Decomposition Formula and Stationary Measures for Stochastic Lotka-Volterra Systems with Applications to Turbulent Convection

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Abstract

Motivated by the remarkable works of Busse and his collaborators in the 1980s on turbulent convection in a rotating layer, we explore the long-run behavior of stochastic Lotka-Volterra (LV) systems both in pull-back trajectories and in stationary measures. A decomposition formula is established to describe the relationship between the solutions of stochastic and deterministic LV systems and the stochastic logistic equation. By virtue of this formula, it can be verified that every pull-back omega limit set is an omega limit set of the deterministic LV system multiplied by the random equilibrium of the stochastic logistic equation. The formula is also used to derive the existence of a stationary measure, its support and ergodicity. We prove the tightness of stationary measures and that their weak limits are invariant with respect to the corresponding deterministic system and supported in the Birkhoff center.

The developed theory is successfully utilized to completely classify three dimensional competitive stochastic LV systems into 37 classes. The expected occupation measures weakly converge to a strongly mixing stationary measure and all stationary measures are obtained for each class except class 27 c). Among them there are two classes possessing a continuum of random closed orbits and strongly mixing stationary measures supported on the cone surfaces, which weakly converge to the Haar measures of periodic orbits as the noise intensity vanishes. The class 27 c) is an exception, almost every pull-back trajectory cyclically oscillates around the boundary of the stochastic carrying simplex characterized by three unstable stationary solutions. The limit of the expected occupation measures is neither unique nor ergodic. These are consistent with symptoms of turbulence.

Résumé

Motivés par le travail de Busse et al. [1] concernant la convection turbulente dans la couche de rotation, nous traiterons le comportement à long-terme des Équations Lotka-Volterra (LV) stochastiques par la trajectoire du produit fibré (*pullback*) et la mesure stationnaire. Nous montrons une formule de décomposition stochastique décrivant la relation entre les Équations LV stochastiques et la solution des Équations LV déterministes et celle de la fonction logistic stochastique. En vertu de cette formule, nous vérifierons que l'ensemble des limites oméga de toute trajectoire du produit fibré est un ensemble des limites oméga des Équations LV déterministes multiplié par l'équilibre aléatoire (*random equilibrium*) de la fonction logistique stochastique. Cette formule sert à dériver l'existence de la mesure stationnaire, son support et son ergodicité. Nous prouverons aussi la tension (*tightness*) pour l'ensemble des mesures stationnaires, et l'invariance pour leurs limites faibles lorsque l'intensité des bruits diminue jusqu'à zéro, dont le support est contenu dans le centre Birkhoff.

La théorie proposée s'applique avec succès à la classification complète des Équations LV stochastiques compétitives tridimensionnelles en 37 classes. Les mesures d'occupation moyennes convergent faiblement vers une mesure stationnaire ergodique dans le domaine d'attraction d'un ensemble des limites oméga en toutes les classes sauf en classe 27 c), parmi lesquelles il y en a deux qui possèdent un continuum d'orbites closes; et les mesures stationnaires ergodiques, supportées dans une surface conique, convergent faiblement vers les mesures de Haar des orbites périodiques lorsque l'intensité des bruits diminue jusqu'à zéro. Dans la classe exceptionnelle, presque chaque trajectoire du produit fibré oscille de manière cyclique autour de la limite d'un simplexe porteur stochastique, caractérisé par trois solutions stationnaires instables. La limite des mesures d'occupations moyennes n'est ni unique ni ergodique, ce qui correspond aux caractéristiques de la turbulence.

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1. Introduction

Turbulent convection in a fluid layer heated from below and rotating about a vertical axis was studied by Busse and his collaborators [1, 2, 3]. They proved that turbulence occurs when both the Rayleigh number and the Taylor number exceed their critical values. This kind of turbulence is understood in terms of a manifold of stationary solutions, each of which is unstable relative to some other solution in the manifold so that the system evolves in time by realizing cyclically the different solutions of the manifold. This cyclically fluctuating solution was observed by May and Leonard [4] in the context of population biology, and was confirmed by experimental observations in [1, 3].

Using the depth d of the layer, the temperature difference between the upper and lower boundaries divided by the Rayleigh number $\frac{(T_2-T_1)}{R}$, and the thermal diffusion time $\frac{d^2}{\kappa}$ as scales for length, temperature and time, respectively, the convection model described as

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above is formulated by the Navier-Stokes equations for the velocity vector $\mathbf{v}$ and the heat equation for the deviation θ of the temperature from the static state:

$$\begin{cases} P^{-1}(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\sqrt{T}}{2}\lambda \times \mathbf{v} = -\nabla\pi + \lambda\theta + \nabla^2\mathbf{v}, \\ \nabla \cdot \mathbf{v} = 0, \\ (\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla)\theta = R\lambda \cdot \mathbf{v} + \nabla^2\theta, \end{cases} \tag{1}$$

where $\lambda = (0, 0, 1)^\tau$. The physical state of the layer is represented in terms of three dimensionless parameters: the Rayleigh, Taylor, and Prandtl numbers

$$R = \frac{g\gamma(T_2 - T_1)d^3}{\kappa\nu}, \quad T = \frac{4\Omega^2 d^4}{\nu^2}, \quad \text{and} \quad P = \frac{\nu}{\kappa}.$$

Here γ, g, κ and ν are the thermal expansion coefficient, the gravitational acceleration constant, the thermal diffusivity and the kinematic viscosity, respectively. Ω is the angular velocity rotating about the vertical axis through the center of the layer. A stress-free condition is applied to the boundaries.

The vertical component of the velocity field in the limit of small amplitudes can be approximately expressed by

$$\mathbf{v}_z = f(z, \alpha, \tau) \sum_{j=-n}^{n} c_j(t)\exp(i\mathbf{k}_j \cdot \mathbf{r}), \tag{2}$$

where z (normalized between $-\frac{1}{2}$ and $\frac{1}{2}$) is the component of the position vector $\mathbf{r}$ in the vertical direction, $\tau = \frac{\sqrt{T}}{2}$, $\mathbf{k}_j$ is the horizontal wave vector with $|\mathbf{k}_j| = \alpha$ and $\mathbf{k}_{-j} = -\mathbf{k}_j$. The equation for $f(z, \alpha, \tau)$ together with the Dirichelet boundary conditions at $z = \pm\frac{1}{2}$ represents an eigenvalue problem for $R_0(\alpha, \tau) = \alpha^{-2}((\pi^2+\alpha^2)^3+\pi^2\tau^2)$ calculated by Küppers and Lorz [5]. At a finite value α_c this function attains its minimum $R_c(\tau)$ at which the onset of convection occurs. The time-dependent amplitudes $c_j(t)$ are subject to the conditions $c_j(t) = -c_j^*(t)$, where $c_j^*(t)$ denotes the complex conjugate of $c_j(t)$ (see [2, 3] for more details). Then it follows from [2] that c_i satisfies the equations

$$M\frac{dc_i}{dt} = c_i\left((R - R_c)\,K - \frac{1}{2}\sum_{j=-n}^{n} T_{ij}|c_j|^2\right), \quad i = 1, 2, ..., n, \tag{3}$$

where the matrix elements T_{ij} obey the symmetry relationships

$$T_{ij} = T_{i-j} = T_{-ij}. \tag{4}$$

When the Rayleigh number R exceeds the critical value R_c depending on the Taylor number T, the static state becomes unstable and convective motions set in. In the case where $n = 3$, by setting $y_i = |c_i|^2$ for $i = 1, 2, 3$ and making suitable rescales, Busse et al. [1, 2, 3]

transformed (3) into the standard *symmetric May-Leonard system* [4]:

$$\begin{aligned}\frac{dy_1}{dt} &= y_1(1 - y_1 - \alpha y_2 - \beta y_3),\\ \frac{dy_2}{dt} &= y_2(1 - \beta y_1 - y_2 - \alpha y_3),\\ \frac{dy_3}{dt} &= y_3(1 - \alpha y_1 - \beta y_2 - y_3),\end{aligned} \tag{5}$$

where $y_i \geq 0,\ i = 1,2,3$. If the Taylor number T exceeds the critical value T_c, then $\alpha+\beta > 2$ and $\alpha < 1$. It is well-known that the May-Leonard system (5) exhibits nonperiodic oscillations of bounded amplitude but ever increasing cycle time. Hence, Busse et al. [1, 2, 3] concluded that the turbulent convection in a rotating layer is approximately described by a manifold of three stationary solutions, all of which are unstable with respect to some other. Such a manifold is spanned by the three axial equilibria and their connecting orbits, and approached rapidly from arbitrary nontrivial initial conditions. Interestingly, the manifold is exactly the *carrying simplex* founded by Hirsch [6] nearly one decade later.

However, Busse and Heikes [1, p.174] addressed that: "small-amplitude disturbances are present at all times $\cdots$. The existence of a noise level prevents the amplitudes y_i from decaying to arbitrary small levels. At the same time it introduces a random element into the time dependence of the system$\cdots$." In the second paragraph of introduction, Heikes and Busse [3] clearly showed that this randomness occurs for Rayleigh number R close to the critical value of the onset of convection, R_c. These statements mean that the deviation of the Rayleigh number from its critical value is an average size in some sense.

This leads us to hypothesize that the Rayleigh number R of the system (1) is disturbed by random noise $\{\xi_t\}_{t\geq 0}$, which may be attributed to wind speed, clouds on solar radiation and other effects. Thus the perturbed system is formulated as

$$\begin{cases} P^{-1}(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\sqrt{T}}{2}\lambda\times\mathbf{v} = -\nabla\pi + \lambda\theta + \nabla^2\mathbf{v},\\ \nabla\cdot\mathbf{v} = 0,\\ (\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla)\theta = R\lambda\cdot\mathbf{v} + \nabla^2\theta + R\lambda\cdot\mathbf{v}\xi_t. \end{cases} \tag{6}$$

Using the same approach in [2], the random force is then transformed to each mode:

$$M\frac{dc_i}{dt} = c_i\left((R - R_c + \xi_t)\,K - \frac{1}{2}\sum_{j=-n}^{n} T_{ij}|c_j|^2\right),\quad i = 1,2,...,n. \tag{7}$$

Let $\xi_t = \dot{B}_t$ and $y_i = |c_i|^2$ for $i = 1,2,\cdots,n$. After rescaling time t by Mt, we get stochastic Lotka-Volterra equations:

$$(\mathrm{E}_\sigma) : dy_i = y_i\left(r + \sum_{j=1}^{n} a_{ij}y_j\right)dt + \sigma y_i dB_t,\ i = 1,2,...,n \tag{8}$$

on the positive orthant, where $r = (R - R_c)K,\ a_{ij} = -T_{ij},\ \sigma$ are parameters and B_t is a Brownian motion. (E_σ) is the so-called Itô stochastic differential equations. Our main purpose is to develop an approach to prove the existence of cyclically fluctuating solutions in path of the system (8) with three modes provided that both the Rayleigh number and the Taylor number exceed their critical values, and that these cyclically fluctuating solutions concentrate around three axes, which shows that turbulence still remains preserved under stochastic disturbances. (E_σ) is the well-known Lotka-Volterra equations with the growth rate r and a white noise perturbation. Lotka-Volterra equations play an important role in population dynamics, game theory, and so on (see [7, 8]).

Since the pioneer work of Khasminskii [9, 10], the existence and uniqueness of regular stationary measures of stochastic ordinary differential equations in $\mathbb{R}^n$ have been extensively studied. However, the feasible domain of (E_σ) is the positive orthant of $\mathbb{R}^n$. Recently, in a general domain of $\mathbb{R}^n$, Huang et al. [11, 12, 13] have systematically investigated stationary measures of stochastic ordinary differential equations via Fokker-Plank equations. In [11], they provided several useful estimates for regular stationary measures in an exterior domain, which can be used to obtain tightness of a family of stationary measures. Their key technique is the level set method, and in particular, the integral identity they derived. This tool is used to give new existence results on stationary measures under Lyapunov-like and weaker regularity conditions in [12]. Bogachev et al. [14] and Shaposhnikov [15] provided examples admitting multiple or even infinite number of stationary measures, the readers may refer to a review article by Bogachev et al.[16] for more related results. For the attractors of random dynamical systems, one can refer to the works of [17, 18, 19] and the references therein. Huang et al. [13] showed that as diffusion matrices vanish, the weak$^*-$limits of stationary measures are invariant measures of the dissipative unperturbed system concentrated on its global attractor.

The purpose of this paper is to explore the long-run behavior of stochastic Lotka-Volterra systems (E_σ) both in limit of pull-back trajectories and in stationary measures. Motivated by [20], we will first investigate the relationship between (E_σ), (E_0) and the stochastic logistic equation

$$dg = g(r - rg)dt + \sigma g dB_t \tag{9}$$

and establish the following stochastic decomposition formula

$$\Phi(t,\omega,y) = g(t,\omega,g_0)\Psi\left(\int_0^t g(s,\omega,g_0)ds, \frac{y}{g_0}\right), \tag{10}$$

where $\Phi(t,\omega,y)$, $\Psi(t,y)$ are the solutions of (E_σ) and (E_0) with the initial point y, respectively, and $g(t,\omega,g_0)$ is the solution of (9) with the initial point $g_0 > 0$.

The stochastic decomposition formula (10) will play an important role in achieving our goal. By virtue of this formula, it can be verified that every pull-back omega limit set of (E_σ) is an omega limit set of (E_0) multiplied by the random equilibrium of the stochastic logistic equation (9). We also investigate the weak convergence of the transition probability function to a strongly mixing stationary measure as time approaches infinity. Using the stochastic decomposition formula (10), the Khasminskii theorem [9, p.65] and the Portmanteau theorem, it can be shown that a bounded orbit of (E_0) deduces the existence of a stationary measure of (E_σ) supported in a lower dimensional cone consisting of all rays connecting the origin and all points in the omega limit set of this orbit. This means that any stationary measure is not regular. If (E_0) is dissipative, then we prove that the set of stationary measures is tight, and that their limiting measures in weak topology are invariant with respect to the flow of (E_0) as the noise intensity σ tends toward zero, with supports contained in the Birkhoff center of (E_0). This means that on the global attractor of (E_0) any limiting measure charges no on the complement of the Birkhoff center. In section 6, we give a complete classification of the three dimensional competitive stochastic Lotka-Volterra system with identical intrinsic growth rate both in pull-back trajectory and in stationary motion. There are exactly 37 scenarios in terms of competitive coefficients, and the expected occupation measures always weakly converge to strongly mixing stationary measures except one class. Each pull-back trajectory in 34 classes asymptotically converges to a random equilibrium, but possibly a different random equilibrium for a different trajectory in the same class. All limiting measures of stationary measures in each of these 34 classes are the convex combination of its Dirac measures at equilibria. Two of the remaining classes possess a family of stochastic closed orbits, and a continuum of strongly mixing stationary measures supported on cone surfaces determined by periodic orbits of (E_0), respectively. These strongly mixing stationary measures weakly converge to the Haar measures of periodic orbits as the noise intensity tends toward zero. Among the above 36 classes, all stationary measures are expressed by these strongly mixing stationary measures via ergodic decomposition theorem. The final class, the most interesting and complicated one, violates this law. The expected occupation measures of a solution do not weakly converge, but have infinite number of limit measures which are not ergodic and supported in three positive axes. As the noise intensity tends toward zero, these stationary measures weakly converge to a convex combination of Dirac measures at three unstable axial equilibria. We will reveal that the essential reason for both peculiar characteristics is that the solutions stay near three equilibria for a very long time (approximately infinite) with probability nearly one. Besides, almost every pull-back trajectory cyclically oscillates around the boundary of the stochastic carrying simplex which is characterized by three unstable stationary solutions. All these are subject to turbulent characteristics. This proves a stochastic version of cyclical fluctuation and that the turbulence in a fluid layer heated from below and rotating about a vertical axis is robust under stochastic disturbances.

2. Probability Preliminaries

In this section we introduce some definitions and notation, establish some conventions, and some key probability results, which will be employed throughout the paper.

A continuous, adapted random process $B = \{B_t, \mathcal{F}_t : t \in \mathbb{R}\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is called a two-sided time *Brownian motion*, if $B_0 = 0$ a.s., for any $t \neq s$, $B_t - B_s$ is normally distributed with mean zero and variance $|t-s|$, and B has independent increments. In a canonical way, we may assume $\Omega = C_0(\mathbb{R}, \mathbb{R})$ endowed with the compact-open topology, and $\mathcal{F}$ is its Borel $\sigma-$algebra, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$, and $B_t(\omega)$ can be regarded as coordinate process $\omega(t)$. We define the shift by

$$\theta_t : \Omega \to \Omega,\ \theta_t\omega(s) := \omega(s+t) - \omega(t),\ s, t \in \mathbb{R}.$$

Then θ_t is a homeomorphism for each t and $(t,\omega) \to \theta_t\omega$ is continuous, hence measurable. Thus the Brownian motion generates an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t : t \in \mathbb{R}\})$ (see Appendix A.3 in Arnold [21] for details).

Definition 2.1 (Random Dynamical System)**.** *A random dynamical system* (RDS) *with one-sided time $\mathbb{R}_+$ and the state space Y over a metric dynamical system $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ is a $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}(Y), \mathcal{B}(Y))$-measurable mapping*

$$\Phi : \mathbb{R}_+ \times \Omega \times Y \mapsto Y, \quad (t, \omega, y) \mapsto \Phi(t, \omega, y),$$

such that
(i) $\Phi(t, \omega, \cdot) : Y \to Y$ *is continuous for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*
(ii) *the mappings $\Phi(t, \omega) := \Phi(t, \omega, \cdot)$ form a cocycle over θ*

$$\Phi(0, \omega) = id, \quad \Phi(t + s, \omega) = \Phi(t, \theta_s\omega) \circ \Phi(s, \omega)$$

for all $t, s \in \mathbb{R}_+$ and $\omega \in \Omega$. Here $\circ$ means composition of mappings.

The pull-back omega limit set $\Gamma_y(\omega)$ of the pull-back trajectory $\Phi(t, \theta_{-t}\omega, y)$ is defined by

$$\Gamma_y(\omega) := \bigcap_{t>0} \overline{\bigcup_{\tau \geq t} \Phi(\tau, \theta_{-\tau}\omega, y)}.$$

Definition 2.2 (Random Equilibrium)**.** *A random variable $u : \Omega \to Y$ is said to be a random equilibrium (or stationary solution) of* RDE (θ, Φ) *if it is invariant under Φ, i.e., if*

$$\Phi(t, \omega, u(\omega)) = u(\theta_t\omega) \quad \textit{for all } t \geq 0 \quad \textit{and all } \omega \in \Omega.$$

First, we state an extension of Itô's formula which is adopted from [22, p.152-153].

Proposition 2.1 (An Extension of Itô's Formula)**.** *Let $X = \{X_t, \mathcal{F}_t : 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be a continuous semimartingale and $G : \mathbb{R} \times (\Omega \times \mathbb{R}_+) \to \mathbb{R}$ such that*
(i) $(x, u) \to G(x, \omega, u)$ *is continuous for every ω;*
(ii) $x \to G(x, \omega, u)$ *is C^2 for every (ω, u);*
(iii) $(\omega, u) \to G(x, \omega, u)$ *is adapted for every x;*
(iv) *for every (x, ω), the map $u \to G(x, \omega, u)$ is of class C^1 and the derivation is continuous in the variable x.*
Then the following extension of Itô's formula holds

$$G(X_t, t) = G(X_0, 0) + \int_0^t \frac{\partial G}{\partial x}(X_u, u) dX_u + \int_0^t \frac{\partial G}{\partial u}(X_u, u) du + \frac{1}{2} \int_0^t \frac{\partial^2 G}{\partial x^2}(X_u, u) d\langle X, X \rangle_u. \quad (11)$$

Throughout of this paper, we will use the notation $\mathbb{R}^n_+ := \{y \in \mathbb{R}^n : y = (y_1, y_2, ..., y_n),\ y_i \geq 0,\ i = 1, 2, ..., n\}$ to denote the positive orthant, and its interior is denoted by $\mathrm{Int}\mathbb{R}^n_+$. Suppose that E is a metric space and $\mathcal{E} := \mathcal{B}(E)$ denotes its Borel $\sigma-$algebra. Then $\mathcal{B}_b(E)$ (resp.

$\mathcal{C}_b(E)$) is the set of all real bounded Borel measurable functions (resp. bounded continuous functions) on E.

Suppose that $\Phi(t,\omega,y)$ is the solution of (E_σ) with the initial state $y \in \mathbb{R}^n_+$. For any $A \in \mathcal{B}(\mathbb{R}^n_+)$, the *transition probability function* is defined by

$$P(t,y,A) := \mathbb{P}(\Phi(t,\omega,y) \in A) \tag{12}$$

which generates a Markovian semigroup $\{P_t, t \geq 0\}$ by

$$P_t f(y) := \int_{\mathbb{R}^n_+} f(z)P(t,y,dz), \quad y \in \mathbb{R}^n_+, \tag{13}$$

where $f \in \mathcal{B}_b(\mathbb{R}^n_+)$. This Markovian semigroup $\{P_t, t \geq 0\}$ is *stochastically continuous* (see [23, Theorem 3.2.6, p.13]) in the sense that

$$\lim_{t\to 0} P_t f(y) = f(y), \text{ for all } f \in \mathcal{C}_b(\mathbb{R}^n_+) \text{ and } y \in \mathbb{R}^n_+,$$

and a *Feller semigroup* in the sense that

$$P_t f \in \mathcal{C}_b(\mathbb{R}^n_+), \text{ for all } f \in \mathcal{C}_b(\mathbb{R}^n_+) \text{ and } t > 0.$$

According to [24, p.29] and [25, p.772], the *expected occupation measure* at the time T for the continuous-time process $\{\Phi_t\}_{t\in\mathbb{R}_+}$ is defined by

$$P^{(T)}(y,A) := \frac{1}{T}\int_0^T P(s,y,A)ds \tag{14}$$

with the initial point $y \in \mathbb{R}^n_+$ and $A \in \mathcal{B}(\mathbb{R}^n_+)$.

A probability measure μ on $\mathcal{B}(\mathbb{R}^n_+)$ is called *stationary* (or *invariant*) with respect to the Markovian semigroup $\{P_t, t \geq 0\}$ if

$$\int_{\mathbb{R}^n_+} P(t,y,A)\mu(dy) = \mu(A) \text{ for any } t \geq 0 \text{ and } A \in \mathcal{B}(\mathbb{R}^n_+). \tag{15}$$

Similar definition may apply to $\mathcal{B}(\mathrm{Int}\mathbb{R}^n_+)$.

The following theorem, due to Khasminskii (see [9, p.65]), gives a criterion for the existence of a stationary measure.

Theorem 2.1 (Khasminskii Theorem)**.** *A necessary and sufficient condition for the existence of a stationary Markov process with the given time-homogeneous stochastically continuous Feller transition probability function $P(t,y,A)$ is that for some point $y \in \mathbb{R}^n_+$*

$$\lim_{R\to\infty} \liminf_{T\to\infty} P^{(T)}(y, B_R^c) = 0, \tag{16}$$

where $B_R := \{y \in \mathbb{R}^n_+ : \|y\| < R\}$ denotes the open ball in $\mathbb{R}^n_+$ with the center at the origin and radius R, B_R^c is its complement set.

(16) implies that there exists a time sequence $\{T_n : n \in \mathbb{N}\}$ tending to infinity such that $P_n := P^{(T_n)}(y, \cdot)$ weakly converges to a stationary measure μ of (E_σ). We will frequently use the Portmanteau theorem (see [26, Theorem 2.1, p.11-12]), which is stated below.

