Integral Kernels for k-hypermonogenic Functions

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Abstract

We consider the modified Cauchy-Riemann operator

$$M_k = \sum_{i=0}^{n} e_i \partial_{x_i} + \frac{k}{x_n} Q'$$

in the universal Clifford algebra $C\ell_{0,n}$ with the basis $e_1, \ldots, e_n$. The null-solutions of this operator are called k-hypermonogenic functions. We calculate the k-hyperbolic harmonic fundamental solutions i.e. solutions to $M_k \overline{M}_k f = 0$ and use these solutions find k-hypermonogenic kernels for a Cauchy-type integral formula in the upper half-space.

1 Introduction

Complex function theory can be generalized to higher dimensions by considering different extensions of the Cauchy-Riemann equations. One possibility is to consider the monogenic functions as has been done in [2]. A second way to is to modify the C-R system so that the integer powers of the variable are included in the class of function satisfying these equations. This approach leads to the hypermonogenic functions and was started by H. Leutwiler and studied in e.g. [3]. This particular modification of the C-R system with the C-R operator M_{n-1} is connected to hyperbolic geometry since the hyperbolic Laplace operator appears in $M_{n-1}\overline{M}_{n-1}$.

The monogenic functions correspond to the null-solutions of M_0 and more generally k-hypermonogenic functions can be defined by $M_k f = 0$. Several properties of these functions were studied in [4] and [5]. An overview of the theory can be found in [6]. We consider the k-hypermonogenic functions in the Poincaré upper half-space. The concept of k-hyperbolic harmonicity can be defined by $M_k \overline{M}_k f = 0$. The solutions of this equation can be also presented with eigenfunctions of the Laplace-Beltrami operator of the upper half-space.

In this article we present the fundamental solutions of $M_k \overline{M}_k$ and prove a Cauchy-type integral formula in $\mathbb{R}^{n+1}_+$ with k-hypermonogenic kernels given in terms of hypergeometric series. We also present the previously known kernels in $\mathbb{R}^3_+$ as special cases of these solutions.

2 Preliminaries

We work with the universal Clifford algebra $C\ell_{0,n}$ with the basis $e_1, \ldots, e_n$ which satisfies $e_i e_j + e_j e_i = -2\delta_{ij}$. The upper half-space $\mathbb{R}^{n+1}_+$ is identified with the elements

$$x = \sum_{i=0}^{n} x_i e_i$$

which are also called paravectors. For the involutions in $C\ell_{0,n}$ we use the following notations:

Main Involution	$a \mapsto a'$	$e_i \mapsto -e_i$ for $i = 1, \ldots, n$	$(ab)' = a'b'$
Reversion	$a \mapsto a^*$	$e_i \mapsto e_i$ for $i = 1 \ldots, n$	$(ab)^* = b^* a^*$
Conjugation	$a \mapsto \overline{a}$		$\overline{a} = (a')^*$
Q-Part	$a \mapsto \hat{a}$	$e_i \mapsto (-1)^{\delta_{in}} e_i$ for $i = 1, \ldots, n$	$\widehat{ab} = \hat{a}\hat{b}$

For an arbitrary element $a \in C\ell_{0,n}$ given by

$$a = p + q e_n$$

we define $Pa = p$ and $Qa = q$. These parts can also be given as

$$Pa = \frac{a + \hat{a}}{2} \quad \text{and} \quad Qa = -\frac{a - \hat{a}}{2} e_n.$$

The (left) Cauchy-Riemann-operators are defined by

$$D = e_0 \partial_{x_0} + \sum_{i=1}^{n} e_i \partial_{x_i}, \quad \overline{D} = e_0 \partial_{x_0} - \sum_{i=1}^{n} e_i \partial_{x_i}$$

and their modifications by

$$M_k = D + \frac{k}{x_n} Q', \qquad \overline{M}_k = \overline{D} - \frac{k}{x_n} Q',$$

where $Q'(a) = Q(a')$. If the variable in the differentiation differs from x it is denoted by superscript e.g. D^a.

