorname{rang}\big(f(\mu)\big) \subset \mu, \big|\{\mu \mid f(\mu) \neq \emptyset\}\big| \leq \lambda\big\},$$

where $f \geq g$ if and only if $f(\mu) \subseteq g(\mu)$ for all $\mu \in \operatorname{reg}(\kappa)$. Let us define $\mathbb{P}_\mu$ and $\mathbb{P}^\mu$ for all μ by: $\mathbb{P}_\mu = \{f \in \mathbb{P} \mid \operatorname{sprt}(f) \subset \mu\}$ and $\mathbb{P}^\mu = \{f \in \mathbb{P} \mid \operatorname{sprt}(f) \subset \kappa\backslash\mu\}$. It is known that all regular $\lambda < \mu \leq \kappa$ satisfy the following:

(i) if $\mu > \lambda^+$, then $\mathbb{P}_\mu$ has the μ-c.c.,
(ii) $\mathbb{P}_\mu$ and $\mathbb{P}^\mu$ are $<\lambda^+$-closed,
(iii) $\mathbb{P} = \mathbb{P}_\kappa \Vdash \lambda^{++} = \check{\kappa}$,
(vi) if $\mu < \kappa$, then $\mathbb{P} \Vdash cf(\check{\mu}) = \lambda^+$,
(v) if $p \in \mathbb{P}$, σ is a name, and $p \Vdash$ "σ is a club in λ^{++}," then there is a club $E \subset \kappa$ such that $p \Vdash \check{E} \subset \sigma$.

Claim 2.11.1 *There is a sequence $\langle S_\gamma\rangle_{\gamma<\lambda^+}$ of disjoint stationary subsets of $S^{\lambda^{++}}_{\lambda^+}$ such that in $V[G]$ $S^{\lambda^{++}}_{\lambda}$ $\diamond$-reflects to S_γ for every $\gamma < \lambda^+$.*

Proof Let G be $\mathbb{P}$-generic over V, and define $G_\mu = G \cap \mathbb{P}_\mu$ and $G^\mu = G \cap \mathbb{P}^\mu$. So G_μ is $\mathbb{P}_\mu$-generic over V, G^μ is $\mathbb{P}^\mu$-generic over $V[G_\mu]$, and $V[G] = V[G_\mu][G^\mu]$. Let S^*_β denote the set of all Π^β_1-indescribable cardinals below κ, and let $S_\beta = S^*_\beta \backslash S^*_{\beta+1}$. We will show that $S^{\lambda^{++}}_\lambda$ $\diamond$-reflects to S^V_β for all $\beta < \lambda^+$. Let us fix $\beta < \lambda^+$ and denote by Y the set S^V_β. By Lemma 2.3 we know that S^V_β is stationary, and by (v) it remains stationary in $V[G]$. By (i) we know that there are no antichains of length μ in $\mathbb{P}_\mu$, and since $|\mathbb{P}_\mu| = \mu$, we conclude that there are at most μ antichains. On the other hand, there are μ^+ many subsets of μ. Hence, there is a bijection

$$h_\mu\colon \mu^+ \to \{\sigma \mid \sigma \text{ is a nice } \mathbb{P}_\mu \text{ name for a subset of } \mu\}$$

for each $\mu \in \operatorname{reg}(\kappa)$ such that $\mu > \lambda^+$, where a nice $\mathbb{P}_\mu$ name for a subset of $\check{\mu}$ is of the form $\bigcup\{\{\check{\alpha}\} \times A_\alpha \mid \alpha \in B\}$ with $B \subset \check{\mu}$ and A_α an antichain in $\mathbb{P}_\mu$. Notice that the nice $\mathbb{P}_\mu$ names for subsets of $\check{\mu}$ are subsets of V_μ. Let us define

$$D_\mu = \begin{cases} [h_\mu([(\bigcup G)(\mu^+)](0))]_G & \text{if this set is stationary,} \\ \mu & \text{otherwise.} \end{cases}$$

We will show that $\langle D_\mu\rangle_{\mu\in Y}$ is the needed $\diamond$-sequence in $V[G]$.

Suppose toward a contradiction that there are a stationary set $S \subset S^{\lambda^{++}}_\lambda$ and a club $C \subset \lambda^{++}$ (in $V[G]$) such that for all $\alpha \in C \cap Y$, $D_\alpha \neq S \cap \alpha$. By (v) there is a club $C_0 \subset C$ such that $C_0 \in V$. Let $\dot{S}$ be a nice name for S, and let p be a condition such that p forces that $\dot{S}$ is stationary. We will show that

$$H = \{q < p \mid q \Vdash D_\mu = \dot{S} \cap \check{\mu} \text{ for some } \mu \in C_0\}$$

is dense below p, which is a contradiction. Let us slightly redefine $\mathbb{P}$.

Let $\mathbb{P}^* = \{q \mid \exists r \in \mathbb{P}\,(r \restriction \mathrm{sprt}(r) = q)\}$. Clearly $\mathbb{P} \cong \mathbb{P}^*$, $\mathbb{P}^* \subseteq V_\kappa$, and $\mathbb{P}^*_\mu = \mathbb{P}^* \cap V_\mu$, where $\mathbb{P}^*_\mu = \{q \mid \exists r \in \mathbb{P}_\mu\,(r \restriction \mathrm{sprt}(r) = q)\}$. It can be verified that the properties mentioned above also hold for $\mathbb{P}^*_\mu$. From now on denote $\mathbb{P}^*_\mu$ by $\mathbb{P}_\mu$. Let r be a condition stronger than p, and let

$$R = \big(\mathbb{P} \times \{0\}\big) \cup \big(\dot{S} \times \{1\}\big) \cup \big(C_0 \times \{2\} \cup \big(\{r\} \times \{3\}\big)\big).$$

Let $\forall A\varphi$ be the formula:

If A is closed and unbounded and $t < r$ are arbitrary, then there exist $q < r$ and $\alpha \in A$ such that $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$.

Clearly, $\forall A\varphi$ says that $r \Vdash$ ($\dot{S}$ is stationary). By (v) it is enough to quantify over club sets in V. Notice that $t < r$, $q < t$, A is a club, and $\alpha \in A$ are first-order expressible using R as a parameter. The definition of $\check{\alpha}$ is recursive in α:

$$\check{\alpha} = \big\{(\check{\gamma}, 1_{\mathbb{P}}) \bigm| \gamma < \alpha\big\}$$

and it is absolute for V_κ. Then $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$ is equivalent to saying that for each $q' < q$ there exists $q'' < q'$ with $(\check{\alpha}, q'') \in \dot{S}$, and this is first-order expressible using R as a parameter. Therefore, $\forall A\varphi$ is a Π_1 property of the structure $(V_\kappa, \in, R)$, even more

$$(\forall A\varphi) \wedge (\kappa \text{ is } \Pi^\beta_1\text{-indescribable})$$

is a Π_1 property of the structure $(V_{\kappa+\lambda^+}, \in, R, (V_{\kappa+\xi} \mid \xi < \lambda^+))$. By reflection, there is $\mu < \kappa$ Π^β_1-indescribable such that $\mu \in C_0$, $r \in \mathbb{P}_\mu$, and

$$\big(V_{\mu+\lambda^+}, \in, R, (V_{\mu+\xi} \mid \xi < \lambda^+)\big) \models \forall A\varphi.$$