Theorem 2.2 (Portmanteau Theorem)**.** *Let P_n and μ be probability measures on $(\mathbb{R}^n_+, \mathcal{B}(\mathbb{R}^n_+))$. These five conditions are equivalent:*
(i) $P_n \overset{w}{\to} \mu$;
(ii) $\lim_{n\to\infty} \int f dP_n = \int f d\mu$ *for all bounded, uniformly continuous real* f;
(iii) $\limsup_{n\to\infty} P_n(F) \leq \mu(F)$ *for all closed* F;
(iv) $\liminf_{n\to\infty} P_n(G) \geq \mu(G)$ *for all open* G;
(v) $\lim_{n\to\infty} P_n(A) = \mu(A)$ *for all* $\mu-$*continiuty sets* A.

We will introduce the concepts of ergodicity and strong mixing for stationary measures and their properties, which are taken from [23].

We note that each probability measure on the Borel $\sigma-$algebra of a separable and locally compact Hausdorff space E is a Radon measure. Thus, the Kolmogorov extension theorem can be applied to construct a probability on infinite product space $\Omega = E^{\mathbb{R}}$ of all the $E-$valued functions (see [27]). With a given Markovian semigroup $\{P_t, t \geq 0\}$, and a stationary measure μ we will associate now, in a unique way, a dynamical system $S^\mu \equiv (\Omega, \mathcal{F}, \mathbb{P}_\mu, \{\theta_t : t \in \mathbb{R}\})$. The canonical (coordinate) process $X(t), t \in \mathbb{R}$, will be Markovian, with transition probabilities $P(t, x, \cdot), t > 0, x \in E$; it will also be stationary and such that $\mathcal{L}(X(t)) = \mu, t \in \mathbb{R}$.

The dynamical system S^μ is called *ergodic* if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}_\mu(\theta_{-t}A \cap B)dt = \mathbb{P}_\mu(A)\mathbb{P}_\mu(B), \text{ for all } A, B \in \mathcal{F}.$$

The dynamical system S^μ is said to be *strongly mixing* if

$$\lim_{t\to\infty} \mathbb{P}_\mu(\theta_{-t}A \cap B) = \mathbb{P}_\mu(A)\mathbb{P}_\mu(B), \text{ for all } A, B \in \mathcal{F}.$$

It is clear that a strongly mixing system is ergodic.

Theorem 2.3. *Let $P_t, t > 0$, be a stochastically continuous Markovian semigroup and μ a stationary measure with respect to $P_t, t > 0$. Then the following conditions are equivalent:*
(i) μ *is strongly mixing;*
(ii) *for arbitrary* $\varphi \in L^2(E, \mu)$ *we have*

$$\lim_{t\to\infty} P_t\varphi = \langle \varphi, 1\rangle, \text{ in } L^2(E, \mu). \tag{17}$$

This theorem is adopted from [23, Theorem 3.4.2, p.35]. The sufficient condition for a stationary measure to be strongly mixing is that the semigroup $P_t, t > 0$ weakly converges to μ, which is stated as follows and is taken from [23, Corollary 3.4.3, p.36] except the uniqueness.

Theorem 2.4. *Let μ be a stationary measure with respect to the semigroup $P_t, t > 0$. Assume that*

$$\lim_{t\to\infty} P(t, x, \cdot) = \mu \text{ weakly}, \ x \in E.$$

Then μ is strongly mixing and the unique stationary measure for the semigroup $P_t, t > 0$.

Now we give the proof of the uniqueness of the stationary measure μ. Let ν be an arbitrary stationary measure on E and $f \in \mathcal{C}_b(E)$. Then

$$\begin{aligned}
& \int_E f(x)\nu(dx) \\
= & \int_E f(x) \int_E P(t, y, dx)\nu(dy) \\
= & \int_E \nu(dy) \int_E f(x)P(t, y, dx) \\
= & \lim_{t\to\infty} \int_E \nu(dy) \int_E f(x)P(t, y, dx) \\
= & \int_E \nu(dy) \lim_{t\to\infty} \int_E f(x)P(t, y, dx) \\
= & \int_E f(x)\mu(dx),
\end{aligned}$$

that is, $\nu = \mu$.

A Markovian semigroup $P_t, t > 0$, is said to be a *strongly Feller* semigroup at time $t_0 > 0$ if for arbitrary $\varphi \in \mathcal{B}_b(E)$, $P_{t_0}\varphi \in \mathcal{C}_b(E)$.

A Markovian semigroup $P_t, t > 0$, is said to be *irreducible* at time $t_0 > 0$ if, for arbitrary non empty open set Γ and all $x \in \Gamma$, $P(t_0, x, \Gamma) > 0$.

Theorem 2.5. *Let $P_t, t > 0$, be a stochastically continuous Markovian semigroup and μ a stationary measure with respect to $P_t, t > 0$. If the Markovian semigroup $P_t, t > 0$, is a strongly Feller semigroup at time $t_0 > 0$ and irreducible at time $s_0 > 0$, then*
(i) *μ is strongly mixing and for arbitrary $x \in E$ and $\Gamma \in \mathcal{E}$,*

$$\lim_{t\to\infty} P(t, x, \Gamma) = \mu(\Gamma);$$

(ii) *μ is the unique stationary measure for the semigroup $P_t, t > 0$.*

Theorem 2.5 is a corollary of Doob's theorem (see [23, Proposition 4.1.1, p.42] and [23, Theorem 4.2.1, p.43]).

We note that Theorems 2.3 and 2.4 can be applied to $E = \mathbb{R}^n_+$ or $\mathrm{Int}\mathbb{R}^n_+$.

Suppose that $(E_i, \mathcal{E}_i)$ $(i = 1, 2)$ are metric spaces, $\Phi_i(t, \omega, x_i)(i = 1, 2)$ are continuous random processes on $\mathbb{R}_+ \times E_i$. Φ_1 and Φ_2 are said to be *conjugate* if there is a homeomorphism $\psi : E_1 \to E_2$, called a *conjugate mapping*, such that

$$\psi(\Phi_1(t, \omega, x_1)) = \Phi_2(t, \omega, \psi(x_1)), \text{ for all } t > 0, \ \omega \in \Omega, \ x_1 \in E_1. \tag{18}$$

Assume that Φ_1 and Φ_2 are Markovian processes. Then we can define transition probability functions $P^i(t, x_i, A_i)$, $A_i \in \mathcal{E}_i$ as in (12) and Markovian semigroups P^i_t $(i = 1, 2)$ as in (13), respectively, which are stochastically continuous Feller semigroups. By (18), we have

$$P^2(t, x_2, A_2) = P^1(t, \psi^{-1}(x_2), \psi^{-1}(A_2)), \text{ for all } t > 0, x_2 \in E_2, \ A_2 \in \mathcal{E}_2. \tag{19}$$

Applying (19) and the change-of-variables formula (see [28, Theorem 3.6.1, p.190]), we get the following result.

Theorem 2.6. *Suppose that the Markovian processes* Φ_1 *and* Φ_2 *are conjugate. Then we have*
(i) μ *is a stationary measure with respect to* P_t^1 *if and only if* $\mu\psi^{-1}(\cdot)$ *is a stationary measure with respect to* P_t^2*;*
(ii) *the Markovian semigroup* $P_t^1, t > 0$ *is strongly Feller (irreducible) at time* $t_0 > 0$ *if and only if the Markovian semigroup* $P_t^2, t > 0$ *is strongly Feller (irreducible) at time* $t_0 > 0$*.*

The following well-known lemma (see [21, Lemma 2.1.5, p.54]) will be employed in section 5.

Lemma 2.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(X, \mathcal{B})$ *a measurable space,* $h : X \times \Omega \to \mathbb{R}$ *bounded measurable,* $\mathcal{C} \subset \mathcal{F}$ *a sub*$-\sigma-$*algebra,* $\xi : \Omega \to X$ $\mathcal{C}, \mathcal{B}$ *measurable, and* $h(x, \cdot)$ *and* $\mathcal{C}$ *independent for each* $x \in X$*. Then*

$$\mathbb{E}[h(\xi(\cdot), \cdot)|\mathcal{C}] = \mathbb{E}[h(\xi(\cdot), \cdot)|\xi] = H \circ \xi, \;\; H(x) := \mathbb{E}h(x, \cdot).$$

3. Stochastic Decomposition Formula

In this section, we will present stochastic decomposition formulae which essentially express the solution of a stochastic Lotka-Volterra system as the product of the solutions of stochastic logistic equation and a deterministic Lotka-Volterra system.

Theorem 3.1 (Stochastic Decomposition Formula of Itô Type)**.** *Let* $\Phi(t, \omega, y)$ *and* $\Psi(t, y)$ *be the solutions of* (E_σ) *and* (E_0)*, respectively. Then*

$$\Phi(t, \omega, y) = g(t, \omega, g_0)\Psi\Big(\int_0^t g(s, \omega, g_0)ds, \frac{y}{g_0}\Big), \;\; y \in \mathbb{R}_+^n, \;\; g_0 > 0, \tag{20}$$

where $g(t, \omega, g_0)$ *is a positive solution of the stochastic logistic equation*

$$dg = g(r - rg)dt + \sigma g dB_t, \;\; g(0, \omega, g_0) = g_0. \tag{21}$$

Proof. Define

$$G_i(x, \omega, u) := x\Psi_i\Big(\int_0^u g(s, \omega, g_0)ds, \frac{y}{g_0}\Big)$$

and let $\tilde{\Phi}(t, \omega, y)$ denote the right hand of (20). Since $g(t, \omega, g_0)$ is a solution of (21), it is continuous and adapted to the filtration $\{\mathcal{F}_t\}$. By the definition of Riemann integral, the integral of $g(t, \omega, g_0)$ with upper limit t is still adapted to the filtration $\{\mathcal{F}_t\}$. Therefore, $G_i(x, \omega, u)$ is adapted with respect to (ω, u), continuous with respect to (x, u), continuously differentiable with respect to u, and linear with respect to x. This means that all conditions (i)-(iv) of Proposition 2.1 hold.

Applying the Itô's formula (11) to G_i, we obtain that for each $i = 1, 2, ..., n$,

$$\begin{aligned} d\tilde{\Phi}_i = \;& \Psi_i(\int_0^t g(s,\omega,g_0)ds, \tfrac{y}{g_0})[g(r-rg)dt + \sigma g dB_t] \\ & + g^2(t,\omega,g_0)\Psi_i(\int_0^t g(s,\omega,g_0)ds, \tfrac{y}{g_0})\times \\ & \Big[r + \sum_{j=1}^n a_{ij}\Psi_j\Big(\int_0^t g(s,\omega,g_0)ds, \frac{y}{g_0}\Big)\Big]dt \\ = \;& \tilde{\Phi}_i(r + \sum_{j=1}^n a_{ij}\tilde{\Phi}_j)dt + \sigma\tilde{\Phi}_i dB_t. \end{aligned}$$

This completes the proof.

□

We can also present a stochastic decomposition formula for Stratonovich stochastic differential equations:

$$dy_i = y_i\Big(r + \sum_{j=1}^n a_{ij}y_j\Big)dt + \sigma y_i \circ dB_t, \ i = 1, 2, ..., n, \tag{22}$$

which reveals the connection between solutions of (22) and those of

$$\frac{dy_i}{dt} = F_i(y) := y_i\Big(r + \sum_{j=1}^n a_{ij}y_j\Big), \ y_i \geq 0, \ i = 1, 2, ..., n, \tag{23}$$

and Stratonovich stochastic logistic equation:

$$dg = g(r - rg)dt + \sigma g \circ dB_t. \tag{24}$$

Here $\circ$ means Stratonovich integral.

Theorem 3.2 (Stochastic Decomposition Formula of Stratonovich Type)**.** *Assume that* $\Phi(t,\omega,y)$ *and* $\Psi(t,y)$ *are the solutions of* (22) *and* (23)*, respectively, and* $g(t,\omega,g_0)$ *is a positive solution of the logistic equation* (24)*. Then*

$$\Phi(t,\omega,y) = g(t,\omega,g_0)\Psi\Big(\int_0^t g(s,\omega,g_0)ds, \frac{y}{g_0}\Big), \ y \in \mathbb{R}_+^n, \ g_0 > 0. \tag{25}$$

Proof. Applying Theorem 2.4.2 in [19, p.72], we know that the Stratonovich stochastic LV system (22) is equivalent to the Itô stochastic LV system:

$$dy_i = y_i\Big(r + \frac{\sigma^2}{2} + \sum_{j=1}^n a_{ij}y_j\Big)dt + \sigma y_i dB_t, \ i = 1, 2, ..., n. \tag{26}$$

Similarly, the Stratonovich stochastic logistic equation (24) is equivalent to the Itô stochastic logistic equation:

$$dg = g(r + \frac{\sigma^2}{2} - rg)dt + \sigma g dB_t. \tag{27}$$

Thus, (25) can be obtained in the same manner as done in Theorem 3.1.

□

Remark 3.1. *The stochastic decomposition formulae* (20) *and* (25) *still hold if g_0 and y are replaced by $\mathcal{F}_0$-measurable random variables.*

4. The Long-Run Behavior of Equation (22)

Let $r > 0$. Then the stochastic logistic equation (24) can be solved explicitly as

$$g(t,\omega,x) = \frac{x\exp\{rt+\sigma B_t(\omega)\}}{1+rx\int_0^t \exp\{rs+\sigma B_s(\omega)\}ds},\ x\ge 0, \tag{28}$$

whose *random equilibrium (or stationary solution)* is

$$u(\omega) = \Big(r\int_{-\infty}^0 \exp\{rs+\sigma B_s(\omega)\}ds\Big)^{-1}. \tag{29}$$

g in (28) has the cocycle property and $u(\theta_t\omega)$ is the unique nontrivial stationary solution, whose probability density function satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\Big(x(r+\frac{\sigma^2}{2}-rx)p\Big)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\Big(x^2p\Big)$$

and can be solved as follows

$$p^\sigma(x) = \frac{(\frac{2r}{\sigma^2})^{\frac{2r}{\sigma^2}}}{\Gamma(\frac{2r}{\sigma^2})}x^{\frac{2r}{\sigma^2}-1}\exp\{-\frac{2r}{\sigma^2}x\},\ \sigma\neq 0,\ x\ge 0, \tag{30}$$

where $\Gamma(\cdot)$ is the $\Gamma-$function, see [29] for details. Using the properties of the $\Gamma-$function, we can see that

$$\mathbb{E}u = \int_0^{+\infty} xp^\sigma(x)dx = 1. \tag{31}$$

The Birkhoff-Khintchin ergodic theorem (see, e.g., Arnold [21, Appendix A.1]) implies that

$$\lim_{|t|\to\infty}\frac{1}{t}\int_0^t u(\theta_s\omega)ds = 1 \tag{32}$$

on a θ-invariant set $\Omega^*\in\mathcal{F}$ of full measure.

From Remark 3.1 it follows that

Corollary 4.1. *Suppose that $\Phi(t,\omega,u(\omega)y)$ and $\Psi(t,y)$ are the solutions of* (22) *and* (23) *through $u(\omega)y$ and y, respectively. Then*

$$\Phi(t,\omega,u(\omega)y) = u(\theta_t\omega)\Psi\Big(\int_0^t u(\theta_s\omega)ds, y\Big),\ y\in\mathbb{R}^n_+. \tag{33}$$

Chueshov [19, p.202] proved the following.

Lemma 4.1. *Every positive pull-back trajectory for stochastic logistic equation* (24) *is exponentially convergent to the random equilibrium* $u(\omega)$*, that is, there is a* $\gamma > 0$ *such that*

$$\lim_{t\to\infty} e^{\gamma t}|g(t,\theta_{-t}\omega,x) - u(\omega)| = 0 \quad \textit{for all } x > 0 \textit{ and } \omega \in \Omega. \tag{34}$$

The following lemma is about the ergodic property of the solutions of stochastic logistic equation (24).

Lemma 4.2. *For any* $x > 0$ *and* $A \in \mathcal{B}(\mathbb{R}_+)$*, we have*

$$\lim_{t\to\infty} \mathbb{P}\{\omega : g(t,\omega,x) \in A\} = \mu_g^\sigma(A), \tag{35}$$

where $\mu_g^\sigma(A) = \mathbb{P}u^{-1}(A)$ *is the unique nontrivial stationary measure for* (24)*, whose density function is given in* (30)*. Moreover,* $\{\mu_g^\sigma : \sigma > 0\}$ *is tight.*

Proof. From (34) and Lebesgue's dominated convergence theorem, it follows that $g(t,\cdot,x)$ converges to $u(\cdot)$ in distribution as $t \to \infty$, that is, for any $x > 0$,

$$\mathbb{P}g(t,\cdot,x)^{-1}(\cdot) \xrightarrow{w} \mu_g^\sigma(\cdot), \text{ as } t \to \infty. \tag{36}$$

It is easy to see that any open subset $G \subset \mathbb{R}_+$ can be decomposed into a union of countable disjoint open intervals $\{I_i \mid i = 1, 2, \cdots\}$ in $\mathbb{R}_+$. Since $\mu_g^\sigma(\cdot)$ has continuous density function (30), and since $\mu_g^\sigma(\partial I_i) = 0$ for each i, we have $\mu_g^\sigma(\partial G) = 0$. The Portmanteau theorem (see Theorem 2.2(v)) implies that (35) holds for any open subset G. Therefore, (35) holds for any Borel subset $A \in \mathcal{B}(\mathbb{R}_+)$ by [24, Proposition 1.4.3, p.7] and the discussion in [24, p.6]. Moreover, it follows from the Chebyshev inequality and (31) that $\{\mu_g^\sigma : \sigma > 0\}$ is tight. □

Lemma 4.3. *For the stochastic logistic equation* (24)*, there is a* $\theta-$*invariant set* Ω^* *of full measure such that for all* $x > 0$ *and* $\omega \in \Omega^*$*, we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(s,\omega,x)ds = 1, \textit{ and} \tag{37}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(s,\theta_{-t}\omega,x)ds = 1. \tag{38}$$

Proof. Recalling the Strong Law of Large Number for Brownian motion (see [30, p.104]), we have

$$\lim_{|t|\to\infty} \frac{B_t(\omega)}{t} = 0, \quad \text{a.s.} \tag{39}$$

Let $\Omega^* = \{\omega \in \Omega | (39) \text{ holds}\}$. Then Ω^* is a $\theta-$invariant set of full measure. In fact, for any $\omega \in \Omega^*$ and $s \in \mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{B_t(\theta_s\omega)}{t} = \lim_{|t|\to\infty} \frac{B_{t+s}(\omega) - B_s(\omega)}{t} = 0.$$

Therefore, Ω^* is a $\theta-$invariant, and $\mathbb{P}(\Omega^*) = 1$ follows from (39).

Now fixing $x > 0$ and $\omega \in \Omega^*$, and applying (28) and (39), we get

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \int_0^t g(s,\omega,x)ds = & \lim_{t\to\infty} \frac{1}{rt} \ln\{1 + rx \int_0^t \exp(rs + \sigma B_s(\omega))\}ds \\ = & \lim_{t\to\infty} \frac{1}{rt} \ln \int_0^t \exp(rs + \sigma B_s(\omega))ds \\ = & 1. \end{aligned}$$

(37) is proved. Similarly,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \int_0^t g(s,\theta_{-t}\omega,x)ds = & \lim_{t\to\infty} \frac{1}{rt} \ln\{1 + rx \int_0^t \exp(rs + \sigma B_s(\theta_{-t}\omega))\}ds \\ = & \lim_{t\to\infty} \frac{1}{rt} \ln \int_0^t \exp(rs + \sigma B_s(\theta_{-t}\omega))ds \\ = & \lim_{t\to\infty} \tfrac{1}{rt} \ln \textstyle\int_{-t}^0 \exp(r(s+t) + \sigma B_s(\omega))ds = 1. \end{aligned}$$

□

Theorem 4.1. (θ, Φ) *is a local random dynamical system. If the domain of* $\Psi(\cdot,\cdot)$ *is* $[0,\infty) \times \mathbb{R}^n_+$*, then* (θ, Φ) *is a global random dynamical system.*

Proof. Applying [21, Theorem 2.3.36], we know that (θ, Φ) is a local random dynamical system. Suppose that the domain of $\Psi(\cdot,\cdot)$ is $[0,\infty) \times \mathbb{R}^n_+$. Then by (37) and the stochastic decomposition formula (25), the forward explosion time of the orbit $\Phi(t,\omega,y)$ is infinity for any $y \in \mathbb{R}^n_+$.

Denote by $v(\omega)$ the random variable in $\mathbb{R}$ such that $v(t,\omega) := v(\theta_t\omega)$ is the *Stationary Ornstein-Uhlenbeck Process* in $\mathbb{R}$ which solves the OU-equation

$$dv = -rvdt + \sigma dB_t, \quad r > 0.$$

Let us introduce a conjugate transformation $T(\omega,\cdot) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ given by

$$T(\omega, y) = y\exp\{-v(\omega)\}, \quad \omega \in \Omega.$$

If we apply the Itô formula to the function $z_i(t,\omega) = \Phi_i(t,\omega,y)\exp\{-v(\theta_t\omega)\}$, then we find that $z(t,\omega) := (z_1(t,\omega), z_2(t,\omega), \cdots, z_n(t,\omega))$ $(t > 0,\ \omega \in \Omega)$ satisfies

$$\frac{dz_i}{dt} = z_i\Big(r(1 + v(\theta_t\omega)) + \exp\{v(\theta_t\omega)\} \sum_{j=1}^n a_{ij} z_j\Big). \tag{40}$$

The RDE (40) generates an RDS (θ, Z), where $Z(t,\omega,z), (\ t > 0, \omega \in \Omega, z \in \mathbb{R}^n_+)$ is the global solution to the system (40) with the initial date $z \in \mathbb{R}^n_+$. Thus, define $\Phi(t,\omega,y) := \exp\{v(\theta_t\omega)\}Z(t,\omega,y\exp\{-v(\omega)\})(\ t > 0,\ \omega \in \Omega,\ y \in \mathbb{R}^n_+)$. Then (θ, Φ) is a global random dynamical system. □

For any $y \in \mathbb{R}_+^n$, let $L(y) := \{\lambda y : \lambda \geq 0\}$ denote the ray joining the origin and y. P is called an equilibrium of (23) if $F(P) = O$, an equilibrium $P = (p_1, \cdots, p_n)$ is said to be positive if $p_i > 0$ for $i = 1, 2, ..., n$. Set by $\omega_F(z)$ the $\omega-$limit set of the trajectory $\Psi(t, z)$. Then we define

$$\mathcal{A}(\omega_F(z)) := \{y \in \mathbb{R}_+^n : \lim_{t\to\infty} \mathrm{dist}(\Psi(t, y), \omega_F(z)) = 0\}$$

to be the attracting domain of $\omega_F(z)$. Let $\mathcal{E}$ and Γ denote the equilibria set and a closed orbit of (23), respectively. Then $\mathcal{A}(P)$ ($P \in \mathcal{E}$) and $\mathcal{A}(\Gamma)$ are the attracting domains of the equilibrium P and the closed orbit Γ, respectively. A subset $S \subset \mathbb{R}_+^n$ is called positively invariant (invariant) set of (23) if $\Psi(t, S) \subset (=)S$ for each $t \geq 0$.

Theorem 4.2 (Cone Invariance). *Let $S \subset \mathbb{R}_+^n$ be a positively invariant set of* (23). *Then the cone set*

$$\Lambda(S) := \{\lambda y : \text{for any } \lambda \geq 0 \text{ and } y \in S\}$$

is positively invariant in the sense that

$$\Phi(t, \omega, \lambda y) \in \Lambda(S) \text{ whenever } \lambda \geq 0, \quad y \in S, \ t > 0, \text{ and } \omega \in \Omega.$$

Similar result holds for pull-back trajectory.

Proof. Take $\lambda \geq 0$, $y \in S$, and $t > 0$. Then it follows from the stochastic decomposition formula (25) that

$$\Phi(t, \omega, \lambda y) = g(t, \omega, \lambda)\Psi\Big(\int_0^t g(s, \omega, \lambda)ds, y\Big).$$

Thus, the conclusion is implied by the positive invariance of S and the definition of $\Lambda(S)$. $\square$

Now we state the main result of this section, which says that the pull-back omega limit set of a trajectory of (22) is the omega limit set of the trajectory of (23) with the same initial date multiplied by the random equilibrium of (24).