We use the Poincaré upper half space model for the hyperbolic geometry. In this model the metric is defined by

$$ds^2 = \frac{\sum_{i=0}^{n} dx_i^2}{x_n^2}.$$

Lemma 2.1. [9] *The hyperbolic distance between points x and a in $\mathbb{R}^{n+1}_+$ is*

$$d_h(x,a) = \cosh^{-1}(\lambda(x,a))$$

where

$$\lambda(x,a) = \frac{|x-a|^2}{2x_n a_n} + 1$$

and $|x-a|$ is the Euclidean distance between x and a in $\mathbb{R}^{n+1}_+$.

Definition 2.2. [4] *Let $\Omega \subset \mathbb{R}^{n+1}$ be open. A continuously differentiable function $f : \Omega \to C\ell_{0,n}$ is called (left) k-hypermonogenic if $M_k = 0$ for any $x \in \{x \in \Omega \mid x_n \neq 0\}$. Special cases of this are monogenic functions if $k = 0$ and hypermonogenic functions if $k = n-1$. A twice continuously differentiable function $f : \Omega \to C\ell_{0,n}$ is called k-hyperbolic harmonic if it satisfies $M_k \overline{M}_k f = 0$ for any $x \in \{x \in \Omega \mid x_n \neq 0\}$.*

There is an important correspondence between the k-hypermonogenic and $-k$-hypermonogenic functions.

Theorem 2.3. [5] *Let $\Omega \subset \mathbb{R}^{n+1} \setminus \{x_n = 0\}$ be open and $k \in \mathbb{R}$. A continuously differentiable function $f : \Omega \to C\ell_{0,n}$ is k-hypermonogenic iff the function $x_n^{-k} f e_n$ is $-k$-hypermonogenic.*

The name hyperbolic harmonic in the definition 2.2 stems from the expression

$$x_n^2 M_{n-1} \overline{M}_{n-1} f = \Delta_h f + (n-1) Q f e_n,$$

where Δ_h is the invariant Laplacian in the upper half-space. For P-and Q-parts the equation $M_k \overline{M}_k f = 0$ becomes [4]

$$\left(x_n^2 \Delta - k x_n \frac{\partial}{\partial x_n} \right) Pf = 0, \tag{1}$$

$$\left(x_n^2 \Delta - k x_n \frac{\partial}{\partial x_n} + k \right) Qf = 0. \tag{2}$$

We transform this by $g = x_n^{\frac{n-1-k}{2}} f$ into two eigenvalue-problems

$$\left(\Delta_h + \frac{1}{4}\left(n^2 - (k+1)^2\right)\right) Pg = 0, \tag{3}$$

$$\left(\Delta_h + \frac{1}{4}\left(n^2 - (k-1)^2\right)\right) Qg = 0. \tag{4}$$

Functions depending only on the hyperbolic distance centered at a are of the form

$$u(x,a) = \tilde{u}(\lambda(x,a)).$$

The eigenfunctions $u(\lambda)$ of the Laplace-Beltrami operator satisfy

$$(\lambda^2 - 1)u''(\lambda) + (n+1)\lambda u'(\lambda) = \gamma u(\lambda). \tag{5}$$

For the solutions of this equation we use the associated Legendre functions.

3 Associated Legendre Functions

The associated Legendre equation in its standard form is

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \left(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right)u = 0.$$

In general it has solutions $P_\nu^\mu(\pm x)$, $P_\nu^{-\mu}(\pm x)$, $Q_\nu^\mu(\pm x)$, $Q_{-\nu-1}^\mu(\pm x)$, where we have used the notation presented in [10]. These associated Legendre functions can be defined with hypergeometric functions for all $\mu, \nu \in \mathbb{C}$ as

$$P_\nu^{-\mu}(x) = \left(\frac{x-1}{x+1}\right)^{\frac{\mu}{2}} \mathbf{F}\left(-\nu, \nu+1; \mu+1; \frac{1-x}{2}\right), \quad |x-1| < 2$$

$$Q_\nu^\mu(x) = \frac{\sqrt{\pi}(x^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}x^{\nu+\mu+1}} \mathbf{F}\left(\frac{1}{2}\mu + \frac{1}{2}\nu + 1, \frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{x^2}\right), \quad |x| > 1.$$