In the same way as in Claim 2.3.1, we can show that there is $\mu < \kappa$ Π^β_1-indescribable that is not $\Pi^{\beta+1}_1$-indescribable, that is, $(\check{\mu}_G \in Y)^{V[G]}$ such that $\mu \in C_0$, $r \in \mathbb{P}_\mu$, and $(V_{\mu+\lambda^+}, \in, R, (V_{\mu+\xi} \mid \xi < \lambda^+)) \models \forall A\varphi$. Notice that $\alpha \in S \cap \mu$ implies that $(\check{\alpha}, \check{q}) \in \dot{S}$ for some $q \in \mathbb{P}_\mu$. Let $\dot{S}_\mu = \dot{S} \cap V_\mu$; thus, $r \Vdash_{\mathbb{P}_\mu}$ ($\dot{S}_\mu$ is stationary). Let us define q as follows: $\mathrm{dom}(q) = \mathrm{dom}(r) \cup \{\mu^+\}$, $q \restriction \mu = r \restriction \mu$ and $q(\mu^+) = f$, $\mathrm{dom}(f) = \{0\}$, and $f(0) = h^{-1}_\mu(\dot{S}_\mu)$. Since $\mathbb{P}^\mu$ is $<\lambda^+$-closed and does not kill stationary subsets of $S^{\lambda^{++}}_\lambda$, $(\dot{S}_\mu)_{G_\mu}$ is stationary in $V[G]$, and by the way we chose μ, $(\dot{S}_\mu)_{G_\mu} = (\dot{S}_\mu)_G$. Therefore $q \Vdash_{\mathbb{P}}$ ($\dot{S}_\mu$ is stationary), and by the definition of D_μ (in $V[G]$) we conclude that $q \Vdash_{\mathbb{P}} \dot{S}_\mu = D_\mu$. Finally, by the way we chose μ, we get that $(\dot{S}_\mu)_G = S \cap \mu$. We conclude that H is dense below p, which is a contradiction. □

From now on in this proof, we will work in $V[G]$. In particular, κ will be λ^{++}.

Claim 2.11.2 *We have $E^{\kappa,\kappa}_{\lambda\text{-club}} \leq_c \Pi_{\lambda^+} E^{2,\kappa}_{\lambda\text{-club}}$.*

Proof Let H be a bijection from κ to 2^{λ^+}. Define $\mathcal{F}\colon \kappa^\kappa \to (2^\kappa)^{\lambda^+}$ by $\mathcal{F}(f) = (f_\gamma)_{\gamma<\lambda^+}$, where $f_\gamma(\alpha) = H(f(\alpha))(\gamma)$ for every $\gamma < \lambda^+$ and $\alpha < \kappa$. Let us show that $\mathcal{F}$ is a reduction of $E^{\kappa,\kappa}_{\lambda\text{-club}}$ to $\Pi_{\lambda^+} E^{2,\kappa}_{\lambda\text{-club}}$.

Clearly, $f(\alpha) = g(\alpha)$ implies that $H(f(\alpha)) = H(g(\alpha))$ and $f_\gamma(\alpha) = g_\gamma(\alpha)$ for every $\gamma < \lambda^+$. Therefore, $f\ E^{\kappa,\kappa}_{\lambda\text{-club}}\ g$ implies that for all $\gamma < \lambda^+$, $f_\gamma\ E^{2,\kappa}_{\lambda\text{-club}}\ g_\gamma$ holds. So $f\ \Pi_{\lambda^+} E^{2,\kappa}_{\lambda\text{-club}}\ g$.

Suppose that for every $\gamma < \lambda^+$ there is C_γ, a λ-club, such that $f_\gamma(\alpha) = g_\gamma(\alpha)$ holds for every $\alpha \in C_\gamma$. Since the intersection of less than κ λ-club sets is a λ-club set, there is a λ-club C on which the functions f_γ and g_γ coincide for every $\gamma < \lambda^+$. Therefore, $H(f(\alpha))(\gamma) = H(g(\alpha))(\gamma)$ holds for every $\gamma < \lambda^+$ and every $\alpha \in C$, so $H(f(\alpha)) = H(g(\alpha))$ for every $\alpha \in C$. Since H is a bijection, we can conclude that $f(\alpha) = g(\alpha)$ for every $\alpha \in C$, and hence $f\ E^{\kappa,\kappa}_{\lambda\text{-club}}\ g$. □

By Claim 2.11.1, there is a sequence $\langle S_\gamma\rangle_{\gamma<\lambda^+}$ of disjoint stationary subsets of $S^\kappa_{\lambda^+}$ such that S^κ_λ $\diamond$-reflects to S_γ for all $\gamma < \lambda^+$. Let $\langle D^\gamma_\alpha\rangle_{\alpha\in S_\gamma}$ be a sequence that witnesses that S^κ_λ $\diamond$-reflects to S_γ.

For every $\eta \in \kappa^\kappa$, define $F(\eta)$ by

$$F(\eta)(\alpha) = \begin{cases} 1 & \text{if there is } \gamma < \lambda^+ \text{ with } \alpha \in S_\gamma \text{ and} \\ & \mathcal{F}(\eta)_\gamma^{-1}[1] \cap D^\gamma_\alpha \text{ stationary in } \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\mathcal{F}(\eta)_\gamma)_{\gamma<\lambda^+} = \mathcal{F}(\eta)$ and where $\mathcal{F}$ is the reduction given by Claim 2.11.2.

Suppose that η, ξ are not $E^{\kappa,\kappa}_{\lambda\text{-club}}$-equivalent. By Claim 2.11.2 there exists $\gamma < \lambda^+$ such that $\mathcal{F}(\eta)_\gamma^{-1}[1]\Delta\mathcal{F}(\xi)_\gamma^{-1}[1]$ is stationary. Therefore, either $\mathcal{F}(\eta)_\gamma^{-1}[1]\backslash \mathcal{F}(\xi)_\gamma^{-1}[1]$ or $\mathcal{F}(\xi)_\gamma^{-1}[1]\backslash\mathcal{F}(\eta)_\gamma^{-1}[1]$ is stationary. Without loss of generality, let us assume that $\mathcal{F}(\eta)_\gamma^{-1}[1]\backslash\mathcal{F}(\xi)_\gamma^{-1}[1]$ is stationary. Since S^κ_λ $\diamond$-reflects to S_γ,

$$A = \left\{\alpha \in S_\gamma \;\middle|\; \left(\mathcal{F}(\eta)_\gamma^{-1}[1]\backslash\mathcal{F}(\xi)_\gamma^{-1}[1]\right) \cap \alpha = D^\gamma_\alpha\right\}$$

is stationary and D^γ_α is stationary in α, and therefore $A \subseteq F(\eta)^{-1}[1]$. On the other hand, for every α in A we have $\mathcal{F}(\xi)_\gamma^{-1}[1] \cap D^\gamma_\alpha = \emptyset$, so $A \cap F(\xi)^{-1}[1] = \emptyset$ and we conclude that $A \subseteq F(\eta)^{-1}[1]\Delta F(\xi)^{-1}[1]$. Therefore $F(\eta)^{-1}[1]\Delta F(\xi)^{-1}[1]$ is stationary, and $F(\eta)$ and $F(\xi)$ are not $E^{2,\lambda^{++}}_{\lambda^+\text{-club}}$-equivalent.

Suppose that $F(\eta)$ and $F(\xi)$ are not $E^{2,\lambda^{++}}_{\lambda^+\text{-club}}$-equivalent, so $F(\eta)^{-1}[1]\Delta F(\xi)^{-1}[1]$ is stationary. Since $\lambda^+ < \kappa$, by Fodor's lemma we know that there exists $\gamma < \lambda^+$ such that $\{\alpha \in S_\gamma \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\}$ is stationary. Hence, the symmetric difference of the sets $\{\alpha \in S_\gamma \mid \mathcal{F}(\eta)_\gamma^{-1}[1] \cap D^\gamma_\alpha \text{ is stationary in } \alpha\}$ and $\{\alpha \in S_\gamma \mid \mathcal{F}(\xi)_\gamma^{-1}[1] \cap D^\gamma_\alpha \text{ is stationary in } \alpha\}$ is stationary. For simplicity, let us denote by A_η the set $\{\alpha \in S_\gamma \mid \mathcal{F}(\eta)_\gamma^{-1}[1] \cap D^\gamma_\alpha \text{ is stationary in } \alpha\}$ and by A_ξ the set $\{\alpha \in S_\gamma \mid \mathcal{F}(\xi)_\gamma^{-1}[1] \cap D^\gamma_\alpha \text{ is stationary in } \alpha\}$. Therefore, either $A_\eta\backslash A_\xi$ or $A_\xi\backslash A_\eta$ is stationary. Without loss of generality, we can assume that $A_\eta\backslash A_\xi$ is stationary. Hence, $\bigcup_{\alpha\in A_\eta\backslash A_\xi}(\mathcal{F}(\eta)_\gamma^{-1}[1] \cap D^\gamma_\alpha)\backslash\mathcal{F}(\xi)_\gamma^{-1}[1]$ is stationary and is contained in $\mathcal{F}(\eta)_\gamma^{-1}[1]\Delta\mathcal{F}(\xi)_\gamma^{-1}[1]$. By Claim 2.11.2, we conclude that η and ξ are not $E^{\kappa,\kappa}_{\lambda\text{-club}}$-equivalent. □