Theorem 4.3. *Suppose that $\Psi(t, y)$ is a bounded solution to equations* (23). *Then the pull-back omega limit set $\Gamma_y(\omega)$ of the trajectory $\Phi(t, \theta_{-t}\omega, y)$ emanating from y is $u(\omega)\omega_F(y)$, whose attracting domain is $\mathcal{A}(\omega_F(y))$.*

Proof. For a given $y \in \mathbb{R}_+^n$, $z \in \omega_F(y)$, the stochastic decomposition formula (25) implies that

$$\begin{aligned}
& \Phi(t, \theta_{-t}\omega, y) - u(\omega)z \\
= \ & g(t, \theta_{-t}\omega, 1)\Psi(\textstyle\int_0^t g(s, \theta_{-t}\omega, 1)ds, y) - u(\omega)z \\
= \ & (g(t, \theta_{-t}\omega, 1) - u(\omega))\Psi(\textstyle\int_0^t g(s, \theta_{-t}\omega, 1)ds, y) \\
& \quad +u(\omega)(\Psi(\textstyle\int_0^t g(s, \theta_{-t}\omega, 1)ds, y) - z),
\end{aligned}$$

which deduces that

$$\begin{aligned}
& \mathrm{dist}(\Phi(t,\theta_{-t}\omega,y),u(\omega)\omega_F(y)) \\
\le\ & |g(t,\theta_{-t}\omega,1)-u(\omega)|\|\Psi(\textstyle\int_0^t g(s,\theta_{-t}\omega,1)ds,y)\| \\
& +u(\omega)\mathrm{dist}(\Psi(\textstyle\int_0^t g(s,\theta_{-t}\omega,1)ds,y),\omega_F(y)) \\
\to\ & 0 \text{ as } t\to\infty
\end{aligned}$$

by Lemma 4.1, (38) and the boundedness of the trajectory $\Psi(\cdot,y)$. This proves

$$\Gamma_y(\omega)\subset u(\omega)\omega_F(y) \text{ for any } \omega\in\Omega.$$

Suppose that $z^*\in\omega_F(y)$. From (38), there exists a sequence of $\{t_n\}$ tending to infinity such that

$$\lim_{n\to\infty}\Psi(\int_0^{t_n} g(s,\theta_{-t_n}\omega,1)ds,y)=z^*.$$

By the stochastic decomposition formula (25), we have

$$\Phi(t_n,\theta_{-t_n}\omega,y)=g(t_n,\theta_{-t_n}\omega,1)\Psi\Big(\int_0^{t_n} g(s,\theta_{-t_n}\omega,1)ds,y\Big). \tag{41}$$

Letting $n\to\infty$ in (41) and using Lemma 4.1, we get that $u(\omega)z^*\in\Gamma_y(\omega)$. In other words, $\Gamma_y(\omega)=u(\omega)\omega_F(y)$.

Let p be in the attracting domain of $\Gamma_y(\omega)$, that is,

$$\lim_{t\to\infty}\mathrm{dist}(\Phi(t,\theta_{-t}\omega,p),u(\omega)\omega_F(y))=0.$$

This means that the trajectory $\Psi(\cdot,p)$ must be bounded. Applying the result proved above, we have $\Gamma_p(\omega)=u(\omega)\omega_F(p)$. Because the pull-back trajectory $\Phi(t,\theta_{-t}\omega,p)$ is attracted by $u(\omega)\omega_F(y)$, $u(\omega)\omega_F(p)\subset u(\omega)\omega_F(y)$, in other words, $\omega_F(p)\subset\omega_F(y)$. The proof is complete. □

5. Stationary Measures, Weak Convergence and Ergodicity

Throughout the rest of the paper, we will assume without further notice that the domain of $\Psi(\cdot,\cdot)$ is $[0,\infty)\times\mathbb{R}^n_+$.

In this section, we will investigate stationary measures and their supports, weak convergence of the transition probability function, and ergodicity of the solutions of (22).

Theorem 5.1 (The Existence of Stationary Solution by Equilibrium)**.** *Suppose that $P\in\mathcal{E}$ is a positive equilibrium of* (23)*. Then the system* (22) *always has stationary solution $U(\omega):=u(\omega)P$, whose support is the ray $L(P)$ and distribution function is*

$$F_P^\sigma(y)=\int_0^{\min\{\frac{y_1}{p_1},\frac{y_2}{p_2},\dots,\frac{y_n}{p_n}\}} p^\sigma(s)ds. \tag{42}$$

Let μ_P^σ denote the probability measure decided by the distribution function F_P^σ. Then for any $A \in \mathcal{B}(\mathbb{R}_+^n)$,

$$\mu_P^\sigma(A) = \mathbb{P}(U \in A) \tag{43}$$

defines a stationary measure of the Markov semigroup $\{P_t, t \geq 0\}$.

Proof. The stochastic decomposition formula (33) implies that the system (22) always has a stationary solution $U(\theta_t\omega) := u(\theta_t\omega)P$, whose support is obviously the ray $L(P)$. Let F_P^σ denote the distribution function of $U(\theta_t\omega)$. Then, by the θ-invariant property with respect to $\mathbb{P}$,

$$\begin{aligned} F_P^\sigma(y) &= \mathbb{P}\{\omega : u(\omega)p_i \leq y_i,\ i = 1, 2, \cdots n\} \\ &= \mathbb{P}\{\omega : u(\omega) \leq \min\{\tfrac{y_1}{p_1}, \tfrac{y_2}{p_2}, ..., \tfrac{y_n}{p_n}\}\} \\ &= \int_0^{\min\{\frac{y_1}{p_1}, \frac{y_2}{p_2}, ..., \frac{y_n}{p_n}\}} p^\sigma(s)ds. \end{aligned}$$

The expressions (42) and (43) are immediate.

In order to prove that $\mu_P^\sigma(\cdot)$ is stationary with respect to $\{P_t, t \geq 0\}$, we need to show that $\mu_P^\sigma(\cdot)$ satisfies (15), that is,

$$P_t\mu_P^\sigma(A) = \int_{\mathbb{R}_+^n} P(t, y, A)\mu_P^\sigma(dy) = \mu_P^\sigma(A) \tag{44}$$

for any $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}_+^n)$.

According to Arnold [21, p.107], the future $\mathcal{F}_+$ and the past $\mathcal{F}_-$ σ-algebras for RDS (θ, Φ) are defined by

$$\mathcal{F}_+ = \sigma\{B_t(\omega) : t \geq 0\}$$

and

$$\mathcal{F}_- = \sigma\{B_t(\omega) : t \leq 0\},$$

respectively. It is easy to see that $\mathcal{F}_+$ and $\mathcal{F}_-$ are independent and

$$u(\omega) = \left(r\int_{-\infty}^0 \exp\{rs + \sigma B_s(\omega)\}ds\right)^{-1}$$

is $\mathcal{F}_-$−measurable. This implies that $U(\omega)$ is $\mathcal{F}_-$−measurable. We denote by $I_A(\cdot)$ the characteristic function of a set $A \in \mathcal{B}(\mathbb{R}_+^n)$. Then by Lemma 2.1 with $h(y, \omega) = I_A(\Phi(t, \omega, y))$, $\mathcal{C} = \mathcal{F}_-$ and $\xi = U$, it yields that for each $t > 0$, a.s.

$$\begin{aligned} & \mathbb{E}[I_A(U(\theta_t\omega))|\mathcal{F}_-] \\ = \ & \mathbb{E}[I_A(\Phi(t, \omega, U(\omega)))|\mathcal{F}_-] \\ = \ & \mathbb{E}[I_A(\Phi(t, \cdot, y))]|_{y=U(\omega)} \end{aligned}$$

where we have used the fact that $I_A(\Phi(t, \omega, y))$ is $\mathcal{F}_+$−measurable for each $y \in \mathbb{R}_+^n$. Therefore,

$$\begin{aligned} \mu_P^\sigma(A) &= \mathbb{E}[I_A(U(\omega))] \\ &= \mathbb{E}[I_A(U(\theta_t\omega))] \\ &= \mathbb{E}[\mathbb{E}[I_A(\Phi(t, \cdot, y))]|_{y=U(\omega)}] \\ &= \mathbb{E}[P(t, y, A)]|_{y=U(\omega)}] \\ &= \int_{\mathbb{R}_+^n} P(t, y, A)\mu_P^\sigma(dy), \end{aligned}$$

that is, (44) holds. This completes the proof. □

Remark 5.1. *The above proof of μ_P^σ being stationary is probabilistic. Instead, we can give a dynamical proof, which is presented in the following.*

The random equilibrium $U(\omega)$ of the random dynamical system $\Theta := (\theta, \Phi)$ generates an invariant measure μ, whose factorization μ_ω is a random Dirac measure, i.e., $\mu_\omega = \delta_{U(\omega)}$. It is easy to see that $\mu_\omega(\cdot)$ is $\mathcal{F}_-$−measurable. Hence, $\mathbb{E}[\mu_\cdot|\mathcal{F}_+] = \mathbb{E}\mu_\cdot = \mu_P^\sigma$. [21, Theorem 2.3.45, p.107] asserts that $\mathbb{P} \times \mu_P^\sigma$ is an invariant measure for Θ. Therefore, for any $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}_+^n)$, we have

$$\begin{aligned} & \textstyle\int_{\mathbb{R}_+^n} P(t,y,A)\mu_P^\sigma(dy) \\ = & \textstyle\int_{\mathbb{R}_+^n}\int_\Omega I_A(\Phi(t,\omega,y))\mathbb{P}(d\omega)\mu_P^\sigma(dy) \\ = & \textstyle\int_\Omega I_\Omega(\theta_t\omega)\int_{\mathbb{R}_+^n} I_A(\Phi(t,\omega,y))\mu_P^\sigma(dy)\mathbb{P}(d\omega) \\ = & \textstyle\int_{\Omega\times\mathbb{R}_+^n} I_{\Omega\times A}(\theta_t\omega, \Phi(t,\omega,y))\mathbb{P}\times\mu_P^\sigma(d\omega,dy) \\ = & \textstyle\int_{\Omega\times\mathbb{R}_+^n} I_{\Omega\times A}(\omega,y)\mathbb{P}\times\mu_P^\sigma(d\omega,dy) \\ = & \mathbb{P}\times\mu_P^\sigma(\Omega\times A) = \mu_P^\sigma(A), \end{aligned}$$

in the fourth equality, we have used the invariance of $\mathbb{P} \times \mu_P^\sigma$ with respect to RDS Θ. *This shows that μ_P^σ is stationary.*

Remark 5.2. *A stationary measure μ of a system of stochastic ordinary differential equations is called regular if it admits a continuous density function v with respect to the Lebesgue measure, i.e., $d\mu(x) = v(x)dx$. We claim that μ_P^σ is not regular.*

Otherwise, assume that the density v is continuous. Let $W = \{y = (y_1, \cdots, y_n) \in \mathrm{Int}\mathbb{R}_+^n : \frac{y_i}{p_i} \neq \frac{y_j}{p_j}, i \neq j, i,j = 1, \cdots, n\}$, which is an open dense subset in $\mathbb{R}_+^n$. Then it follows that

$$\int_0^{y_1}\int_0^{y_2}...\int_0^{y_n} v(s_1,s_2,...,s_n)ds_1ds_2...ds_n = \int_0^{\frac{y_k}{p_k}} p^\sigma(s)ds \ , \text{ for some } k.$$

Differentiating on both sides of the above equation, we obtain that $v(y_1, y_2, ..., y_n) = 0$ on W. Together with the continuity of v, we have $v \equiv 0$ on $\mathbb{R}_+^n$. This implies that $\mu_P^\sigma = 0$, which is impossible from (43).

Observing from (42), we have

$$F(sP) = \int_0^s p^\sigma(\tau)d\tau,$$

which is defined as the distribution function of μ_P^σ on the ray $L(P)$, its density function is p^σ.

Theorem 5.2 (Weak Convergence and Strong Mixing). (i) $\mu_P^\sigma(\cdot) \xrightarrow{w} \delta_P(\cdot)$ *as* $\sigma \to 0$.

(ii) *Suppose that P is a globally asymptotically stable equilibrium of* (23) *in* $\mathrm{Int}\mathbb{R}_+^n$. *Then for each $y \in \mathrm{Int}\mathbb{R}_+^n$, $P(t,y,\cdot) \to \mu_P^\sigma(\cdot)$ weakly as $t \to \infty$, μ_P^σ is the unique stationary measure with respect to the Markovian semigroup P_t in* $\mathrm{Int}\mathbb{R}_+^n$, *it is still strongly mixing on* $\mathbb{R}_+^n$.

Proof. (i) In order to prove $\mu_P^\sigma(\cdot) \to \delta_P(\cdot)$ weakly as $\sigma \to 0$, we only need to verify $\mu_P^\sigma(\cdot) \to \delta_P(\cdot)$ in vague topology as $\sigma \to 0$ because $\mu_P^\sigma(\cdot)$, $\delta_P(\cdot)$ are all probability measures. Equivalently, for arbitrary $f \in C_c(\mathbb{R}_+^n)$, we need to prove

$$\lim_{\sigma \to 0} \int_{\mathbb{R}_+^n} f(y)\mu_P^\sigma(dy) = f(P), \tag{45}$$

where $C_c(\mathbb{R}_+^n)$ denotes the set of all continuous functions with compact supports in $\mathbb{R}_+^n$. In particular, $\lim_{y\to\infty} f(y) = 0$ for any $f \in C_c(\mathbb{R}_+^n)$.

For any nonnegative integers $m_1, m_2, \cdots, m_n$, let $\widetilde{f}(y) = \exp(-\sum_{i=1}^n m_i y_i)$ and $\alpha = \frac{2r}{\sigma^2}$. Then

$$\begin{aligned} \int_{\mathbb{R}_+^n} \widetilde{f}(y)\mu_P^\sigma(dy) = &\ \int_{\mathbb{R}_+^n} \exp(-\textstyle\sum_{i=1}^n m_i y_i)\mu_P^\sigma(dy) \\ = &\ \int_{L(P)} \exp(-\textstyle\sum_{i=1}^n m_i y_i)\mu_P^\sigma(dy) \\ = &\ \int_0^\infty \exp(-s\textstyle\sum_{i=1}^n m_i p_i)p^\sigma(s)ds \\ = &\ \int_0^\infty \frac{\alpha^\alpha}{\Gamma(\alpha)} s^{\alpha-1}\exp\{-(\textstyle\sum_{i=1}^n m_i p_i + \alpha)s\}ds \\ = &\ \frac{\alpha^\alpha}{(m_1p_1+\cdots+m_np_n+\alpha)^\alpha} \\ \to &\ \exp(-\textstyle\sum_{i=1}^n m_i p_i) = \widetilde{f}(P) \end{aligned}$$

as $\sigma \to 0$. This shows (45) holds for such exponential functions, hence it still holds for linear combination for these exponential functions.

For any $f \in C_c(\mathbb{R}_+^n)$, take the transformation $t_i = e^{-y_i}$, $i = 1, 2, \cdots, n$. Then $t_i \in (0, 1], i = 1, 2, \cdots, n$ and

$$g(t_1, \cdots, t_n) := f(-\ln t_1, \cdots, -\ln t_n)$$

is continuous on $(0, 1]^n$. By the assumption on f, $\lim_{t_i \to 0} g(t_1, \cdots, t_n) = 0$. Define $g(t_1, \cdots, t_n) = 0$ if there is at least an i with $t_i = 0$. Then g is continuous on $[0, 1]^n$. By the Weierstrass Theorem, for any $\epsilon > 0$, there is a polynomial P_m on $[0, 1]^n$ such that

$$\max_{[0,1]^n} |g(t_1, \cdots, t_n) - P_m(t_1, \cdots, t_n)| < \frac{\epsilon}{3}.$$

In particular,

$$|P_m(e^{-p_1}, \cdots, e^{-p_n}) - f(P)| < \frac{\epsilon}{3}.$$

The last paragraph has shown that there is a σ_0 such that as $|\sigma| < \sigma_0$,

$$\Big| \int_{\mathbb{R}_+^n} P_m(e^{-y_1}, e^{-y_2}, \cdots, e^{-y_n})\mu_P^\sigma(dy) - P_m(e^{-p_1}, e^{-p_2}, \cdots, e^{-p_n}) \Big| < \frac{\epsilon}{3}.$$

Thus, when $|\sigma| < \sigma_0$,

$$\begin{aligned} &\ |\int_{\mathbb{R}_+^n} f(y)\mu_P^\sigma(dy) - f(P)| \\ \le &\ |\int_{\mathbb{R}_+^n} \Big(f(y) - P_m(e^{-y_1}, e^{-y_2}, \cdots, e^{-y_n})\Big)\mu_P^\sigma(dy)| \\ &\ +|\int_{\mathbb{R}_+^n} P_m(e^{-y_1}, e^{-y_2}, \cdots, e^{-y_n})\mu_P^\sigma(dy) - P_m(e^{-p_1}, e^{-p_2}, \cdots, e^{-p_n})| \\ &\ +|P_m(e^{-p_1}, e^{-p_2}, \cdots, e^{-p_n}) - f(P)| \\ < &\ \epsilon. \end{aligned}$$

This proves (45).

(ii) Assume that P is a globally asymptotically stable equilibrium of (23) in $\mathrm{Int}\mathbb{R}_+^n$. Let f be a bounded continuous function on $\mathbb{R}_+^n$ and fix $y \in \mathrm{Int}\mathbb{R}_+^n$,

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathbb{R}_+^n} f(z)P(t,y,dz) =& \ \lim_{t\to\infty}\textstyle\int_\Omega f(\Phi(t,\omega,y))\mathbb{P}(d\omega)\\ =& \ \lim_{t\to\infty}\textstyle\int_\Omega f(\Phi(t,\theta_{-t}\omega,y))\mathbb{P}(d\omega)\\ =& \ \textstyle\int_\Omega f(u(\omega)P)\mathbb{P}(d\omega)\\ =& \ \textstyle\int_{\mathbb{R}_+^n} f(z)\mu_P^\sigma(dz),\end{aligned}$$

by the assumption of (ii), the Lebesgue dominated convergence theorem and Theorem 4.3. This deduces that $P(t,y,\cdot) \xrightarrow{w} \mu_P^\sigma(\cdot)$ as $t\to\infty$. It follows from Theorem 2.4 that μ_P^σ is the unique stationary measure for the Markov semigroup P_t in $\mathrm{Int}\mathbb{R}_+^n$.

Since for each $y \in \mathrm{Int}\mathbb{R}_+^n$, $P(t,y,\cdot)\to\mu_P^\sigma(\cdot)$ weakly as $t\to\infty$, Theorem 2.4 implies that the restriction of μ_P^σ to $\mathrm{Int}\mathbb{R}_+^n$ is strongly mixing. It follows from (17) that for arbitrary $\varphi \in L^2(\mathrm{Int}\mathbb{R}_+^n, \mu_P^\sigma)$ we have

$$\lim_{t\to\infty} P_t\varphi = \langle\varphi, 1\rangle, \text{ in } L^2(\mathrm{Int}\mathbb{R}_+^n, \mu_P^\sigma). \tag{46}$$

Since $P(t,y,\partial\mathbb{R}_+^n) = \mu_P^\sigma(\partial\mathbb{R}_+^n) = 0$ for $y\in\mathrm{Int}\mathbb{R}_+^n$ and $t>0$, it follows from (46) that for arbitrary $\varphi\in L^2(\mathbb{R}_+^n,\mu_P^\sigma)$,

$$\lim_{t\to\infty} P_t\varphi = \langle\varphi, 1\rangle, \text{ in } L^2(\mathbb{R}_+^n, \mu_P^\sigma).$$

Applying Theorem 2.3, we know that μ_P^σ is strongly mixing on $\mathbb{R}_+^n$. This completes the proof. □

In the case that P is a nontrivial boundary equilibrium of (23), Theorems 5.1 and 5.2 still hold, which are stated as follows and can be proved by the same arguments.

Theorem 5.3. *Suppose that P is any nonzero equilibrium of* (23). *Then the system* (22) *always has stationary solution $U(\omega) := u(\omega)P$, whose support is the ray $L(P)$ and distribution function is*

$$F_P^\sigma(y) = \int_0^{\min\{\frac{y_i}{p_i}:\ p_i\neq 0\}} p^\sigma(s)ds, \quad y\in\mathbb{R}_+^n. \tag{47}$$

Let μ_P^σ denote the probability measure decided by the distribution function F_P^σ. Then for any $A\in\mathcal{B}(\mathbb{R}_+^n)$,

$$\mu_P^\sigma(A) = \mathbb{P}(U\in A) \tag{48}$$

defines a stationary measure, and $\mu_P^\sigma(\cdot)\xrightarrow{w}\delta_P(\cdot)$ as $\sigma\to 0$.

In addition, for each $y\in\mathcal{A}(P)$, $P(t,y,\cdot)\to\mu_P^\sigma(\cdot)$ weakly as $t\to\infty$, μ_P^σ is the unique stationary measure with respect to the Markov semigroup P_t in $\mathcal{A}(P)$, and it is still strongly mixing on $\mathbb{R}_+^n$.

Theorems 5.1, 5.2 and 5.3 help us to provide examples having a continuum of stationary motions, which comes from a continuum of equilibria of deterministic systems.

Example 5.1. Consider three-dimensional stochastic competitive LV system:

$$\begin{aligned} dy_1 &= y_1(1 - y_1 - y_2 - y_3)dt + \sigma y_1 \circ dB_t, \\ dy_2 &= y_2(1 - y_1 - y_2 - y_3)dt + \sigma y_2 \circ dB_t, \\ dy_3 &= y_3(1 - y_1 - y_2 - y_3)dt + \sigma y_3 \circ dB_t. \end{aligned} \tag{49}$$

The standard simplex $\Delta := \{(y_1, y_2, y_3) : y_1 + y_2 + y_3 = 1, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}$ is the nonzero equilibria set of the corresponding system without noise. Strongly mixing stationary motions of (49) are $\{u(\omega)P : P \in \Delta \bigcup \{O\}\}$.

Example 5.2. Consider three-dimensional stochastic competitive LV system:

$$\begin{aligned} dy_1 &= y_1(1 - 2y_1 - y_2 - y_3)dt + \sigma y_1 \circ dB_t, \\ dy_2 &= y_2(1 - y_1 - 2y_2 - y_3)dt + \sigma y_2 \circ dB_t, \\ dy_3 &= y_3(1 - \frac{3}{2}y_1 - \frac{3}{2}y_2 - y_3)dt + \sigma y_3 \circ dB_t. \end{aligned} \tag{50}$$

The nonzero equilibria set of the corresponding system without noise is

$$\mathcal{E} = \{(\alpha, \alpha, 1 - 3\alpha) : 0 \leq \alpha \leq \frac{1}{3}\}.$$

Strongly mixing stationary motions of (50) are $\{u(\omega)P : P \in \mathcal{E}\}$.

The ergodic stationary measures discussed above originate from equilibria of system (23) via the decomposition formula (25). We will investigate other types of stationary measures coming from nontrivial omega limit sets of (23).