In addition, we use the notational convention

$$\mathbf{F}(a,b;c;x) = \frac{1}{\Gamma(c)} {}_2F_1(a,b;c;x).$$

The function $P_\nu^{-\mu}(x)$ is now given by the previous series expansion in the domain $|x-1| < 2$. We transform this into the domain $|x| > 0$ by the transformation (24) in [1] p.129 and obtain

$$P_\nu^{-\mu}(x) = \frac{(x^2-1)^{\frac{\mu}{2}}}{2^\mu x^{\mu-\nu}} \mathbf{F}\left(\frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}; \mu+1; 1 - \frac{1}{x^2}\right).$$

We now have the solutions $P_\nu^{-\mu}(x)$ which is zero at $x = 1$ and $Q_\nu^\mu(x)$ with a singularity at $x = 1$. The functions $Q_\nu^\mu(x)$ satisfy

$$\lim_{x \to 1^+} \left[Q_\nu^\mu(x) \left(\frac{2}{x-1} \right)^{-\frac{\mu}{2}} \right] = \frac{\Gamma(\mu)}{2\Gamma(\nu + \mu + 1)},$$

in which the gamma functions cause some restrictions on the values of μ and ν. In particular $\Re\mu > 0$ and $\nu + \mu \notin \mathbb{Z}_-$. The limit can be deduced by noting that according to the formula 15.8.1 in [11] hypergeometric functions satisfy

$$\mathbf{F}(a, b; c; z) = (1 - z)^{c-a-b} \mathbf{F}(c - a, c - b; c; z)$$

if $|z| < 1$ and thus

$$\left(\frac{2}{x-1} \right)^{-\frac{\mu}{2}} Q_\nu^\mu = \frac{\sqrt{\pi}(x+1)^{\frac{-\mu}{2}}}{2^{\nu+1+\frac{\mu}{2}} x^{\nu-\mu+1}} \mathbf{F}\left(\frac{1}{2}\nu - \frac{1}{2}\mu + 1, \frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{x^2} \right)$$

which gives at $x = 1$

$$\begin{aligned} \lim_{x \to 1^+} \left[Q_\nu^\mu(x) \left(\frac{2}{x-1} \right)^{-\frac{\mu}{2}} \right] &= \frac{\sqrt{\pi}}{2^{\mu+\nu+1}} \frac{\Gamma(\mu)}{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + 1\right)} \\ &= \frac{\Gamma(\mu)}{2\Gamma(\nu + \mu + 1)}. \end{aligned}$$

We used the formula (15.4.23) in [11] for ${}_2F_1$ and in the last equality above we have used the duplication formula for the gamma function.

We collect next some identities and particular cases concerning associated Legendre functions.

Lemma 3.1. *Associated Legendre functions of the second type satisfy the differentiation formulas* ($|x| > 1$)

$$\begin{aligned} (x^2 - 1)\frac{d}{dx} Q_\nu^\mu(x) &= (\nu + \mu + 1)(\nu - \mu + 1) Q_{\nu+1}^\mu(x) - (\nu + 1) x Q_\nu^\mu(x), \\ (x^2 - 1)\frac{d}{dx} Q_\nu^\mu(x) &= -Q_{\nu-1}^\mu(x) + \nu x Q_\nu^\mu(x). \end{aligned}$$

The first of these differentiation formulas follows from the definition of Q_ν^μ and the differentiation formula for the hypergeometric function. The second formula is a modification of the formula (14.10.5) in [11] for the function Q_ν^μ valid for $\nu + \mu \notin \mathbb{Z}_-$.