Note that Theorem 2.11 implies the consistency of

$$E^{2,\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{\lambda^{++},\lambda^{++}}_{\lambda\text{-club}} \leq_c E^{2,\lambda^{++}}_{\lambda^+\text{-club}} \leq_c E^{\lambda^{++},\lambda^{++}}_{\lambda^+\text{-club}}.$$

In particular, for $\lambda = \omega$ we get the expression

$$E^{2,\omega_2}_{\omega\text{-club}} \leq_c E^{\omega_2,\omega_2}_{\omega\text{-club}} \leq_c E^{2,\omega_2}_{\omega_1\text{-club}} \leq_c E^{\omega_2,\omega_2}_{\omega_1\text{-club}}.$$

Question 2.12 Is it consistent that

$$E^{2,\kappa}_{\gamma\text{-club}} \lneq_c E^{\kappa,\kappa}_{\gamma\text{-club}} \lneq_c E^{2,\kappa}_{\lambda\text{-club}}$$

holds for all $\gamma, \lambda < \kappa$ and $\gamma < \lambda$?

We will finish this section by showing that the reduction $E^{\omega_2,\omega_2}_{\omega\text{-club}} \leq_c E^{\omega_2,\omega_2}_{\omega_1\text{-club}}$ can be obtained using other reflection principles. Specifically, full reflection implies this reduction. For stationary subsets S and A of κ, we say that S *reflects fully* in A if the set $\{\alpha \in A \mid S \cap \alpha$ is nonstationary in $\alpha\}$ is nonstationary. Note that if $S \subset S^\kappa_\gamma$ reflects fully in S^κ_λ, then the set $\{\alpha \in S^\kappa_\lambda \mid S \cap \alpha$ is stationary in $\alpha\}$ is a stationary set.

Theorem 2.13 (Jech and Shelah [7, Theorem 1.3]) *Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$ and such that*

1. *every stationary set $S \subset S^{\omega_2}_\omega$ reflects fully in $S^{\omega_2}_{\omega_1}$;*
2. *for every $2 < n$ and every $0 \leq k \leq n-3$, every stationary set $S \subset S^{\omega_n}_{\omega_k}$ reflects fully in $S^{\omega_n}_{\omega_{n-1}}$.*

In the generic extension of Theorem 2.13 it holds that $\omega_i^{<\omega_i} = \omega_i$ for all $i < \omega$ (see [7, Theorem 1.3]).

Corollary 2.14 *Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$ and such that: $E^{\omega_2,\omega_2}_{\omega\text{-club}} \leq_c E^{\omega_2,\omega_2}_{\omega_1\text{-club}}$, and for every $n > 2$ and every $0 \leq k \leq n-3$, $E^{\omega_n,\omega_n}_{\omega_k\text{-club}} \leq_c E^{\omega_n,\omega_n}_{\omega_{n-1}\text{-club}}$.*

In [7] it was also proved that Theorem 2.13(ii) is optimal, in the sense that it cannot be improved to include the case $k = n-2$ (see [7, Proposition 1.6]). The best possible reduction we can get using only full reflection is the one in Corollary 2.14. By a Σ^1_1-completeness result, it is known that the following is consistent (see Theorem 3.1 below):

$$\forall k < n-1 \quad (E^{\omega_n,\omega_n}_{\omega_k\text{-club}} \leq_c E^{\omega_n,\omega_n}_{\omega_{n-1}\text{-club}}).$$

3 Σ^1_1-Completeness

An equivalence relation E on $X \in \{\kappa^\kappa, 2^\kappa\}$ is Σ^1_1 if E is the projection of a closed set in $X^2 \times \kappa^\kappa$ and it is Σ^1_1-*complete* if every Σ^1_1-equivalence relation is Borel reducible to it. The study of Σ^1_1 and Σ^1_1-complete equivalence relations is an important area of generalized descriptive set theory because, for example, the isomorphism relation on classes of models is always Σ^1_1. The same holds, in fact, in classical descriptive set theory, but the behavior of Σ^1_1-complete relations there is different. For example, in the classical setting ($\kappa = \omega$) the isomorphism relation is never Σ^1_1-complete, while in generalized descriptive set theory this is often the case (see, e.g., [2], [6]).

Theorem 3.1 ([6, Theorem 7]) *Suppose that $V = L$ and $\kappa > \omega$. Then $E^{\kappa,\kappa}_{\mu\text{-club}}$ is Σ^1_1-complete for every regular $\mu < \kappa$.*

We know that $E^{\kappa,\kappa}_{\lambda\text{-club}} \restriction \alpha$ is an equivalence relation for every $\alpha < \kappa$ with $cf(\alpha) > \lambda$. Let us define the following relation:

$$(\eta, \xi) \in E^{\kappa,\kappa}_{\text{reg}} \restriction \alpha \Leftrightarrow \big\{\beta \in \text{reg}(\alpha) \bigm| \eta(\beta) \neq \xi(\beta)\big\} \text{ is not stationary.}$$

It is easy to see that $E^{\kappa,\kappa}_{\text{reg}} \restriction \alpha$ is an equivalence relation.

Definition 3.2 (Weakly compact diamond) This notion was originally defined in Sun [9]. Let $\kappa > \omega$ be a cardinal. The *weakly compact ideal* is generated by the sets of the form $\{\alpha < \kappa \mid \langle V_\alpha, \in, U \cap V_\alpha\rangle \models \neg\varphi\}$, where $U \subset V_\kappa$ and φ is a

Π^1_1-sentence such that $\langle V_\kappa, \in, U\rangle \models \varphi$. One can define a diamond principle with respect to this ideal (rather than the nonstationary ideal). A set $A \subset \kappa$ is said to be *weakly compact* if it does not belong to the weakly compact ideal. Note that κ is weakly compact if and only if there exists $A \subset \kappa$ which is weakly compact, that is, the weakly compact ideal is proper. For weakly compact $S \subset \kappa$, the S-*weakly compact diamond*, $\mathrm{WC}_\kappa(S)$, is the statement that there exists a sequence $(A_\alpha)_{\alpha<\kappa}$ such that for every $A \subset S$ the set

$$\{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$$

is weakly compact. We denote $\mathrm{WC}_\kappa = \mathrm{WC}_\kappa(\kappa)$.

For a survey on weakly compact diamonds, see Hellsten [5].

Fact 3.3 *The main facts that we will use are the following:*

1. *If κ is weakly compact and $V = L$, then* WC_κ *holds.*
2. *If κ is weakly ineffable (same as almost ineffable), then* WC_κ *holds.*

See [5] for proofs and references.

Lemma 3.4 *Let κ be a weakly compact cardinal. The weakly compact diamond* WC_κ *implies the following principle* WC^*_κ. *There exists a sequence $\langle f_\alpha\rangle_{\alpha\in\mathrm{reg}(\kappa)}$ such that*

(a) $f_\alpha \colon \alpha \to \alpha$,
(b) *for all $g \in \kappa^\kappa$ and stationary $Z \subset \kappa$ the set*

$$\left\{\alpha \in \mathrm{reg}(\kappa) \;\middle|\; g \restriction \alpha = f_\alpha \wedge \alpha \cap Z \text{ is stationary}\right\}$$

is stationary.

Proof For the sake of this proof we view functions $f \colon \alpha \to \alpha$ as subsets of $\alpha \times \alpha$.