Theorem 5.4 (The Existence of Stationary Solution by Limit Set)**.** *Suppose that* $\Psi(t, y)$ *is a bounded trajectory of* (23) *for* $y \in \mathbb{R}^n_+$. *Then the system* (22) *admits a stationary measure. Furthermore, if the origin* O *is a repeller and initial value* $y \neq O$, *then this stationary measure is not the Dirac measure at the origin.*

Proof. Since the trajectory of $\Psi(t, y)$ of (23) is bounded, there exists a positive constant N such that

$$\|\Psi(t, y)\| \leq N \text{ for all } t > 0. \tag{51}$$

By Theorem 2.1 and Chebyshev inequality, to prove the existence of stationary measure of system (22), it will suffice to prove that there exists a constant M such that

$$\mathbb{E}\|\Phi(t, \omega, y)\|^2 \leq M \text{ for all } t \geq 0. \tag{52}$$

It follows from the stochastic decomposition formula (25) that

$$\Phi(t, \omega, y) = g(t, \omega, 1)\Psi(\int_0^t g(s, \omega, 1)ds, y).$$

Therefore, $\|\Phi(t,\omega,y)\| \le Ng(t,\omega,1)$. We only need to prove $\mathbb{E}|g(t,\omega,1)|^2$ is bounded. Applying the Itô formula to (27), we obtain that

$$g^2(t,\omega,1) = 1 + 2(r+\sigma^2)\int_0^t g^2(s,\omega,1)ds - 2r\int_0^t g^3(s,\omega,1)ds + 2\sigma\int_0^t g^2(s,\omega,1)dB_s.$$

Taking the mathematical expectation in the two sides of the above equation and utilizing the Fubini theorem, we have

$$\mathbb{E}g^2(t,\omega,1) = 1 + 2(r+\sigma^2)\int_0^t \mathbb{E}g^2(s,\omega,1)ds - 2r\int_0^t \mathbb{E}g^3(s,\omega,1)ds.$$

Differentiating the above equality, we get that

$$\frac{d}{dt}\mathbb{E}g^2(t,\omega,1) = 2(r+\sigma^2)\mathbb{E}g^2(t,\omega,1) - 2r\mathbb{E}g^3(t,\omega,1).$$

By the Hölder inequality, we have $\mathbb{E}g^2(t,\omega,1) \le (\mathbb{E}g^3(t,\omega,1))^{\frac{2}{3}}$, which deduces that

$$\frac{d}{dt}\mathbb{E}g^2(t,\omega,1) \le 2(r+\sigma^2)\mathbb{E}g^2(t,\omega,1) - 2r(\mathbb{E}g^2(t,\omega,1))^{\frac{3}{2}}.$$

The differential inequality theory implies that

$$\mathbb{E}g^2(t,\omega,1) \le (1+\frac{\sigma^2}{r})^2. \tag{53}$$

This shows that (52) holds. Theorem 2.1 asserts that there exists a stationary measure ν_y^σ of the Markovian semigroup $\{P_t, t \ge 0\}$.

If in addition the origin O is a repeller, then there is a positive constant $k > 0$ such that

$$k \le \|\Psi(t,y)\| \text{ for all } t > 0. \tag{54}$$

Recall the proof of Theorem 2.1 (see [9, p.66]), there is a sequence $\{T_n\}$ tending to infinity such that the sequence of expected occupation measures:

$$P^{(T_n)}(y,\cdot) := \frac{1}{T_n}\int_0^{T_n} P(s,y,\cdot)ds \tag{55}$$

converges weakly to the stationary measure ν_y^σ. Finally, we will prove that ν_y^σ is not the Dirac measure at the origin.

In fact, let $B_R := \{y \in \mathbb{R}_+^n : \|y\| < R\}$ denote the open ball in $\mathbb{R}_+^n$ with the center at the origin and radius R. Then it follows from the fact $P^{(T_n)}(y,\cdot) \xrightarrow{w} \nu_y^\sigma$ and Theorem 2.2 (iv) that

$$\nu_y^\sigma(B_R) \le \liminf_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} P(t,y,B_R)dt, \tag{56}$$

where $P(t,y,B_R) = \mathbb{P}(\|\Phi(t,\omega,y)\| < R) \le \mathbb{P}(g(t,\omega,1) < \frac{R}{k})$ by the stochastic decomposition formula (25) and (54). Using (35), we have

$$\lim_{t\to\infty} \mathbb{P}(g(t,\omega,1) < \frac{R}{k}) = \int_0^{\frac{R}{k}} p^\sigma(s)ds.$$

Thus, we obtain that

$$\nu_y^\sigma(B_R) \le \int_0^{\frac{R}{k}} p^\sigma(s)ds. \tag{57}$$

As a result, $\nu_y^\sigma(\{O\}) = \lim_{R\to 0} \nu_y^\sigma(B_R) = 0$, in other words, ν_y^σ is not the Dirac measure at the origin. □

Remark 5.3. *Note that the pull-back omega limit set $\Gamma_y(\omega)$ of the trajectory $\Phi(t,\theta_{-t}\omega,y)$ is $u(\omega)\omega_F(y)$ from Theorem 4.3, which implies that the difference between $\Phi(t,\omega,y)$ and $u(\theta_t\omega)\omega_F(y)$ converges to zero in probability as $t\to\infty$. This is the evidence to encourage us to conjecture that the support of ν_y^σ is contained in the cone $\Lambda(\omega_F(y))$. The following assertion shows that this is true.*

Theorem 5.5 (The Support of Stationary Measure)**.** *Suppose that $\Psi(t,y)$ is a bounded trajectory of* (23) *with $y \neq O$. Then the support of the stationary measure ν_y^σ is contained in the cone $\Lambda(\omega_F(y))$.*

Proof. For $y \in \mathbb{R}_+^n$ with $y \neq O$, assume that ν_y^σ is a limit point of expected occupation measure family $\{\frac{1}{T}\int_0^T P(t,y,\cdot)dt : T > 0\}$ for $T \to \infty$ in the topology of weak convergence. We shall prove that

$$\nu_y^\sigma\Big(\Lambda(\omega_F(y))\Big) = 1. \tag{58}$$

In order to prove (58), it suffices to show that

$$\nu_y^\sigma(\Lambda(U_\epsilon^c)) = 0 \quad \text{for} \quad 0 < \epsilon \ll 1 \tag{59}$$

where $U_\epsilon(\omega_F(y)) := \{x \in \mathbb{R}_+^n : \mathrm{dist}(x,\omega_F(y)) \le \epsilon\}$ and U_ϵ^c denotes its complement.

It follows from Theorem 4.3 that

$$\lim_{t\to\infty} \mathrm{dist}(\Phi(t,\theta_{-t}\omega,y),\Lambda(\omega_F(y)) = 0,$$

which means that for any fixed $\omega \in \Omega$, $\Phi(t,\theta_{-t}\omega,y) \in \Lambda(U_\epsilon) := \Lambda(U_\epsilon(\omega_F(y)))$ for sufficiently large t. Applying the Fatou Lemma, we have

$$\limsup_{t\to\infty} P(t,y,\Lambda(U_\epsilon^c)) = \limsup_{t\to\infty} \mathbb{E} I_{\Lambda(U_\epsilon^c)}(\Phi(t,\theta_{-t}\omega,y)) \le \mathbb{E}\limsup_{t\to\infty} I_{\Lambda(U_\epsilon^c)}(\Phi(t,\theta_{-t}\omega,y)) = 0.$$

Then it follows from $P^{(T_n)}(y,\cdot) \xrightarrow{w} \nu_y^\sigma$ and Theorem 2.2 (iv) that

$$\nu_y^\sigma(\Lambda(U_\epsilon^c)) \le \liminf_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} P(t,y,\Lambda(U_\epsilon^c))dt = \limsup_{t\to\infty} P(t,y,\Lambda(U_\epsilon^c)) = 0.$$

This completes the proof of (59). □

Remark 5.4. *Suppose that* $\Psi(t, y)$ *is a nontrivial periodic orbit* Γ *of* (23). *Then* $\Lambda(\Gamma)$ *is a cone surface with the origin as vertex. It follows from [20, Proposition 4.13] that* $\omega_F(x) = \Gamma$ *for all* $x \in \Lambda(\Gamma) \setminus \{O\}$*, that is,* Γ *is a global attractor when the flow* Ψ *is restricted to* $\Lambda(\Gamma) \setminus \{O\}$*. By the cone invariance,* $\Lambda(\Gamma)$ *is invariant for both* $\Phi(t, \omega, \cdot)$ *and* $\Phi(t, \theta_{-t}\omega, \cdot)$*. Applying Theorem 4.3, we know that* $u(\omega)\Gamma$ *is a global attractor for pull-back flow* $\Phi(t, \theta_{-t}\omega, \cdot)$ *restricted on* $\Lambda(\Gamma) \setminus \{O\}$*. In three dimensional stochastic competitive Lotka-Volterra system,* ν_y^σ *is a unique nontrivial stationary measure (in Theorem 7.8). We guess that* ν_y^σ *is a unique nontrivial stationary measure supported on* $\Lambda(\Gamma)$ *and* $u(\theta_t\omega)\Psi(\int_0^t u(\theta_s\omega)ds, y)$ *is just such a stationary process in this general situation, but we cannot prove it. Here we leave it an open problem. However, in the following, we are able to show that* ν_y^σ *converges weakly to the Haar measure supported on* Γ *as* $\sigma \to 0$ (*see Theorem 6.2 and Corollary 6.2*)*. A similar problem can be proposed for a quasiperiodic orbit* $\Psi(t, y)$*.*

It is easy to see that all these stationary measures are not regular.

Example 5.3. Consider the following three-dimensional prey-predator LV system:

$$\begin{aligned}\frac{dy_1}{dt} &= y_1(1 - y_1 + 2y_2 - 3y_3),\\ \frac{dy_2}{dt} &= y_2(1 - 3y_1 - y_2 + y_3),\\ \frac{dy_3}{dt} &= y_3(1 + y_1 - 4y_2 - y_3).\end{aligned} \tag{60}$$

It is easy to see that the system (60) has a unique positive equilibrium $E_0 = (\frac{3}{8}, \frac{1}{4}, \frac{3}{8})$. [20, Example 3.1] has shown that (60) admits a family of invariant cone surfaces $\Lambda(h)$:

$$\frac{y_1 y_2 y_3}{(2y_1 + 3y_2 + 2y_3)^3} \equiv h, \quad 0 < h \le \frac{1}{324}$$

on which there is no equilibrium except $h = \frac{1}{324}$. Hence, every trajectory on $\Lambda(h)$ will converge to a periodic orbit on it. These periodic orbits must lie on the center manifold of P_0, which is transversal to each $\Lambda(h)$ and intersects with $\Lambda(h)$ on the unique closed orbit $\Gamma(h)$.

Now we study the noise disturbed system:

$$\begin{aligned}dy_1 &= y_1(1 - y_1 + 2y_2 - 3y_3)dt + \sigma y_1 \circ dB_t,\\ dy_2 &= y_2(1 - 3y_1 - y_2 + y_3)dt + \sigma y_2 \circ dB_t,\\ dy_3 &= y_3(1 + y_1 - 4y_2 - y_3)dt + \sigma y_3 \circ dB_t.\end{aligned} \tag{61}$$

Applying Theorems 5.5, 5.1, and 4.3, we conclude that there exists a stationary measure ν_h^σ supported on $\Lambda(h)(0 < h \le \frac{1}{324})$ and every nontrivial pull-back trajectory on $\Lambda(h)$ tends to $u(\omega)\Gamma(h)$ as $t \to \infty$. The subsequent theorem will show that ν_h^σ converges weakly to the Haar measure on the closed orbit $\Gamma(h)$ as $\sigma \to 0$.

Example 4.3 illustrates that (61) has a family of stationary measures coming from a continuum of periodic orbits of (60). Such stationary processes are not isolated. The following gives an example to possess as least three isolated stationary processes.

Example 5.4. Consider four-dimensional white noise perturbed prey-predator Lotka-Volterra system:

$$\begin{aligned} dy_1 &= y_1(2 - \frac{3}{4}y_1 + y_2 - \frac{3}{2}y_3 - 2y_4)dt + \sigma y_1 \circ dB_t, \\ dy_2 &= y_2(2 + 3y_1 - 3y_2 - \frac{33}{2}y_3 - 4y_4)dt + \sigma y_2 \circ dB_t, \\ dy_3 &= y_3(2 + \frac{2959}{4000}y_1 - \frac{9}{2}y_3 - \frac{989}{125}y_4)dt + \sigma y_3 \circ dB_t \\ dy_4 &= y_4(2 + \frac{1}{2}y_1 - y_2 - 3y_3 - 6y_4)dt + \sigma y_4 \circ dB_t. \end{aligned} \tag{62}$$

The deterministic system without noise in each equation was investigated in [20, Example 3.2]. This deterministic system has a unique equilibrium P and at least two limit cycles Γ_1 and Γ_2. It follows from Theorems 5.4, 5.5 and 5.1 that (62) admits at least three isolated stationary measures, named by ν_1^σ, ν_2^σ, and μ_P^σ, which support on $\Lambda(\Gamma_1)$, $\Lambda(\Gamma_2)$, and $L(P)$, respectively.

In the language of dynamics, the stationary measures $\{\nu_h^\sigma\}$ in Example 4.3 are degenerate, while ν_1^σ, ν_2^σ, and μ_P^σ in Example 4.4 are hyperbolic.

6. Limit Measures of Stationary Measures and Their Supports

In this section, we will explore the weak convergence of stationary measures as the noise intensity σ tends to zero. The paper [31] has established the frame to study limiting behavior of stationary measures with small noise intensity. According to the frame, the study is divided into three steps: the first step is to prove that the solution of (22) converges in probability to the solution of (23) uniformly on any compact set as $\sigma \to 0$; the second step is to prove the tightness of stationary measures and then to show that any limiting measure is an invariant measure of the deterministic system (23); the third step is to deduce that any limiting measure is supported in the Birkhoff center of (23).

Let us start with the first step. Before that, we will present the dissipation assumption.

The system (23) is said to be *dissipative*, if there is a compact invariant set D, called the *fundamental attractor*, which uniformly attracts each compact set of initial values.

Throughout this section, we assume that system (23) is dissipative. Because we are concerned with the variation of the solution $\Phi(t, \omega, y)$ as $\sigma \to 0$, we let $\Phi^\sigma(t, \omega, y)$ denote the solution of (22) from now on, similarly for $g^\sigma(t, \omega, 1)$.

Proposition 6.1. *Let $K \subset \mathbb{R}_+^n$ be a compact set and $T > 0$ an arbitrary number. Then there is a constant C depending on K and T, such that*

$$\sup_{y \in K} \mathbb{E}[\|\Phi^\sigma(T, \omega, y) - \Psi(T, y)\|] \le C|\sigma|, \tag{63}$$

which implies that for any $\delta > 0$,

$$\lim_{\sigma\to 0}\sup_{y\in K}\mathbb{P}\{\|\Phi^\sigma(T,\omega,y)-\Psi(T,y)\|\geq\delta\}=0. \tag{64}$$

Proof. Utilizing the stochastic decomposition formula (25), we get that

$$\begin{aligned}
& \Phi^\sigma(t,\omega,y)-\Psi(t,y)\\
= & (g^\sigma(t,\omega,1)-1)\Psi(\textstyle\int_0^t g^\sigma(s,\omega,1)ds,y)+(\Psi(\int_0^t g^\sigma(s,\omega,1)ds,y)-\Psi(t,y))\\
= & (g^\sigma(t,\omega,1)-1)\Psi(\textstyle\int_0^t g^\sigma(s,\omega,1)ds,y)\\
& +\textstyle\int_0^1 F(\Psi(\lambda\int_0^t g^\sigma(s,\omega,1)ds+(1-\lambda)t,y)d\lambda\int_0^t(g^\sigma(s,\omega,1)-1)ds.
\end{aligned}$$

From the compactness of K, the dissipation of Ψ and the continuity of F, it follows that there is a constant C_0 such that

$$\mathbb{E}\|\Phi^\sigma(T,\omega,y)-\Psi(T,y)\|\leq C_0[\mathbb{E}|g^\sigma(T,\omega,1)-1|+\int_0^T\mathbb{E}|g^\sigma(t,\omega,1)-1|dt]. \tag{65}$$

By the Hölder inequality, we have for any $t\in[0,T]$,

$$\begin{aligned}
& \mathbb{E}|g^\sigma(t,\omega,1)-1|\\
= & \mathbb{E}|(g^\sigma(t,\omega,1))^{-1}-1|g^\sigma(t,\omega,1)\\
\leq & \sqrt{\mathbb{E}|(g^\sigma(t,\omega,1))^{-1}-1|^2}\sqrt{\mathbb{E}|g^\sigma(t,\omega,1)|^2}.
\end{aligned}$$

From (53) it follows that

$$\mathbb{E}|g^\sigma(t,\omega,1)-1|\leq(1+\frac{\sigma^2}{r})\sqrt{\mathbb{E}|(g^\sigma(t,\omega,1))^{-1}-1|^2}. \tag{66}$$

Let $h^\sigma(t,\omega,1):=(g^\sigma(t,\omega,1))^{-1}$. Then we need to estimate $\mathbb{E}|h^\sigma(t,\omega,1)-1|^2$.

Using (27) and the Itô formula, we derive that

$$dh_t^\sigma=[r+(\frac{\sigma^2}{2}-r)h_t^\sigma]dt-\sigma h_t^\sigma dB_t. \tag{67}$$

Applying the Itô formula to $(h_t^\sigma)^2$, and then taking the mathematical expectation in the two sides, we obtain that

$$\mathbb{E}(h_t^\sigma)^2=1+2r\int_0^t\mathbb{E}h_s^\sigma ds+2(\sigma^2-r)\int_0^t\mathbb{E}(h_s^\sigma)^2ds,$$

which implies that

$$\begin{aligned}
\tfrac{\mathbb{E}(h_t^\sigma)^2}{dt} = & 2r\mathbb{E}h_t^\sigma+2(\sigma^2-r)\mathbb{E}(h_t^\sigma)^2\\
\leq & 2\sqrt{\mathbb{E}(h_t^\sigma)^2}[r-(r-\sigma^2)\sqrt{\mathbb{E}(h_t^\sigma)^2}].
\end{aligned}$$

This shows that
$$\mathbb{E}(h_t^\sigma)^2 \le (\frac{r}{r-\sigma^2})^2. \tag{68}$$
It follows from (67) that
$$h_t^\sigma - 1 = r\int_0^t (1-h_s^\sigma)ds + \frac{\sigma^2}{2}\int_0^t h_s^\sigma ds - \sigma\int_0^t h_s^\sigma dB_s.$$
Let $T>0$ and any $t\in[0,T]$. Then
$$\begin{aligned}
&\mathbb{E}[\sup_{0\le s\le t}(h_s^\sigma-1)^2]\\
\le\ & 3\{r^2T\textstyle\int_0^t \mathbb{E}[\sup_{0\le l\le s}(h_l^\sigma-1)^2]ds + T\frac{\sigma^4}{4}\int_0^t \mathbb{E}(h_s^\sigma)^2 ds + \sigma^2\mathbb{E}[\sup_{0\le s\le t}(\int_0^s h_l^\sigma dB_l)^2]\}\\
\le\ & 3\{r^2T\textstyle\int_0^t \mathbb{E}[\sup_{0\le l\le s}(h_l^\sigma-1)^2]ds + (T\frac{\sigma^4}{4}+4\sigma^2)\int_0^t \mathbb{E}(h_s^\sigma)^2 ds\}\\
\le\ & 3\{r^2T\textstyle\int_0^t \mathbb{E}[\sup_{0\le l\le s}(h_l^\sigma-1)^2]ds + T(T\frac{\sigma^4}{4}+4\sigma^2)(\frac{r}{r-\sigma^2})^2\}.
\end{aligned}$$
Here in the second inequality, we have used Doob's maximal inequality ([30, p.14]) and the Itô isometry ([30, p.137]), and in the third inequality, we have applied (68). The Grownwall inequality is applied here so that we conclude that for all $t\in[0,T]$,
$$\mathbb{E}[\sup_{0\le s\le t}(h_s^\sigma-1)^2] \le 3T(T\frac{\sigma^4}{4}+4\sigma^2)(\frac{r}{r-\sigma^2})^2\exp(3r^2T^2). \tag{69}$$
(63) follows from (65), (66), and (69) immediately, and the Chebyshev inequality implies (64). □

Let $\mathcal{I}^\sigma$ denote the set of all stationary measures of (22) and $\mathcal{I} := \bigcup_{\sigma>0}\mathcal{I}^\sigma$.

Theorem 6.1. *Suppose that Ψ is dissipative. Then $\mathcal{I}$ is tight.*

Proof. Since Ψ is dissipative, there exists $t_0>0$ such that for all $t\ge t_0$ and $y\in\mathbb{R}_+^n\setminus\{O\}$, we have
$$\Psi(t,\frac{y}{\|y\|}) \subset D_1 = \{z\in\mathbb{R}_+^n : \mathrm{dist}(z,D)<1\}. \tag{70}$$
Here D is the fundamental attractor of Ψ. For any $y\in\mathbb{R}_+^n\setminus\{O\}$, we define a stopping time
$$\tau(\omega,\|y\|,t_0) = \inf\{t>0 : \int_0^t g^\sigma(s,\omega,\|y\|)ds > t_0\}, \tag{71}$$
which obviously satisfies that
$$\int_0^{\tau(\omega,\|y\|,t_0)} g^\sigma(s,\omega,\|y\|)ds = t_0, \quad \text{for all} \quad \omega\in\Omega^*. \tag{72}$$

From (37) it follows that $\tau(\omega, \|y\|, t_0) < \infty$. Therefore,

$$\lim_{t\to\infty} \mathbb{P}(\tau(\cdot, \|y\|, t_0) > t) = 0. \tag{73}$$

For any constant $R > 0$, it follows from the decomposition formula (25) that

$$\begin{aligned} & \mathbb{P}\{\omega : \|\Phi^\sigma(t,\omega,y)\| > R^2\} \\ = \; & \mathbb{P}\{\omega : \|g^\sigma(t,\omega,\|y\|)\Psi(\textstyle\int_0^t g^\sigma(s,\omega,\|y\|)ds, \frac{y}{\|y\|})\| > R^2\} \\ \leq \; & \mathbb{P}\{\omega : g^\sigma(t,\omega,\|y\|) > R\} + \mathbb{P}\{\omega : \|\Psi(\textstyle\int_0^t g^\sigma(s,\omega,\|y\|)ds, \frac{y}{\|y\|})\| > R\}. \end{aligned}$$

The ergodicity property (35) of g^σ implies that $\lim_{t\to+\infty} \mathbb{P}\{\omega : g^\sigma(t,\omega,\|y\|) > R\} = \mu_g^\sigma((R,+\infty))$. We can choose R large enough such that $U_R \supset D_1$, where $U_R = \{z \in \mathbb{R}_+^n : \|z\| \leq R\}$. From (37), (70)-(73), we have

$$\begin{aligned} & \mathbb{P}\{\omega : \|\Psi(\textstyle\int_0^t g^\sigma(s,\omega,\|y\|)ds, \frac{y}{\|y\|})\| > R\} \\ = \; & \mathbb{P}\{\omega : \tau(\omega,\|y\|,t_0)) \leq t, \|\Psi(\textstyle\int_0^t g^\sigma(s,\omega,\|y\|)ds, \frac{y}{\|y\|})\| > R\} \\ & + \mathbb{P}\{\omega : \tau(\omega,\|y\|,t_0)) > t, \|\Psi(\textstyle\int_0^t g^\sigma(s,\omega,\|y\|)ds, \frac{y}{\|y\|})\| > R\} \\ \leq \; & 0 + \mathbb{P}\{\omega : \tau(\omega,\|y\|,t_0)) > t\} \to 0 \quad \text{as } t \to +\infty. \end{aligned}$$

Thus, for any constant $R > 0$ such that $U_R \supset D_1$, we get

$$\begin{aligned} & \limsup_{t\to+\infty} \mathbb{P}\{\omega : \|\Phi^\sigma(t,\omega,y)\| > R^2\} \\ \leq \; & \limsup_{t\to+\infty} \mathbb{P}\{\omega : g^\sigma(t,\omega,\|y\|) > R\} \\ & + \limsup_{t\to+\infty} \mathbb{P}\{\omega : \|\Psi(\textstyle\int_0^t g^\sigma(s,\omega,\|y\|)ds, \frac{y}{\|y\|})\| > R\} \\ = \; & \mu_g^\sigma((R,+\infty)) \end{aligned}$$

for all $y \in \mathbb{R}_+^n \setminus \{O\}$. Moreover, $\Phi^\sigma(t,\omega,O) \equiv O$. Therefore, the above inequality holds for all $y \in \mathbb{R}_+^n$. By the Fatou lemma, for any $U_R \supset D_1$ and $\mu \in \mathcal{I}$, say $\mu \in \mathcal{I}^\sigma$, we have

$$\begin{aligned} \mu(U_{R^2}^c) = \; & \lim_{t\to+\infty} \textstyle\int_{\mathbb{R}_+^n} P^\sigma(t,y,U_{R^2}^c)\mu(dy) \\ \leq \; & \textstyle\int_{\mathbb{R}_+^n} \limsup_{t\to+\infty} P^\sigma(t,y,U_R^c)\mu(dy) \\ \leq \; & \mu_g^\sigma((R,+\infty)). \end{aligned}$$