Also the following formulas can be found in [11]. We have the connection formula

$$\frac{P_\nu^{-\mu}}{\Gamma(\nu-\mu+1)} + \frac{2\sin(\mu\pi)}{\pi} Q_\nu^{-\mu} = \frac{P_\nu^{\mu}}{\Gamma(\nu+\mu+1)} \tag{6}$$

and there are also simple expressions for certain values of μ and ν. In the three-dimensional case relevant formulas are

$$\begin{aligned} P_\nu^{\frac{1}{2}}(\cosh r) &= \left(\frac{2}{\pi\sinh r}\right)^{\frac{1}{2}} \cosh\left(\left(\nu+\frac{1}{2}\right) r\right), \\ Q_\nu^{\pm\frac{1}{2}}(\cosh r) &= \left(\frac{\pi}{2\sinh r}\right)^{\frac{1}{2}} \frac{e^{-\left(\nu+\frac{1}{2}\right)r}}{\Gamma\left(\nu+\frac{3}{2}\right)}, \end{aligned}$$

and in the hypermonogenic case

$$P_\nu^{-\nu}(\cosh r) = \frac{(\sinh r)^\nu}{2^\nu \Gamma(\nu+1)}.$$

4 k-Hyperbolic Harmonic Kernels

We transform the problem (5) to the standard form by setting $g = (\lambda^2-1)^{-\alpha} u$ with $\alpha = \frac{1-n}{4}$. The equation for g becomes

$$(1-\lambda^2)g'' - 2\lambda g' + \left(\gamma + \frac{n^2-1}{4} - \frac{(n-1)^2}{4(1-\lambda^2)}\right) g = 0.$$

We use this to solve the problem (3). We now have $\gamma = -\frac{n^2-(k+1)^2}{4}$ so the coefficients of the corresponding Legendre equation are

$$\begin{aligned} \nu(\nu+1) = &\frac{(k+1)^2-1}{4} \quad \Longleftrightarrow \quad \nu = \frac{\pm|k+1|-1}{2}, \\ \mu = &\pm\frac{n-1}{2}. \end{aligned}$$

We take the coefficients $\mu, \nu \geq 0$. After reversing the transformations we've done the solutions to the original problem (1) for positive values of k become

$$\begin{aligned} f_k &= x_n^{\frac{k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} Q_{\frac{k}{2}}^{\frac{n-1}{2}}, \\ g_k &= x_n^{\frac{k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} P_{\frac{k}{2}}^{\frac{1-n}{2}}. \end{aligned}$$

We'll need the solution also for negative k. We use the notation $-k$ so the parameter k itself is always non-negative. With the substitution $k \mapsto -k$ we get

$$\nu = \frac{|1-k|-1}{2} = \begin{cases} \frac{k}{2}-1 & \text{if } k > 1 \\ -\frac{k}{2} & \text{if } k \leq 1. \end{cases}$$

The solutions for $-k$ are thus

$$f_{-k} = x_n^{\frac{-k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} Q_{\frac{k}{2}-1}^{\frac{n-1}{2}},$$
$$g_{-k} = x_n^{\frac{-k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} P_{\frac{k}{2}-1}^{\frac{1-n}{2}}.$$

This works also in the case $0 < k < 1$ since we may then use another solution with $-\nu-1 = \frac{k}{2}-1$ in place of ν. We multiply these solutions with suitable constants and get the k-hyperbolic harmonic fundamental solutions

$$H_k = 2^{\frac{1-n}{2}} \frac{\Gamma\left(\frac{k+n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} a_n^{\frac{k+1-n}{2}} x_n^{\frac{k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} Q_{\frac{k}{2}}^{\frac{n-1}{2}},$$
$$H_{-k} = 2^{\frac{1-n}{2}} \frac{\Gamma\left(\frac{k+n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} a_n^{\frac{-k+1-n}{2}} x_n^{\frac{-k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} Q_{\frac{k}{2}-1}^{\frac{n-1}{2}}.$$

5 Kernels in The Case of $\mathbb{R}^3_+$

Previously known kernels can be obtained as special cases of the fundamental solutions. If we consider the space $\mathbb{R}^3_+$ we have the parameters $n = 2$ and $\mu = \frac{1}{2}$. The familiar kernel which was studied in [7] can be given as a sum of the solutions H_k and

$$K_k = 2^{-\frac{n+1}{2}} \pi \frac{\Gamma\left(\frac{k+n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{k-n+3}{2}\right)} a_n^{\frac{k+1-n}{2}} x_n^{\frac{k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} P_{\frac{k}{2}}^{\frac{1-n}{2}}$$