Let $(A_\alpha)_{\alpha<\kappa}$ be the WC_κ-sequence, and let $\pi \colon \kappa \times \kappa \to \kappa$ be a bijection. Let C_π be the set $\{\alpha < \kappa \mid \pi[\alpha \times \alpha] = \alpha\}$. It is standard to verify that C_π is a club. For all $\alpha \in \mathrm{reg}(\kappa)$, let $f_\alpha = \pi^{-1}[A_\alpha]$ if $\alpha \in C_\pi$ and $\pi^{-1}[A_\alpha]$ is a function (i.e., for all $\beta < \alpha$ there exists exactly one γ such that $(\beta,\gamma) \in \pi^{-1}[A_\alpha]$) and otherwise set f_α to be arbitrary. Let us show that this sequence is as desired. Let $g \in \kappa^\kappa$ be a function, and let Z be stationary. Let C_g be the set $\{\alpha < \kappa \mid g[\alpha] \subset \alpha\}$ which is again a club. The set

$$\left\{\alpha < \kappa \;\middle|\; \pi[g] \cap \alpha = A_\alpha\right\}$$

is weakly compact and so is

$$\left\{\alpha \in C_g \cap C_\pi \;\middle|\; \pi[g] \cap \alpha = A_\alpha\right\}.$$

But since $\alpha \in C_\pi \cap C_g$, we have $\pi[g] \cap \alpha = \pi[g \cap (\alpha \times \alpha)]$, so this set is equal to

$$\begin{aligned} S &= \left\{\alpha \in C_g \cap C_\pi \;\middle|\; g \cap (\alpha \times \alpha) = \pi^{-1}[A_\alpha]\right\} \\ &= \{\alpha \in C_g \cap C_\pi \mid g \restriction \alpha = f_\alpha\}. \end{aligned}$$

By the weak compactness of S, the stationarity of Z is reflected to a stationary subset $S' \subset S$, so $Z \cap \alpha$ is stationary for all $\alpha \in S'$. $\square$

Theorem 3.5 *Suppose that $S = S^\kappa_\lambda$ for some λ regular cardinal, or $S = \mathrm{reg}(\kappa)$ and κ is a weakly compact cardinal. If κ has the weakly compact diamond, then $E^{\kappa,\kappa}_S \le_c E^{2,\kappa}_{\mathrm{reg}}$.*

Proof Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence that witnesses WC^*_κ of Lemma 3.4. Let $g_\alpha\colon\kappa\to\kappa$ be the function defined by $g_\alpha\restriction\alpha = f_\alpha$ and $g_\alpha(\beta)=0$ for all $\beta\geq\alpha$. Let us define $F\colon\kappa^\kappa\to 2^\kappa$ by

$$F(\eta)(\alpha)=\begin{cases}1 & \text{if } \alpha\in\mathrm{reg}(\kappa),\ E_S^{\kappa,\kappa}\restriction\alpha \text{ is an equivalence relation, and}\\ & (\eta,g_\alpha)\in E_S^{\kappa,\kappa}\restriction\alpha,\\ 0 & \text{otherwise.}\end{cases}$$

(Recall Definition 2.7 for $E_S^{\kappa,\kappa}\restriction\alpha$.) Let us prove that if $(\eta,\xi)\in E_S^{\kappa,\kappa}$, then $(F(\eta),F(\xi))\in E^{2,\kappa}_{\mathrm{reg}}$. Suppose that $(\eta,\xi)\in E_S^{\kappa,\kappa}$. Note that $F(\eta)(\alpha)=F(\xi)(\alpha)=0$ for all $\alpha\notin\mathrm{reg}(\kappa)$, so it is sufficient to show that the set

$$\{\alpha\in\mathrm{reg}(\kappa)\mid F(\eta)(\alpha)\neq F(\xi)(\alpha)\}$$

is nonstationary. Now, there is a club D such that $D\cap\{\alpha\in S\mid\eta(\alpha)\neq\xi(\alpha)\}$ is nonstationary. So by letting C be the club of the limit points of D, it holds that for all $\alpha\in C\cap\mathrm{reg}(\kappa)$, the functions η and ξ are $E_S^{\kappa,\kappa}\restriction\alpha$-equivalent. Thus, by the definition of F, at the points of the set $C\cap\mathrm{reg}(\kappa)$ the functions $F(\eta)$ and $F(\xi)$ will get the same value.

Now let us prove that if $(\eta,\xi)\notin E_S^{\kappa,\kappa}$, then $(F(\eta),F(\xi))\notin E^{2,\kappa}_{\mathrm{reg}}$. Suppose that $(\eta,\xi)\notin E_S^{\kappa,\kappa}$. Then there is a stationary $Z\subset S$ on which $\eta(\alpha)\neq\xi(\alpha)$. By Lemma 3.4, there is a stationary set $A\subseteq\mathrm{reg}(\kappa)$ such that for all $\alpha\in A$ we have that $Z\cap\alpha$ is stationary and $\eta\restriction\alpha=f_\alpha$. This means that

$$\{\beta<\alpha\mid\eta(\beta)\neq\xi(\beta)\}$$

is stationary, and so $(\eta,\xi)\notin E_S^{\kappa,\kappa}\restriction\alpha$ holds for all $\alpha\in A$. However, $\eta\restriction\alpha=f_\alpha$ implies that $(\eta,g_\alpha)\in E_S^{\kappa,\kappa}\restriction\alpha$, and so by transitivity $(\xi,g_\alpha)\notin E_S^{\kappa,\kappa}\restriction\alpha$. Hence, we get that $F(\eta)(\alpha)=1$, but $F(\xi)(\alpha)=0$. This holds for all $\alpha\in A$ and A is stationary, so $(F(\eta),F(\xi))\notin E^{2,\kappa}_{\mathrm{reg}}$. □

Corollary 3.6 *Suppose that $V=L$ and κ is weakly compact. Then $E^{2,\kappa}_{\mathrm{reg}}$ is Σ^1_1-complete.*

Proof This follows from Theorem 3.1, Fact 3.3, and Theorem 3.5. □

Corollary 3.7 *Suppose that κ is a weakly ineffable cardinal. Then $E^{\kappa,\kappa}_{\mathrm{reg}}\leq_c E^{2,\kappa}_{\mathrm{reg}}$.*

Proof The result follows from Theorem 3.5 and Fact 3.3. □

Theorem 3.8 *If κ is a Π^1_2-indescribable cardinal, then $E^{\kappa,\kappa}_{\mathrm{reg}}$ is Σ^1_1-complete.*

Remark Here the notion of Π^1_2-indescribability is the usual one, not to be confused with the Π^λ_1-indescribability from Definition 2.2.

Proof Let E be a Σ^1_1-equivalence relation on κ^κ. Then there is a closed set C on $\kappa^\kappa\times\kappa^\kappa\times\kappa^\kappa$ such that $\eta\ E\ \xi$ if and only if there exists $\theta\in\kappa^\kappa$ such that $(\eta,\xi,\theta)\in C$. Let us define $U=\{(\eta\restriction\alpha,\xi\restriction\alpha,\theta\restriction\alpha)\mid(\eta,\xi,\theta)\in C\wedge\alpha<\kappa\}$, and for every $\gamma<\kappa$ define

$$C_\gamma=\{(\eta,\xi,\theta)\in\gamma^\gamma\times\gamma^\gamma\times\gamma^\gamma\mid\forall\alpha<\gamma\ (\eta\restriction\alpha,\xi\restriction\alpha,\theta\restriction\alpha)\in U\}.$$

Let $E_\gamma\subset\gamma^\gamma\times\gamma^\gamma$ be the relation defined by $(\eta,\xi)\in E_\gamma$ if and only if there exists $\theta\in\gamma^\gamma$ such that $(\eta,\xi,\theta)\in C_\gamma$. Notice that E_γ is not necessarily an equivalence relation. Let us define the reduction by

$$F(\eta)(\alpha) = \begin{cases} f_\alpha(\eta) & \text{if } E_\alpha \text{ is an equivalence relation and } \eta \restriction \alpha \in \alpha^\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_\alpha(\eta)$ is a code in $\kappa \backslash \{0\}$ for the E_α-equivalence class of η.