The result immediately follows from the tightness of $\{\mu_g^\sigma \mid \sigma > 0\}$ proved in Lemma 4.2. $\square$

Proposition 6.2. *Let $\mu^i \in \mathcal{I}^{\sigma^i}$, $i = 1, 2, \cdots$. Assume that $\mu^i \xrightarrow{w} \mu$ as $\sigma^i \to 0$, $i \to \infty$. Then μ is an invariant measure of Ψ, that is, $\mu \circ \Psi^{-1}(T,\cdot) = \mu$ for any $T > 0$.*

Proof. Let $\mu^i \xrightarrow{w} \mu$ as $i \to \infty$. It suffices to prove that for any nonzero $g \in C_b(\mathbb{R}_+^n)$ and $T > 0$,

$$\int g(y)\mu \circ \Psi_T^{-1}(dy) = \int g(y)\mu(dy), \tag{74}$$

equivalently,

$$\int g(\Psi(T,y))\mu(dy) = \int g(y)\mu(dy).$$

$\{\mu^i\}$ is tight by Theorem 6.1. For every $\eta > 0$, there exists a compact set $K \subset \mathbb{R}^n_+$ such that $\inf_i \mu^i(K) \geq 1 - \frac{\eta}{\|g\|}$.

$$\begin{aligned}
& |\textstyle\int g(y)\mu^i \circ \Psi(T,\cdot)^{-1}(dy) - \int g(y)\mu^i(dy)| \\
= \;& |\textstyle\int g(\Psi(T,y))\mu^i(dy) - \int \mathbb{E}g(\Phi^{\sigma^i}(T,\omega,y))\mu^i(dy)| \\
\leq \;& \textstyle\int \mathbb{E}|g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y))|\mu^i(dy) \\
= \;& \textstyle\int I_K(y)\mathbb{E}|g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y))|\mu^i(dy) \\
& + \textstyle\int I_{K^c}(y)\mathbb{E}|g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y))|\mu^i(dy) \\
\leq \;& \textstyle\int \mathbb{E}|I_K(y)[g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y))]|\mu^i(dy) + 2\eta.
\end{aligned}$$

It is easy to see that $G := \Psi(T,K) \subset \mathbb{R}^n_+$ is a compact set. Hence, there is a $\delta > 0$ such that $|g(y) - g(z)| < \eta$ whenever $y \in G, z \in \mathbb{R}^n_+$ with $\|y - z\| < \delta$. Thus, one can derive that

$$\begin{aligned}
& \textstyle\int \mathbb{E}|I_K(y)[g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y))]|\mu^i(dy) \\
= \;& \textstyle\int_K \mathbb{E}|I_{\{\|\Psi(T,y)-\Phi^{\sigma^i}(T,\omega,y)\|\geq\delta\}}(\omega)[g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y)))]|\mu^i(dy) \\
& + \textstyle\int_K \mathbb{E}|I_{\{\|\Psi(T,y)-\Phi^{\sigma^i}(T,\omega,y)\|<\delta\}}(\omega)[g(\Psi(T,y)) - g(\Phi^{\sigma^i}(T,\omega,y))]|\mu^i(dy) \\
\leq \;& 2\|g\| \sup_{y\in K} \mathbb{P}(\|\Psi(T,y) - \Phi^{\sigma^i}(T,\omega,y)\| \geq \delta) + \eta \\
< \;& 2\eta
\end{aligned}$$

for i sufficiently large. Here we have used Proposition 6.1. As a consequence, we have proved that for any $\eta > 0$,

$$|\int g(y)\mu^i \circ \Psi(T,\cdot)^{-1}(dy) - \int g(y)\mu^i(dy)| < 4\eta$$

for all sufficiently large i. Letting $i \to \infty$, we obtain that

$$|\int g(y)\mu \circ \Psi(T,\cdot)^{-1}(dy) - \int g(y)\mu(dy)| \leq 4\eta.$$

(74) follows from η being arbitrary. The proof is complete. □

By the Poincaré recurrence theorem (see, e.g., Mañé [32, Theorem 2.3, p. 29]), we can obtain the following consequence immediately.

Proposition 6.3. *Assume that μ is an invariant probability measure of the flow Ψ. Let* $\mathrm{supp}(\mu)$ *denote the support of μ and $B(\Psi)$ the* Birkhoff*'s center of Ψ. Then the support of μ is contained in the* Birkhoff*'s center of Ψ, i.e.,*

$$\mathrm{supp}(\mu) \subset B(\Psi),$$

where $B(\Psi) = \overline{\{y \in \mathbb{R}^n_+ : y \in \omega_F(y)\}}$.

The main result in this section is summarized as follows.

Theorem 6.2. *Let Ψ be dissipative. Then $\mathcal{I}$ is tight. If $\mu^i \in \mathcal{I}^{\sigma^i}$, $i = 1, 2, \cdots$ satisfying $\sigma^i \to 0$ as $i \to \infty$, and $\mu^i \xrightarrow{w} \mu$ as $i \to \infty$, then μ is an invariant measure of Ψ, whose support is contained in the Birkhoff's center of Ψ.*

Proof. The proof follows from Propositions 6.1-6.3 and Theorem 6.1. □

Before finishing this section, we will present applications to Stratonovich stochastic competitive differential equations:

$$dy_i = y_i\Big(r - \sum_{j=1}^{n} a_{ij} y_j\Big)dt + \sigma y_i \circ dB_t,\ i = 1, 2, ..., n, \tag{75}$$

whose corresponding system without noise is

$$\frac{dy_i}{dt} = y_i\Big(r - \sum_{j=1}^{n} a_{ij} y_j\Big),\ i = 1, 2, ..., n, \tag{76}$$

where $r > 0, a_{ij} > 0,\ i, j = 1, 2, \cdots, n$.

Theorem 6.3. (Hirsch [6]) *The system* (76) *admits an invariant hypersurface Σ (called carrying simplex), homeomorphic to the closed unit simplex $S_n = \{y \in \mathbb{R}^n_+ : \sum_i y_i = 1\}$ by radial projection, such that every trajectory in $\mathbb{R}^n_+ \setminus \{O\}$ is asymptotic to one in Σ. In particular, the system is dissipative, and the fundamental attractor D is surrounded by Σ and the boundary of $\partial\mathbb{R}^n_+$.*

Combining the stochastic decomposition formula and Hirsch's carrying simplex theorem, we immediately obtain the following.

Corollary 6.1 (Stochastic Carrying Simplex)**.** *Stochastic competitive LV system* (75) *possesses a stochastic carrying simplex $\Sigma(\omega) := u(\omega)\Sigma$, which is invariant for pull-back flow $\Phi(t, \theta_{-t}\omega, y)$ and attracts any nontrivial pull-back trajectory.*

Theorem 6.3 tells us that every trajectory of (76) in $\mathbb{R}^n_+ \setminus \{O\}$ is asymptotic to one in Σ, that is, for any $y \in \mathbb{R}^n_+ \setminus \{O\}$, there is a point $z \in \Sigma$ such that

$$\lim_{T\to\infty} \|\Psi(T, y) - \Psi(T, z)\| = 0.$$

In the following, we shall prove that this result still holds for the expected occupation measures $\{P^{(T)}(y, \cdot) \mid T > 0\}$ generated by the solution $\Phi^\sigma(t, \omega, y)$ of (75). We call $\{P^{(T)}(y, \cdot) \mid T > 0\}$ and $\{P^{(T)}(z, \cdot) \mid T > 0\}$ *weakly asymptotic* if for all $f \in C_c(\mathbb{R}^n_+)$,

$$\lim_{T\to\infty} \Big|\int_{\mathbb{R}^n_+} f(x)P^{(T)}(y, dx) - \int_{\mathbb{R}^n_+} f(x)P^{(T)}(z, dx)\Big| = 0. \tag{77}$$

This means that the set $\mathcal{I}^\sigma_y$ of all weak limit measures for $\{P^{(T)}(y, \cdot) \mid T > 0\}$ coincides with $\mathcal{I}^\sigma_z$ of $\{P^{(T)}(z, \cdot) \mid T > 0\}$.

Theorem 6.4. *For any $y \in \mathbb{R}^n_+ \setminus \{O\}$, there is a point $z \in \Sigma$ such that both expected occupation measures $\{P^{(T)}(y, \cdot) \mid T > 0\}$ and $\{P^{(T)}(z, \cdot) \mid T > 0\}$ are weakly asymptotic.*

Proof. By Theorem 6.3, for any $y \in \mathbb{R}^n_+ \setminus \{O\}$, there is a $z \in \Sigma$ such that the solutions $\Psi(t, y)$ and $\Psi(t, z)$ are asymptotic, that is, for any $\epsilon > 0$, there is a t_0 such that as $t \geq t_0$,

$$\|\Psi(t, y) - \Psi(t, z)\| < \epsilon. \tag{78}$$

Firstly, we will show that (77) holds for $f(x) = \exp(-\sum_{i=1}^n m_i x_i)$ with any nonnegative integers $m_1, m_2, \cdots, m_n$. Obviously, there is a constant B such that $\|\nabla f(x)\| \leq B$.

From (73), it follows that for any $\epsilon > 0$, there is a $T_0 = T_0(\epsilon)$ such that

$$\mathbb{P}(\tau(\cdot, 1, t_0) > T_0) < \epsilon. \tag{79}$$

$$\begin{aligned}
& \limsup_{T\to\infty} \Big| \int_{\mathbb{R}^n_+} f(x) P^{(T)}(y, dx) - \int_{\mathbb{R}^n_+} f(x) P^{(T)}(z, dx) \Big| \\
= \; & \limsup_{T\to\infty} \Big| \frac{1}{T} \int_{T_0}^{T} \Big[\mathbb{E} f(\Phi^\sigma(t, \omega, y)) - \mathbb{E} f(\Phi^\sigma(t, \omega, z)) \Big] dt \Big| \\
\leq \; & \limsup_{T\to\infty} \frac{B}{T} \int_{T_0}^{T} \mathbb{E}\Big[g^\sigma(t, \omega, 1) \|\Psi(\int_0^t g^\sigma(s, \omega, 1) ds, y) - \Psi(\int_0^t g^\sigma(s, \omega, 1) ds, z)\| \Big] dt \\
\leq \; & \limsup_{T\to\infty} \frac{\bar{B}}{T} \int_{T_0}^{T} \Big(\mathbb{E} \|\Psi(\int_0^t g^\sigma(s, \omega, 1) ds, y) - \Psi(\int_0^t g^\sigma(s, \omega, 1) ds, z)\|^2 \Big)^{\frac{1}{2}} dt \\
= \; & \limsup_{T\to\infty} \frac{\bar{B}}{T} \int_{T_0}^{T} \Big(\mathbb{E} I_{\{\tau \leq T_0\}} \|\Psi(\int_0^t g^\sigma(s, \omega, 1) ds, y) - \Psi(\int_0^t g^\sigma(s, \omega, 1) ds, z)\|^2 \Big)^{\frac{1}{2}} dt \\
+ \; & \limsup_{T\to\infty} \frac{\bar{B}}{T} \int_{T_0}^{T} \Big(\mathbb{E} I_{\{\tau > T_0\}} \|\Psi(\int_0^t g^\sigma(s, \omega, 1) ds, y) - \Psi(\int_0^t g^\sigma(s, \omega, 1) ds, z)\|^2 \Big)^{\frac{1}{2}} dt \\
\leq \; & \bar{B}(1 + B_{yz})\epsilon
\end{aligned}$$

where $\bar{B} = B(1 + \frac{\sigma^2}{r})$, in the second inequality, we have used the Hölder inequality and (53), and B_{yz} is a constant, depending on the bounds of the trajectories $\Psi(t, y)$ and $\Psi(t, z)$. Since ϵ is arbitrary, (77) holds, hence it still holds for linear combination of these exponential functions. (77) follows from the Stone-Weierstrass Theorem immediately. $\square$

Corollary 6.2. *$\mathcal{I}$ is tight. Let $\mu^i \in \mathcal{I}^{\sigma^i}_{y^i_0}$ with $y^i_0 \neq O$, $i = 1, 2, \cdots$. Assume that $\mu^i \xrightarrow{w} \mu$ as $\sigma^i \to 0$, $i \to \infty$. Then μ is an invariant measure of Ψ, whose support is contained in its Birkhoff's center. Moreover, $\mu(\Sigma) = 1$.*

Proof. It is only necessary to show that $\mu(\{O\}) = 0$, others follow from Theorem 6.2. From Theorem 6.4, we know that for every $y \neq O$, there exists $z \in \Sigma$ such that $\mathcal{I}^\sigma_y = \mathcal{I}^\sigma_z$. Without loss of generality, we may assume that $y^i_0 \in \Sigma$ for $i = 1, 2, \cdots$. Since Σ is invariant, there is a constant $k > 0$ such that

$$\inf_i \|\Psi(t, z^i_0)\| \geq k, \text{ for all } t > 0.$$

Let $R < k$ in (57). Then we use Theorem 2.2(iv) and (v) and (57) to get that

$$\mu(B_R) \leq \liminf_{i\to\infty} \mu^i(B_R) \leq \lim_{\sigma^i\to 0} \int_0^{\frac{R}{k}} p^{\sigma^i}(s)ds = \delta_1([0, \frac{R}{k}]) = 0,$$

where δ_1 is the Dirac measure at $\{1\}$ on $\mathbb{R}_+$. This implies that $\mu(\{O\}) = 0$. □

7. The Complete Classification for 3-Dim Stochastic Competitive LV System

This section focuses on three dimensional Stratonovich stochastic competitive LV equations:

$$\begin{aligned}
dy_1 &= y_1(r - a_{11}y_1 - a_{12}y_2 - a_{13}y_3)dt + \sigma y_1 \circ dB_t, \\
dy_2 &= y_2(r - a_{21}y_1 - a_{22}y_2 - a_{23}y_3)dt + \sigma y_2 \circ dB_t, \\
dy_3 &= y_3(r - a_{31}y_1 - a_{32}y_2 - a_{33}y_3)dt + \sigma y_3 \circ dB_t.
\end{aligned} \tag{80}$$

Here $r > 0, a_{ij} > 0,\ i, j = 1, 2, 3$. We will classify the long-run behavior of stochastic system (80) both in pull-back trajectories and in stationary measures. To achieve this goal, we have to introduce the classification results for the corresponding deterministic three dimensional competitive LV equations:

$$\begin{aligned}
\frac{dy_1}{dt} &= y_1(r - a_{11}y_1 - a_{12}y_2 - a_{13}y_3), \\
\frac{dy_2}{dt} &= y_2(r - a_{21}y_1 - a_{22}y_2 - a_{23}y_3), \\
\frac{dy_3}{dt} &= y_3(r - a_{31}y_1 - a_{32}y_2 - a_{33}y_3),
\end{aligned} \tag{81}$$

which are given in [20].

7.1. Review of the Classification for 3-Dim Deterministic Competitive LV System

Zeeman [33] classified the stable nullcline classes for general three dimensional competitive LV equations, which permit different intrinsic growth rates. The stable nullcline class means that their boundary equilibria are hyperbolic and have the same local dynamics on Σ after a permutation of the indices $\{1, 2, 3\}$. She got that general three dimensional competitive LV equations admit in total 33 stable nullcline classes. Nevertheless, among the same stable nullcline class, two systems may have different dynamics, global dynamics is unknown for six stable nullcline classes. However, in the case of the identical intrinsic growth rate, global dynamics for all stable nullcline classes can be classified in the competitive parameters a_{ij}, as done in [20].

Theorem 7.1. ([20, Theorem 4.12]) *There are exactly* 37 *dynamical classes in* 33 *stable nullcline classes for system* (81)*. Each class is given by inequalities in competitive coefficients permitting permutation of indices, all trajectories tend to equilibria for classes* 1-25, 26 a)*,* *26 c), 27 a) and 28-33, a center on Σ only occurs in 26 b) and 27 b), and the heteroclinic cycle attracts all orbits except $L(P)$ in class 27 c). All are depicted on Σ and presented in Table 1 of Appendix A.*

Let us explain what the notations on Σ in Table 1 mean and how to get global dynamical behavior from the pictures in Table 1. By Hirsch's Theorem 6.3, the carrying simplex Σ is homeomorphic to the closed unit simplex S_3 by radial projection. So we regard S_3 as Σ and draw pictures on the standard simplex S_3, where three vertexes $\{R_1, R_2, R_3\}$ represent three axial equilibria of (81). Let us take the class 14 in Appendix A (see Fig.1) as an example to explain the notations and their meanings. A closed dot $\bullet$ denotes an attracting equilibrium (see R_2, V_2) on Σ, an open dot $\circ$ denotes the repelling one (see R_1) on Σ, and the intersection of stable and unstable manifolds is a saddle on Σ (see R_3, V_1). The asymptotic behavior of every trajectory on Σ is clearly seen from Fig.1.

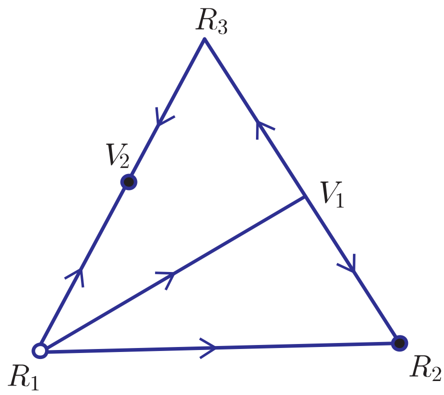

Figure 1: The dynamics in Σ.

Let $\mathcal{A}^{\Sigma}(Q)$ denote the attracting domain of an equilibrium $Q \in \mathcal{E}$ on Σ. It follows from [20, Proposition 4.13] that any pair of nonzero points on $L(y)$ have the same omega limit set. We can obtain the attracting domain of Q as follows

$$\mathcal{A}(Q) = \bigcup \{L(y) \setminus \{O\} : y \in \mathcal{A}^{\Sigma}(Q)\}. \tag{82}$$

Therefore, the attracting domain of a given equilibrium Q can be derived by $\mathcal{A}^{\Sigma}(Q)$ drawn in Table 1 and (82). This has given precise long-term behavior for 34 classes :1-25, 26 a), 26 c), 27 a) and 28-33 in Table 1.

It remains to describe the remaining three classes: class 26 b), class 27 b), and class 27 c). For this aim, define

$$\alpha_i = a_{i+1,i+1} - a_{i,i+1}, \quad \beta_i = a_{i,i-1} - a_{i-1,i-1}, \quad i \mod 3, \text{ and} \tag{83}$$

$$\theta := \prod_{i=1}^{3}(a_{i,i-1} - a_{i-1,i-1}) - \prod_{i=1}^{3}(a_{i+1,i+1} - a_{i,i+1}) = \beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3. \tag{84}$$

The system (81) admits nontrivial periodic orbits if and only if $\theta = 0$ (see [20, Theorem 4.3]), which only occurs in class 26 b) and class 27 b). Both classes possess heteroclinic cycle connecting three equilibria, interior of which on Σ a continuum of periodic orbits $\{\Gamma(h) : h \in I\}$ are full of. Each closed orbit $\Gamma(h)$ is the intersection of the carrying simplex Σ and invariant cone surface given by

$$\Lambda(h) : V(y) := y_1^{\mu} y_2^{\nu} y_3^{\omega} (\beta_2\alpha_3 y_1 + \alpha_1\alpha_3 y_2 + \beta_1\beta_2 y_3) \equiv h, \tag{85}$$

where $\mu = -\beta_2\beta_3/D^*$, $\nu = -\alpha_1\alpha_3/D^*$, $\omega = -\alpha_1\beta_2/D^*$, $D^* = (\beta_2\beta_3 + \beta_2\alpha_1 + \alpha_1\alpha_3)$, and α_i, β_i are given in (83). We depict typical closed orbit and its attracting cone surface for these two classes in Fig.2 and Fig.3. The readers are referred to [20, Theorem 4.13] for details.

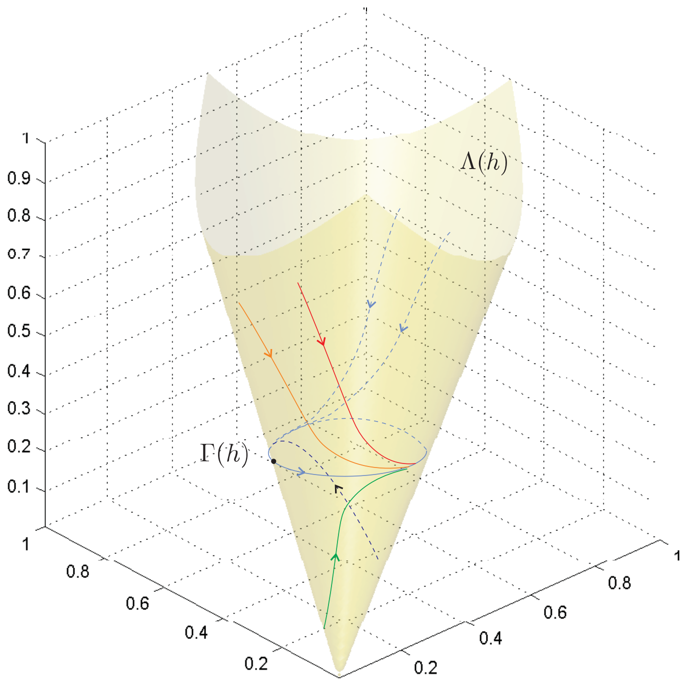

Figure 2: The attracting domain of the closed orbit $\Gamma(h)$ is a cone $\Lambda(h)$.

Now we summarize the long-run behavior for these three classes as follows.

Theorem 7.2. (Chen, Jiang, and Niu [20])

(a) *Let the competitive parameters satisfy inequalities in class* 26 b) *besides* $\theta = 0$. *Then the unique positive equilibrium* P *attracts* $L(P) \setminus \{O\}$*; the closed orbit* $\Gamma(h)$ *attracts* $\Lambda(h) \setminus \{O\}$*; all other trajectories converge an equilibrium.*

(b) *Let the competitive parameters satisfy inequalities in class* 27 b) *besides* $\theta = 0$. *Then the unique positive equilibrium* P *attracts* $L(P) \setminus \{O\}$*; the closed orbit* $\Gamma(h)$ *attracts* $\Lambda(h) \setminus \{O\}$.

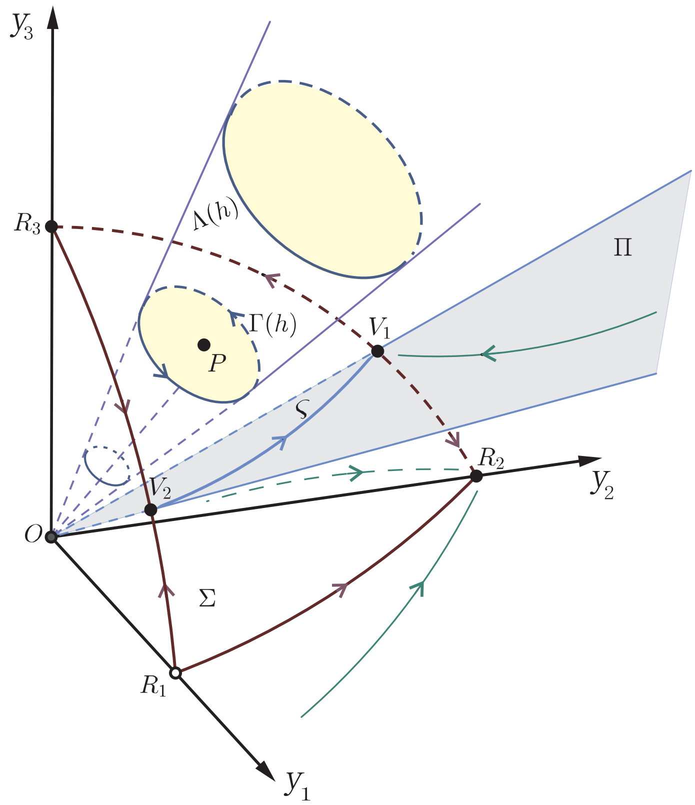

Figure 3: The global phase portraits of a system in class 26 b).