with the restriction $n < k+3$. The function K_k is just a suitable constant multiple of the solution g_k, considered above, which vanishes at $\lambda = 1$. The connection formula (6) implies now

$$\begin{aligned} H_k + K_k &= \frac{2^{-\frac{n+1}{2}}\pi}{\Gamma\left(\frac{n+1}{2}\right)} a_n^{\frac{k+1-n}{2}} x_n^{\frac{k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} P_{\frac{k}{2}}^{\frac{n-1}{2}}(\lambda) \\ &= \sqrt{\frac{\pi}{2}} a_2^{\frac{k-1}{2}} x_2^{\frac{k-1}{2}} (\lambda^2-1)^{-\frac{1}{4}} P_{\frac{k}{2}}^{\frac{1}{2}}(\lambda) \\ &= a_2^{\frac{k-1}{2}} x_2^{\frac{k-1}{2}} \frac{\cosh\left(\frac{k+1}{2}r\right)}{\sinh r}. \end{aligned}$$

where we have substituted $n = 2$ and $\lambda = \cosh r$ with r denoting the distance $d_h(x, a)$. In the case $k = n - 1 = 1$ we have the kernel

$$H_{-1} = \frac{1}{a_2 x_2 \sqrt{\lambda^2 - 1}} = \frac{1}{a_2 x_2 \sinh r}$$

and the kernel for hypermonogenic functions [8] becomes

$$H_1 = 2(\coth r - 1).$$

6 Integral Formulas

We recall the Stokes' formula for the operator M_k.

Theorem 6.1. [5] *Let Ω be an open subset of $\mathbb{R}^{n+1} \setminus \{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in C^1(\Omega, C\ell_n)$, then*

$$\int_{\partial\Omega} f d\sigma_k g = \int_{\Omega} (f M_k) g + f (M_k g) - \frac{k}{x_n} P(g f') e_n dm_k.$$

We also have the formulas separately for the P- and Q-parts:

$$\int_{\partial\Omega} P(f d\sigma_k g) = \int_{\Omega} P\left((f M_k)\, g + f\,(M_k g) \right) dm_k,$$
$$\int_{\partial\Omega} Q(f d\sigma g) = \int_{\Omega} Q\left((f M_{-k})\, g + f\,(M_k g) \right) dm.$$

We show separately for both parts that the formulas reproduce k-hypermonogenic functions with the function g replaced by the fundamental solution.

Theorem 6.2. *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ and $K \subset \Omega$ be a smoothly bounded compact set with the unit normal ν. If f is k-hypermonogenic in Ω and $a \in K$, then*

$$Pf(a) = -\frac{1}{\omega_n} \int_{\partial K} P(\overline{M}_k H_k d\sigma_k f).$$

For the Q-part we calculate similarly.

Theorem 6.3. *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ and $K \subset \Omega$ be a smoothly bounded compact set with the unit normal ν. If f is k-hypermonogenic in Ω and $a \in K$, then*

$$Qf(a) = -\frac{a_n^k}{\omega_n} \int_{\partial\Omega} Q(\overline{M}_{-k} H_{-k} d\sigma f).$$

Proof. Using the previous notations, the constant appearing in the kernel H_{-k} is $-\frac{C}{s'}$. We obtain

$$\begin{aligned}\overline{D}H_{-k} =& -\frac{C}{s'}s'a_n^{s'}e_n x_n^{s'-1}(\lambda^2-1)^{\frac{1-n}{4}}Qb\\ &-\frac{C}{s'}a_n^{s'}x_n^{s'}(\lambda^2-1)^{\frac{1-n}{4}-1}\cdot(-ss'Qa+s'\lambda Qb)\,\overline{D}(\lambda)\\ =& -Ca_n^{s'}e_n x_n^{s'-1}(\lambda^2-1)^{\frac{1-n}{4}}Qb\\ &+Ca_n^{s'}x_n^{s'}(\lambda^2-1)^{\frac{1-n}{4}-1}\cdot(sQa-\lambda Qb)\,\overline{D}(\lambda)\end{aligned}$$

where we have use the differentiation formula

$$(\lambda^2-1)\partial_\lambda Q_\nu^\mu = (\nu+\mu+1)(\nu-\mu+1)Q_{\nu+1}^\mu - (\nu+1)\lambda Q_\nu^\mu,$$

that is,

$$(\lambda^2-1)\partial_\lambda Qb = -s'sQa - \frac{k}{2}\lambda Qb.$$

Just as in the previous case we get

$$\lim_{\lambda\to 1^+}\int_{\partial B(a,r)} Q(\overline{D}(H_{-k})\nu f)dS = -\frac{\omega_n}{a_n^k}Qf(a).$$