Let us prove that if $(\eta, \xi) \in E$, then $(F(\eta), F(\xi)) \in E^{\kappa,\kappa}_{\text{reg}}$. Suppose that $(\eta, \xi) \in E$. Then there is $\theta \in \kappa^\kappa$ such that $(\eta, \xi, \theta) \in C$ and for all $\alpha < \kappa$ we have that $(\eta \restriction \alpha, \xi \restriction \alpha, \theta \restriction \alpha) \in U$. On the other hand, we know that there is a club D such that for all $\alpha \in D \cap \text{reg}(\kappa)$, $\eta \restriction \alpha$, $\xi \restriction \alpha$, $\theta \restriction \alpha \in \alpha^\alpha$. We conclude that for all $\alpha \in D \cap \text{reg}(\kappa)$, if E_α is an equivalence relation, then $(\eta, \xi) \in E_\alpha$. Therefore, for all $\alpha \in D \cap \text{reg}(\kappa)$, $F(\eta)(\alpha) = F(\xi)(\alpha)$, so $(F(\eta), F(\xi)) \in E^{\kappa,\kappa}_{\text{reg}}$. Let us prove that if $(\eta, \xi) \notin E$, then $(F(\eta), F(\xi)) \notin E^{\kappa,\kappa}_{\text{reg}}$. Suppose that $\eta, \xi \in \kappa^\kappa$ are such that $(\eta, \xi) \notin E$. We know that there is a club D such that for all $\alpha \in D \cap \text{reg}(\kappa)$, $\eta \restriction \alpha$, $\xi \restriction \alpha \in \alpha^\alpha$.

Note that because C is closed $(\eta, \xi) \notin E$ is equivalent to

$$\forall \theta \in \kappa^\kappa \quad \big(\exists \alpha < \kappa \; (\eta \restriction \alpha, \xi \restriction \alpha, \theta \restriction \alpha) \notin U\big),$$

so the sentence $(\eta, \xi) \notin E$ is a Π^1_1 property of the structure $(V_\kappa, \in, U, \eta, \xi)$. On the other hand, the sentence $\forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[((\zeta_1, \zeta_2) \in E \wedge (\zeta_2, \zeta_3) \in E) \to (\zeta_1, \zeta_3) \in E]$ is equivalent to the sentence $\forall \zeta_1, \zeta_2, \zeta_3, \theta_1, \theta_2 \in \kappa^\kappa[\exists \theta_3 \in \kappa^\kappa(\psi_1 \vee \psi_2 \vee \psi_3)]$, where ψ_1, ψ_2, and ψ_3 are, respectively, the formulas $\exists \alpha_1 < \kappa \; (\zeta_1 \restriction \alpha_1, \zeta_2 \restriction \alpha_1, \theta_1 \restriction \alpha_1) \notin U$, $\exists \alpha_2 < \kappa \; (\zeta_2 \restriction \alpha_2, \zeta_3 \restriction \alpha_2, \theta_2 \restriction \alpha_2) \notin U$, and $\forall \alpha_3 < \kappa \; (\zeta_1 \restriction \alpha_3, \zeta_3 \restriction \alpha_3, \theta_3 \restriction \alpha_3) \in U$. Therefore, the sentence $\forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[((\zeta_1, \zeta_2) \in E \wedge (\zeta_2, \zeta_3) \in E) \to (\zeta_1, \zeta_3) \in E]$ is a Π^1_2 property of the structure $(V_\kappa, \in, U)$.

The sentence $\forall \zeta_1, \zeta_2 \in \kappa^\kappa[(\zeta_1, \zeta_2) \in E \to (\zeta_2, \zeta_1) \in E]$ is equivalent to the sentence $\forall \zeta_1, \zeta_2, \theta_1 \in \kappa^\kappa[\exists \theta_2 \in \kappa^\kappa(\psi_1 \vee \psi_2)]$, where ψ_1 and ψ_2 are, respectively, the formula $\exists \alpha_1 < \kappa \; (\zeta_1 \restriction \alpha_1, \zeta_2 \restriction \alpha_1, \theta_1 \restriction \alpha_1) \notin U$ and the formula $\forall \alpha_2 < \kappa \; (\zeta_2 \restriction \alpha_2, \zeta_1 \restriction \alpha_2, \theta_2 \restriction \alpha_2) \in U$. Therefore, the sentence $\forall \zeta_1, \zeta_2 \in \kappa^\kappa[(\zeta_1, \zeta_2) \in E \to (\zeta_2, \zeta_1) \in E]$ is a Π^1_2 property of the structure $(V_\kappa, \in, U)$.

The sentence $\forall \zeta \in \kappa^\kappa[(\zeta, \zeta) \in E]$ is equivalent to the sentence

$$\forall \zeta \in \kappa^\kappa \quad \big[\exists \theta \in \kappa^\kappa\big(\forall \alpha < \kappa \; (\zeta \restriction \alpha, \zeta \restriction \alpha, \theta \restriction \alpha) \in U\big)\big].$$

Therefore, the sentence $\forall \zeta \in \kappa^\kappa[(\zeta, \zeta) \in E]$ is a Π^1_2 property of the structure $(V_\kappa, \in, U)$.

It follows that the sentence

$$(D \text{ is unbounded in } \kappa) \wedge \big((\eta, \xi) \notin E\big) \wedge (E \text{ is an equivalence relation}) \wedge (\kappa \text{ is regular})$$

is a Π^1_2 property of the structure $(V_\kappa, \in, U, \eta, \xi)$. By Π^1_2 reflection, we know that there are stationary many $\gamma \in \text{reg}(\kappa)$ such that γ is a limit point of D, E_γ is an equivalence relation, and $(\eta \restriction \gamma, \xi \restriction \gamma) \notin E_\gamma$. We conclude that there are stationary many $\gamma \in \text{reg}(\kappa)$ such that $f_\gamma(\eta) \neq f_\gamma(\xi)$, and hence $(F(\eta), F(\eta)) \notin E^{\kappa,\kappa}_{\text{reg}}$. □

Corollary 3.9 *Suppose that κ is an ineffable cardinal, or weakly ineffable and Π^1_2-indescribable. Then $E^{2,\kappa}_{\text{reg}}$ is Σ^1_1-complete.*

Proof An ineffable cardinal is both weakly ineffable and Π^1_2-indescribable. So the result follows by combining Corollary 3.7 and Theorem 3.8. □

We will finish this article with a model-theoretic result.

Theorem 3.10 *Let DLO be the theory of dense linear orderings without end-points. If κ is a Π^1_2-indescribable cardinal, then $\cong_{\text{DLO}}$ is Σ^1_1-complete.*

Proof By Theorem 3.8 it is enough to show that $E^{\kappa,\kappa}_{\text{reg}} \leq_c \cong_{\text{DLO}}$. To show this, first we will construct models of DLO, $\mathcal{A}^{\mathcal{F}(f)}$, for every $f:\kappa\to\kappa$, such that $f\ E^{\kappa,\kappa}_{\text{reg}}\ g$ if and only if $\mathcal{A}^{\mathcal{F}(f)} \cong \mathcal{A}^{\mathcal{F}(g)}$. After that we construct the reduction of $E^{\kappa,\kappa}_{\text{reg}}$ to $\cong_{\text{DLO}}$.

Let us take the language $\mathcal{L}' = \{L, C, <, R\}$, with L and C as unary predicates, and $<$ and R as binary relations. Let K be the class of $\mathcal{L}'$-structures $\mathcal{A} = (\text{dom}(\mathcal{A}), L, C, <, R)$ that satisfy the following conditions:

(1) $L \cap C = \emptyset$.
(2) $L \cup C = \text{dom}(\mathcal{A})$.
(3) $< \subseteq L \times L$ is a dense linear order without endpoints on L.
(4) $R \subseteq L \times C$.
(5) Let us denote by $R^-(y,x)$ the formula $\neg R(y,x)$. For all $x \in C$, it holds that $R(\mathcal{A},x) \cup R^-(\mathcal{A},x) = L$, $R(\mathcal{A},x)$ has no largest element, and $R^-(\mathcal{A},x)$ has no least element and they are nonempty.