(c) *Let the competitive parameters inequalities in class* 27 c) *hold. Then* $\mathcal{A}(\mathcal{H}) = \mathbb{R}^3_+ \setminus L(P)$*, where* $\mathcal{H}$ *is the heteroclinic cycle .*

7.2. The Complete Classification for Long-Run Behavior via Pull-Back Trajectory

Combing Theorems 4.3, 7.1 and 7.2, we can completely classify the long-run behavior of pull-back trajectories of three dimensional stochastic competitive LV system (80).

Theorem 7.3. *Among classes* 1-25, 26 a), 26 c), 27 a) *and* 28-33*, each pull-back trajectory* $\Phi(t, \theta_{-t}\omega, y)$ *converges a random equilibrium. More precisely, for a given equilibrium* $Q \in \mathcal{E}$, $\Phi(t, \theta_{-t}\omega, y) \to u(\omega)Q$ *as* $t \to \infty$ *for all* $y \in \mathcal{A}(Q)$*. The same result hold for the remaining three classes when* y *is located in an attracting domain of an equilibrium of* (81).

Theorem 7.4. *Assume that* $\theta = 0$ *and the competitive parameters inequalities in class* 26 b) *or class* 27 b) *hold. Then the pull-back omega limit set* $\Gamma_y(\omega)$ *of the trajectory* $\Phi(t, \theta_{-t}\omega, y)$ *is* $u(\omega)\Gamma(h)$ *if and only if* $y \in \Lambda(h) \setminus \{O\}$.

Theorem 7.5. *Assume that* $\theta > 0$ *and the competitive parameters inequalities in class* 27 c) *hold. Then the pull-back omega limit set* $\Gamma_y(\omega)$ *of the trajectory* $\Phi(t, \theta_{-t}\omega, y)$ *emanating from* y *is* $u(\omega)\mathcal{H}$ *if and only if* $y \notin L(P)$ *with* y_i *being positive,* $i = 1, 2, 3$*, where* $\mathcal{H}$ *is the heteroclinic cycle of* (81).

7.3. The Classification via Stationary Measures

First, let us consider the case for trajectory of (81) to converge to an equilibrium.

Theorem 7.6. *Let $Q \in \mathcal{E}$. Then for each $y \in \mathcal{A}(Q)$, $P(t, y, \cdot) \to \mu_Q^\sigma(\cdot)$ weakly as $t \to \infty$, μ_Q^σ is the unique stationary measure with respect to the Markov semigroup P_t in $\mathcal{A}(Q)$, and it is still strongly mixing on $\mathbb{R}_+^n$. Moreover, as $\sigma \to 0$, $\mu_Q^\sigma(\cdot) \xrightarrow{w} \delta_Q(\cdot)$. These results are available for classes* 1-25, 26 a), 26 c), 27 a) *and* 28-33 *as well as any equilibrium of* (81) *in classes* 26 b), 27 b) *and* 27 c) *when we restrict the state space on its stable manifold.*

Proof. For a given equilibrium $Q \in \mathcal{E}$, it follows from the cone invariance that $\Phi(t, \omega, y) \in \mathcal{A}(Q)$ for any $y \in \mathcal{A}(Q)$. Then the probability distribution function $P(t, y, \cdot)$ is supported in $\mathcal{A}(Q)$ if $y \in \mathcal{A}(Q)$. Thus, replacing $\mathrm{Int}\mathbb{R}_+^n$ by $\mathcal{A}(Q)$, we can verify this theorem in the quite same manner as that of Theorem 5.2. $\square$

Theorem 7.7. *Suppose that* (81) *is one of systems in classes* 1-25, 26 a), 26 c), 27 a) *and* 28-33 *and that $\mathcal{E}$ is finite. Then all its stationary measures are the convex combinations of strongly mixing stationary measures $\{\mu_Q^\sigma : Q \in \mathcal{E}\}$. As $\sigma \to 0$, all their limiting measures are the convex combinations of the Dirac measures $\{\delta_Q(\cdot) : Q \in \mathcal{E}\}$.*

Proof. Assume that (81) is one of systems of the given 34 classes. Then $\mathbb{R}_+^3 = \bigcup\{\mathcal{A}(Q) : Q \in \mathcal{E}\}$. Let $Q \in \mathcal{E}$ and suppose that ν is an arbitrary stationary measure for the Markov semigroup P_t in $\mathbb{R}_+^3$. Then we shall prove

$$\sum_{Q \in \mathcal{E}} \nu(\mathcal{A}(Q))\mu_Q^\sigma(\cdot) = \nu(\cdot), \tag{86}$$

where $\sum_{Q \in \mathcal{E}} \nu(\mathcal{A}(Q)) = \nu(\mathbb{R}_+^3) = 1$.

By the definition of stationary measure, for any $t > 0$, one has

$$\int_{\mathbb{R}_+^3} \nu(dy)P(t, y, \cdot) = \nu(\cdot),$$

that is,

$$\sum_{Q \in \mathcal{E}} \int_{\mathcal{A}(Q)} \nu(dy)P(t, y, \cdot) = \nu(\cdot).$$

For any $f \in \mathcal{C}_b(\mathbb{R}_+^n)$,

$$\int_{\mathbb{R}_+^n} f(z) \sum_{Q \in \mathcal{E}} \int_{\mathcal{A}(Q)} \nu(dy)P(t, y, dz) = \int_{\mathbb{R}_+^n} f(z)\nu(dz),$$

in other words,

$$\sum_{Q \in \mathcal{E}} \int_{\mathcal{A}(Q)} \nu(dy) \int_{\mathbb{R}_+^n} f(z)P(t, y, dz) = \int_{\mathbb{R}_+^n} f(z)\nu(dz). \tag{87}$$

From Theorem 5.3, we know that for each $Q \in \mathcal{E}$ and $y \in \mathcal{A}(Q)$, $P(t, y, \cdot) \xrightarrow{w} \mu_Q^\sigma(\cdot)$ as $t \to \infty$. Letting t tend to infinity in (87), we get that

$$\int_{\mathbb{R}_+^n} f(z) \sum_{Q \in \mathcal{E}} \nu(\mathcal{A}(Q)) \mu_Q^\sigma(dz) = \int_{\mathbb{R}_+^n} f(z) \nu(dz)$$

This shows that (86) holds, as a result, ν is the convex combination of $\{\mu_Q^\sigma : Q \in \mathcal{E}\}$. The remaining result follows from Theorem 7.6 immediately. □

Theorem 7.8. *Assume that $\theta = 0$ and the competitive parameters inequalities in class* 26 b) *or class* 27 b) *hold. Then there exists a unique and strongly mixing nontrivial stationary measure ν_h^σ supporting on the cone*

$$\Lambda(h) : V(y) := y_1^\mu y_2^\nu y_3^\omega (\beta_2 \alpha_3 y_1 + \alpha_1 \alpha_3 y_2 + \beta_1 \beta_2 y_3) \equiv h \in I, \tag{88}$$

where $\mu = -\beta_2 \beta_3 / D^*$, $\nu = -\alpha_1 \alpha_3 / D^*$, $\omega = -\alpha_1 \beta_2 / D^*$, $D^* = (\beta_2 \beta_3 + \beta_2 \alpha_1 + \alpha_1 \alpha_3)$, α_i, β_i are given in (83)*, and I is the feasible image interval for V, and*

$$\lim_{t \to \infty} P(t, y, A) = \nu_h^\sigma(A) \text{ for any } y \in \Lambda(h) \setminus \{O\} \text{ and } A \in \mathcal{B}(\Lambda(h) \setminus \{O\}). \tag{89}$$

Besides, ν_h^σ converges weakly to the Haar measure on the closed orbit $\Gamma(h)$ as $\sigma \to 0$.

Proof. Fix $h \in I$ and $y_0 \in \Gamma(h)$, define $\varphi(y) = \inf\{t > 0,\ \Psi(t, y_0) = y\}$ for any $y \in \Gamma(h)$, and denote by $\Upsilon = \varphi(y_0)$ the period of the orbit $\Psi(t, y_0)$. Let $S := \mathbb{R}_+ \bmod \Upsilon$ denote a circle. Then it is not difficult to see that $\varphi : \Gamma(h) \to S$ is a homeomorphism. By Theorem 6.3, for any $y \in \Lambda(h) \setminus \{O\}$, there are unique $\lambda > 0$ and $z \in \Gamma(h)$ such that $y = \lambda z$. Define $\psi :\ \Lambda(h) \setminus \{O\} \to \mathbb{R} \times S$ by

$$\psi(y) := \Big(\ln \lambda,\ \varphi(z) \Big),\ y \in \Lambda(h) \setminus \{O\}$$

where $y = \lambda z$ with $\lambda > 0$ and $z \in \Gamma(h)$. It is easy to see that $\psi : \Lambda(h) \setminus \{O\} \to \mathbb{R} \times S$ is a homeomorphism, its inverse is $\psi^{-1}(x, \tau) = e^x \Psi(\tau, y_0)$.

For any $y = \lambda z \in \Lambda(h)$ with $\lambda > 0$ and $z \in \Gamma(h)$, it follows from (25) that

$$\Phi(t, \omega, y) = g(t, \omega, \lambda) \Psi(\int_0^t g(s, \omega, \lambda) ds, z).$$

Obviously, $\Psi(\int_0^t g(s, \omega, \lambda) ds, z) \in \Gamma(h)$. Set

$$H(t, \omega, H_0) := \ln (g(t, \omega, \lambda)),\ T(t, \omega, H_0, T_0) := \varphi\Big(\Psi(\int_0^t g(s, \omega, \lambda) ds, z) \Big),$$

where $H_0 := \ln \lambda$ and $T_0 := \varphi(z)$. Then applying the Itô formula, we have

$$\begin{aligned} H(t, H_0) &= H_0 + r \int_0^t (1 - e^{H(s, H_0)}) ds + \int_0^t \sigma dB_s, \\ T(t, H_0, T_0) &= (T_0 + \int_0^t e^{H(s, H_0)} ds) \bmod \Upsilon. \end{aligned} \tag{90}$$

By the definition,

$$\psi(\Phi(t,\omega,y)) = \Big(\ln\left(g(t,\omega,\lambda)\right),\ \varphi(\Psi(\int_0^t g(s,\omega,\lambda)ds, z))\Big) = \Big(H(t,\omega,H_0),\ T(t,\omega,H_0,T_0)\Big).$$

Therefore, Φ on $\Lambda(h)\backslash\{O\}$ and (H,T) on $\mathbb{R}\times S$ are conjugate and ψ is a conjugate mapping.

Now, we prove that the Markov semigroup associated with (H,T) on $\mathbb{R}\times S$ is strongly Feller and irreducible at any time $t>0$.

For any $(H_0,\widetilde{T}_0)\in\mathbb{R}^2$, consider the equations

$$\begin{aligned} H(t,H_0) &= H_0 + r\int_0^t (1-e^{H(s,H_0)})ds + \int_0^t \sigma dB_s,\\ \widetilde{T}(t,H_0,\widetilde{T}_0) &= \widetilde{T}_0 + \int_0^t e^{H(s,H_0)}ds. \end{aligned} \tag{91}$$

By Theorem 4.2 in [34], the semigroup $(\widetilde{P}_t)_{t\geq 0}$ associated with (91) is strongly Feller on $\mathbb{R}^2$ at any $t>0$, i.e., for any $t>0$, $f\in\mathcal{B}_b(\mathbb{R}^2)$,

$$(H_0,\widetilde{T}_0)\in\mathbb{R}^2\to\mathbb{E}f(H(t,H_0),\widetilde{T}(t,H_0,\widetilde{T}_0)) \text{ is continuous.}$$

Hence, for any $F\in\mathcal{B}_b(\mathbb{R}\times S)$, set $f_F(H,\widetilde{T}) := F(H,\widetilde{T} \bmod \Upsilon)$, we have $f_F\in\mathcal{B}_b(\mathbb{R}^2)$, and then

$$\begin{aligned} (H_0,T_0)\in\mathbb{R}\times S\to\quad & \mathbb{E}F(H(t,H_0),T(t,H_0,T_0))\\ &= \mathbb{E}f_F(H(t,H_0),\widetilde{T}(t,H_0,T_0)) \text{ is continuous.} \end{aligned}$$

This implies that (H,T) is a strongly Feller diffusion on $\mathbb{R}\times S$ at any $t>0$.

Now we prove that (H,T) is irreducible on $\mathbb{R}\times S$. We only need to prove that for any $a,b\in\mathbb{R}$ with $a<b$, $c,d\in S$ with $c<d$ and $A:=(a,b)\times(c,d)$,

$$\mathbb{P}\Big((H(t,H_0),T(t,H_0,T_0))\in A\Big)>0,\quad \text{for any } t>0 \text{ and } (H_0,T_0)\in\mathbb{R}\times S.$$

Set

$$\mathcal{A}(c,d;T_0,\Upsilon) := \bigcup_{n=0}^{\infty}\Big(c+n\Upsilon-T_0, d+n\Upsilon-T_0\Big),\ \tilde{A} := (e^a,e^b)\times\mathcal{A}(c,d;T_0,\Upsilon).$$

Define the map $\mathbb{L}: C([0,t],\mathbb{R}_+)\to\mathbb{R}^2$ by

$$\mathbb{L}(f) := \Big(\frac{f(t)}{e^{-H_0}+r\int_0^t f(s)ds},\ \int_0^t \frac{f(s)}{e^{-H_0}+r\int_0^s f(l)dl}ds\Big).$$

Then $\mathbb{L}$ is continuous, and by (28) and the definition of H,

$$\mathbb{P}\Big((H(t,H_0),T(t,H_0,T_0))\in A\Big) = \mathbb{P}\Big(\mathbb{L}(e^{r\cdot+\sigma B_\cdot})\in\tilde{A}\Big). \tag{92}$$

Denote

$$B := \Big\{ f \in C([0,t], \mathrm{Int}\mathbb{R}_+) : f(0) = 1,\ \mathbb{L}(f) \in \tilde{A} \Big\}.$$

We claim that $B \neq \emptyset$. In fact, let

$$\widetilde{B} := \{ h \in C([0,t], \mathrm{Int}\mathbb{R}_+) :\ h(0) = e^{H_0},\ \Big(h(t), \int_0^t h(s)ds \Big) \in \tilde{A} \}.$$

Then we first show that $\widetilde{B} \neq \emptyset$.

Since $\frac{e^{H_0}+e^b}{2}t$ is a given constant, we define $\widetilde{n} := \inf\{ n : c + n\Upsilon - T_0 \geq \frac{e^{H_0}+e^b}{2}t \}$, which exists. Choose a constant $\widetilde{h}$ such that the area in the shadow domain of Fig.4 is the mean value of $c + \widetilde{n}\Upsilon - T_0$ and $d + \widetilde{n}\Upsilon - T_0$. Thus $\widetilde{h} = \frac{c+d+2\widetilde{n}\Upsilon - 2T_0}{t} - \frac{e^{H_0}}{2} - \frac{e^a+e^b}{4}$. Let h be defined as the broken line in Fig.4. Then it is easy to see that $h(0) = e^{H_0}, h(t) = \frac{e^a+e^b}{2} \in (e^a, e^b)$ and the integral $\int_0^t h(s)ds = \frac{c+d}{2} + \widetilde{n}\Upsilon - T_0 \in (c + \widetilde{n}\Upsilon - T_0, d + \widetilde{n}\Upsilon - T_0)$. This implies that $h \in \widetilde{B}$.

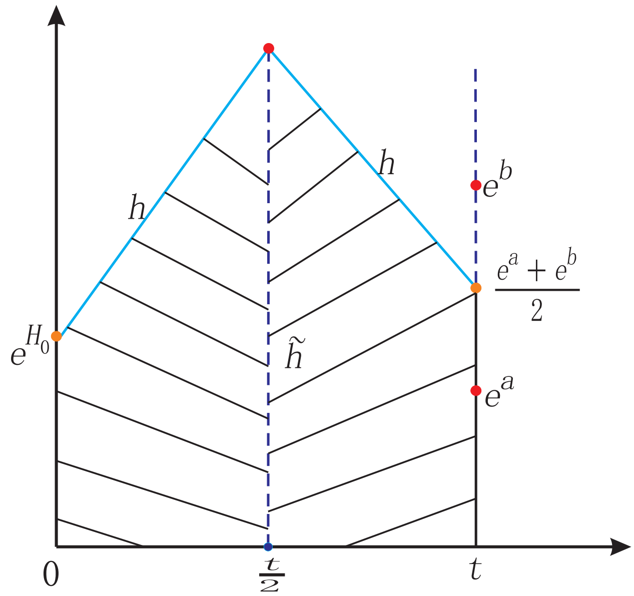

Figure 4: The image of h.

Take $h \in \widetilde{B}$, and let

$$f(s) := e^{-H_0} h(s) e^{r \int_0^s h(\tau)d\tau},\ s \in [0,t].$$

Then $f \in C([0,t], \mathrm{Int}\mathbb{R}_+)$ and $r\int_0^s f(l)dl = e^{-H_0}(e^{r\int_0^s h(\tau)d\tau} - 1)$. It is clear that $f(0) = h(0)e^{-H_0} = 1$, $\frac{f(t)}{e^{-H_0}+r\int_0^t f(s)ds} = h(t) \in (e^a, e^b)$ and

$$\int_0^t \frac{f(s)}{e^{-H_0} + r\int_0^s f(l)dl} ds = \int_0^t h(s)ds \in \mathcal{A}(c, d; T_0, \Upsilon).$$

This implies that $f \in B$, that is, $B \neq \emptyset$.

Take $\widetilde{f} \in B$. Thus the set $U = \mathbb{L}^{-1}(\tilde{A})$ is an open set containing $\widetilde{f}$. This shows that there exists $\epsilon > 0$ such that

$$C_\epsilon^{\widetilde{f}} = \{g \in C([0,t], \mathbb{R}_+),\ g(0) = 1,\ \sup_{s\in[0,t]} |g(s) - \widetilde{f}(s)| < \epsilon\} \subset U.$$

Then there exists an open set D in the space $\{p \in C([0,t], \mathbb{R}),\ p(0) = 0\}$ with sup norm such that

$$e^{r\cdot+\sigma p(\cdot)} \in C_\epsilon^{\widetilde{f}}, \quad \forall p \in D.$$

By (92),

$$\mathbb{P}\Big((H(t,H_0), T(t,H_0,T_0)) \in A\Big) \geq \mathbb{P}\Big(B(\cdot,\omega) \in D\Big) > 0. \tag{93}$$

The second inequality follows from the fact of classical Wiener space (see e.g. [35, 36]). This implies that (H,T) is irreducible on $\mathbb{R} \times S$.

Applying Theorem 2.6(ii), we conclude that the Markovian semigroup $P_t, t > 0$ associated with Φ is both strongly Feller and irreducible at any $t > 0$.

Theorem 6.3 tells us that (81) is dissipative and the origin is a repeller. Applying Theorem 5.4, we obtain that for any $y \in \Lambda(h) \setminus \{O\}$, there exists a stationary measure ν_h^σ of Φ supported on the cone surface $\Lambda(h) \setminus \{O\}$. When we restrict our attention to $\Lambda(h) \setminus \{O\}$ and use Theorem 2.5, we obtain that ν_h^σ is strongly mixing and unique on $\Lambda(h) \setminus \{O\}$, and that (89) holds. Replacing $\mathrm{Int}\mathbb{R}_+^n$ by $\Lambda(h) \setminus \{O\}$ and using the same manner as done in the last paragraph of the proof of Theorem 5.2, we can verify that ν_h^σ is strongly mixing on $\mathbb{R}_+^3$.

Finally, applying Corollary 6.2, we conclude that ν_h^σ converges weakly to the Haar measure on the closed orbit $\Gamma(h)$ as $\sigma \to 0$. □

Theorems 7.6 and 7.8 have given all ergodic stationary measures for all classes except class 27c). From ergodic decomposition theorem [37, §1.2], every stationary measure is expressed by ergodic stationary measures, which is stated in the following.

Theorem 7.9. *Assume that $\theta = 0$ and the competitive parameters inequalities in class* 26 b) *or class* 27 b) *hold. Let $\mathcal{E}^{26} = \{O, P, V_1, V_2, R_1, R_2, R_3\}$ and $\mathcal{E}^{27} = \{O, P, R_1, R_2, R_3\}$ denote the equilibria set of the classes* 26 *and* 27*, respectively. Then the set of all ergodic stationary measures is*

$$\mathcal{M}^e(\Phi) = \{\nu_h^\sigma : h \in I\} \bigcup \{\mu_Q^\sigma : Q \in \mathcal{E}^i\}, \quad i = 26, 27,$$

and there exists a probability measure ν_μ on $\mathcal{M}^e(\Phi)$ such that

$$\mu(\cdot) = \int_{\mathcal{M}^e(\Phi)} \eta(\cdot) d\nu_\mu(\eta)$$

for any stationary measure μ of Φ.