□

Combining these P- and Q-parts we have the integral formula

$$\begin{aligned}f(a) =& -\frac{1}{\omega_n}\int_{\partial\Omega}\left[x_n^{-k}P(\overline{M}_kH_k)+a_n^kQ(\overline{M}_{-k}H_{-k})e_n\right]P(d\sigma f)\\ &-\frac{1}{\omega_n}\int_{\partial\Omega}\left[a_n^kP(\overline{M}_{-k}H_{-k})+x_n^{-k}Q(\overline{M}_kH_k)e_n\right]Q'(d\sigma f).\end{aligned}\tag{7}$$

Furthermore, we get a Cauchy-type formula for k-hypermonogenic functions.

Theorem 6.4. *Let Ω be an open subset of $\mathbb{R}_+^{n+1}$ and $K\subset\Omega$ be a smoothly bounded compact set with the unit normal ν. If f is k-hypermonogenic in Ω and $a\in K$, then*

$$f(a) = \frac{1}{\omega_n}\int_{\partial K} h_k^a(x,a)P(d\sigma f) + h_{-k}^a(x,a)Q'(d\sigma f),$$

where

$$h_k^a(x,a) = x_n^{-k}\overline{D}^a H_k(x,a)$$

and

$$h_{-k}^a(x,a) = a_n^k\overline{D}^a H_{-k}(x,a)e_n$$

are the k-hypermonogenic kernels with respect to a.

Proof. We show that

$$a_n^k \partial_{x_n} H_{-k} + x_n^{-k} \partial_{a_n} H_k = 0,$$
$$a_n^k \partial_{a_n} H_{-k} + x_n^{-k} \partial_{x_n} H_k = 0$$

so the formula (7) yields the result. We use the same notation as in the previous theorems. We start with the first equation and take the x_n derivative from $\overline{D}H_{-k}$ computed previously. Dividing by the constant

$$-\frac{C}{s'} = 2^{\frac{1-n}{2}} \frac{\Gamma\left(\frac{k+n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

the first term on the right becomes

$$\begin{aligned} -\frac{s'}{C} \cdot a_n^k \partial_{x_n} H_{-k} =& s' a_n^s x_n^{s'-1} (\lambda^2 - 1)^{\frac{1-n}{4}} Qb \\ &+ a_n^s x_n^{s'} (\lambda^2 - 1)^{\frac{1-n}{4} - 1} \cdot (s' \lambda Qb - s' s Qa)\, \partial_{x_n} \lambda. \end{aligned}$$

The a_n-derivative in the first equation is calculated in a similar fashion as in the proof of the integral formula for the P-part. We thus find

$$\begin{aligned} \frac{1}{C} \cdot x_n^{-k} \partial_{a_n} H_k =& s a_n^{s-1} x_n^{s'} (\lambda^2 - 1)^{\frac{1-n}{4}} Qa \\ &+ a_n^s x_n^{s'} (\lambda^2 - 1)^{\frac{1-n}{4} - 1} \cdot (s \lambda Qa - Qb)\, \partial_{a_n} \lambda. \end{aligned}$$

Finally, using

$$\partial_{x_n} \lambda = \frac{1}{a_n} - \frac{\lambda}{x_n} \quad \text{and} \quad \partial_{a_n} \lambda = \frac{1}{x_n} - \frac{\lambda}{a_n}$$

we find

$$a_n^k \partial_{x_n} H_{-k} + x_n^{-k} \partial_{a_n} H_k = 0.$$

The second equation

$$a_n^k \partial_{a_n} H_{-k} + x_n^{-k} \partial_{x_n} H_k = 0$$

can be proven similarly. □

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