Let us define the following partial order $\preceq$ on K. We say that $\mathcal{A} \preceq \mathcal{B}$ if and only if

(i) $\mathcal{A} \subseteq \mathcal{B}$,
(ii) for all $x \in C^{\mathcal{A}}$, $R(\mathcal{B},x) = \{y \in L^{\mathcal{B}} \mid \exists z \in R(\mathcal{A},x), y < z\}$ and $R^-(\mathcal{B},x) = \{y \in L^{\mathcal{B}} \mid \exists z \in R^-(\mathcal{A},x), z < y\}$,
(iii) for all $x \in C^{\mathcal{B}}\backslash C^{\mathcal{A}}$ there are $y \in R(\mathcal{B},x)$ and $z \in R^-(\mathcal{B},x)$ such that for all $a \in L^{\mathcal{A}}$, $a < y \vee a > z$.

Notice that it is possible to have a chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \cdots$ of length α in K, and a structure $\mathcal{C} \in K$, such that $\bigcup_{i<\alpha} \mathcal{A}_i \in K$, $\mathcal{A}_i \preceq \mathcal{C}$ holds for all $i < \alpha$, and $\bigcup_{i<\alpha} \mathcal{A}_i \not\preceq \mathcal{C}$. But all other requirements of abstract elementary classes are satisfied, as one can easily see, in particular for every chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \cdots$ of length α in K, $\bigcup_{i<\alpha} \mathcal{A}_i \in K$.

Claim 3.10.1 *$(K,\preceq)$ has the amalgamation property and the joint embedding property.*

Proof The joint embedding property is easily seen to follow from the amalgamation property. For the amalgamation property, let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ be such that $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{A} \preceq \mathcal{C}$ hold. Without loss of generality, we can assume that $\text{dom}(\mathcal{B}) \cap \text{dom}(\mathcal{C}) = \text{dom}(\mathcal{A})$. Let us construct $\mathcal{D}$ with $\text{dom}(\mathcal{D})$ equal to $\text{dom}(\mathcal{B}) \cup \text{dom}(\mathcal{C})$, $L^{\mathcal{D}} = L^{\mathcal{B}} \cup L^{\mathcal{C}}$, and $C^{\mathcal{D}} = C^{\mathcal{B}} \cup C^{\mathcal{C}}$. To define $<^{\mathcal{D}}$ and $R^{\mathcal{D}}$, first define $<' = <^{\mathcal{B}} \cup <^{\mathcal{C}}$. For every two elements $b, c \in L^{\mathcal{D}}$ define $b <^{\mathcal{D}} c$ if either $b <' c$, or there is $a \in L^{\mathcal{A}}$ such that $b <' a <' c$, or $b \in L^{\mathcal{B}}$, $c \in L^{\mathcal{C}}$ and there is no $a \in L^{\mathcal{A}}$ such that $c <' a <' b$. For every $x \in C^{\mathcal{A}}$, $R(\mathcal{D},x) = R(\mathcal{B},x) \cup R(\mathcal{C},x)$. For all $x \in C^{\mathcal{B}}\backslash C^{\mathcal{A}}$, $y \in R(\mathcal{D},x)$ if and only if there exists $z \in L^{\mathcal{B}}$ such that $z \in R(\mathcal{B},x)$ and $y <^{\mathcal{D}} z$. For all $x \in C^{\mathcal{C}}\backslash C^{\mathcal{A}}$, $y \in R(\mathcal{D},x)$ if and only if there exists $z \in L^{\mathcal{C}}$ such that $z \in R(\mathcal{C},x)$ and $y <^{\mathcal{D}} z$. It is clear that $\mathcal{D} \in K$, and $\mathcal{B} \preceq \mathcal{D}$ and $\mathcal{C} \preceq \mathcal{D}$. □

Let us denote by $\mathcal{A}_1 \oplus_{\mathcal{A}_0} \mathcal{A}_2$ the structure $\mathcal{D}$, in Claim 3.10.1, that witnesses the amalgamation property for the structures $\mathcal{A}_0 \preceq \mathcal{A}_1$ and $\mathcal{A}_0 \preceq \mathcal{A}_2$. For every ordinal α, let us denote by α^* the set α ordered by the reverse order $<^*$, that is, $\beta <^* \gamma$ if $\gamma \in \beta$. Let us order the members of $\mathbb{Q} \times \alpha^*$ by: $(r_1,\alpha_1) <^{*\alpha} (r_2,\alpha_2)$ if and only if $\alpha_1 <^* \alpha_2$, or $\alpha_1 = \alpha_2$ and $r_1 <^{\mathbb{Q}} r_2$.

Let $K_{<\kappa}$ be the collection of all members of K of size less than κ. For every $\mathcal{A} \in K_{<\kappa}$, denote by $\{\mathcal{A}(i)\}_{i<\kappa}$ an enumeration of all the strong extensions of $\mathcal{A}$, that is, $\mathcal{A} \preceq \mathcal{B}$, of size less than κ (up to isomorphism over $\mathcal{A}$). Let $\Pi\colon \kappa \to \kappa \times \kappa$, $\Pi(\alpha) = (pr_1(\Pi(\alpha)), pr_2(\Pi(\alpha)))$ be a bijection, where pr stands for projection, such that $pr_1(\Pi(i)) \le i$ for all i. Given a function $f\colon \kappa \to \mathrm{reg}(\kappa)$, let us construct the following sequence of models:

(i) $\mathcal{A}_0^f = (\mathbb{Q}, \emptyset, <, \emptyset)$.

(ii) For a successor ordinal, let $\mathcal{D} = \mathcal{A}_i^f \oplus_{\mathcal{A}^f_{pr_1(\Pi(i))}} \mathcal{A}^f_{pr_1(\Pi(i))}(pr_2(\Pi(i)))$. Define $L^{\mathcal{A}^f_{i+1}} = L^{\mathcal{D}} \cup \mathbb{Q}$, $C^{\mathcal{A}^f_{i+1}} = C^{\mathcal{D}}$, $<^{\mathcal{A}^f_{i+1}} = <^{\mathcal{D}} \cup <^{\mathbb{Q}} \cup \{(x,y) \mid x \in L^{\mathcal{D}} \wedge y \in \mathbb{Q}\}$, and $R^{\mathcal{A}^f_{i+1}} = R^{\mathcal{D}}$. Clearly $\mathcal{A}^f_{i+1} \in K$.

(iii) For i a limit ordinal, let $\mathcal{D} = \bigcup_{j<i} \mathcal{A}_j^f$. Define $L^{\mathcal{A}_i^f} = L^{\mathcal{D}} \cup (\mathbb{Q} \times f(i)^*)$, $C^{\mathcal{A}_i^f} = C^{\mathcal{D}} \cup \{x\}$, $<^{\mathcal{A}_i^f} = <^{\mathcal{D}} \cup <^{*f(i)} \cup \{(a,b) \mid a \in L^{\mathcal{D}} \wedge b \in \mathbb{Q} \times f(i)^*\}$, and $R^{\mathcal{A}_i^f} = R^{\mathcal{D}} \cup \{(y,x) \mid y \in L^{\mathcal{D}}\}$. Clearly $\mathcal{A}_i^f \in K$.

Define $\mathcal{A}_\kappa^f$ by $\bigcup_{j<\kappa} \mathcal{A}_j^f$. Then $\mathcal{A}^f = (L^{\mathcal{A}_\kappa^f}, <^{\mathcal{A}_\kappa^f})$ is a model of DLO.