Remark 7.1. *We can express all stationary measures more precisely.*

Define $L^\sigma: \ I\bigcup\mathcal{E} \to \mathcal{M}^e(\Phi)$ *as*

$$L^\sigma: \qquad \begin{array}{l} h \in I \to \nu_h^\sigma \\ Q \in \mathcal{E} \to \mu_Q^\sigma. \end{array}$$

Then L^σ *is a bijective mapping. Set*

$$\mathcal{A} = \Big\{\{\vartheta \in I\bigcup\mathcal{E}, \ L^\sigma(\vartheta) \in O\}, \quad \forall O \in \mathcal{B}(\mathcal{M}^e(\Phi))\Big\}.$$

For the above probability measure ν_μ *on* $\mathcal{M}^e(\Phi)$*, let*

$$m_\mu\Big(\{\vartheta \in I\bigcup\mathcal{E}, \ L^\sigma(\vartheta) \in O\}\Big) := \nu_\mu(O), \quad \forall O \in \mathcal{B}(\mathcal{M}^e(\Phi)).$$

Then m_μ *is a probability measure on* $(I\bigcup\mathcal{E}, \mathcal{A})$*, and*

$$\mu(\cdot) = \int_{I\bigcup\mathcal{E}} L^\sigma(\vartheta)(\cdot) m_\mu(d\vartheta).$$

Theorem 7.10. *Assume that* $\theta = 0$ *and the competitive parameters inequalities in class* 26 b) *or class* 27 b) *hold. Let* $\mu^i := \nu_{h^i}^{\sigma^i}, \ i = 1, 2, \cdots$ *satisfy* $\sigma^i \to 0$ *and* $\mu^i \xrightarrow{w} \mu$ *as* $i \to \infty$*, where* $\nu_{h^i}^{\sigma^i}$ *is the unique strongly mixing nontrivial stationary measure supported on the cone surface* $\Lambda(h^i)$*. Suppose that each* $\Gamma(y_0^i)$ *is the closed orbit generating the cone surface* $\Lambda(h^i), \ i = 1, 2, \cdots$ *and that* $y_0^i \to y_0$ *as* $i \to \infty$*. Then if* y_0 *lies in the interior of the heteroclinic cycle* $\mathcal{H}$*, then* μ *is the Haar measure on* $\Gamma(y_0)$ *for* $y_0 \neq P$*, or the Dirac measure* $\delta_P(\cdot)$ *at* P *for* $y_0 = P$*. If* $y_0 \in \mathcal{H}$*, then*

$$\mu(\{E_1, E_2, E_3\}) = 1, \tag{94}$$

where E_1, E_2, E_3 *are three equilibria of heteroclinic cycle* $\mathcal{H}$ *in class* 26 b) *or class* 27 b)*.*

Proof. Let $\mu^i := \nu_{h^i}^{\sigma^i}, \ i = 1, 2, \cdots$ satisfy $\sigma^i \to 0$ and $\mu^i \xrightarrow{w} \mu$ as $i \to \infty$. Suppose that each $\Gamma(y_0^i)$ is the closed orbit generating the cone surface $\Lambda(h^i), \ i = 1, 2, \cdots$ and that $y_0^i \to y_0$ as $i \to \infty$. We first consider the case that y_0 lies in the interior of $\mathcal{H}$ on Σ with $y_0 \neq P$. If there is a subsequence of $\{y_0^i\}$ lying on $\Gamma(y_0)$, then Theorem 7.8 implies that μ is the Haar measure on $\Gamma(y_0)$. Otherwise, we suppose that all points in $\{y_0^i\}$ are different. If $\{y_0^i\}$ are in the interior of $\Gamma(y_0)$ on Σ, then we may assume that y_0^i lies in the interior of $\Gamma(y_0^{i+1})$ on Σ for $i = 1, 2, \cdots$. Thus, the first part result deduces that $\mu^i\Big(\Lambda(\Gamma(y_0^i))\Big) = 1$ for $i = 1, 2, \cdots$. Let D_i and D_0 denote the interior of the closed orbits $\Gamma(y_0^i)$ and $\Gamma(y_0)$ on Σ, respectively. Then $\mu^k(\Lambda(D_i)) = 0$ for $1 \leq i \leq k$. However, $\Lambda(D_i) \setminus \{O\}$ is an open subset in $\mathbb{R}_+^3$. For each $i \geq 1$, it follows Theorem 2.2(iv) that

$$\mu(\Lambda(D_i) \setminus \{O\}) \leq \liminf_{k\to\infty} \mu^k(\Lambda(D_i) \setminus \{O\}) = 0. \tag{95}$$

In addition, $\mu(\{O\}) = 0$ by Corollary 6.2. This proves that $\mu(\Lambda(D_i)) = 0$ for each $i \geq 1$. Using the continuity of probability measure, we have $\mu(\Lambda(D_0)) = 0$. Again utilizing Theorem 2.2(iii), we get that $\mu(\Lambda(\overline{D_0})) = 1$. Hence $\mu\Big(\Lambda(\Gamma(y_0))\Big) = 1$. Since the set of recurrent points on $\Lambda(\Gamma(y_0))$ is $\Gamma(y_0) \cup \{O\}$ and $\mu(\{O\}) = 0$, μ is the Haar measure on $\Gamma(y_0)$. The case that y_0 lies outside of $\Gamma(y_0)$ on Σ can be treated analogously.

Secondly, we assume that $y_0 = P$, $y_0^i \neq P$ for each i, and that y_0^i lies in the interior of $\Gamma(y_0^{i-1})$ on Σ for $i = 2, 3, \cdots$. Then $\mu^k(\Lambda(\overline{D_i})) = 1$ for $1 \leq i \leq k$. Theorem 2.2(iii) implies that $\mu(\Lambda(\overline{D_i})) = 1$ for each i. $\mu(L(P)) = 1$ follows from the continuity of the probability measure μ, and $\mu = \delta_P(\cdot)$ from $\mu(\{O\}) = 0$.

Thirdly, suppose $y_0 \in \mathcal{H}$. Then without loss of generality, we may assume that $y_0^i \in D_{i+1}$ for each i. By a similar way, we can obtain (95) and $\mu(\Lambda(D_i)) = 0$ for $i = 1, 2, \cdots$. Let D^* denote the interior of $\mathcal{H}$ on Σ. Then $\mu(\Lambda(D^*)) = 0$. It is easy to see $\mu(\Lambda(D^*) \cup \Lambda(\mathcal{H})) = 1$, and hence that $\mu(\Lambda(\mathcal{H})) = 1$. It is not difficult to see that the recurrent points on $\Lambda(\mathcal{H})$ are $\{E_1, E_2, E_3, O\}$. Consequently, (94) follows from Corollary 6.2. The proof is complete. $\square$

Let $\mathbb{R}_j^+ := \{(y_1, y_2, y_3) \in \mathbb{R}_+^3 | y_k = 0 \text{ for } k \neq j\}$ denote the nonnegative y_j-axis for $j = 1, 2, 3$.

Theorem 7.11. *Assume that* $\theta > 0$ *and the competitive parameters inequalities in class* 27 c) *hold. Then* $\nu_y^\sigma \in \mathcal{I}_y^\sigma$ *will be supported on the three nonnegative axes for any* $y \in \mathrm{Int}\mathbb{R}_+^3 \backslash L(P)$ *and*

$$\nu_y^\sigma(A) = \sum_{j=1}^{3} \lambda_j \mu_g^\sigma(A \cap \mathbb{R}_j^+),\ \lambda_j = \nu_y^\sigma(\mathbb{R}_j^+),\ \textit{for any } A \in \mathcal{B}(\mathbb{R}_+^3). \tag{96}$$

Let $\mu^i := \nu_{y_0^i}^{\sigma^i} \in \mathcal{I}_{y_0^i}^{\sigma^i}$, $i = 1, 2, \cdots$. *If* $\mu^i \xrightarrow{w} \mu$ *as* $\sigma^i \to 0$, $i \to \infty$, *then*

$$\mu(\{R_1, R_2, R_3\}) = 1, \tag{97}$$

where R_1, R_2, R_3 *are three axial equilibria of* (81).

Proof. By Theorem 5.5, $\mathrm{supp}(\nu_y^\sigma) \subset \partial\mathbb{R}_+^3$. In the following, we shall show that $\mathrm{supp}(\nu_y^\sigma) = \cup_{j=1}^3 \mathbb{R}_j^+$. For this purpose, we only need to prove

$$\nu_y^\sigma(\partial\mathbb{R}_+^3 \setminus \cup_{j=1}^3 \mathbb{R}_j^+) = 0. \tag{98}$$

Suppose that $p(p_1, p_2, 0)$ and $q(q_1, q_2, 0)$ lie on $\mathcal{H}$ such that p is close to R_1 and q is close to R_2 as near as we wish. Let C denote the trajectory from p to q and s denote the time length for the trajectory to run from p to q. Assume that $\epsilon > 0$ is sufficiently small and

$$B_\epsilon(C) := \{x \in \mathbb{R}_+^3 : \mathrm{dist}(x, C) < \epsilon\}.$$

Since $\Psi(t, y)$ is asymptotic to the heteroclinic cycle $\mathcal{H}$, $\Psi(t, y)$ will enter and then go out of $B_\epsilon(C)$ with infinitely many times. By the continuity of Ψ with respect to initial points, the time length from entering $B_\epsilon(C)$ to going out of $B_\epsilon(C)$ for the trajectory $\Psi(t, y)$ is approximately s. However, since R_1, R_2 and R_3 are saddles, the time for $\Psi(t, y)$ to spend in the vicinity of R_j is proportional to the total time elapsed up to that stage t (see the detailed estimation in [4]).

Define $t_1^1 = \inf\{t \geq 0,\ \Psi(t,y) \in B_\epsilon(C)\}$, $t_2^1 = \inf\{t \geq t_1^1,\ \Psi(t,y) \notin B_\epsilon(C)\}$, $t_1^n = \inf\{t \geq t_2^{n-1},\ \Psi(t,y) \in B_\epsilon(C)\}$, $t_2^n = \inf\{t \geq t_1^n,\ \Psi(t,y) \notin B_\epsilon(C)\}$, for $n \geq 2$. Denote $T_2 := \{t \geq 0 :\ \Psi(t,y) \in B_\epsilon(C)\}$. Then

$$T_2 = \cup_{n=1}^{\infty}[t_1^n, t_2^n].$$

By the above discussion, we have

$$\lim_{T\to\infty} \frac{L(T_2 \cap [0,T])}{L([0,T])} = 0, \tag{99}$$

where L denotes the Lebesgue measure on $\mathbb{R}$. Define

$$T_2^S(\omega) := \{t \geq 0 :\ \int_0^t g(s,\omega,g_0)ds \in T_2\} = \{t \geq 0 :\ \Psi(\int_0^t g(s,\omega,g_0)ds, y) \in B_\epsilon(C)\}.$$

Since $\int_0^t g(s,\omega,g_0)ds$ is monotonously increasing,

$$T_2^S(\omega) = \cup_{n=1}^{\infty}[t_1^{S,n}(\omega), t_2^{S,n}(\omega)]$$

where $t_i^{S,n}(\omega) := \tau(\omega, g_0, t_i^n)$ is given in (71) and hence $\int_0^{t_i^{S,n}(\omega)} g(s,\omega,g_0)ds = t_i^n, \quad i = 1,2.$ It is easy to see that $t_i^n, t_i^{S,n} \to \infty$ as $n \to \infty$.

For any $\delta > 0$, set $T_g^\delta(\omega) = \{t \geq 0 :\ g(t,\omega,g_0) \in (0,\delta]\}$, by the ergodic property (35) of g, we have

$$\lim_{T\to\infty} \mathbb{E}\frac{L(T_g^\delta(\omega) \cap [0,T])}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}(I_{(0,\delta]}(g(s,\omega,g_0)))ds = \mu_g^\sigma((0,\delta]) \tag{100}$$

where μ_g^σ is the nontrivial stationary measure of g.

We have

$$\begin{aligned}
& L([t_1^{S,n}(\omega), t_2^{S,n}(\omega)]) \\
= \ & L([t_1^{S,n}(\omega), t_2^{S,n}(\omega)] \cap T_g^\delta(\omega)) + L([t_1^{S,n}(\omega), t_2^{S,n}(\omega)] \cap (T_g^\delta(\omega))^c) \\
\leq \ & L([t_1^{S,n}(\omega), t_2^{S,n}(\omega)] \cap T_g^\delta(\omega)) + (t_2^n - t_1^n)/\delta.
\end{aligned} \tag{101}$$

For any fixed $T > 0$, let $N := \max\{n : t_2^{S,n} \leq T\}$. Then

$$T_2^S(\omega) \cap [0,T] = \cup_{n=1}^N [t_1^{S,n}, t_2^{S,n}] \cup [t_1^{S,N+1}, t_2^{S,N+1} \wedge T].$$

Applying (101), we have

$$\begin{aligned}
& L(T_2^S(\omega) \cap [0,T]) \\
= \ & \textstyle\sum_{n=1}^N L([t_1^{S,n}, t_2^{S,n}]) + L([t_1^{S,N+1}, t_2^{S,N+1} \wedge T]) \\
\leq \ & \textstyle\sum_{n=1}^N L([t_1^{S,n}, t_2^{S,n}] \cap T_g^\delta(\omega)) + L([t_1^{S,N+1}, t_2^{S,N+1} \wedge T] \cap T_g^\delta(\omega)) \\
+ \ & \textstyle\frac{1}{\delta}[\sum_{n=1}^N (t_2^n - t_1^n) + \max\{0, \int_0^T g(s,\omega,g_0)ds - t_1^{N+1}\}] \\
\leq \ & L([0,T] \cap T_g^\delta(\omega)) + \textstyle\frac{1}{\delta} L([0, \int_0^T g(s,\omega,g_0)ds] \cap T_2).
\end{aligned}$$

For any $T_0 > 0$, denote

$$\Omega_{T_0} = \{\omega \in \Omega : \sup_{t\in[T_0,\infty)} |\frac{1}{t}\int_0^t g(s,\omega,g_0)ds - 1| \le 1\}.$$

Ω_{T_0} is increasing with respect to T_0. By (37),

$$\lim_{T_0\to\infty} \mathbb{P}(\Omega_{T_0}) = 1. \tag{102}$$

For any $\omega \in \Omega_{T_0}$, $T \ge T_0$,

$$\int_0^T g(s,\omega,g_0)ds \le 2T.$$

By the above estimations, for any $\omega \in \Omega_{T_0}$, $T \ge T_0$, we have

$$\begin{aligned}
& \frac{1}{T}\int_0^T I_{B_\epsilon(C)}\Big(\Psi(\int_0^t g(s,\omega,g_0)ds, y)\Big)dt \\
= \ & \frac{L(T_2^S(\omega)\cap[0,T])}{T} \\
\le \ & \frac{L([0,T]\cap T_g^\delta(\omega))}{T} + \frac{L([0,\int_0^T g(s,\omega,g_0)ds]\cap T_2)}{\delta T} \\
\le \ & \frac{L([0,T]\cap T_g^\delta(\omega))}{T} + \frac{L([0,2T]\cap T_2)}{\delta T}.
\end{aligned}$$

Then

$$\begin{aligned}
& \mathbb{E}\Big(\frac{1}{T}\int_0^T I_{B_\epsilon(C)}\Big(\Psi(\int_0^t g(s,\omega,g_0)ds, y)\Big)dt\Big) \\
= \ & \mathbb{E}\Big(\frac{L(T_2^S(\omega)\cap[0,T])}{T}\Big) \\
\le \ & \mathbb{P}\Big((\Omega_{T_0})^c\Big) + \mathbb{E}\Big(\frac{L([0,T]\cap T_g^\delta(\omega))}{T}\Big) + \frac{L([0,2T]\cap T_2)}{\delta T}.
\end{aligned}$$

Combining this with (99), (100) and (102), we get that

$$\limsup_{T\to\infty} \mathbb{E}\Big(\frac{1}{T}\int_0^T I_{B_\epsilon(C)}\Big(\Psi(\int_0^t g(s,\omega,g_0)ds, y)\Big)dt\Big) \le \mu_g^\sigma((0,\delta]), \tag{103}$$

which implies that

$$\lim_{T\to\infty} \mathbb{E}\Big(\frac{1}{T}\int_0^T I_{B_\epsilon(C)}\Big(\Psi(\int_0^t g(s,\omega,g_0)ds, y)\Big)dt\Big) = 0.$$

We obtain that $\nu_y^\sigma(\Lambda(B_\epsilon(C))) = 0$, which implies that ν_y^σ charges no on the interior of nonnegative $(y_1, y_2)-$plane. Similarly, ν_y^σ charges no on the interiors of other two nonnegative planes. This proves that $\mathrm{supp}(\nu_y^\sigma) = \cup_{j=1}^3 \mathbb{R}_j^+$.

It follows that $\Phi(t, \omega, y) \in \mathbb{R}_+^3 \setminus \mathbb{R}_j^+$ if $y \in \mathbb{R}_+^3 \setminus \mathbb{R}_j^+$ for any $t \geq 0$ and $\omega \in \Omega$. Therefore, for any $t > 0$, $y \in \mathbb{R}_+^3 \setminus \mathbb{R}_j^+$ and $I \in \mathcal{B}(\mathbb{R}_j^+)$, $j = 1, 2, 3$,

$$P(t, y, I) = 0, \text{ for any } y \in \mathbb{R}_+^3 \setminus \mathbb{R}_j^+ \text{ and } I \in \mathcal{B}(\mathbb{R}_j^+).$$

By the definition of stationary measure, for any $I \in \mathcal{B}(\mathbb{R}_j^+)$ and $t > 0$, we have

$$\nu_y^\sigma(I) = \int_{\mathbb{R}_+^3} \nu_y^\sigma(dz) P(t, z, I) = \int_{\mathbb{R}_j^+} \nu_y^\sigma(dz) P(t, z, I).$$

This shows that the restriction of ν_y^σ on $\mathbb{R}_j^+$ is stationary. However, μ_g^σ is the unique nontrivial probability stationary measure on $\mathbb{R}_j^+$, we get that $\nu_y^\sigma(I) = \nu_y^\sigma(\mathbb{R}_j^+)\mu_g^\sigma(I)$. This proves (96) with $\lambda_j = \nu_y^\sigma(\mathbb{R}_j^+)$, $j = 1, 2, 3$.

Suppose that σ^i, y_0^i and μ^i satisfy the conditions in the theorem. Then $\mu^i(\cup_{j=1}^3 \mathbb{R}_j^+) = 1$. Applying Theorem 2.2(iii), we derive that

$$\mu(\cup_{j=1}^3 \mathbb{R}_j^+) \geq \limsup_{i \to \infty} \mu^i(\cup_{j=1}^3 \mathbb{R}_j^+) = 1.$$

All recurrent points of Ψ on $\cup_{j=1}^3 \mathbb{R}_j^+$ are $\{O, R_1, R_2, R_3\}$. By Corollary 6.2, $\mu(\{O\}) = 0$. So we conclude that $\mu(\{R_1, R_2, R_3\}) = 1$ by Corollary 6.2. The proof is complete. $\square$

8. Conclusions and Discussion

Heikes and Busse [3] found that the Rayleigh number of (1) is affected by noise. This is the reason why we have established the random system (6). Using the same calculating procedure as Busse and his collaborators did in [1, 2, 3], we get the stochastic Lotka-Volterra system (7) representing $n-$modes. This paper provides a stochastic decomposition formula. That is, every solution process of the stochastic Lotka-Volterra system with identical intrinsic growth rate is expressed in terms of a solution of the corresponding deterministic Lotka-Volterra system without noise perturbation multiplied by an appropriate solution process of the scalar logistic equation with the same type of noise perturbation. Using this decomposition formula, we have shown that every pull-back omega limit set of the stochastic Lotka-Volterra system is an omega limit set of the corresponding deterministic Lotka-Volterra system multiplied by the random equilibrium of the scalar stochastic logistic equation with the same type of noise. This illustrates the dynamics of a trajectory of the deterministic Lotka-Volterra system is preserved if the identical intrinsic growth rate is perturbed by a white noise. Employing the stochastic decomposition formula, the Khasminskii theorem and the Portmanteau theorem, it is shown that a bounded orbit of the deterministic Lotka-Volterra system implies the existence of a stationary measure of the stochastic Lotka-Volterra system supported in a lower dimensional cone which consists of all rays connecting the origin and all points in the omega limit set of this orbit. In particular, an equilibrium Q of the deterministic Lotka-Volterra system produces a strongly mixing stationary measure μ_Q^σ of the stochastic Lotka-Volterra system supported in a ray connecting the origin and the equilibrium Q, which has a continuous distribution function and weakly converges to the Dirac measure at Q as σ vanishes by the Weierstrass theorem. In addition, a trajectory $\Psi(t, y)$ converging to Q is equivalent to the pull-back trajectory through y converging to the random equilibrium $u(\omega)Q$. A closed orbit $\Psi(t, y)$ of the deterministic Lotka-Volterra system deduces the existence of a stationary measure ν_y^σ of the stochastic Lotka-Volterra system supported in a two dimensional cone surface with the origin as the vertex determined by this closed orbit, which weakly converges to the Haar measure on the closed orbit as σ vanishes. The solutions of the stochastic Lotka-Volterra system are invariant with respect to the cone surface. Therefore, none of stationary measures are regular. This paper reveals the strong connection between the dynamics of the deterministic Lotka-Volterra system and the long-run behavior of the stochastic Lotka-Volterra system. This allows us to construct many examples that possess a continuum of strongly mixing stationary measures or multiple isolated strongly mixing stationary measures or even others.

Suppose that the deterministic Lotka-Volterra system (E_0) is dissipative. Then we prove that the set of all stationary measures is tight, and that their limiting measures in weak topology are invariant with respect to the flow of (E_0) as the noise intensity σ tends to zero, whose supports are contained in the Birkhoff center of (E_0). This means that on the global attractor of (E_0) any limiting measure charges no on the complement of the Birkhoff center. In the case that (E_0) is competitive, the global attractor is the compact invariant set surrounded by the carrying simplex Σ and the boundary of $\mathbb{R}_+^n$. However, the Birkhoff center consists of all recurrent points in the carrying simplex and the origin. This means that our result on the support of limiting measures is much more precise than that of Huang et al .[13].

Finally, we provide a complete classification of dynamics for three dimensional competitive system (E_σ) both in pull-back trajectory and in stationary motion. There are exactly 37 dynamic scenarios in terms of competitive coefficients. Among them, each pull-back trajectory in 34 classes (classes 1-25, 26 a), 26 c), 27 a) and 28-33 in Appendix A) is asymptotically stationary, but there are possibly different stationary solutions for different trajectories in the same class. For any given system in these 34 classes, all its stationary measures are the convex combinations of $\{\mu_Q^\sigma : Q \in \mathcal{E}\}$, each of which is strongly mixing. As $\sigma \to 0$, all their limiting measures are the convex combinations of the Dirac measures $\{\delta_Q(\cdot) : Q \in \mathcal{E}\}$. Two of the remaining classes (classes 26 b) and 27 b)) possess a family of stochastic closed orbits, and there exists a continuum of invariant cone surfaces $\Lambda(h)$ determined by the origin and the closed orbits of the corresponding deterministic Lotka-Volterra system. For each $\Lambda(h)$, the system admits a unique nontrivial strongly mixing stationary measure ν_h^σ supported on it, which weakly converges to the Haar measure of the periodic orbit as the noise intensity tends to zero. In addition, any limiting measure for a sequence of stationary measures $\nu_{y_0^i}^{\sigma^i}$, $i = 1, 2, \cdots$, satisfying $\sigma^i \to 0$ and $y_0^i \to \mathcal{H}$ with $y_0^i \in \mathrm{Int}\mathbb{R}_+^3$, will be supported at the three equilibria on $\mathcal{H}$.

In order to explain that the final class, the class 27 c), corresponds to turbulence, we introduce the concept of turbulence proposed by Busse et al. [1, 2, 3]. In the introduction of [3], they defined turbulence *"as a manifold of stationary solutions, all of which are unstable to some other solution in the manifold, such that the realization of the system moves constantly from the neighborhood of one solution to that of another."* Here stationary solutions are equilibria of some $n-$modes system. For the system (1), Busse and his collaborators calculated up to $n = 60$ modes and found that the essential features can actually be seen in the May-Leonard system (5) representing three modes. If both the Rayleigh number and the Taylor number exceed their critical values, then $\alpha + \beta > 2$ and $\alpha < 1$. In this case, the five equilibria of (5) are all unstable, leading to the behavior exhibited in Figure 5. Figure 5 (a) shows that trajectories cyclically fluctuate around the heteroclinic cycle $\mathcal{H}$, which spanns the carrying simplex Σ, and Figure 5 (b) illustrates quantitative evolution of solutions of (5) taken from [1, 2, 3]. Figure 5 (b) shows y_1, starting at some arbitrary amplitude much larger than y_2 and y_3, is unstable to y_3 and is eventually replaced by y_3. However, y_3 is unstable to y_2, y_2 grows to replace y_3. Using the concept of turbulence formulated by Heikes and Busse [3], they concluded that turbulence occurs when both the Rayleigh number and the Taylor number exceed their critical values, where the manifold in the definition is the carrying simplex Σ.

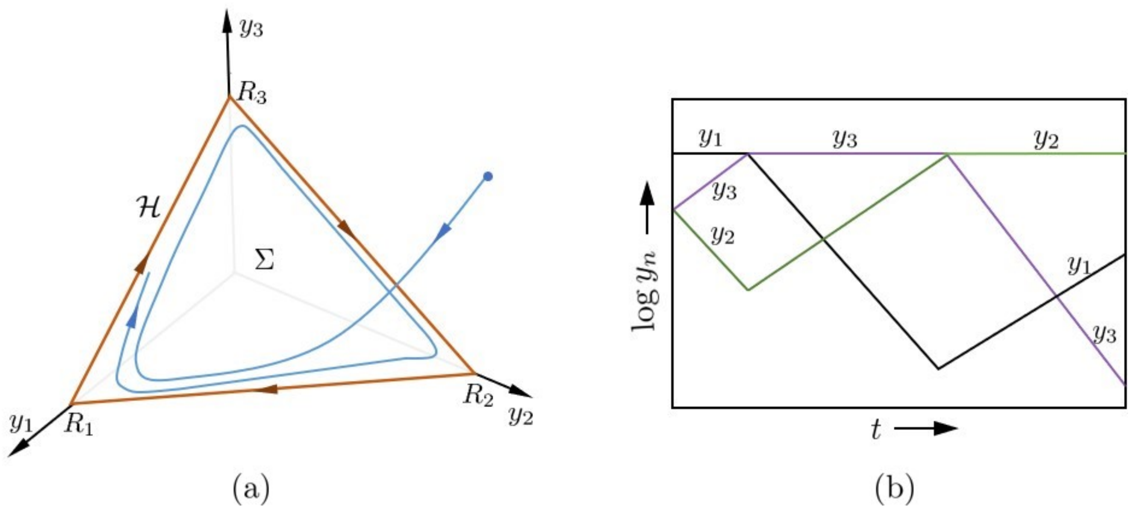

Figure 5: (a) heteroclinic cycle of (5); (b) cyclical fluctuating solution of (5).