Notice that if $i < \kappa$ and $\mathcal{C} \in K$, $|\mathcal{C}| < \kappa$, are such that $\mathcal{A}_i^f \preceq \mathcal{C}$, then there is $j < \kappa$ such that $\mathcal{A}_i^f(j) = \mathcal{C}$. Therefore, there is $l < \kappa$ such that $\Pi(l) = (i,j)$, $\mathcal{A}^f_{pr_1(\Pi(l))} = \mathcal{A}_i^f$, and $\mathcal{A}^f_{pr_1(\Pi(l))}(pr_2(\Pi(l))) = \mathcal{C}$. We conclude that if $i < \kappa$ and $\mathcal{C} \in K_{<\kappa}$ are such that $\mathcal{A}_i^f \preceq \mathcal{C}$, then there are $j < \kappa$ and a strong embedding $F\colon \mathcal{C} \to \mathcal{A}_j^f$ such that $F(\mathcal{C}) \preceq \mathcal{A}_j^f$ and $F \restriction \mathcal{A}_i^f = \mathrm{id}$. Now we will show that if f and g are functions from κ into $\mathrm{reg}(\kappa)$ such that $f \restriction (\kappa \setminus \mathrm{reg}(\kappa)) = g \restriction (\kappa \setminus \mathrm{reg}(\kappa))$, then $f\ E^{\kappa,\kappa}_{\mathrm{reg}}\ g$ if and only if $\mathcal{A}^f \cong \mathcal{A}^g$. First of all, let us prove that $(f,g) \in E^{\kappa,\kappa}_{\mathrm{reg}}$ implies $\mathcal{A}^f \cong \mathcal{A}^g$. Suppose that $(f,g) \in E^{\kappa,\kappa}_{\mathrm{reg}}$. Then there is a club C such that for all $\alpha \in C \cap \mathrm{reg}(\kappa)$, $f(\alpha) = g(\alpha)$. Since $f \restriction (\kappa \setminus \mathrm{reg}(\kappa)) = g \restriction (\kappa \setminus \mathrm{reg}(\kappa))$, we have that for all $\alpha \in C$, $f(\alpha) = g(\alpha)$. By the way the models $\mathcal{A}_\alpha^f$ and $\mathcal{A}_\alpha^g$ were constructed for α a limit ordinal, we know that if α is such that $f(\alpha) = g(\alpha)$ and there is an isomorphism $F\colon \bigcup_{i<\alpha} \mathcal{A}_i^f \to \bigcup_{i<\alpha} \mathcal{A}_i^g$, then there is an isomorphism $G\colon \mathcal{A}_\alpha^f \to \mathcal{A}_\alpha^g$ such that $F \subseteq G$. For all $i < \kappa$, construct $\alpha_i < \kappa$ and a strong embedding F_i such that the following hold:

(i) For every $i < \kappa$ there is some $\gamma \in C$ such that $\alpha_i < \gamma < \alpha_{i+1}$.

(ii) For all $i < j < \kappa$, $f_i \subseteq f_j$.

(iii) The following hold for every limit ordinal $\beta < \kappa$:

  (a) for every even $0 < i < \omega$, $\mathrm{dom}(F_{\beta+i}) = \mathcal{A}^f_{\alpha_{\beta+i}}$, and $F_{\beta+i}(\mathcal{A}^f_{\alpha_{\beta+i}}) \preceq \mathcal{A}^g_{\alpha_{\beta+i+1}}$,

  (b) for every odd $0 < i < \omega$, $\mathrm{rang}(F_{\beta+i}) = \mathcal{A}^g_{\alpha_{\beta+i}}$, and $F^{-1}_{\beta+i}(\mathcal{A}^g_{\alpha_{\beta+i}}) \preceq \mathcal{A}^f_{\alpha_{\beta+i+1}}$,

  (c) $\alpha_\beta = \bigcup_{i<\beta} \alpha_i$, $\mathrm{dom}(F_\beta) = \mathcal{A}^f_{\alpha_\beta}$, and $\mathrm{rang}(F_\beta) = \mathcal{A}^g_{\alpha_\beta}$.

We will construct these sequences by induction. For $i = 0$, take $\alpha_0 = 0$ and $F_0 = \mathrm{id}$.

Successor case: Suppose that β is a limit ordinal or zero, and suppose that $0 \le i < \omega$ are such that $\alpha_{\beta+i}$ and $F_{\beta+i}$ are constructed such that conditions (i), (ii), and (iii) are satisfied. Let us start with the case when i is odd. Choose $\alpha_{\beta+i+1}$ such that (i) holds. Since $F^{-1}(\mathcal{A}^g_{\alpha_{\beta+i}}) \preceq \mathcal{A}^f_{\alpha_{\beta+i+1}}$, there are $\mathcal{C} \in K_{<\kappa}$ and $F \supseteq F_{\beta+i}$ such that $\mathcal{A}^g_{\alpha_{\beta+i}} \preceq \mathcal{C}$ and $F\colon \mathcal{A}^f_{\alpha_{\beta+i+1}} \to \mathcal{C}$ is an isomorphism. By the observation we made above, there are $j < \kappa$ and a strong embedding $G\colon \mathcal{C} \to \mathcal{A}_j^g$ such that

$G(\mathcal{C}) \preceq \mathcal{A}^g_j$ and $G \restriction \mathcal{A}^g_{\alpha_{\beta+i}} = \mathrm{id}$. Define $F_{\alpha_{\beta+i+1}} = G \circ F_{\alpha_{\beta+i}}$. Clearly $F_{\alpha_{\beta+i+1}}$ satisfies conditions (ii) and (iii). The case when i is even is similar to the odd case.

Limit case: Suppose that β is a limit ordinal such that for all $i < \beta$, α_i and F_i are constructed such that conditions (i), (ii), and (iii) are satisfied. By (i), we know that $\alpha_\beta = \bigcup_{i<\beta} \alpha_i$ is a limit point of C, so $f(\alpha_\beta) = g(\alpha_\beta)$. On the other hand, by conditions (ii) and (iii) we know that

$$\bigcup_{i<\beta} F_i \colon \bigcup_{i<\beta} \mathcal{A}^f_{\alpha_i} \to \bigcup_{i<\beta} \mathcal{A}^g_{\alpha_i}$$

is an isomorphism. Therefore, there is an isomorphism $G\colon \mathcal{A}^f_\alpha \to \mathcal{A}^g_\alpha$ such that $\bigcup_{i<\beta} F_i \subseteq G$. We conclude that $F_{\alpha_\beta} = G$ satisfies (ii) and (iii).

Finally, note that

$$\bigcup_{i<\kappa} F_i \colon \bigcup_{i<\kappa} \mathcal{A}^f_{\alpha_i} \to \bigcup_{i<\kappa} \mathcal{A}^g_{\alpha_i}$$

is an isomorphism. We conclude that $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.

Let us prove that $\mathcal{A}^f \cong \mathcal{A}^g$ implies $(f,g) \in E^{\kappa,\kappa}_{\mathrm{reg}}$. Suppose toward a contradiction that $(f,g) \notin E^{\kappa,\kappa}_{\mathrm{reg}}$ and there is an isomorphism $F\colon \mathcal{A}^f \to \mathcal{A}^g$. Since F is an isomorphism, there is a club C such that $F(\bigcup_{i<\alpha} \mathcal{A}^f_i) = \bigcup_{i<\alpha} \mathcal{A}^g_i$ holds for all $\alpha \in C$. Since $(f,g) \notin E^{\kappa,\kappa}_{\mathrm{reg}}$, $C \cap \{\alpha \in \mathrm{reg}(\kappa) \mid f(\alpha) \neq g(\alpha)\}$ is nonempty. Take $\alpha \in C \cap \{\gamma \in \mathrm{reg}(\kappa) \mid f(\gamma) \neq g(\gamma)\}$. We know that $F(\bigcup_{i<\alpha} \mathcal{A}^f_i) = \bigcup_{i<\alpha} \mathcal{A}^g_i$ and $f(\alpha) \neq g(\alpha)$. Hence, the coinitiality of $\{a \in \mathcal{A}^f \mid \forall b \in \bigcup_{i<\alpha} \mathcal{A}^f_i\, (b <^{\mathcal{A}^f} a)\}$ with respect to $<^{\mathcal{A}^f}$ is $f(\alpha)$. Since F is an isomorphism and $F(\bigcup_{i<\alpha} \mathcal{A}^f_i) = \bigcup_{i<\alpha} \mathcal{A}^g_i$, the coinitiality of $\{a \in \mathcal{A}^g \mid \forall b \in \bigcup_{i<\alpha} \mathcal{A}^g_i\, (b <^{\mathcal{A}^g} a)\}$ with respect to $<^{\mathcal{A}^g}$ is also $f(\alpha)$. We conclude that $f(\alpha) = cf(g(\alpha))$, so $f(\alpha) = g(\alpha)$, which is a contradiction. To finish with the construction of the models, let us define $\mathcal{A}^{\mathcal{F}(f)}$ for all $f\colon \kappa \to \kappa$. Fix a bijection $G\colon \kappa \to \mathrm{reg}(\kappa)$. Define $\mathcal{F}\colon \kappa^\kappa \to \kappa^\kappa$ by

$$\mathcal{F}(f)(\alpha) = \begin{cases} G(f(\alpha)) & \text{if } \alpha \in \mathrm{reg}(\kappa), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f\ E^{\kappa,\kappa}_{\mathrm{reg}}\ g$ if and only if $\mathcal{F}(f)\ E^{\kappa,\kappa}_{\mathrm{reg}}\ \mathcal{F}(g)$, and $\mathcal{F}(f)\ E^{\kappa,\kappa}_{\mathrm{reg}}\ \mathcal{F}(g)$ if and only if $\mathcal{A}^{\mathcal{F}(f)}$ and $\mathcal{A}^{\mathcal{F}(g)}$ are isomorphic. Now we will construct a reduction of $E^{\kappa,\kappa}_{\mathrm{reg}}$ to $\cong_{\mathrm{DLO}}$ by coding the models $\mathcal{A}^{\mathcal{F}(f)}$ by functions $\eta\colon \kappa \to \kappa$.

Clearly the models $\mathcal{A}^{\mathcal{F}(f)}$ satisfy that

$$\mathcal{F}(f) \restriction \alpha = \mathcal{F}(g) \restriction \alpha \Leftrightarrow \mathcal{A}^{\mathcal{F}(f)}_\alpha = \mathcal{A}^{\mathcal{F}(g)}_\alpha.$$

For every $f \in \kappa^\kappa$ define $C_f \subseteq \mathrm{Card} \cap \kappa$ such that for all $\alpha \in C_f$, it holds that for every $\beta < \alpha$, $|\mathcal{A}^{\mathcal{F}(f)}_\beta| < |\mathcal{A}^{\mathcal{F}(f)}_\alpha|$. For every $f \in \kappa^\kappa$ and $\alpha \in C_f$ choose a bijection $E^\alpha_f\colon \mathrm{dom}(\mathcal{A}^{\mathcal{F}(f)}_\alpha) \to |\mathcal{A}^{\mathcal{F}(f)}_\alpha|$ such that for all $\beta < \alpha$ in C_f, it holds that $E^\beta_f \subseteq E^\alpha_f$. Then $\bigcup_{\alpha \in C_f} E^\alpha_f = E_f$ is such that $E_f\colon \mathrm{dom}(\mathcal{A}^{\mathcal{F}(f)}) \to \kappa$ is a bijection, and for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$ the following holds: if $\mathcal{F}(f) \restriction \alpha = \mathcal{F}(g) \restriction \alpha$, then $E_f \restriction \mathrm{dom}(\mathcal{A}^{\mathcal{F}(f)}_\alpha) = E_g \restriction \mathrm{dom}(\mathcal{A}^{\mathcal{F}(g)}_\alpha)$. Let π be the bijection in Definition 1.4. Define the function $\mathcal{G}$ by:

$$\mathcal{G}\big(\mathcal{F}(f)\big)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \pi(m, a_1, \dots, a_n) \text{ and} \\ & \mathcal{A}^{\mathcal{F}(f)} \models P_m(E_f^{-1}(a_1), \dots, E_f^{-1}(a_n)), \\ 0 & \text{in the other case.} \end{cases}$$

To show that $\mathcal{G}$ is continuous, let $[\eta \restriction \alpha]$ be a basic open set, and let $\xi \in \mathcal{G}^{-1}[[\eta \restriction \alpha]]$. There is $\beta \in C_\xi$ such that for all $\gamma < \alpha$, if $\gamma = \pi(m, a_1, a_2, \dots, a_n)$, then $E_\xi^{-1}(a_i)$ is an element of $\mathrm{dom}(\mathcal{A}_\beta^\xi)$ for all $i \leq n$. Since for all $\zeta \in [\xi \restriction \beta]$ it holds that $\mathcal{A}_\beta^\xi = \mathcal{A}_\beta^\zeta$, for every $\gamma < \alpha$ such that $\gamma = \pi(m, a_1, a_2, \dots, a_n)$, it holds that

$$\mathcal{A}^\xi \models P_m\big(E_\xi^{-1}(a_1), E_\xi^{-1}(a_2), \dots, E_\xi^{-1}(a_n)\big)$$

if and only if

$$\mathcal{A}^\zeta \models P_m\big(E_\zeta^{-1}(a_1), E_\zeta^{-1}(a_2), \dots, E_\zeta^{-1}(a_n)\big).$$

We conclude that $\mathcal{G}(\zeta) \in [\eta \restriction \alpha]$, and $\mathcal{G} \circ \mathcal{F}$ is a continuous reduction of $E_{\mathrm{reg}}^{\kappa,\kappa}$ to $\cong_{\mathrm{DLO}}$. □

4 Further Research

In this article we established the Σ_1^1-completeness of a range of equivalence relations in various circumstances. Some of these theorems are proved in ZFC, some are consistency results, and some are relative consistency results. In particular, the equivalence relation modulo the nonstationary ideal is Σ_1^1-complete if κ is an ineffable cardinal. This and related equivalence relations play a role in model theory as exemplified by Theorem 3.10, which shows how generalized descriptive set theory is different from the classical study where $\kappa = \omega$ and the isomorphism relation of countable structures is never Σ_1^1-complete. This was also the original motivation for studying such fine-grained questions as whether $E_{\mu\text{-club}}^{\kappa,\kappa}$ can be reduced to $E_{\mu\text{-club}}^{2,\kappa}$ for some $\mu < \kappa$. How much more can one prove in ZFC for $\kappa > \omega$? For successor cardinals the answer is partially known (see Friedman, Wu, and Zdomskyy [4]) starting from $V = L$ for every successor cardinal κ there exists a GCH and cardinal-preserving forcing notion such that in the extension the equivalence relation modulo the nonstationary ideal is not Σ_1^1-complete. The following questions remain open.

Question 4.1 Is it consistent that the isomorphism relation on graphs or dense linear orders is not Σ_1^1-complete for some $\kappa > \omega$? Of course, κ cannot be Π_2^1-indescribable by Theorem 3.10.

Question 4.2 Is it consistent for some cardinal κ and a regular $\mu < \kappa$ that $E_\mu^{\kappa,\kappa}$ is not reducible to $E_\mu^{2,\kappa}$? Note: it has been shown in [2] that it is consistent that $E_S^{2,\kappa}$ is not reducible to $E_{S'}^{2,\kappa}$ for $S' \setminus S$ stationary, which implies the consistency of, for example, $E_\mu^{\kappa,\kappa} \not\leq_B E_{\mu'}^{2,\kappa}$ for $\mu \neq \mu'$.

Question 4.3 Is it consistent that κ is inaccessible and $E_S^{2,\kappa}$ is not Σ_1^1-complete for some stationary $S \subset \kappa$? What about κ being weakly compact and $S = S_\mu^\kappa$ for some regular $\mu < \kappa$? Note: it follows from the main result of [4], Theorem 1.1, that it is consistent that $E_\kappa^{2,\kappa}$ is not Σ_1^1-complete (in fact Δ_1^1) for successor κ.

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