The class 27 c) includes the stochastic May-Leonard system representing three modes of (6) with $\alpha+\beta > 2$ and $\alpha < 1$. Theorem 7.5 illustrates that almost every pull-back trajectory cyclically oscillates around the boundary of the stochastic carrying simplex $u(\omega)\Sigma$, which is spanned by three unstable stationary solutions $u(\theta_t\omega)R_i,\ i = 1, 2, 3$ in class 27 c). The actual realized state wanders from a neighborhood of one of the stationary solutions to that of the next. These are subject to the characteristics of Busse et al's turbulence definition. Besides, Theorem 7.11 illustrates that the weak limit measures of the expected occupation measures of a solution are neither unique nor ergodic and that they are convex combinations of three axial stationary measures. These reflect that stochastic May-Leonard system also has the characteristics of Busse et al's turbulence definition from the viewpoint of time-averaging. We will reveal in Appendix B that the essential reason for these peculiar characteristics is that pull-back trajectories stay close to $u(\omega)R_1, u(\omega)R_2, u(\omega)R_3$ for a very long time (approximately infinite) with a probability nearly one. This proves that the turbulence in a fluid layer heated from below and rotating about a vertical axis is robust under stochastic disturbances.

The stochastic decomposition formula plays an important role in completely classfying long-run behavior. This technique is still valid for other systems, e.g., Lotka-Volterra reaction-diffusion systems (see [38]), Gompertzian model and stochastic systems when nonlinear terms of drift in (E_σ) are replaced by homogeneous functions of degree $k > 1$. We leave these for future consideration. If the degenerate noise in (80) is replaced by mutually independent white noises, then many researchers have given sufficient conditions to guarantee that there is a unique strongly mixing stationary measure supported in the interior of the orthant (see [39, 40, 41]). Usually, these results are only used to those systems whose corresponding deterministic systems (81) are permanent. Thus, the existing results are only valid for classes 27 a), 29, 31, and 33. However, for degenerate noise system (80), our results are complete on stationary measures, we have obtained all strongly mixing stationary measures combining all stationary measures.

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9. Appendix A. The Complete Dynamical Classification for both Autonomous and Stochastic Three Dimensional Competitive LV Systems with Identical Intrinsic Growth Rate on the Carrying Simplex

Table 1. Description How to Understand the Dynamics on the Carrying Simplex.

Autonomous Case: The total of 37 dynamical classes among the 33 stable nullcline equivalence classes for (81), where the parameters a_{ij} and Σ are given by a representative system of that class. The notation $\bullet$ and $\circ$ denote an attractor and a repeller on Σ, respectively, while a saddle on Σ is the intersection of its stable and unstable manifolds (We refer to [20]).
Stochastic Perturbation Case: The carrying simplex Σ in autonomous case is replaced by the fiber $u(\omega)\Sigma$ ($\omega \in \Omega$); an equilibrium Q, a closed orbit Γ and a heteroclinic cycle $\mathcal{H}$ are understood as $u(\omega)Q$, $u(\omega)\Gamma$ and $u(\omega)\mathcal{H}$, respectively. All trajectories are understood pull-back ones.

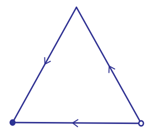
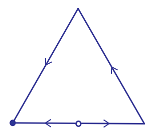
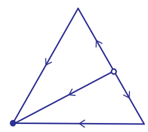
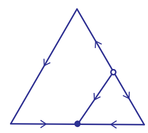
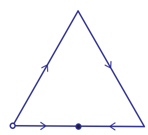
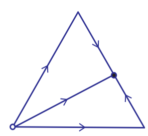
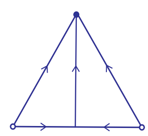
Class	The Corresponding Parameters	Phase Portrait in Σ
1	$a_{11} < a_{21}, a_{11} < a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$	
2	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
3	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} > a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$	
4	(i) $a_{11} > a_{21}, a_{11} < a_{31}, a_{22} > a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
5	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} < a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
6	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$	
7	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} < a_{23}$ (ii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	

Table 1: (continued)

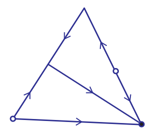
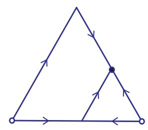
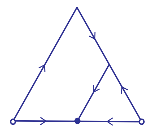
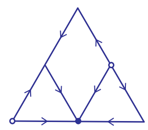
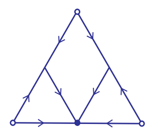
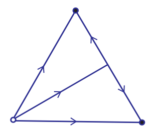
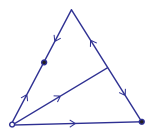
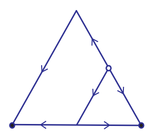
Class	The Corresponding Parameters	Phase Portrait in Σ
8	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$	
9	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	
10	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
11	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
12	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
13	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} < a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$	
14	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) > 0$	
15	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	

Table 1: (continued)

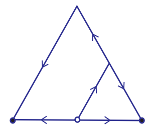
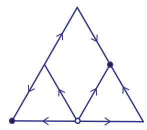
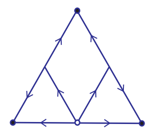
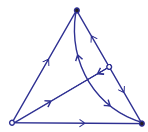
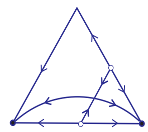
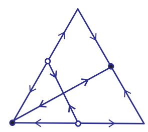
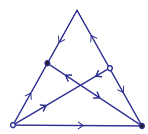
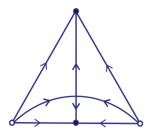
Class	The Corresponding Parameters	Phase Portrait in Σ
16	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
17	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
18	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} < a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
19	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} < a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$	
20	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
21	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) > 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
22	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) > 0$	
23	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} < a_{23}$ (ii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	

Table 1: (continued)

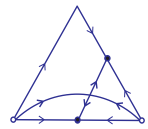
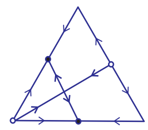
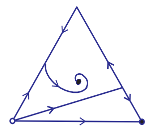
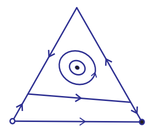
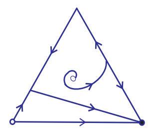
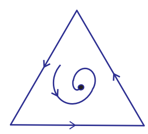
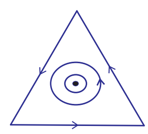
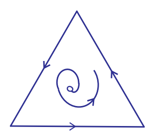
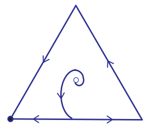
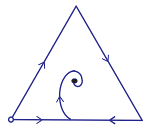
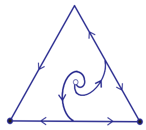
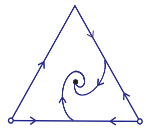
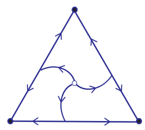
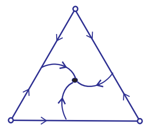
Class	The Corresponding Parameters	Phase Portrait in Σ
24	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
25	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) > 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) > 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) > 0$	
26 a)	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (a) $\theta < 0$	
26 b)	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (b) $\theta = 0$	
26 c)	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (c) $\theta > 0$	
27 a)	(i) $a_{11} > a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (a) $\theta < 0$	
27 b)	(i) $a_{11} > a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (b) $\theta = 0$	
27 c)	(i) $a_{11} > a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (c) $\theta > 0$	

Table 1: (continued)

Class	The Corresponding Parameters	Phase Portrait in Σ
28	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	
29	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} < a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	
30	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} > a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	
31	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} < a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	
32	(i) $a_{11} < a_{21}, a_{11} < a_{31}, a_{22} < a_{12}, a_{22} < a_{32}, a_{33} < a_{13}, a_{33} < a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	
33	(i) $a_{11} > a_{21}, a_{11} > a_{31}, a_{22} > a_{12}, a_{22} > a_{32}, a_{33} > a_{13}, a_{33} > a_{23}$ (ii) $a_{12}(a_{33} - a_{23}) + a_{13}(a_{22} - a_{32}) - (a_{22}a_{33} - a_{23}a_{32}) < 0$ (iii) $a_{21}(a_{33} - a_{13}) + a_{23}(a_{11} - a_{31}) - (a_{11}a_{33} - a_{13}a_{31}) < 0$ (iv) $a_{31}(a_{22} - a_{12}) + a_{32}(a_{11} - a_{21}) - (a_{11}a_{22} - a_{12}a_{21}) < 0$	

10. Appendix B. Turbulent Characteristics: Nonuniqueness and Nonergodicity in Limit for the Expected Occupation Measures

The expected occupation measures of each solution weakly converge to a strongly mixing stationary measure for (80) in all classes except class 27 c). But class 27 c) is quite different. If $y \in \text{Int}\mathbb{R}^3_+ \backslash L(P)$, the corresponding family of the expected occupation measures has infinite weak limit points, which are not ergodic. We will reveal that the essential reason for both peculiar characteristics is that the solutions stay close to R_1, R_2, R_3 very long time (approximately infinite) with probability nearly one.

Theorem 6.4 tells us that for any $y \in \mathbb{R}^3_+ \setminus \{O\}$, there is a point $z \in \Sigma$ such that $\mathcal{I}^\sigma_y = \mathcal{I}^\sigma_z$. So we fix $y \in \Sigma$ with $y \neq P$. In order to prove these by specific estimations, we will consider the May-Leonard system (5) with $\alpha = 0.8$ and $\beta = 1.3$.

Firstly, we will prove that limit point of the family $\{\frac{1}{T}\int_0^T \delta_{\Psi(t,y)}(\cdot)dt\}_{T>0}$ as $T \to \infty$, which is the invariant measure for deterministic system (5), is not unique. That is, the weak limit of

$$\frac{1}{T}\int_0^T \delta_{\Psi(t,y)}(\cdot)dt \xrightarrow{w} \mu(\cdot) \tag{104}$$

is not unique.

Let

$$A_i = \{y = (y_1, y_2, y_3) \in \Sigma : \|y - R_i\| < \frac{1}{2}\}$$

denote the neighborhood of R_i $(i = 1, 2, 3)$ on Σ. Then $\Psi(t, y)$ will be spirally asymptotic to $\mathcal{H}$ as the time goes to infinity. Hence, $\Psi(t, y)$ will enter and then depart A_i with infinite times.

For $n \geq 2$, define

$$\begin{aligned}
T^1_{\text{in}} &= \inf\{t \geq 0,\ \Psi(t,y) \in A_1\}, & T^1_{\text{out}} &= \inf\{t \geq T^1_{\text{in}},\ \Psi(t,y) \notin A_1\},\\
T^n_{\text{in}} &= \inf\{t \geq T^{n-1}_{\text{out}},\ \Psi(t,y) \in A_1\}, & T^n_{\text{out}} &= \inf\{t \geq T^n_{\text{in}},\ \Psi(t,y) \notin A_1\},\\
S^1_{\text{in}} &= \inf\{t \geq T^1_{\text{out}},\ \Psi(t,y) \in A_2\}, & S^1_{\text{out}} &= \inf\{t \geq S^1_{\text{in}},\ \Psi(t,y) \notin A_2\},\\
S^n_{\text{in}} &= \inf\{t \geq S^{n-1}_{\text{out}},\ \Psi(t,y) \in A_2\}, & S^n_{\text{out}} &= \inf\{t \geq S^n_{\text{in}},\ \Psi(t,y) \notin A_2\}.
\end{aligned}$$

Similarly, we denote by τ^n_{in} and τ^n_{out} the time entering and exiting A_3 in n-th spiral cycle (see Fig.6). By the continuity of Ψ, $\tau^n_{\text{in}} - T^n_{\text{out}}$, $S^n_{\text{in}} - \tau^n_{\text{out}}$ and $T^{n+1}_{\text{in}} - S^n_{\text{out}}$ are approximately constants independent of n. May and Leonard [4] showed that the time spent in the neighborhood of R_i is proportional to the total time elapsed up to that stage t. Imitating their estimates, we give the following:

$$T^n_{\text{out}} - T^n_{\text{in}} \simeq 0.42T^n_{\text{out}},\ \tau^n_{\text{out}} - \tau^n_{\text{in}} \simeq 0.42\tau^n_{\text{out}},\ S^n_{\text{out}} - S^n_{\text{in}} \simeq 0.42S^n_{\text{out}}. \tag{105}$$

Choosing two subsequences $\{T^n_{\text{out}}\}$ and $\{S^n_{\text{out}}\}$, therefore for sufficiently large n, we have

$$\frac{1}{T^n_{\text{out}}}\int_0^{T^n_{\text{out}}} \delta_{\Psi(t,y)}(A_1)dt = \frac{1}{T^n_{\text{out}}}\sum_{i=1}^n (T^i_{\text{out}} - T^i_{\text{in}}) \geq \frac{T^n_{\text{out}} - T^n_{\text{in}}}{T^n_{\text{out}}} = 0.42 > 0, \tag{106}$$

$$\frac{1}{S^n_{\text{out}}}\int_0^{S^n_{\text{out}}} \delta_{\Psi(t,y)}(A_1)dt = \frac{1}{S^n_{\text{out}}}\sum_{i=1}^n (T^i_{\text{out}} - T^i_{\text{in}}) \leq \frac{T^n_{\text{out}}}{S^n_{\text{out}}} \leq (0.58)^2 \leq 0.34. \tag{107}$$

Here we have used the property that $S^n_{\text{in}} - T^n_{\text{out}} \simeq 0.42S^n_{\text{in}}$, which holds from (105) and the continuity of Ψ. From (106), (107) and Proposition 6.3, it is easily to show that the limit of (104) is not unique.

Subsequently, we consider stochastic May-Leonard system with $\alpha = 0.8$ and $\beta = 1.3$. We will analyze the weak limit of

$$\Big\{Q(T, A_1) := \frac{1}{T}\int_0^T I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds, y)\Big)dt\Big\}_{T>0}, \quad \text{as } T \to \infty.$$

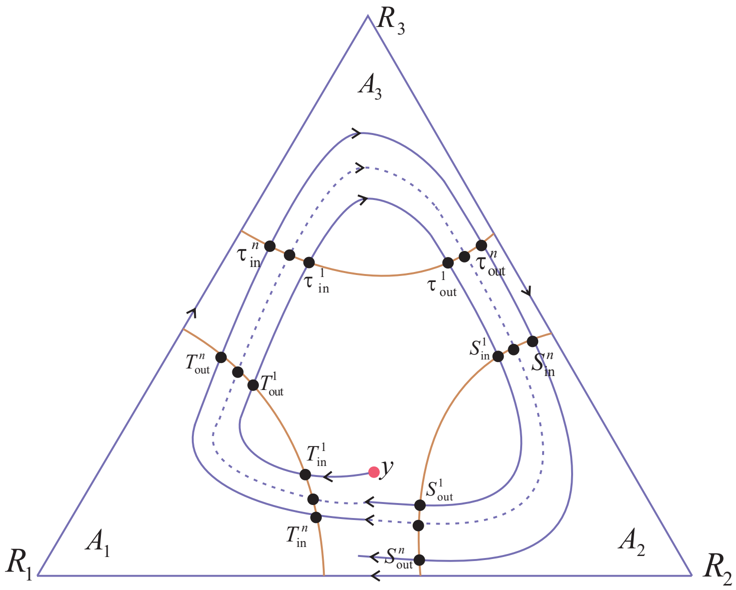


Figure 6

Let $\epsilon = 0.0001$ and $\Omega^{\epsilon}_T := \{\omega : \sup_{t\in[T,\infty)} |\frac{1}{t}\int_0^t g(s,\omega,1)ds - 1| \leq \epsilon\}$. Then (37) implies that $\Omega^{\epsilon}_T \uparrow$ with respect to T and $\lim_{T\to\infty}\mathbb{P}(\Omega^{\epsilon}_T) = 1$. Thus for $\eta = 0.9999$, there exists $T_0 > 0$ such that

$$\mathbb{P}(\Omega^{\epsilon}_T) \geq \eta, \quad \forall T \geq T_0.$$

Define $t^n_1(\omega) := \tau(\omega, 1, T^n_{\text{in}})$ and $t^n_2(\omega) := \tau(\omega, 1, T^n_{\text{out}})$ as given in (71). Set $\Omega^n_{T_0} := \{\omega : t^n_1(\omega) \geq T_0\}$. Then $\Omega^n_{T_0} \uparrow$ with respect to n and $\lim_{n\to\infty}\mathbb{P}(\Omega^n_{T_0}) = 1$. Thus there exists an N_0 such that

$$\mathbb{P}(\Omega^n_{T_0}) \geq \eta, \quad \forall n \geq N_0.$$

Step 1. Set $T_n = T^n_{\text{out}}$. Then we will analyze $Q(T_n, A_1)$.

For any n satisfying $n \geq N_0$ and $T_n \geq T_0$, choosing any $\omega \in \Omega^n_{T_0} \cap \Omega^{\epsilon}_{T_0}$, we have the following:

- $(1-\epsilon)T^n_{\text{out}} = (1-\epsilon)T_n \leq \int_0^{T_n} g(s,\omega,1)ds \leq (1+\epsilon)T_n = (1+\epsilon)T^n_{\text{out}}$,
- $t^n_2(\omega) \geq t^n_1(\omega) \geq T_0$,
- $(1-\epsilon)t^n_1(\omega) \leq \int_0^{t^n_1(\omega)} g(s,\omega,1)ds = T^n_{\text{in}} \leq (1+\epsilon)t^n_1(\omega)$,
- $(1-\epsilon)t^n_2(\omega) \leq \int_0^{t^n_2(\omega)} g(s,\omega,1)ds = T^n_{\text{out}} \leq (1+\epsilon)t^n_2(\omega)$.

Combining the fact that $T^n_{\text{out}} - T^n_{\text{in}} \simeq 0.42T^n_{\text{out}}$, we have

$$t^n_1(\omega) < T^n_{\text{out}} = T_n, \quad t^n_2(\omega) \geq \frac{T^n_{\text{out}}}{1+\epsilon} = \frac{T_n}{1+\epsilon}, \quad \frac{T^n_{\text{in}}}{1-\epsilon} \geq t^n_1(\omega).$$

Hence,

$$Q(T_n, A_1) \geq \frac{t_2^n(\omega) \bigwedge T_n - t_1^n(\omega)}{T_n} \geq \frac{\frac{T_{\text{out}}^n}{1+\epsilon} - \frac{T_{\text{in}}^n}{1-\epsilon}}{T_{\text{out}}^n} \geq 0.419.$$

Then

$$\liminf_{n\to\infty} Q(T_n, A_1) \geq 0.419\mathbb{P}(\Omega_{T_0}^{N_0} \cap \Omega_{T_0}^{\epsilon}) \geq 0.419 \times 0.9998 \geq 0.41. \tag{108}$$

Step 2. Let $S_n = S_{\text{out}}^n$. Then we will analyze $Q(S_n, A_1)$.

For any n satisfying $n \geq N_0$ and $S_n \geq T_0$, choosing any $\omega \in \Omega_{T_0}^n \cap \Omega_{T_0}^{\epsilon}$, we have the following:

- $(1-\epsilon)S_{\text{out}}^n = (1-\epsilon)S_n \leq \int_0^{S_n} g(s,\omega,1)ds \leq (1+\epsilon)S_n = (1+\epsilon)S_{\text{out}}^n$,
- $t_2^{n+1}(\omega) \geq t_1^{n+1}(\omega) \geq t_2^n(\omega) \geq t_1^n(\omega) \geq T_0$,
- $(1-\epsilon)t_1^i(\omega) \leq \int_0^{t_1^i(\omega)} g(s,\omega,1)ds = T_{\text{in}}^i \leq (1+\epsilon)t_1^i(\omega), \quad i = n, n+1$,
- $(1-\epsilon)t_2^i(\omega) \leq \int_0^{t_2^i(\omega)} g(s,\omega,1)ds = T_{\text{out}}^i \leq (1+\epsilon)t_2^i(\omega), \quad i = n, n+1$,
- $T_{\text{in}}^{n+1} \simeq S_{\text{out}}^n$, $S_{\text{out}}^n - S_{\text{in}}^n \simeq 0.42S_{\text{out}}^n$, $S_{\text{in}}^n - T_{\text{out}}^n \simeq 0.42S_{\text{in}}^n$.

Hence,

- $(1-\epsilon)t_2^n(\omega) \leq T_{\text{out}}^n \simeq 0.58S_{\text{in}}^n \simeq 0.58^2 S_{\text{out}}^n = 0.58^2 S_n \Rightarrow t_2^n(\omega) \leq S_n$,
- $$\begin{aligned} & (1-\epsilon)t_1^{n+1}(\omega) \leq T_{\text{in}}^{n+1} \simeq S_{\text{out}}^n = S_n \\ \leq \; & (1+\epsilon)t_1^{n+1}(\omega) \leq \frac{1+\epsilon}{1-\epsilon}T_{\text{in}}^{n+1} \simeq 0.58\frac{1+\epsilon}{1-\epsilon}T_{\text{out}}^{n+1} \\ \leq \; & 0.58\frac{(1+\epsilon)^2}{1-\epsilon}t_2^{n+1}(\omega) < t_2^{n+1}(\omega), \end{aligned}$$

  that is,
  $$(1-\epsilon)t_1^{n+1}(\omega) \leq S_n \leq (1+\epsilon)t_1^{n+1}(\omega) < t_2^{n+1}(\omega).$$

Hence

$$\begin{aligned} & Q(S_n, A_1) \\ = \; & \frac{1}{S_n}\sum_{i=1}^{\infty}\Big(t_2^i(\omega) \bigwedge S_n - t_1^i(\omega) \bigwedge S_n\Big) \\ = \; & \frac{1}{S_n}\Big[\sum_{i=1}^{n}\Big(t_2^i(\omega) - t_1^i(\omega)\Big) + \Big(S_n - t_1^{n+1}(\omega) \bigwedge S_n\Big)\Big] \\ \leq \; & \frac{t_2^n(\omega) + S_n - t_1^{n+1}(\omega) \bigwedge S_n}{S_n} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{S_n}\Big(\frac{0.58^2}{1-\epsilon}S_n + S_n - \frac{S_n}{1+\epsilon}\Big) \\
&= \frac{0.58^2}{1-\epsilon} + \frac{\epsilon}{1+\epsilon} < 0.34.
\end{aligned}
$$

Then

$$
\limsup_{n\to\infty} Q(S_n, A_1) \leq 0.34\mathbb{P}(\Omega_{T_0}^{N_0} \cap \Omega_{T_0}^{\epsilon}) + \mathbb{P}\Big[(\Omega_{T_0}^{N_0} \cap \Omega_{T_0}^{\epsilon})^c\Big] \leq 0.342. \tag{109}
$$

(108) and (109) imply that $\frac{1}{T}\int_0^T \mathbb{E}I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1),y)\Big)dt$ does not have unique limit as $T\to\infty$. Equivalently, using Theorem 2.2 (iii) and (iv) and Theorem 7.11, we obtain that $\frac{1}{T}\int_0^T \mathbb{E}I_{\Lambda(A_1)}\Big(\Phi(t,\omega,y)\Big)dt$ does not have unique limit as $T\to\infty$.

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